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OLIGOPOLY PRICE DISCRIMINATION:
THE ROLE OF INVENTORY CONTROLS

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Abstract

Inventory controls, used most notably by airlines, are sales limits assigned to individual prices. While typically viewed as a tool to manage demand uncertainty, we argue that inventory controls can also facilitate intertemporal price discrimination in oligopoly. In our model, competing firms first choose quantity and then choose prices in a series of advance-purchase markets. When demand becomes less elastic over time, as is the case in airline markets, a monopolist can easily price discriminate; however, we show that oligopoly firms generally cannot. We also show that using inventory controls allows oligopoly firms to set increasing prices, regardless of whether or not demand is uncertain.

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1 Introduction

In many markets, such as in the airline, hotel, sport and entertainment industries, firms have fixed capacity and compete on price. Seminal research by Kreps and Scheinkman (1983) and Davidson and Deneckere (1986) analyze sequential oligopoly games that reflect key features of these industries. These papers are important because they describe when the Cournot model is a valid prediction of the perhaps more realistic sequential quantity-price game. However, one limitation of these works is that they only consider one pricing period, yet, the most widely cited examples of sequential quantity-price games are ones in which consumers purchase in advance and firms adjust their prices over time. Airline markets in particular are known for sharp price increases in the final weeks before departure.[1] While this pattern of prices is consistent with theoretical models of demand uncertainty (Prescott 1975, Eden 1990, Dana 1999), recent empirical work on airline pricing finds that intertemporal price discrimination is the dominant reason for increasing prices (Puller, Sengupta, and Wiggins 2012).

We explore whether intertemporal price discrimination arises in an oligopoly model of sequential quantity-price games with multiple sales periods and a perishability date. We primarily focus on the case in which demand becomes more inelastic over time as it maps well to the case of airlines, and it is an environment in which a monopolist would clearly charge higher prices to late-arriving, price-insensitive consumers. Our main contribution is to show that strong competitive forces prevent oligopoly firms from utilizing intertemporal price discrimination. This is true even in a simple extension with uncertain demand.

The intuition behind our main result is that once quantity is fixed, firms have very strong incentives to raise their prices in early, more-elastic periods in order to shift low-price, early-period sales to their rivals and, as a consequence, to shift high-price, late-period

[1] A report produced for Expedia, Airlines Reporting Corporation (2015), suggests that airline fares are lowest 57 days before departure and increase dramatically within the last 21 days.

sales to themselves. This results in equilibrium prices that are flat over time even though later arrivals have higher willingness to pay. This pricing pattern is inconsistent with observed patterns in airlines. We reconcile this inconsistency by showing that if competing firms commit to the use of inventory controls, or limits on unit sales assigned to prices, they can profitably engage in intertemporal price discrimination. Such controls have been used by airlines for decades and studied extensively in the context of demand uncertainty (Talluri and Van Ryzin 2006, McGill and Van Ryzin 1999). We argue they can be useful for a different purpose: they can used to facilitate oligopoly price discrimination, resulting in higher profits, regardless as to whether or not there is uncertainty about demand.

In our baseline model, firms sell a homogeneous good and have no private information. We primarily focus on the case in which demand becomes more inelastic over time because this pattern is consistent with the empirical literature on airlines. In particular, Lazarev (2013) and Williams (2018) quantify the effects of intertemporal price discrimination in monopoly markets. At the deadline, no further sales can take place. We assume that there is a continuum of consumers who are each assigned to one of the sequential markets. This assumption can easily be relaxed when demand becomes more inelastic since consumers are small and will not wait if prices are increasing.

For tractability, we analyze a model with two advance-purchase sales periods, although we also discuss extending the analysis to any finite number of periods. The challenge in solving our game, and the games studied by Kreps and Scheinkman (1983) and Davidson and Deneckere (1986), is that quantity-constrained price competition generates mixed-strategy equilibrium. Solving our game is even more challenging because we consider more than one sales period. We make the analysis simpler and more intuitive by focusing on high costs of capacity; this focus eliminates mixed strategies on and off the equilibrium path.

Our main result is that there exist strong competitive forces that prevent intertemporal price discrimination. That is, under mild conditions, equilibrium prices are flat over time, even though consumers who arrive later have higher willingness to pay. No increasing price equilibrium exists because an individual firm has an incentive to raise price in the early period, shifting demand to its competitors, in order to sell more in the later period, when consumers are less price-sensitive and the equilibrium price is higher. Other firms have similar incentives, which results in uniform prices arising as the unique pure-strategy equilibrium outcome. We also show under similar, but less general conditions, equilibrium prices can be flat when the elasticity of demand is increasing over time; that is, early arrivals are less price sensitive. We show there exists an important asymmetry in what is required for deviations to be profitable under these two scenarios.

We extend the analysis by solving the model when unit-sales limits, or inventory controls, are used in conjunction with price setting. We show that setting inventory controls provides the commitment necessary for firms to increase prices over time as they limit firms' ability to shift demand to competitors in the early, less desirable, period. This result holds even in a simple extension of the model that incorporates demand uncertainty. Thus, while inventory controls have largely been studied in the context of demand uncertainty, we show that they also facilitate intertemporal price discrimination in oligopoly markets.

We also discuss a version of the model with product differentiation. When products are differentiated, prices are no longer uniform across time as firms benefit from the inability to shift all of the demand using very small price changes. However, the strategic incentives explored in this paper are still present. We show that products in a commonly used demand systems to study airlines—discrete choice logit demand of differentiated products—must be highly differentiated for prices to increase substantially across periods. For this reason, we postulate that inventory controls, pioneered by airlines, are particularly valuable in this context because products are typically close substitutes and inventory controls allow firms to earn higher profits by setting higher prices to their less price-sensitive customers.

1.1 Related Literature

This paper contributes to three strands of the economics literature. First, we analyze a model of price competition with capacity constraints (Levitan and Shubik 1972, Allen and Hellwig 1986, Osborne and Pitchik 1986, Klemperer and Meyer 1986). As in Kreps and Scheinkman (1983) and Davidson and Deneckere (1986), our firms choose capacity and then price, but unlike earlier research, we consider more than one pricing period. Our results are also related to Van den Berg, Bos, Herings, and Peters (2012), who consider a two-period quantity game with capacity constraints, with and without commitment. However, our main focus is on the way firms use prices to shift rivals sales from the higher-price period to the lower-price period, which does not happen in their sequential quantity game because quantity decisions do not affect the way their rivals' capacity is allocated across periods, as prices decisions do in our model.

Second, we analyze intertemporal price discrimination. Stokey (1979) is a seminal paper that shows that monopoly intertemporal price discrimination is not always feasible. Several more-recent papers find that price adjustments over time are profitable in environments with deadlines and limited capacity (Gallego and van Ryzin 1994, Su 2007, Board and Skrzypacz 2016, Dilme and Li 2017). This is particularly true when consumers learn their preferences over time, as in Akan, Ata, and Dana (2015) and Ata and Dana (2015).

Important empirical contributions to the literature on intertemporal price discrimination, and more generally price adjustments over time, include Nair (2007) on video games, Sweeting (2012) on stadium seats, and Hendel and Nevo (2013) on storable goods. Much of our focus in on prices that increase as a deadline draws closer (e.g., event or departure time), a pattern that has been found in several airline studies (Lazarev 2013, McAfee and te Velde 2006, Williams 2018). We argue that this pattern is unlikely to exist in competitive markets unless firms use inventory controls.

Finally, our work is related to the literature on inventory controls (see Littlewood (1972), Belobaba (1987), Belobaba (1989) and Weatherford and Bodily (1992), and surveys by Talluri and Van Ryzin (2006), McGill and Van Ryzin (1999) and Stole (2003)). While prior research views inventory controls as a tool for managing aggregate demand uncertainty, our paper shows that they also facilitate intertemporal price discrimination in oligopoly markets.

2 The Model

Consider an oligopoly with n firms selling a homogeneous good to a continuum of consumers in a series of advance-purchase sales markets. For simplicity, we consider just two selling periods, $t = 1, 2$. Even though firms may charge different prices, the market demand in each period is given by continuous functions of a single price, $D_1(p)$ and $D_2(p)$. Let $D_{Tot}(p) = D_1(p) + D_2(p)$ denote the total demand at a uniform price p, and let $p_1(q)$, $p_2(q)$ and $p_{Tot}(q)$ denote the inverses of $D_1(p)$, $D_2(p)$ and $D_{Tot}(p)$, respectively. The price elasticity of demand is given by $\eta_t(p) = D_t'(p)p/D_t(p)$. We assume that the revenue function, $p_t(q)q$, is concave in each period.

We analyze a three-stage game. In the first stage (stage zero), firms simultaneously choose their capacities, $K^i > 0, \forall i = 1, \ldots, n$. The cost per unit of capacity is $c \geq 0$ for all firms. In stages one and two, firms sell their capacity in two sequential advance-purchase sales periods. Sales in both of these periods are from a common capacity constraint (e.g., seats on the same flight). For simplicity, we assume that the marginal cost of each sale is zero (the cost of putting a passenger in an otherwise empty airline seat is zero). This is a game of complete information, so capacities, prices and sales are all observable in all periods.

We assume that consumers cannot delay their purchasing decisions but, instead, are exogenously assigned to purchase in either the first or second sales period. In our setting, this assumption can easily be relaxed by allowing consumers to learn their preferences over time and to choose when to purchase, as consumers will purchase as early as possible to avoid price increases.[2]

Products are homogeneous, so consumers purchase at the lowest price available, as long as their valuation exceeds the price. If the firms set different prices, then a firm with a higher price can have positive sales only after all of the firms with lower prices have sold all of their capacity. If two or more firms charge the same price, then we assume that firms divide the sales equally, subject to their capacity constraints. How much the firm with the higher price sells–that is, the firm's residual demand–depends on the rationing rule. The residual demand function is $RD_t\left(p; \mathbf{p}^{-i}, \mathbf{K}^{-i}\right)$, where the arguments are firm i's own price, p, and vectors of all of the other firms' prices and capacities. Our results hold for both the efficient rationing rule and the proportional rationing rule. Recall that the residual demand for the efficient rationing rule is

$$RD_t(p; \mathbf{p}^{-i}, \mathbf{q}^{-i}) = D_t(p) - \sum_{j \neq i; p^j < p} q^j, t = 1, 2, \tag{1}$$

and the residual demand for the proportional rationing rule is

$$RD_t(p; \mathbf{p}^{-i}, \mathbf{q}^{-i}) = D_t(p) \left[1 - \sum_{j \neq i; p^j < p} \frac{q^j}{D_t(p^j)} \right], t = 1, 2, \tag{2}$$

if no other firm charges p, and the demand is shared if any of firm i's rivals is charging p.

If there were just one pricing period, then we know from Kreps and Scheinkman (1983), who analyze efficient rationing, and Davidson and Deneckere (1986), who analyze proportional rationing, that the pricing subgame would have a unique Nash equilibrium. These papers characterize profits for all capacity levels and show that the price game has a mixed-strategy equilibrium when capacities are sufficiently large. Because we have two pricing periods, characterizing the equilibrium profits is considerably more challenging.

[2] In this case, some consumers prefer to purchase in the second period because they do not know their demand until the second period. Other consumers prefer to purchase in the first period, even with the option to wait, because they know their demands early and because they rationally anticipate that the firms' prices will be higher if they wait (see, for example, Dana (1998) and Akan, Ata, and Dana (2015)).

By focusing on large capacity costs, we avoid this challenge and ensure that the pricing game has pure-strategy equilibria. Similar results may hold for low capacity costs, but we do not solve the mixed-strategy equilibria needed to prove that the results generalize. The assumption that capacity costs are large seems particularly reasonable as more than 75% of airlines' costs do not vary with the number of passengers served, and the remainder (reservations and sales expense and passenger service) includes mainly labor costs that do not with the number of passengers served.[3]

Our assumption on capacity costs allows us to consider both efficient and proportional rationing. Recall that in both Kreps and Scheinkman (1983) and Davidson and Deneckere (1986), when capacity is small—for example, smaller than a monopolist's output if capacity were free—the price is equal to the market-clearing price and does not depend on the rationing rule. This is because marginal revenue is positive in the pricing stage even when firms collude, which implies that marginal revenue is positive for every firm. Thus, firms can never increase their profits by setting a price above the market clearing price.

To simplify our proofs, we make two additional assumptions. The first assumption is similar, but slightly stronger, then the assumption that guarantees market clearing in the final stage game in Kreps and Scheinkman (1983) and Davidson and Deneckere (1986). It is stronger because it guarantees that marginal revenue is positive in the second pricing period, even if firms choose not to sell any of their capacity in the first period.

Assumption 1. *The competitive output, $D_{Tot}(c)$, is smaller than the monopoly output, $q_2^m(0)$, produced when the monopolist has zero capacity costs.*

Assumption 1 is clearly satisfied if the cost of capacity c is sufficiently large. We also place a mild restriction on firm strategies. We assume that the firms' total equilibrium capacity does not exceed the capacity that would be produced if the market were perfectly competitive, $D_{Tot}(c)$.[4]

[3] Calculations based on DOT Form 41 filings and reported in ICAO, Airline Operating Costs and Productivity, February 2017, https://www.icao.int/.

[4] Note that Assumption 2 is a restriction on the entire vector of firms' strategies. Readers who find this

Assumption 2. *Firms' capacities are less than the capacities in a perfectly competitive market, or* $\sum_i K^i \leq D_{Tot}(c)$.

The interpretation of Assumption 2 is similar to that of an equilibrium refinement. That is, we characterize the unique equilibrium within this restricted strategy set and show that the equilibrium is unique within the set and that its is in the interior of the set, i.e., $K_i < D_{Tot}(c)/n, \forall i$, so $\sum_i K^i < D_{Tot}(c)$. While we do not formally show that every deviation outside this set is unprofitable, we do show that profits are zero and decreasing in capacity for all i on the border of the set (defined by $\sum_i K^i = D_{Tot}(c)$), so it is reasonable to think that deviations to strategy vectors outside this set are not profitable and that no equilibrium exists outside this strategy set.

In our model, it is reasonable to expect industry profits to be negative for any capacities not satisfying Assumption 2 because capacity is chosen simultaneously. In some dynamic games unused capacity might be an effective off-the-equilibrium-path instrument for punishment, but in our model firms do not value holding capacity that they will never utilize. If we could show that firms' profit functions were everywhere concave or quasi-concave in capacity, we would not need Assumption 2 and we could relax or drop Assumption 1. However, proving concavity everywhere is difficult because it is hard to characterize the firms' profits and price strategies when capacities are large. Instead, we prefer to impose Assumption 1 and 2; they imply that every on- or off-the-equilibrium-path pricing subgame has a pure-strategy equilibrium and make the proofs much easier.

Finally, for some of our analysis, we assume demand becomes less inelastic, or more elastic, over time:

Assumption 3. *Demand becomes more inelastic over time, so* $|\eta_2(p)| < |\eta_1(p)|$ *for all p.*

assumption unsatisfying, because firm i's strategy set depends on other firms' capacities, might be reassured by knowing that we can replace this assumption with the stronger assumption that each firm's capacity is less than $D_{Tot}(c)/n$ and obtain the same results.

3 Equilibrium Characterization

We now solve the full model as described in Section 2. We solve the three-stage game (capacity in stage 0, price in stage 1, and price in stage 2) by backwards induction. All proofs appear in the Appendix.

3.1 The Pricing Subgames

We begin by characterizing prices in the second pricing period. Lemma 1 states that in the second period, firms set prices to clear the market. This result is well known from Kreps and Scheinkman (1983) for efficient rationing and from Davidson and Deneckere (1986) for proportional rationing.

Lemma 1. *Under either the efficient or the proportional rationing rule, if Assumption 1 and Assumption 2 hold, then in any subgame perfect equilibrium (SPE) of the three-stage game, the price in the second selling period clears the market.*

Lemma 1 allows us to easily characterize all of the subgame perfect equilibria of the pricing subgame. We say that the equilibrium is unique when all the equilibria of the game have the same outcomes and payoffs for all players.

First, no equilibrium exists in the pricing subgame in which $p_1 \neq p_2$. If all firms charged prices p_1 and p_2, and $p_1 < p_2$, then a firm could deviate to a slightly higher price in period 1. The firm's period 1 sales would fall discretely (perhaps to zero); its period 2 sales would rise discretely; and prices would change by, at most, an arbitrarily small amount. Its profits would be strictly higher. And if all firms charged prices p_1 and p_2, and $p_1 > p_2$, then a firm could deviate to a slightly lower price in period 1. The firm's period 1 sales would rise discretely (perhaps to its capacity); its period 2 sales would fall discretely; and prices would change by, at most, an arbitrarily small amount. Its profits would again be strictly higher.

Next, we characterize the pricing subgame equilibria. Proposition 1, below, shows that there are two types of subgame perfect equilibria in the pricing subgame. In a uniform-price equilibria, prices are the same across firms and periods. Since the market clears in the second period (Lemma 1), any uniform-price equilibrium must satisfy $D_1(p^*) + D_2(p^*) = \sum_i K^i$, so the uniform price is unique. In asymmetric-price equilibria, a single firm sells in the first period; the first-period price is lower than the second-period price; and all other firms sell only in the second period.

Asymmetric-price equilibria have a particular form. Only one firm, firm i, sells in the first period. Let p_1^i and q_1^i denote its first-period price and quantity, where

$$p_1^i = \underset{p \in [p_1(K^i), \infty]}{\operatorname{argmax}} \, pD_1(p) + p_2\left(\sum_i K^i - D_1(p)\right)\Big(K^i - D_1(p)\Big), \tag{3}$$

or, equivalently,

$$q_1^i = \underset{q \in [0, K^i]}{\operatorname{argmax}} \, p_1(q)q + p_2\left(\sum_i K^i - q\right)\Big(K^i - q\Big). \tag{4}$$

In both expressions, the firm's output is constrained so that first-period sales do not exceed K^i. The second-period price is higher than p_1^i and is given by

$$p_2 = p_2\left(\sum_i K^i - D_1(p_1^i)\right). \tag{5}$$

Proposition 1. *Under either the efficient or the proportional rationing rule, if Assumptions 1 and 2 hold, then every pure-strategy SPE of the pricing subgame is either a uniform-price equilibrium or an asymmetric-price equilibrium satisfying Equations (3), (4) and (5). Also,*

1. *when a uniform-price equilibrium exists, it is the unique pure-strategy SPE;*
2. *when an asymmetric-price equilibrium exists, no uniform-price equilibrium exists;*
3. *here are, at most, n asymmetric-price equilibria; and*

4. *Finally, if $\eta_1 = \eta_2$ then a uniform-price equilbrium exists; if $\eta_1 > \eta_2$ (Assumption 3), then either a uniform price or an asymmetric-price equilibria exist; and if $\eta_1 < \eta_2$ then a uniform-equilibria may exist, and an asymmetric-price equilibria never exists.*

Intuitively, asymmetric-price equilibria exist because a lower price in the first pricing period increases sales in the first period, leading to a higher second-period price. But a firm can increase its profit in this way only if the elasticity is decreasing and if it has sufficient capacity to meet all of the demand in first period and has additional capacity to sell at the higher second-period price. Other firms free ride and sell only in the second pricing period at the higher price.

Asymmetric-price equilibria are more likely to exist when one firm has more capacity than its rivals. The incentive to deviate to a lower price is increasing in the deviating firm's capacity, decreasing in the rival firms' capacity, increasing in the elasticity of first-period demand, and decreasing in the size of first-period demand.

Proposition 1 holds whether the elasticity is increasing or decreasing. That is, the fact that one market opens before the other does have impacts on the exact equilibrium strategies, but just as if the two markets were open simultaneously, price competition puts pressure on firms to equalize prices across the two markets. In the discussion that follows we focus on the case in which the elasticity is decreasing. We will emphasize an important fundamental asymmetry between increasing and decreasing elasticity models later on.

While asymmetric-price equilibria exist in some instances, Assumption 4 below implies that only a uniform-price equilibrium exists when the elasticity is decreasing over time. This relatively weak condition implies that no firm has enough capacity to profitably deviate from the symmetric uniform-price equilibrium.

Assumption 4 requires that demand in period 2 not be too inelastic relative to demand in period 1. Demand in period 2 is less elastic by assumption, but not inelastic that unilaterally cutting price in the first period in order to drive up price in the second period would be profitable for a individual firm.

Assumption 4. *The elasticities of demand and capacities satisfy*

$$\frac{\eta_2(p)}{\eta_1(p)} > \frac{K^i}{\sum_{j=1}^{n} K^j}, \forall p, i. \tag{6}$$

The next proposition (Proposition 2) shows that under Assumption 3 and Assumption 4, the unique equilbrium of the two-period pricing subgame is the uniform-price equilibrium.

Proposition 2. *When Assumptions 1-4 hold, the unique subgame-perfect pure-strategy Nash equilibrium of the pricing subgame is a uniform-price equilibrium.*

Intuitively, when the elasticity is decreasing, deviating to a lower price from a uniform price is profitable for a monopolist if it raises the second-period profit by more than it lowers the first-period profit. However, since rivals free ride and sell only in period 2, an oligopoly firm that deviates from the uniform price, by lowering its first-period price, earns at most $1/n$th of the second-period industry profits. The oligopoly firm that deviates cannot increase its profit unless it can increase the second-period industry profits by at least n times the decrease in its first-period profit. For such a deviation to be profitable, the first-period demand must be at least n times more elastic than the second-period demand. Assumption 4 guarantees that such a deviation is not profitable.

Strong competitive pressures also exist when demand becomes more elastic over time; however, the existence of a unique uniform pricing equilibrium is more nuanced. Proposition 3 describes two sufficient conditions for the existence of uniform-price equilibrium:

Proposition 3. *When Assumptions 1-2 hold, and when demand becomes more elastic over time, the unique subgame-perfect pure-strategy Nash equilibrium of the pricing subgame is a uniform-price equilibrium when*

$$D_1(p) < \sum_{j \neq i} K^j, \forall i, \tag{7}$$

under both proportional or efficient rationing, or,

$$\frac{\eta_1(p)}{\eta_2(p)} > \max_{i=1,\ldots,n} \frac{K^i}{\sum_{j\neq i}^n K^j}, \forall p, i, \tag{8}$$

under efficient rationing.

The first condition in Proposition 3 guarantees that no firm has sufficient market power to act as a residual monopolist in the first period, and hence, a price increase does not have any effect on any firm's profit. The latter case is analogous to Assumption 4 and guarantees that no profitable deviation exists, but only under efficient rationing. Under proportional rationing, a uniform-price equilibrium may not exist when first-period demand is large relative to second-period demand and a firm with sufficient capacity can profitably deviate to a higher price in period one. If a uniform-price equilibrium does not exist, then no pure-strategy equilibrium of the pricing subgame exists.

Importantly, our results reveal an asymmetry between the case when elasticity is decreasing and the case when elasticity is increasing. When the elasticity of demand is decreasing, the only profitable deviation from a uniform price equilibrium is to lower price in the first period; when the elasticity of demand is increasing, the only possibly profitable deviation is to increase price in the first period. However, in the former case, deviating is only profitable when a firm's capacity is sufficiently large compared to rival firms so it can have significant market share in the second period when prices are higher. In the latter case, deviating is only profitable if it induces all rival firms to sell all of their capacity in the first period.

3.2 The Initial Capacity Choice

In many respects, Proposition 2 is the most interesting result of the paper. It specifies that for any allocation of initial capacity satisfying Assumption 4, oligopoly firms cannot price discriminate when a monopolist clearly would.

We now ask what happens when firms choose their initial capacity optimally. We replace Assumption 4, which is a restriction on capacities, with Assumption 5, which is a restriction on the elasticities. Assumption 5 is weaker. It is equivalent to Assumption 4 when the firms' capacities are symmetric. Proposition 4 establishes that Assumption 5 is sufficient to guarantee that when the firms choose capacity (the full game), the unique subgame perfect equilibrium is a uniform-price equilibrium.

Assumption 5. *The elasticity of demand satisfies*

$$\frac{\eta_2(p)}{\eta_1(p)} > \frac{1}{n}.$$

Proposition 4. *Under Assumptions 1-3 and Assumption 5, then the unique pure-strategy subgame perfect Nash equilibrium of the full game is a uniform-price equilibrium, and equilibrium capacity and profits are equal to the Cournot capacity and profits given demand $D_1(p) + D_2(p)$.*

Proposition 4 implies that we should not expect to see intertemporal price discrimination in oligopoly markets, such as in airline markets. Unless the decrease in the elasticity of demand is very large, firms will choose symmetric capacities, and prices will be uniform over time. However, prices typically rise as the departure time approaches in the airline industry, and empirical work suggests that these increases are related to changes in the elasticity of demand, which is inconsistent with Proposition 4.

The next section of the paper suggests that inventory controls are a way to reconcile this inconsistency. Again, if the elasticity increases over time, then the equilibrium need not be uniform even under Assumptions 1, 2, and 5. The sufficient conditions for a uniform-price equilibrium to exist are significantly stronger because firms can no longer easily break an increasing price equilibrium by deviating to a higher price and forcing rivals to sell more low-priced units. However, even with inventory controls, competition is still an obstacle to obtaining a symmetric equilibria with declining prices since firms can still cut price when the price is high to take all rival sales in the first period.

4 Inventory Controls

In the previous section, we showed that when the elasticity of demand is decreasing, firms produce the same Cournot capacity and set the Cournot price in both periods as they would if there were just one period with demand $D_1(p) + D_2(p)$. This is true even though profits would be higher if firms price discriminate.

Before continuing to our formal analysis of inventory controls, we illustrate the impact of inventory controls on prices and profits in an example. Suppose that the firms could choose capacity each period as if the two periods were separate markets. For example, suppose demand is linear, $p_i = a_i - b_i q_i$, and constant costs c. Cournot profits with price discrimination are given by

$$\Pi\,(\text{price discrimination}) = \frac{(b_2(a_1-c)^2 + b_1(a_2-c)^2)}{(b_1 b_2 (n+1)^2)}$$

and the Cournot profits with uniform pricing are

$$\Pi\,(\text{uniform pricing}) = \frac{(b_2(a_1-c) + b_1(a_2-c))(\frac{b_2 a_1 + b_1 a_2}{b_1+b_2} - c)}{(b_1 b_2 (n+1)^2)}.$$

And profits are higher in the sequential Cournot model (see the Appendix for more details),

$$\begin{aligned}
\Pi(\text{PD}) - \Pi(\text{UP}) &= b_2(a_1-c)^2 + b_1(a_2-c)^2 - (b_2(a_1-c) + b_1(a_2-c))\left(\frac{b_2 a_1 + b_1 a_2}{b_1+b_2} - c\right) \\
&= \frac{1}{b_1+b_2}\,(b_2(a_1-c)(b_1(a_1-a_2) + b_1(a_2-c)b_2(a_2-a_1)) \\
&= \frac{b_1 b_2}{b_1+b_2}\,((a_1-c)(a_1-a_2) + (a_2-c)(a_2-a_1)) = \frac{b_1 b_2}{b_1+b_2}(a_1-a_2)^2 > 0.
\end{aligned}$$

This result holds more generally, and we show that inventory controls make it possible for firms to charge the sequential Cournot prices as long as the elasticity is decreasing over time. We model inventory controls as a game in which firms first choose their capacity and then, in each of the two subsequent periods, simultaneously choose their price and choose an inventory control, which is an upper bound on the quantity sold. In other words, a firm can limit the number of units available at p_1 to exactly the number of units it expects to sell in period 1. With inventory controls, a firm can insure that if a rival firm deviates to a higher price in period 1, its own sales will be unchanged.

This highlights another natural asymmetry that arises between increasing and decreasing elasticity of demand. Inventory controls can can prevent a rival from increasing our sales by raising price, but they cannot prevent a rival from lowering our sales by deviating to a lower price.

Proposition 5. *Under Assumptions 1-3 and Assumption 5, then under either the efficient or the proportional rationing rule, a subgame perfect Nash equilibrium of the model with inventory controls exists in which all firms set the Cournot price and set inventory controls equal to the Cournot quantity in each selling period. Profits are strictly higher than the uniform-price equilibrium.*

Note that in the equilibrium described in the above proposition, inventory controls are set equal to each firm's equilibrium first period output. Inventory controls do not actually restrict output when firms charge equilibrium prices, but they do restrict output if another firm deviates from the equilibrium price to a higher price. The expectation that rivals set inventory controls makes it possible for firms to charge prices that increase over time because inventory controls prevent firms from being able to profitably deviate to a higher price in the first period.

The model with inventory controls has other equilibria. In particular, the symmetric capacity, uniform-price equilibrium characterized in Proposition 4 may still be a subgame perfect equilibrium of the inventory control game. Even when it is not, there are many increasing price paths that can be supported with inventory controls. We think that firms might naturally coordinate on the Cournot quantities, but the main point is that they can earn higher profits using inventory controls.

Inventory controls could also be modeled other ways, including allowing firms to commit to inventory controls before setting price. If firms could commit to inventory controls before announcing prices, inventory controls serve two functions. First, they prevent rival firms from raising their price in order to increase our sales when the price is low. And second, they limit our own sales in period 1. The later is important and impacts equilibrium strategies, but collectively firms want to set increasing prices and sell *more* in period 1 than they do in the uniform-price equilibrium, so commitment does not help firms to unilaterally increase profits in obvious ways.

Probably the most realistic model, at least in the context of airlines, would be a repeated game in which price was observable and inventory controls were unobservable when set, but observable ex post when they were binding. While a repeated game is beyond the scope of this paper, we think that our simpler model is suggestive of what might happen in this setting.

If firms could announce and commit to their inventory controls each period before any firm sets price, then Proposition 4 still holds. In this case, if each firm set an inventory control equal to the Cournot output, this would result in the Cournot prices, and no unilateral inventory control deviation would effect the subsequent prices. But this timing may also eliminate uniform price equilibrium. In a duopoly model a unilateral inventory control would curtail the rival's incentives to raise price and cause the rival to equate marginal revenue across the two periods, even when the rival hadn't set an inventory control itself.

5 Model Extensions

5.1 Product Differentiation

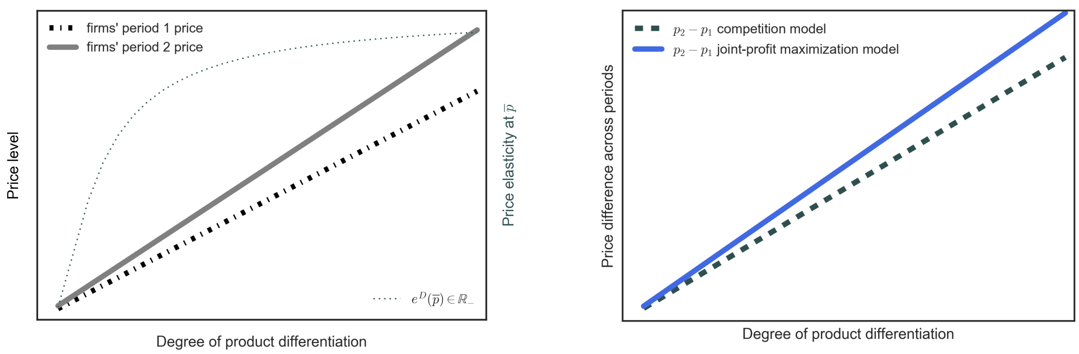
Our first extension is to consider product differentiation. Differentiation does not alter firms' incentive to attempt to shift demand to competitors in the early period. However, product differentiation makes it more costly to shift demand. With undifferentiated products, a small price change shifts all of the demand. With differentiated products, the firm's price increase must be larger, and have a first-order effect on its profits, in order to have a significant impact on a rival's sales.

Product differentiation also introduces increased complexity, so we focus our attention on two firms in a symmetric environment, and give intuition instead of analyzing the equilibrium of the model. We also maintain the assumption that capacity is sufficiently small so that firms always set market-clearing prices in the second period.

Figure 1: Intertemporal Price Discrimination as a Function of Product Differentiation

(a) Prices Across Periods

(b) Competition vs. Joint-Profit Maximization



Notes: Example constructed using a random utility model (logit) with two firms and two periods. Product differentiation is increasing towards the right of the plots. (a) The light dashed line corresponds to the own-price elasticity for a constant price offered by both firms. As products become increasingly differentiated, the difference between p_1 and p_2 increases. (b) Shows the change in price ($p_2 - p_1$) of competition model versus the joint-profit maximization model. Prices are flatter in the competition model, as the gap between the two models grows with the degree of differentiation.

Product differentiation results in equilibrium subgame prices that are no longer uniform over time; however, prices are flatter – as a function of the degree of product differentiation – than joint-profit-maximizing prices (see Figure 1 for an example, where the left plot shows increasing differences in prices across periods as product differentiation increases). To see this, consider two firms, A and B, and let the inverse demand functions be $P_1^A(q_1^A, q_1^B)$, $P_1^B(q_1^A, q_1^B)$, $P_2^A(q_2^A, q_2^B)$, and $P_2^B(q_2^A, q_2^B)$. Joint-profit-maximizing firms would set marginal revenue equal to the shadow cost of capacity in each of the four product markets, so $\frac{\partial p_i^j(q_t^j, q_t^{-j})}{\partial q_t^j} q_t^j + p_t^j(q_t^j, q_t^{-j}) = \lambda, \forall t = 1, 2; j = A, B$. Suppose that the joint-profit-maximizing prices are increasing over time.

Contrast these prices with the prices that would be set by two competing firms given the same initial capacity. If Firm A sets a higher price than the joint-profit-maximizing firm, it will sell less in the first period and, hence, more in the second period. Sales for Firm B are higher in the first period, and it has less to sell in the second period; thus, in the second period, its price is higher and Firm A's demand is higher. Because it ignores the loss for Firm B, Firm A has an incentive to set a higher first-period price than the joint-profit-maximizing monopolist. Firm B has a similar incentive, and, in equilibrium, both firms' prices will be flatter relative to joint-profit-maximizing prices (see the right panel in Figure 1). It is also worth noting that prices might still be perfectly flat if sufficiently many consumers were indifferent between the firms – a symmetric increasing price equilibrium does not exist because either firm could strictly increase profits with an arbitrarily small price increase.

5.2 Aggregate Demand Uncertainty

Inventory controls are generally described as a tool for managing demand uncertainty, so it is important to describe how the model can be extended to include such uncertainty. To generate intuition, we describe an extension in which just first-period demand is uncertain. A monopolist sets the first-period price before learning the first-period demand and sets a second-period price to clear the market.

A simple way to add uncertainty to the model is assume realized demand can be high or low in the first period, but is known to be high in the second period. In this case a monopolist choosing capacity optimally would set a lower price (based on expected demand) in the first period.

However, the monopoly prices are not an equilibrium with competing firms, even if the firms have the same capacity as the monopolist. Because the monopoly prices increase in expectation, competing firms prefer to sell more of their capacity in the second period, when the expected price is higher. And any firm can shift a discrete amount of its first-period sales to its rival through an arbitrarily small price increase in period 1. Thus, expected prices must be equal in the two periods in any symmetric pure-strategy equilibrium.

5.3 Many Periods

An obvious limitation of the paper is that we consider only two pricing periods. The challenge to extending Propositions 2 and 3 to many periods is that it is more difficult to describe assumptions under which firms play pure strategies for all histories of the game, and it is difficult to analytically bound profits in subgames with mixed-strategy equilibria.

However, it is easy to see that the a symmetric increasing price equilibrium still will not exist. Obviously, on the equilibrium path of the final two periods of the many-periods game is equivalent to our study above, so prices must be equal in the final two periods. And the intuition that firms can profit from shifting lower-price sales to the rivals still holds.

Also, with more than two periods, firms will not only consider large price cuts in order to increase future prices, but may also consider smaller price cuts in order to induce rivals to make a larger price cut in the future. Taking sales away from rivals makes their capacity share large in the next period, which could make it profitable for them to make a large price cut.

6 Conclusion

We establish that inventory controls can facilitate intertemporal price discrimination in oligopoly. To do so, we consider an advance-purchase, sequential-pricing model with complete information. If there is just a single firm in the market and demand becomes more inelastic over time, a monopolist can clearly charge higher prices to last-minute consumers.

However, in oligopoly, strong competitive forces arise. Firms have an incentive to shift sales to their rivals in early periods when consumers have lower willingness to pay in order to have increased market power in later periods when consumers have higher willingness to pay. Consequently, we show that firms will compete on price until prices are equalized across the selling periods.

In order for firms to coordinate price increases when later arrivals have higher willingness to pay, they must shield themselves from these strong competitive forces. By committing to a sales limit in each of the sequential markets—adopting inventory controls—firms can coordinate on price increases. There is extensive research in economics and operations research on the use of inventory controls as a tool to manage uncertain demand, but here their use is to facilitate intertemporal price discrimination.

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A Appendix

Proof of Lemma 1:

Proof. Suppose not. Then either some firm is charging a price below the market-clearing price or some firm is charging a price above the market clearing price.

Clearly, a firm charging below the market-clearing price must be selling all of its output and increasing its price would not reduce its sales, so its profit would increase if it raised its price which is a contradiction.

If a firm charges a price above the market-clearing price, then either it does not sell all of its output or some higher-priced firm does not sell all of its output, but there must exist some firm charging a price p greater than the market-clearing price that does not sell all of its output.

Suppose that the firm is not the only firm charging p. Then, the firm can decrease price by an arbitrarily small amount and its sales will increase discretely, so its profits would increase, which is a contradiction.

If no other firm is charging the price p; then the firm's profit is $pRD_2(p; \mathbf{p}^{-i}, \mathbf{q}^{-i})$, where $\mathbf{p}^{-i}$ and $\mathbf{q}^{-i}$ are the other firms' prices and remaining capacities.

Under the efficient rationing rule, the derivative of profit with respect to price is $RD_2(p; \mathbf{p}^{-i}, \mathbf{q}^{-i}) + pD_2'(p)$, which is negative because $RD_2(p; \mathbf{p}^{-i}, \mathbf{q}^{-i}) < D_2(p)$ and because $pD_2'(p) + D_2(p) < 0$. This is true because, by Assumption 1 and Assumption 2, $D_2(p)$ is less than the zero-cost monopoly output. So, lowering price increases profit, which is a contradiction.

Under proportional rationing, the derivative of profit with respect to price is $RD_2(p) + pRD_2'(p) = \left(pD_2'(p) + D_2(p)\right)\left[1 - \sum_{j \neq i} \frac{q^j}{D_2(p^j)}\right]$, which is negative because $pD_2'(p) + D_2(p) < 0$. This is true because, by Assumption 1 and Assumption 2, $D_2(p)$ is less than the zero-cost monopoly output. So lowering price increases profit, which is a contradiction. ■

Proof of Proposition 1:

Define $p_L = \min_i p_1^i$ to be the lowest equilibrium price offered in period 1.

By Lemma 1, under Assumption 1 and Assumption 2, for any history of the game, all firms with positive remaining capacity in the final period charge the market-clearing price in the second-period subgame.

The proof of the proposition proceeds as a series of eight claims.

1) In any equilibrium of the pricing subgame, $p_L \leq p_2$.

Suppose not, so $p_L > p_2$. Suppose also that some firm has zero sales in period 1. Since $p_L > p_2$, the firm with zero sales would be strictly better off setting a first-period price just below p_L. By deviating to $\hat{p}$, this firm increases its sales in period 1 and decreases its sales in period 2 by the same amount. The firm's profits are strictly higher because the price is higher in period 1, and because its deviation may also increase the second-period price. This is a contradiction.

Now, suppose, instead, that $p_L > p_2$ but that every firm has positive sales in period 1. It follows that every firm must be charging p_L. Otherwise, some firm j has positive sales and is charging a price $p^j > p_L$. Let $\hat{K} = \sum_{i|p_i=p_L} K^i$ denote the total capacity at price p_L. Clearly, $D_1(p_L) > \hat{K}$, because firm j's residual demand is positive. But this implies that there exists a strictly positive ϵ such that a firm i charging p_L can deviate to a higher price, $p_L + \epsilon$, and still sell all of its capacity, which is a contradiction.

If $p_L > p_2$ and all firms are charging p_L in period 1, then any firm that has excess capacity in period 1 could strictly increase its profit by deviating to a first-period price of $p_L - \epsilon$, for sufficiently small ϵ. At this price, the deviating firm sells strictly more in period 1, and strictly less in period 2. The deviating firm sells more at a first-period price that is arbitrarily close to p_L and sells less at a second-period price that is arbitrarily close to p_2, and $p_L > p_2$, so profits are higher. So, $p_L \leq p_2$.

2) In any equilibrium of the pricing subgame, when p_L is offered by two or more firms in period 1, then $p_L = p_2$.

Suppose not. So, $p_L < p_2$, and p_L is offered by two or more firms offering p_L. Suppose that firm i is one of those firms. Then, firm i's profit can be written as $p_L x_i + p_2\left(K^i - x_i\right)$, where $x_i = \min\left\{RD_1\left(p_L; p_L, \sum_{j\neq i|p^j=p_L} K^j\right), K^i\right\}$ is firm i's sales at p_L.

At the slightly higher price $p_L + \epsilon$, the firm i's profit is

$$(p_L + \epsilon)\min\left\{RD_1\left(p_L + \epsilon; p_L, \sum_{j\neq i|p^j=p_L} K^j\right), K^i\right\} \\ + \hat{p}_2(\cdot)\max\left\{K^i - RD_1\left(p_L + \epsilon; p_L, \sum_{j\neq i|p^j=p_L} K^j\right), 0\right\}, \quad (9)$$

which is clearly greater than the firm i's profit at p_L when $x^i = K^i$, since $p_L + \epsilon > p_L$ and $\hat{p}_2(\cdot) > p_L$. Thus, all of firm i's sales are at a higher price, and its sales volume doesn't change.

If, on the other hand, $RD_1(p_L; p_L, \sum_{j\neq i|p^j=p_L} K^j) < K^i$, so that $x^i < K^i$, then the deviation is still profitable for firm i because

$$\lim_{\epsilon\to 0} RD_1^i\left(p_L + \epsilon; p_L, \sum_{j\neq i|p^j=p_L} K^j\right) \leq RD_1^i(p_L; p_L, \sum_{j\neq i|p^j=p_L} K^j) < K^i,$$

since RD is decreasing (for either rationing rule), and the limit of (9) as ϵ goes to 0 is

$$p_L \lim_{p \downarrow p_L} RD_1^i(p; p_L, \sum_{j \neq i | p^j = p_L} K^j) + p_2 \left(K^i - \lim_{p \downarrow p_L} RD_1^i(p; p_L, \sum_{j \neq i | p^j = p_L} K^j) \right),$$

so for sufficiently small ϵ, it again follows that profits are higher because the firm sells more units at p_2 and fewer units at (or near) p_L and $p_2 > p_L$. So a deviation is profitable, which is a contradiction, so either $p_L = p_2$, or only one firm charges p_L.

3) If $p_L = p_2$, then the pricing equilibrium is a uniform-price equilibrium.

Suppose that some firm j sets a price $p^j > p_L = p_2$ in period 1 and has strictly positive sales. The residual demand at p^j is strictly positive, which implies that the residual demand in a neighborhood of p_L must also be strictly positive. Therefore, if a firm, say firm i, deviated from p_L to any price $p_L + \epsilon$, a price in a neighborhood of p_L (but below any higher-priced firm's price), it would be able to sell all of its capacity at that price.

This is because when firm i removes its capacity K^i at p_L, it increases the residual demand in a neighborhood of p_L by K^i. This is clearly true for either rationing rule. So any firm charging p_L could strictly increase its profits by increasing its price since its total sales would not be affected.

4) There exists, at most, one uniform-price equilibrium of the pricing subgame (the total sales and the transaction prices in each period are unique).

Given the capacity, the sales and volume of sales in a uniform-price equilibrium are uniquely defined, because only one price satifies $D_1(p) + D_2(p) = \sum_i K^i$.

5) A uniform-price equilibrium exists as long no firm wants to deviate to a higher or lower price in period 1, which is true for many values of K^i.

Consider any equilibrium in which $p_L = p_2$, and no firm has positive sales at any period 1 price other than p_L. The equilibrium price level is clearly unique.

Deviating to a higher price could be profitable, but not if the elasticity of demand is lower in the second pricing period. In this case, setting a higher price in the first period lowers industry profit – given the industry capacity and Assumption 4, industry profit is clearly lower when the first-period price is higher than the second-period price – and the deviator's share of first-period revenue falls, and its share of second-period revenue rises, so the change in revenue for the deviator must be smaller than for other firms, and the deviator's profit must fall.

Deviating to a higher price could be profitable if the elasticity is increasing over time, but such a deviation cannot be profitable unless $\sum_{j\neq i} K^j < D_1(p)$ for some i. Otherwise the deviation has no effect on any firm's profits.

Deviating to a lower price might also be profitable. Clearly, though, such a deviation cannot be profitable unless $K^i > D_1(p)$ so that the firm has positive sales in period 2. Otherwise, all of the deviating firm's sales would be at a lower price. And since $D(p)$ is decreasing, this implies that $K^i > D_1(p_L)$ is a necessary condition for a deviation to a lower price to be profitable, and $K^i < D_1(p_L)$ for all i is a sufficient condition for a uniform-price equilibrium to be the unique equilibrium of the pricing subgame.

6) When a uniform-price equilibrium of the pricing subgame does not exist, then an asymmetric-price equilibrium exists in which exactly one firm offers $p_L < p_2$ and all other firms have zero sales in period 1.

Suppose that a uniform-price equilibrium does not exist. Then, a deviation is profitable for some firm, and, clearly, it must be profitable for the firm with the largest capacity. For any deviation, that firm loses the same profit in period one from the price decrease, but gains more from the associated price increase in period 2.

Let firm i denote the firm with the largest capacity and p_1^i denote the firm's profit-maximizing deviation and $\hat{p}_2$ the resulting second-period price. That is, let p_1^i denote the prices that maximizes the profit function

Then, p_1^i and $\hat{p}_2$ clearly represent an asymmetric-price equilibrium. All firms except firm i sell only in period 2. Firm i sells in both periods. And no firm wants to undercut firm i in period 1 because it would sell more at the low price and less at the high price in period 2. And if it could increase the price and its profits by charging less than p_1^i, then so could firm i, in which case p_1^i is not firm i's profit-maximizing price, which is a contradiction.

7) A uniform-price equilibrium exists if and only if an asymmetric-price equilibrium does not exist.

Recall that (3) has a unique maximum and is concave. Consider the unique candidate uniform-price equilibrium with a price equal to p^*. This equilibrium exists unless some firm i wants to deviate to a lower price in the first period.

Suppose that firm i wants to deviate. Firm i's profit, (3), is maximized at the same price p_1^i. If $p_1^i < p^*$, then the deviation is profitable for firm i, and the uniform-price equilibrium does not exist. Moreover, an asymmetric-price equilibrium clearly exists, because (3) and (5) define an asymmetric-price equilibrium. Firm i cannot increase its profits by changing its price, and all of the other firms are strictly better off free riding and selling at $\hat{p}_2 > p_1^i$ rather than deviating to a first-period price below p_1^i. So, an asymmetric-price equilibrium exists and no uniform-price equilibrium exists.

Now suppose that firm i does not want to deviate. Then a uniform-price equilibrium exists. Now consider any asymmetric-price equilibrium in which firm i charges p. Clearly,

$p < \hat{p}_2$ by definition of an asymmetric-price equilibrium, which implies that p is less than p^*, the uniform-price equilibrium price. Firm i's profits in the asymmetric-price equilibrium are given by equation (3).

Notice that (3) is concave. Clearly, (3) is concave if (4) is concave, and the second derivative of (4) with respect to q_1 is

$$p_1''(q_1)q_1 + 2p_1'(q_1) + p_2''\left(\sum_i K^i - q_1\right)\left(K^i - q_1\right) + 2p_2'\left(\sum_i K^i - q_1\right), \tag{10}$$

which is clearly negative because $K^i - q_1 < \sum_i K^i - q_1$ and because because both revenue functions, $p_j(x)x$, are concave.

Now, if $p > p_1^i$, then firm i can profitably deviate to p_1^i, and the price then maximizes (3). And, if $p < p_1^i$, then because (3) is concave and maximized at p_1^i it follows that firm i is strictly better off increasing its price. So, no asymmetric-price equilibrium exists.

8) There are at most n asymmetric-price equilibria.

We show that there exists, at most, one asymmetric-price equilibrium in which firm i is the low-priced firm in period one (or, more strictly speaking, such equilibria differ only in the prices of firms with zero sales).

In an asymmetric-price equilibrium, if firm i is the low-price firm, then it is the only firm with positive sales in period 1. Let p denote firm i's price.

If $p > p_1^i$, then firm i can profitably deviate to p_1^i. If $p < p_1^i$, then because $\pi(p)$ is concave and maximized at p_1^i, it follows that firm i is strictly better off increasing its price. So, p does not describe a situation in which an asymmetric-price equilibrium exists.

Therefore, the only asymmetric-price equilibrium that exists in which firm i is the low-price firm in the first period is given by (3) and (5).

Proof of Proposition 2:

Proof. Let K^i denote each firm's capacity, and let $\tilde{p}$ denote the unique uniform price defined by $D_{tot}(\tilde{p}) = D_1(\tilde{p}) + D_2(\tilde{p}) = \sum_{i=1}^n K^i$.

Consider a deviation to a lower price in the first pricing period. If $D_1(\tilde{p}) \geq \max_i K^i$, then a deviation to a lower price is not profitable, because any firm that cuts its price in period 1 will sell all of its capacity at the lower deviation price and hence earn strictly lower profits.

If $D_1(\tilde{p}) < \max_i K^i$, then for any firm i such that $K^i \leq D_1(\tilde{p})$, a deviation to a lower price is not profitable by the same argument. If $K^i > D_1(\tilde{p})$, then a deviation could be profitable, but not if demand is becoming more elastic over time.

But a deviation to a lower price could be profitable if demand is becoming less elasticity over time. In this case the firm's problem is to choose a price $p < \tilde{p}$, or equivalently, a quantity $q = D_1(p)$ to maximize

$$\hat{\pi}^i(q;\tilde{p},\mathbf{K}) = qp_1(q) + p_2\left(\sum_{i=1}^{n} K^i - q\right)\left(K^i - q\right), \tag{11}$$

subject to $q \in \left(D_1(\tilde{p}), K^i\right]$. Higher output levels are not feasible and lower output levels are inconsistent with a lower first period price. The first-order condition is

$$\frac{d\hat{\pi}(q;\tilde{p},\mathbf{K})}{dq} = p_1(q) + qp_1'(q) - p_2\left(\sum_{i=1}^{n} K^i - q\right) - p_2'\left(\sum_{i=1}^{n} K^i - q\right)(K^i - q) = 0, \tag{12}$$

or

$$\begin{aligned}\frac{d\hat{\pi}(q;\tilde{p},\mathbf{K})}{dq} &= p_1(q)\left(1 + \frac{1}{\eta_1(p_1(q))}\right) \\ &\quad - p_2\left(\sum_{i=1}^{n} K^i - q\right)\left(1 + \frac{1}{\eta_2\left(p_2\left(\sum_{i=1}^{n} K^i - q\right)\right)}\frac{K^i - q}{\sum_{i=1}^{n} K^i - q}\right) = 0.\end{aligned} \tag{13}$$

Clearly, the objective function, equation (11), is concave, so a deviation to a lower price is profitable if and only if $\lim_{q\downarrow D_1(\tilde{p})} \frac{d\hat{\pi}(q;\tilde{p},\mathbf{K})}{dq} > 0$, or equivalently, $\lim_{p\uparrow\tilde{p}} \frac{d\hat{\pi}(D_1(p);\tilde{p},\mathbf{K})}{dq} > 0$ (again lower output is inconsistent with a deviation to a lower price). But clearly

$$\begin{aligned}\lim_{p\uparrow\tilde{p}} \frac{d\hat{\pi}(D_1(p);\tilde{p},\mathbf{K})}{dq} &< p_1(D_1(\tilde{p}))\left(1 + \frac{1}{\eta_1(p_1(D_1(\tilde{p}))}\right) \\ &\quad - p_2\left(\sum_{i=1}^{n} K^i - D_1(\tilde{p})\right)\left(1 + \frac{1}{\eta_2\left(p_2\left(\sum_{i=1}^{n} K^i - D_1(\tilde{p})\right)\right)}\frac{K^i}{\sum_{i=1}^{n} K^i}\right)\end{aligned}$$

because $\frac{K^i - q}{(\sum_{i=1}^{n} K^i - q)} < \frac{K^i}{\sum_{i=1}^{n} K^i}$. Since $p_1(D_1(\tilde{p})) = p_2\left(\sum_{i=1}^{n} K^i - D_1(\tilde{p})\right) = \tilde{p}$, it follow that a deviation to a lower price is not profitable if

$$\frac{1}{\eta_1(p_1(D_1(\tilde{p})))} - \frac{1}{\eta_2\left(p_2\left(\sum_{i=1}^{n} K^i - D_1(\tilde{p})\right)\right)}\frac{K^i}{\sum_{i=1}^{n} K^i} < 0 \iff \frac{\eta_2(\tilde{p})}{\eta_1(\tilde{p})} > \frac{K^i}{\sum_{i=1}^{n} K^i}, \tag{14}$$

or, equivalently, if Assumption 4 holds. If demand in the second period is too much more inelastic, a deviation will be profitable.

Now consider a deviation to a higher price. If $D_1(\tilde{p}) < \sum_{j\neq i} K^j$, for all i then no firm's deviation to a higher price can have any effect on first or second period sales. The firm's that don't deviate can meet all of the demand at the price $\tilde{p}$.

If $D_1(\tilde{p}) > \sum_{j \neq i} K^j$, for some i, then some firm or firms can deviate to a higher price and have positive sales, however even a monopolist would not find such a deviation profitable when demand is become less elastic over time, so no firm will increase price. ■

Proof of Proposition 3

Let K^i denote each firm's capacity, and let $\tilde{p}$ denote the unique uniform price defined by $D_{tot}(\tilde{p}) = D_1(\tilde{p}) + D_2(\tilde{p}) = \sum_{i=1}^{n} K^i$.

Suppose that Assumptions 1 and 2 hold, but suppose that demand becomes more elastic over time.

First consider a deviation to a lower price in the first period. When demand becomes more elastic over time, even a monopolist does not find this profitable, so no firm can increase its profits by selling at a lower price, even if it raises the second period price for some of its sales.

Now consider a deviation to a higher price in the first period. If $D_1(\tilde{p}) \leq \sum_{j \neq i} K^j$, then if firm i deviates to a higher price, it has zero sales in period 1 and still sells all of its output at the uniform price in the second pricing period, so its profits are the unchanged.

So $D_1(\tilde{p}) \leq \sum_{j \neq i} K^j \forall i$ is a sufficient condition for a uniform-price equilibria to exist.

If $D_1(\tilde{p}) > \sum_{j \neq i} K^j$, for some i then when firm i deviates to a higher price its rivals sell all of their output at a price $\tilde{p}$, so its rivals' profits are unchanged. In this case, firm i acts like a residual monopolist.

When the elasticity is increasing over time, then firm i may be able to increase its profit by raising its price, but only if $\sum_{j \neq i} K^j < D_1(\tilde{p})$. We can think of the firm as a residual monopolist selling to consumers who aren't served at the price $\tilde{p}$. So the firm's problem is to choose a price $p > \tilde{p}$ to maximize

$$\hat{\pi}^i(p; \tilde{p}, \mathbf{K}) = pRD_1(p, \tilde{p}, K^{-i}) + p_2\left(K^i - RD_1(p, \tilde{p}, K^{-i})\right)\left(K^i - RD_1(p, \tilde{p}, K^{-i})\right). \tag{15}$$

or equivalently a quantity q to maximize

$$\hat{\pi}^i(q; \tilde{p}, \mathbf{K}) = qp_1^r\left(q, \sum_{j \neq i} K^j\right) + p_2\left(K^i - q\right)\left(K^i - q\right). \tag{16}$$

where p_1^r is the inverse of the residual demand function.

The objective function is concave for both rationing rules, so a price increase is profitable if and only if

$$\lim_{p \downarrow \tilde{p}} \left.\frac{d\hat{\pi}(q; \tilde{p}, \mathbf{K})}{dq}\right|_{q = D_1(p) - \sum_{j \neq i} K^j} < 0. \tag{17}$$

This implies that profits increase as the firm restricts its output and drives price up above

$\tilde{p}$. This derivative is

$$\frac{d\hat{\pi}(q;\tilde{p},\mathbf{K})}{dq} = p_1^r(q) + q\frac{dp_1^r(q)}{dq} - p_2\left(K^i - q\right) - p_2'\left(K^i - q\right)(K^i - q), \tag{18}$$

or

$$\frac{d\hat{\pi}(q;\tilde{p},\mathbf{K})}{dq} = p_1^r(q)\left[1 + \frac{q}{p_1^r(q)}\frac{dp_1^r(q)}{dq}\right] - p_2\left(K^i - q\right)\left[1 + p_2'\left(K^i - q\right)\frac{K^i - q}{p_2\left(K^i - q\right)}\right] \tag{19}$$

so for efficient rationing, which implies that $\frac{dp_1^r(q)}{dq} = \frac{dp_1(q)}{dq}$, it follows that

$$\lim_{p\downarrow\tilde{p}} \frac{d\hat{\pi}(q;\tilde{p},\mathbf{K})}{dq}\bigg|_{q=D_1(p)-\sum_{j\neq i}K^j} = \tilde{p}\left[\frac{1}{\eta_1(\tilde{p})}\frac{D_1(\tilde{p}) - \sum_{j\neq i}K^j}{D_1(\tilde{p})} - \frac{1}{\eta_2(\tilde{p})}\right] \tag{20}$$

Hence, if $\frac{\eta_1(\tilde{p})}{\eta_2(\tilde{p})} > \frac{D_1(\tilde{p})-\sum_{j\neq i}K^j}{D_1(\tilde{p})}$ then no deviation is profitable. And clearly $\frac{\eta_1(\tilde{p})}{\eta_2(\tilde{p})} > \frac{K^i}{\sum_{j\neq i}K^j}$ is sufficient for a uniform-price equilibrium to exist. That is, if demand in the first period is too inelastic compared to demand in the second period, a deviation will be profitable.

Proof of Proposition 4:

Under Assumptions 1, 2 and 5, if a subgame perfect equilibrium exists in which every firm chooses K^* units of capacity, then, by Proposition 2, the unique subgame perfect equilibrium of the pricing subgame is a uniform-price equilibrium. Moreover, for all firm capacities in a neighborhood of K^*, Assumption 5 and Proposition 2 imply that the unique subgame perfect equilibrium of the pricing subgame is a uniform-price equilibrium, so the first-stage profit function for firm i can be written as

$$\Pi^u(K^i;K^{-i}) = \left(p_{tot}\left(\sum_j K^j\right) - c\right)K^i, \tag{21}$$

where K^{-i} is the capacity of the other firms.

Firm i's capacity, K^i, maximizes firm i's profits only if $K^i = K^*$ is the solution to

$$\frac{\partial\Pi^u(K^i;K^*)}{\partial K^i} = p_{tot}((n-1)K^* + K^i) - c + p_{tot}'((n-1)K^* + K^i)K^i = 0, \tag{22}$$

which is concave and has a unique solution, $K^i(K^*)$, which is decreasing in K^*. So, (22) uniquely defines a symmetric solution K^*, and it is easy to see that K^* must be exactly equal to the Cournot quantity associated with n firms, production cost c, and demand $D_{tot}(p)$.

We have shown that $K^i = K^*$ is local best response. Next, we show that $K^i = K^*$ is the global best response when rival firms choose K^*.

Suppose that $K^i < K^*$. If a uniform price equilibrium exists when firm i chooses K^i and other firms choose K^*, then firm i's profits are given by (21), and so firm i's profits at K^i are strictly lower than at K^*.

If a uniform-price equilibrium does not exist, then an asymmetric-price equilibrium must exist. Under Assumption 5, firm i cannot profit by deviating from the uniform-price equilibrium even if its capacity is K^*, so firm i is not the low-priced firm in the first period. The only asymmetric-price equilibrium that can exist is one in which one of firm i's rivals is the firm that sells at the low price in the first period. There are $n-1$ such equilibria because any of the $n-1$ firms with capacity K^* could set the low price in the first period.

Firm i's profit in all of these asymmetric-price equilibria is

$$\Pi^a(K^i; K^*) = \left[p_2\left((n-1)K^* + K^i - D_1(p_1)\right) - c\right] K^i, \tag{23}$$

where p_1 is the price charged in the first period, and so p_1 maximizes

$$D_1(p_1)p_1 + p_2\left((n-1)K^* + K^i - D_1(p_1)\right)(K^* - D_1(p_1)). \tag{24}$$

Firm i's first order-condition is

$$p_2'\left((n-1)K^* + K^i - D_1(p_1)\right)\left(1 - D_1'(p_1)\frac{dp_1}{dK^i}\right) + p_2\left((n-1)K^* + K^i - D_1(p_1)\right) - c = 0. \tag{25}$$

Because $p_1 < p_2$, $D(p_1)$ is greater than first-period sales at the uniform price. This implies that $n-1$ firms are each selling less than $K^* - D(\tilde{p})/n$ in period 2, where $\tilde{p}$ is the uniform price. In this case, ignoring the impact of K^i on p_1, firm i's best response is greater than $K^* - D(\tilde{p})/n$, which implies that $K^i > K^*$, which is a contradiction. And, as K^i increases, the optimal first-period price falls ($dp_1/dK^i < 0$). Thus, ignoring the impact of K^i on p_1 does not alter the result. This is still a contradiction.

Now suppose that $K^i > K^*$. Again, the equilibrum of the pricing subgame may be an asymmetric-price equilibrium or a uniform-price equilibrium. If it is a uniform-price equilibrium, then by the same argument, profits are strictly lower, which is a contradiction.

If it is an asymmetric-price equilibrium, then it must be an asymmetric-price equilibrium in which firm i sets a low price in the first period. This is because an asymmetric-price equilibrium exists only if a firm wants to deviate from the uniform-price equilibrium, and equation (14) tells us that a firm wants to deviate only if $\eta_2(p)/\eta_1(p)$ exceeds its share of capacity. But by Assumption 5, this happens only if the capacity share exceeds $1/n$ and only firm i's share of capacity exceeds $1/n$.

So, if firm i deviates to $K^i > K^*$, then its profit must be

$$\max_{p_1} D_1(p_1)\, p_1 + p_2\left((n-1)K^* + K^i - D_1(p_1)\right)\left(K^i - D_1(p_1)\right).$$

Rewriting this as a function of quantity yields

$$\max_{q_1} p_1(q_1)q_1 + p_2\left((n-1)K^* + K^i - q_1\right)\left(K^i - q_1\right). \tag{26}$$

Thus, the firm's profit in stage 1 is

$$\max_{q_1} p_1(q_1)q_1 + p_2\left((n-1)K^* + K^i - q_1\right)\left(K^i - q_1\right) - cK^i, \tag{27}$$

and its maximized stage 1 profit is

$$\max_{q_1,K_1} p_1(q_1)q_1 + p_2\left((n-1)K^* + K^i - q_1\right)\left(K^i - q_1\right) - cK^i, \tag{28}$$

which we can rewrite using a change of variables ($q_2 = K^i - q_1$) as

$$\max_{q_1,q_2} p_1(q_1)q_1 - cq_1 + p_2\left((n-1)K^* + q_2\right)q_2 - cq_2. \tag{29}$$

Therefore, q_1 is the first-period monopoly output. and q_2 is the second-period best response to $(n-1)K^*$. But this is not an equilibrium unless $p_1 < p_2$, or equivalently the Lerner index in the first period is smaller than the Lerner index in period 2, or

$$\frac{p_1'(q_1)q_1}{p_1(q_1)} < \frac{p_2'\left((n-1)K^* + q_2\right)q_2}{p_2\left((n-1)K^* + q_2\right)} \tag{30}$$

$$\frac{1}{\eta_1(p_1)} < \frac{1}{\eta_2(p_2)}\frac{q_2}{((n-1)K^* + q_2} \tag{31}$$

or

$$\frac{\eta_2(p_2)}{\eta_1(p_1)} < \frac{q_2}{((n-1)K^* + q_2}, \tag{32}$$

which violates Assumption 4 because $q_2 < K^*$. So, this is a contradiction and no global deviation is profitable.

Proof of Proposition 5:

Proof. In what follows, in order to distinguish the firm's inventory controls from the firm's actual sales, we denote the inventory control for firm i in period t by k_t^i.

Consider an equilibrium in which, on the equilibrium path, firms choose capacity equal to the sum of the Cournot capacity in each period, $q_1^C + q_2^C$, and then set the Cournot price, p_t^C, and set inventory controls equal to the Cournot output in each selling period, $k_1^i = q_1^C$. Off of the equilibrium path, they set the market-clearing price in the last period.

Clearly, no deviation is profitable in the final period. Lemma 1 holds, and the second-period prices for all firms are equal to the market-clearing prices – the presence of inventory controls does not change this result.

Consider a deviation by firm i to a lower price. in the first selling period.

FIX: Assumption 3 implies that $p_1^C < p_2^C$, so a small decrease in price discontinuously increases firm i's first-period sales, decreases firm i's second-period sales, and decreases firm i's profits. More importantly, profits cannot be higher than profits in a uniform-price equilibrium because an asymmetric-price equilibrium does not exist by Proposition 2, and profits in the candidate equilibrium are even higher than profits in the uniform price equilibrium. So, no deviation to a lower price is profitable.

Suppose, instead, that firm i deviates to a higher price in the first selling period. Under the efficient rationing rule, the residual demand function facing the deviating firm is $RD_1^i(p; p_1^C, q_1^C) = D_1(p) - (n-1)q_1^C$. Note that the rival firms' quantities are no longer their capacities, but, instead, are their inventory controls.

Since the shadow cost of capacity is c on the equilibrium path (and more generally is the same in both periods), firm i's first-period profit function is $(D_1(p) - (n-1)q_1^C)(p-c)$ or, equivalently, $(p_1((n-1)q_1^C + q) - c)q$. Thus, the optimal price deviation is given by the first-order condition, which is

$$p_1'\left((n-1)q_1^C + q\right)q + p_1\left((n-1)q_1^C + q\right) = c.$$

But this implies that $q = q_1^C$ and that the optimal price and quantity is the first-period Cournot output (or. more generally. the output that equalizes the marginal revenue across the two periods), so no deviation to a higher price is profitable.

Under proportional rationing, the residual demand function facing the deviating firm is $RD_1^i(p; p_1^C, q_1^C) = D_1(p)\left[1 - \frac{(n-1)q_1^C}{D_1(p_1^C)}\right] = \frac{1}{n}D_1(p)$ since $D_1(p_1^C) = nq_1^C$. The shadow cost of capacity is c on the equilibrium path (but, more generally, is equalized in the two periods), so firm i's first-period profit function is $\frac{1}{n}D_1(p)(p-c)$, or equivalently, $p_1(nq) - c)q$. The first-order condition is $p_1(nq) + p_1'(nq)\,q = c$, which implies that $q = q_1^C$, so no deviation to a higher price is profitable.

In stage 0, firms choose capacity expecting to equalize marginal revenue across periods 1 and 2. It is easy to see that $K^i = q_1^C + q_2^C$ is a best response to $K^j = q_1^C + q_2^C$ for all $j \neq i$. ■

Cournot Model with and without discrimination

First, suppose that $p = a - bq$, firms have constant cost c, and that there are n frms. Cournot output for each of n firms is $(a-c)/b(n+1)$, so the total Cournot output is $(a-c)n/b(n+1)$, the Cournot price is $(a+nc)/(n+1)$, and the Cournot profit of each firm is $(a-c)^2/b(n+1)^2$.

Now consider two markets and suppose firms sell in two markets and demands are $p_1 = a_1 - b_1q_1$ and $p_2 = a_2 - b_2q_2$. Then if the demands are combined into one with the same price, demand is $q^{Tot} = a_1/b_1 + a_2/b_2 - p\left(\frac{1}{b_1} + \frac{1}{b_2}\right)$ or $b_1b_2q^{Tot} = b_2a_1 + b_1a_2 - p\,(b_1 + b_2)$

or $p = \frac{b_2 a_1 + b_1 a_2}{b_1 + b_2} - \frac{b_1 b_2}{b_1 + b_2} q^{Tot}$, so Cournot profit is $(\frac{b_2 a_1 + b_1 a_2}{b_1 + b_2} - c)^2 (b_1 + b_2)/(b_1 b_2 (n+1)^2)$ or $(b_2 a_1 + b_1 a_2 - (b_1 + b_2)c)(\frac{b_2 a_1 + b_1 a_2}{b_1 + b_2} - c)/(b_1 b_2 (n+1)^2)$ or $(b_2(a_1 - c) + b_1(a_2 - c))(\frac{b_2 a_1 + b_1 a_2}{b_1 + b_2} - c)/(b_1 b_2 (n+1)^2)$

If the markets are separate and firms set different quantities (and prices) in each market, then the Cournot profits are $(a_1 - c)^2/(b_1(n+1)^2) + (a_2 - c)^2/(b_2(n+1)^2)$ or equivalently $\left(b_2(a_1 - c)^2 + b_1(a_2 - c)^2\right)/(b_1 b_2 (n+1)^2)$.