INFORMATION AND MARKET POWER

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Abstract

We consider demand function competition with a finite number of agents and private information. We analyze how the structure of the private information shapes the market power of each agent and the price volatility. We show that any degree of market power can arise in the unique equilibrium under an information structure that is arbitrarily close to complete information. In particular, regardless of the number of agents and the correlation of payoff shocks, market power may be arbitrarily close to zero (so we obtain the competitive outcome) or arbitrarily large (so there is no trade in equilibrium). By contrast, price volatility is always less than the variance of the aggregate shock across agents across all information structures, hence we can provide sharp and robust bounds on some but not all equilibrium statistics.

We then compare demand function competition with a different uniform price trading mechanism, namely Cournot competition. Interestingly, in Cournot competition, the market power is uniquely determined while the price volatility cannot be bounded by the variance of the aggregate shock.

JEL CLASSIFICATION: C72, C73, D43, D83, G12.

KEYWORDS: Demand Function Competition, Supply Function Competition, Price Impact, Market Power, Incomplete Information, Bayes Correlated Equilibrium, Volatility, Moments Restrictions, Linear Best Responses.

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1 Introduction

1.1 Motivation and Results

Models of demand function competition (or equivalently, supply function competition) are a cornerstone to the analysis of markets in industrial organization and finance. Economic agents submit demand functions and an auctioneer chooses a price that clears the market. Demand function competition is an accurate description of many important economic markets, such as treasury auctions or electricity markets. In addition, it can be seen as a stylized representation of many other markets, where there may not be an actual auctioneer but agents can condition their bids on market prices and markets clear at equilibrium prices.

Under complete information, there is a well known multiplicity of equilibria under demand function competition (see Klemperer and Meyer (1989)). In particular, under demand function competition, the degree of market power – which measures the distortion of the allocation as a result of strategic withholding of demand – is indeterminate. This indeterminacy arises because, under complete information, an agent is indifferent about what demand to submit at prices that do not arise in equilibrium. Making the realistic assumption that there is incomplete information removes the indeterminacy because every price can arise with positive probability in equilibrium. We therefore analyze demand function competition under incomplete information (Vives (2011)). We consider a setting where a finite number of agents have linear-quadratic preferences over their holdings of a divisible good, and the marginal utility of an agent is determined by a payoff shock; we restrict attention to symmetric environments (in terms of payoff shocks and information structures) and symmetric linear Nash equilibria.

The outcome of demand function competition under incomplete information will depend on the fundamentals of the economic environment - the number of agents and the distribution of payoff shocks - but also on which information structure is assumed. However, it will rarely be clear what would be reasonable assumptions to make about the information structure. We therefore examine if it is possible to make predictions about outcomes under demand function competition in a given economic environment that are robust to the exact modelling of the information structure.

Our first main result establishes the impossibility of robust predictions about market power. We show that any degree of market power can arise in the unique equilibrium under an information structure that is arbitrarily close to complete information. In particular, regardless of the number of agents and the correlation of payoff shocks, market power may be arbitrarily close to zero (so we obtain the competitive outcome) or arbitrarily large (so there is no trade in equilibrium). The reason is that, when there is incomplete information, prices convey information to agents. The slope of the demand function an agent submits will then depend on what information is being revealed, and this will pin down market power in equilibrium.

While the information structures giving rise to extremal outcomes are special, we document that the sensitivity to fine details of the information structure arises for very natural information structures. We give one illustration here. We can always decompose agents' payoff shocks into idiosyncratic and common components. If there was common knowledge of the common component, but agents observed noisy signals of their idiosyncratic component, there would be a unique equilibrium and we can identify the market power as the noise goes to zero. If instead there was common knowledge of the idiosyncratic components, but each agent observed a different noisy signal of the common component, there will be a different unique equilibrium and a different market power in the limit as the noise goes to zero. In the latter case, unlike in the former case, higher prices will reveal positive information about the value of the good to agents and, as a result, agents will submit less price elastic demand functions and there will be high market power. More generally, if agents have distinct noisy but accurate signals of the idiosyncratic and common components of payoff shocks of the other traders, then market power will be determined by the relative accuracy of the signals, even when all signals are very accurate.

Given the sharp indeterminacy in the level of market power induced by the information structure, it is natural to ask what predictions— if any— hold across all information structures.

Our second main result shows that –for any level of market power – price volatility is always (that is, regardless of the information structure) less than the price volatility that is achieved by an equilibrium under complete information. A direct corollary of our result is that price volatility is less than the variance of the average shock across agents across all information structures. Hence, we show that it is possible to provide sharp bounds on some equilibrium statistics, which hold across all information structures.

The first two results in our paper focus on market power and price volatility. There are two natural questions that follow: (i) to what extent can we study other possible statistics of an equilibrium outcome?, and (ii) how many statistics of an equilibrium outcome are necessary to consider in order to fully determine an equilibrium outcome?

The third main result of our paper characterizes the set of outcomes that can be achieved in demand function competition in terms of necessary and sufficient conditions. We show that any distribution of outcomes — that is, any distribution of quantities, payoff shocks and prices— that is an equilibrium outcome is fully determined by only 3 statistics. The first two statistic are essentially the level of market power and price volatility, while the third statistic is the dispersion in the quantities bought by agents. Once these three statistics have been determined all other moments of an equilibrium outcome are uniquely pinned down by the equilibrium conditions and the payoff structure of the game.

The methodology used to study volatility bounds can also be used to compare the set of outcomes of different trading mechanisms across all information structures. We define a distribution of outcomes as the joint distribution of quantities, payoff shocks and price that is induced by an equilibrium outcome. A distribution of outcomes provides a description of the outcome of demand function competition that allows the analyst to abstract from the strategies used in equilibrium and the precise description of the information structure. The key conceptual innovation is to describe the outcomes of the demand function competition game not in terms of the strategies used by the agents (that is, the demand functions), but instead, in terms of the induced economic outcomes (purchased quantity and price) and payoff shocks.

A critical advantage of the focus on the distribution of outcomes is that it can be easily compared with the distribution of outcomes induced by any other trading mechanisms. In the paper we focus our analysis in comparing demand function competition with Cournot competition, as a particular instance of what we call uniform price mechanism. The set of possible first moments under demand function competition has one more degree of freedom than under Cournot competition, while the set of possible second moments under demand function competition has one less degree of freedom than under Cournot competition. This apparently abstract description of the two mechanisms allow us to conclude that price volatility is bounded by the size of aggregate shocks in demand function competition, while in Cournot competition price volatility cannot be bounded by the size of the aggregate shocks. By contrast, the first moment, the market power, or the average volume of trade is uniquely determined in the Cournot competition (unlike in demand function competition).

1.2 Related Literature

The multiplicity of equilibria in demand function competition under complete information was identified by Wilson (1979), Grossman (1981) and Hart (1985), see also Vives (1999) for a more detailed account. Klemperer and Meyer (1989) emphasized that the complete information multiplicity was driven by the fact that agents' demand at non-equilibrium prices was indeterminate. They showed that introducing noise that pinned down best responses lead to a unique equilibrium and thus determinate market power. And they showed that the equilibrium selected was independent of shape of the noise, as the noise became small. They were thus able to offer a compelling prediction about market power. Our results show that their results rely on a maintained private values assumption, implying that agents cannot learn from prices. We replicate the Klemperer and Meyer (1989) finding that small perturbations select a unique equilibrium but - by allowing for the possibility of a common value component of values - we can say nothing about market power in the perturbed equilibria.

Vives (2011) pioneered the study of asymmetric information under demand function competition, and we work in his setting of linear-quadratic payoffs and interdependent values. He studied a particular class of information structures where each trader observes a noisy signal of his own payoff type. We study what happens for all information structures. We show that the impact of asymmetric information on the equilibrium market power can even be larger than the ones derived from the one-dimensional signals studied in Vives (2011). Our results overturn some of the comparative statics and bounds that are found using the specific class of one-dimensional signal structures. In particular, in this paper but not in Vives (2011) market power can be large even when any of the following conditions is satisfied: (*i*) the amount of asymmetric information is small, (*ii*) the number of players is large, or (*iii*) payoff shocks are independently distributed.

Bergemann and Morris (2016) described a general approach for finding equilibria under all information structures in a given game and Bergemann, Heumann, and Morris (2015b) used this methodology in the context of a symmetric game with quadratic payoffs and normal uncertainty. An innovation of this paper with respect to this earlier literature is that we characterize economic outcomes arising in equilibrium (quantities and prices), abstracting from strategic choices (i.e., demand functions). This methodological extension allows a novel comparison of alternative mechanisms, i.e., demand function and Cournot competition.

Our "anything goes" result for market power has the same flavor as abstract game theory results establishing that fine details of the information structure can be chosen to select among multiple rationalizable or equilibrium outcomes of complete information games (Rubinstein (1989) and Weinstein and Yildiz (2007)). Our result is an illustration of the practical importance of these ideas. Demand function competition under complete information is a game with a large degree of indeterminacy built in. Our results show that in this context very natural perturbations lead to very dramatic equilibrium selection. In particular, we do not make an assumption analogous to the "richness" assumption in Weinstein and Yildiz (2007), which in our context would require the strong assumption that there exist "types" with a dominant strategy to submit particular demand functions.

2 Model

Payoff Environment There are N agents who have demand for a divisible good. The utility of agent $i \in \{1, ..., N\}$ who buys $q \in \mathbb{R}$ units of the good at price $p \in \mathbb{R}$ is given by:

$$u_i(\theta_i, q_i, p) \triangleq \theta_i q_i - p q_i - \frac{1}{2} q_i^2, \tag{1}$$

where $\theta_i \in \mathbb{R}$ is the payoff shock of agent i. The payoff shock θ_i describes the marginal willingness to pay of agent i for the good at $q = 0$. The payoff shocks are symmetrically and normally distributed across the agents, and for any i, j:

$$\begin{pmatrix} \theta_i \\ \theta_j \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_\theta \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 \\ \rho_{\theta\theta}\sigma_\theta^2 & \sigma_\theta^2 \end{pmatrix} \right),$$

where $\rho_{\theta\theta}$ is the correlation coefficient between the payoff shocks θ_i and θ_j.

The realized average payoff shock among all the agents is denoted by:

$$\overline{\theta} = \frac{1}{N} \sum_{i \in N} \theta_i, \tag{2}$$

and the corresponding joint distribution of θ_i and $\overline{\theta}$ is given by

$$\begin{pmatrix} \theta_i \\ \overline{\theta} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_\theta \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \frac{1+(N-1)\rho_{\theta\theta}}{N}\sigma_\theta^2 \\ \frac{1+(N-1)\rho_{\theta\theta}}{N}\sigma_\theta^2 & \frac{1+(N-1)\rho_{\theta\theta}}{N}\sigma_\theta^2 \end{pmatrix} \right).$$

The supply of the good is given by an exogenous supply function $S(p)$ as represented by a linear inverse supply function with $\alpha, \beta \in \mathbb{R}_+$:

$$p(q) = \alpha + \beta q. \tag{3}$$

For notational simplicity, we normalize the intercept α of the affine supply function to zero.

Information Structure Each agent i observes a multi-dimensional signals $s_i \in \mathbb{R}^J$ about the payoff shocks:

$$s_i \triangleq (s_{i1}, ..., s_{ij}, ..., s_{iJ}).$$

The joint distribution of signals and payoff shocks

$$(s_1, ..., s_N, \theta_1, ..., \theta_N)$$

is symmetrically and normally distributed. We discuss specific examples of multivariate normal information structures in the following sections.

Demand Function Competition The agents compete via demand functions. Each agent i submits a demand function $x_i : \mathbb{R}^{J+1} \to \mathbb{R}$ that specifies the demanded quantity as a function of the received signal s_i and the market price p, denoted by $x_i(s_i, p)$. The Walrasian auctioneer sets a price p^* such that the market clears for every signal realization s:

$$p^* = \beta \sum_{i \in N} x_i(s_i, p^*) \tag{4}$$

We study the Nash equilibrium of the demand function competition game. The strategy profile $(x_1^*, ..., x_N^*)$ forms a Nash equilibrium if:

$$x_i^* \in \underset{\{x_i : \mathbb{R}^{J+1} \to \mathbb{R}\}}{\arg\max} \; \mathbb{E}\left[\theta_i x_i(s_i, p^*) - p^* x_i(s_i, p^*) - \frac{x_i(s_i, p^*)^2}{2}\right],$$

where

$$p^* = \beta(x_i(s_i, p^*) + \sum_{j \neq i} x_j^*(s_j, p^*)).$$

We say a Nash equilibrium $(x_1^*, ..., x_N^*)$ is linear and symmetric if there exists $(c_0, ..., c_J, m) \in \mathbb{R}^{J+2}$ such that for all $i \in N$:

$$x_i(s_i, p) = c_0 + \sum_{j \in J} c_j s_{ij} - mp.$$

Throughout the paper we focus on symmetric linear Nash equilibria and so hereafter we drop the qualifications "symmetric" and "linear". When we say an equilibrium is unique, we refer to uniqueness within this class of equilibria. In a linear-quadric setting like ours, Du and Zhu (2017) show that there does not exist a nonlinear ex post equilibrium.[1]

Equilibrium Statistics: Market Power and Price Volatility We analyze the set of equilibrium outcomes in demand function competition under incomplete information. We frequently describe the equilibrium outcome through two central statistics of the equilibrium: *market power* and *price volatility.*

The marginal utility of agent i from consuming the q_i-th unit of the good is $\theta_i - q_i$. We define the market power of agent i as the difference between the agent's marginal utility and the equilibrium price divided by the equilibrium price:

$$l_i \triangleq \frac{\theta_i - q_i - p}{p}.$$

This is the natural demand side analogue of the supply side price markup defined by Lerner (1934), commonly referred to as the "Lerner's index". We define the (expected) *equilibrium market power* by:

$$l \triangleq \mathbb{E}\left[\frac{1}{N}\sum_{i \in N} l_i\right] = \frac{1}{N}\mathbb{E}\left[\frac{\sum_{i \in N}(\theta_i - q_i - p)}{p}\right]. \tag{5}$$

[1] They focus on a model in which the agents observe one-dimensional signals and the supply of the asset is inelastic.

The market power l is defined as the expected average of the Lerner index across all agents. If the agents were price takers, then the market power would be $l = 0$.

A second equilibrium statistic of interest is *price volatility*, the variance of the equilibrium price, which we denote by:

$$\sigma_p^2 \triangleq \text{var}(p). \tag{6}$$

Price volatility measures the ex ante uncertainty about the equilibrium price. In the subsequent analysis we find that market power is proportional to the aggregate demand, and that price volatility is proportional to the variance of aggregate demand. Thus, these two equilibrium statistics will represent the first and second moments of the aggregate equilibrium demand.

3 Complete Information: A Review

We first review what happens in demand function competition with complete information. That is, every agent i observes the entire vector of payoff shocks $(\theta_1, ..., \theta_N)$ before submitting his demand $x_i(\theta, p)$. This is a natural starting point to understand the essential elements of demand function competition and allows us to introduce some key ideas. The set of equilibrium outcomes under complete information will play a key role in identifying what happens under incomplete information.

The residual supply faced by agent i will be determined by the demand functions of all the agents other than i. We suppress the dependence of the demand function on the vector θ in this section for notational simplicity, and thus $x_i(\theta, p) \triangleq x_i(p)$.

$$r_i(p) \triangleq S(p) - \sum_{j \neq i} x_j(p). \tag{7}$$

Agent i can then be viewed as a monopsonist over his residual supply. That is, if agent i submits demand $x_i(p)$, then the equilibrium price p^* satisfies $x_i(p^*) = r_i(p^*)$ for every i. Hence, agent i only needs to determine what is the optimal point along the curve $r_i(p)$; this will determine the quantity that agent i purchases and the equilibrium price.

To compute the first order condition for agent i's demand, it is useful to define the *price impact* λ_i of agent i:

$$\frac{1}{\lambda_i} \triangleq \frac{\partial r_i(p)}{\partial p}.$$

The price impact determines the rate at which the price increases when the quantity bought by agent i increases:

$$\lambda_i = \frac{\partial p}{\partial r_i(p)}$$

The first order condition of agent i determines the equilibrium demand of agent i:

$$x_i(p^*) = \frac{\theta_i - p^*}{1+\lambda_i}.$$

It is easy to check that λ_i determines how much demand agent i withholds to decrease the price at which he purchases the good. For example, if $\lambda_i = 0$, then agent i behaves as a price taker. As λ_i increases, agent i withholds more demand to decrease the equilibrium price. Hence, λ_i determines the incentive of agent i to withhold demand to decrease the price.

In the complete information setting, there is a well known indeterminacy of equilibrium price impact. If agent j submits a sufficiently elastic demand, then the price impact of agent i will be close to 0; any increase in the quantity bought by agent i will be offset by a decrease in the quantity bought by agent j, keeping the equilibrium price unchanged. If agent j submits a sufficiently inelastic demand, then the price impact of agent i may be arbitrarily large; any increase in the quantity bought by agent i will be reinforced by an increase in the quantity bought by agent j, leading to arbitrarily large changes in the equilibrium price.

We characterize the set of symmetric linear Nash equilibria. In this class of Nash equilibria all agents have the same price impact, and the price impact is independent of the realization of the shocks $(\theta_1, ..., \theta_N)$. We focus on the equilibrium price impact and the equilibrium price.

Proposition 1 (Equilibrium with Complete Information)
For every $\lambda \geq -1/2$, there exists a symmetric linear equilibrium where the price impact is λ and the equilibrium price is:

$$p^* = \frac{\beta}{1+\beta N+\lambda}\sum_{i\in N}\theta_i. \tag{8}$$

Proposition 1 characterizes the price impact and equilibrium price in a continuum of equilibria parametrized by the price impact λ. As the price impact λ increases, every agent withholds more demand to lower the price. This leads to a lower equilibrium price. It is easy to check that, for every $\lambda \geq -1/2$, the equilibrium quantity bought by agent i is given by:

$$q_i = \frac{1}{1+\beta N+\lambda}\frac{1}{N}\sum_{j\in N}\theta_j + \frac{1}{1+\lambda}(\theta_i - \frac{1}{N}\sum_{j\in N}\theta_j).$$

Thus, as the price impact λ increases, not only does the price decrease, but also the differences between the quantity bought by agent i and agent j decreases. Thus, as price impact increases, the equilibrium becomes less efficient because the total quantity demanded by all agents is too small (which leads to a lower price) and the quantities are inefficiently allocated across agents.

It is informative to describe the symmetric linear Nash equilibria in terms of the equilibrium statistics, market power l and price volatility σ_p^2, as defined earlier.

Corollary 1 (Equilibrium Statistics with Complete Information)
In the symmetric linear Nash equilibrium under complete information with price impact $\lambda \geq -\frac{1}{2}$*, market power and the price volatility are given by:*

$$l = \frac{\lambda}{\beta N} \text{ and } \sigma_p^2 = \frac{(\beta N)^2}{(1 + \beta N + \lambda)^2}\sigma_\theta^2, \tag{9}$$

Market power is a linear function of price impact as the price impact determines how much an agent withholds demand in order to lower prices. Similarly, the equilibrium price (8) is decreasing in the level of price impact. As the price impact increases, agents buy less, which leads to less volatility as a function of the payoff shocks.

By looking at (9), we can see that there is a direct relation between price volatility and market power. Thus we know that market power l is only bounded from below by

$$l \geq -\frac{1}{2\beta N}, \tag{10}$$

and that the price volatility can be directly expressed in terms of the market power:

$$\sigma_p^2 = \frac{(\beta N)^2}{(1 + \beta N(1 + l))^2}\sigma_\theta^2. \tag{11}$$

In Figure 1 we plot all feasible equilibrium pairs of market power and price volatility that can be a achieved under complete information. The equilibrium outcome that would be attained under complete information if we selected the outcome using the equilibrium selection proposed by Klemperer and Meyer (1989) is depicted in Point A. As we study other information structures, we will appeal to a graphic representation of all possible pairs of market power and price volatility similar to Figure 1.

The reason for multiple equilibria is that each agent has multiple best responses. In particular, there are multiple affine functions $x_i(p)$ that intercept with $r_i(p)$ at the same point. Agent i is indifferent between the multiple demand functions that intercept with $r_i(p)$ at the same point. Yet, the slope of $x_i(p)$ determines the slope of $r_j(p)$, which is important for agent j; a more inelastic demand of agent i leads to a higher price impact for agent j. By changing the slope of the demands that each agent submits, it is possible to generate different equilibria that lead to different outcomes.

The multiplicity is an artifact of the complete information assumption. With incomplete information, agents' best responses will typically be pinned down everywhere and there will be a unique equilibrium for any given information structure.

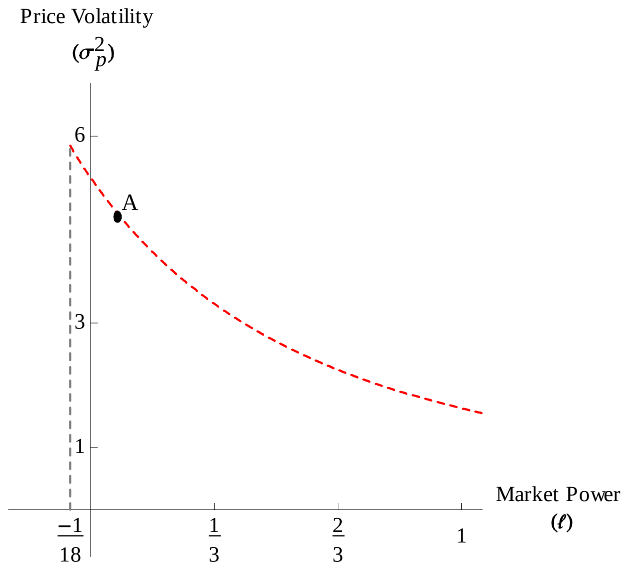

Figure 1: Set of equilibrium pairs $\left(l, \sigma_p^2\right)$ of market power and price volatility with complete information $(\beta = 1, N = 3)$.

4 Market Power and Price Volatility

We now study the set of equilibrium outcomes in demand function competition under incomplete information, focussing on two statistics of equilibrium outcomes: market power and price volatility. Our approach in this paper is to ask what can happen for all information structures. But we illustrate our main results by studying the equilibrium outcomes induced by "natural" information structures — that is, information structures that have a straightforward interpretation and have appeared in the earlier literature. These examples will illustrate how a given information structure dramatically impacts the structure of the equilibrium and provide some initial intuition for where the bounds come from.

4.1 Robust Predictions about Market Power and Price Volatility

With incomplete information, market power and price volatility will be uniquely pinned down given a specific information structure. What robust predictions can be made then that do not depend on the fine details of the information structure? We will show that we cannot make any robust predictions about market power: any positive market power can arise as the unique equilibrium even when we restrict to arbitrarily small amounts of incomplete information. But we can make a sharp prediction about price volatility: no matter the amount of incomplete information, it cannot be higher than what happens in complete information.

We say that an information structure is $\varepsilon-$close to complete information if the conditional variance of the estimate of each payoff shock θ_j is small given the signal received by agent i:

$$\forall i, j \in N, \qquad \text{var}(\theta_j | s_i) < \varepsilon. \tag{12}$$

In an information structure that is ε-close to complete information an agent can observe his own payoff shock and the payoff shock of the other agents with a residual uncertainty of at most ε. If an information structure is $\varepsilon-$close to complete information for a sufficiently small ε, then the information structure will effectively be a perturbation of complete information. We now show that any equilibrium under complete information can be selected as the unique equilibrium in a perturbation of complete information.

Theorem 1 (Equilibrium Selection)
For every $\varepsilon > 0$, and for every pair of market power and price volatility (l, σ_p^2) such that:

$$l \geq -\frac{1}{2}\frac{1}{\beta N} \text{ and } \sigma_p^2 = \frac{(\beta N)^2}{(1 + \beta N(1 + l))^2}\sigma_{\theta}^2,$$

there exists an information structure that is $\varepsilon-$close to complete information and induces (l, σ_p^2) as the unique equilibrium.

Theorem 1 shows that all combinations of market power and price volatility that can be achieved as an equilibrium under complete information can also be achieved as a unique equilibrium in an information structure that is close to complete information. In fact, the result is stronger, every equilibrium outcome under complete information is the unique equilibrium outcome of an information structure that is close to complete information.

The proof of Theorem 1, relegated to the Appendix, uses a class of information structures that we refer to as *noise-free signals*. In the next section, we augment our understanding of how private information determines price volatility and market power using information structures that appeared in earlier work.

Theorem 1 shows that, (i) all equilibrium outcomes under complete information can turn into unique equilibrium outcomes under incomplete information, and (ii) restricting attention to information structures close to complete information do not allow us to provide sharper predictions about market power and price volatility. The large indeterminacy in the set of possible outcomes suggests that it is difficult to offer robust predictions for market power under demand function competition. By contrast, it is possible to provide sharp predictions regarding price volatility with demand function competition.

Theorem 2 (Equilibria Under All Information Structures)
There exists an information structure that induces a pair of market power and price volatility (l, σ_p^2) *if and only if:*

$$l \geq -\frac{1}{2}\frac{1}{\beta N} \text{ and } \sigma_p^2 \leq \frac{(\beta N)^2}{(1+\beta N(1+l))^2}\sigma_{\overline{\theta}}^2. \tag{13}$$

Moreover, all feasible pairs (l, σ_p^2) *are induced by a unique equilibrium for some information structure.*

Theorem 2 provides a sharp bound on all possible equilibrium outcomes. It shows that the equilibrium outcome is bounded by the outcomes that are achieved under complete information. Thus the outcomes that arise under complete information can be seen as the "upper boundary" of the set of outcomes that can arise under all information structures.

The "if" part of the statement closely resembles the proof of Theorem 1. In particular, the set of market power and price volatility that satisfy (13) would be achieved under complete information if one could reduce the variance of the aggregate shocks (i.e. by making $\mathrm{var}(\frac{1}{N}\sum_{i\in N}\theta_i)$ smaller). By decomposing the payoff shocks into an observable and a non-observable component, we can effectively achieve the same outcomes as if there was complete information but the variance of the shocks was smaller.

The "only if" part of the statement is economically more interesting because it uses the restrictions that arise from agents' first order condition. The proof establishes that the equilibrium price has to satisfy:

$$p^* = \frac{\beta}{1+\beta N(1+l)}\sum_{i\in N}\mathbb{E}[\theta_i|s_i, p^*]. \tag{14}$$

That is, the equilibrium price is proportional to the average of the agents' expected payoff shocks. It is crucial that the expected payoff shock of agent i is computed conditional on the equilibrium price — this is an implication of the fact that agents compete in demand functions and hence agent i can condition the quantity he buys on the equilibrium price. The fact that an agent can condition on the equilibrium price disciplines beliefs, which ultimately allows us to bound the price volatility. As we discuss in Section 6, in Cournot competition agents cannot condition the quantity they buy on the equilibrium price, which may result in unbounded volatility even if the volatility of the average shock is arbitrarily small.

4.2 How Private Information Determines Market Power and Price Volatility

We now study three different parametrized classes of information structures: (i) noisy one-dimensional signals, (ii) multi-dimensional signals, and (iii) confounding signals. We study market power and price volatility under these three information structures and use this to provide an intuition of different elements that come into play in our main results, Theorem 1 and Theorem 2.

Under noisy one-dimensional signals, market power always increases with the amount of incomplete information, and large market power can only be induced by a large amount of incomplete information. These are the key findings of Vives (2011), but we will see that they are special to this information structure and in particular will not hold for the others that we consider in this section. If agents observe multidimensional signals, the equilibrium outcomes closely track —within some range— the set of outcomes under complete information. We use these signals to provide an intuition of why small amounts of incomplete information can lead to large variations in market power. Finally, the confounding signals provide a set of information structures in which market power is less than the one induced by the complete information selection proposed by Klemperer and Meyer (1989), which leads to a higher price volatility.

One-Dimensional Noisy Signals The first information structure consists of *one-dimensional noisy signals*. Each agent observes his payoff state with conditionally independent noise. That is, agent i observes the *noisy one-dimensional* signal

$$s_i = \theta_i + \gamma\varepsilon_i, \tag{15}$$

where the noise terms $\{\varepsilon_i\}_{i\in N}$ are independent standard normal. Vives (2011) uses a noisy one-dimensional signal to study the impact of incomplete information on market power. The *one-dimensional noisy signals* are parametrized by a one-dimensional parameter: the standard deviation of the noise term $\gamma \in [0, \infty)$. For every γ, there is a unique linear Nash equilibrium.

In Figure 2 we plot in a yellow curve the set of market power and price volatility that are achieved by one-dimensional noisy signals for all $\gamma \in \mathbb{R}$ (the red dashed curve is the set of outcomes under complete information). Point A corresponds to the outcome when $\gamma = 0$: an agent knows his own payoff shock but remains uncertain about the payoff shock of other agents. Market power is increasing in γ and price volatility is decreasing in γ. Market power increases with γ because — as the signals becomes more noisy — relative to s_i, signal s_j becomes more informative about θ_i. So agent i wants to buy a larger quantity when agent j observes a high signal. For this reason, agent i submits a more inelastic demand; this increases the correlation between the quantity he buys and the quantity bought by agent j. This in turn increase the market power of agent j.

The price volatility decreases because market power increases (as in complete information equilibria) but also because the price becomes less correlated with the average payoff shock of agents. Hence, price

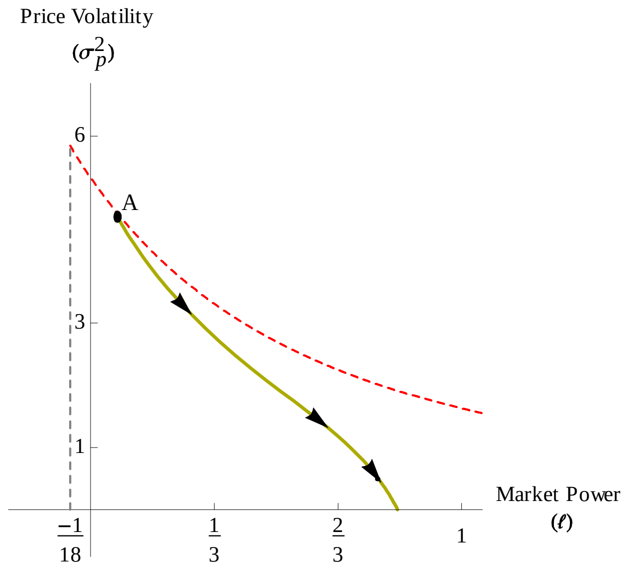

Figure 2: Set of equilibrium pairs $\left(l, \sigma_p^2\right)$ of market power and price volatility under noisy one-dimensional signals.

volatility decreases at a faster rate (as a function of market power) than under complete information. Therefore, there is a tight link between market power and a price that is less informative and less volatile.

We assumed that the individual payoff shocks θ_i and θ_j were positively but not perfectly correlated. The most natural reason for this is that they reflect common and idiosyncratic components. This suggests that we decompose the payoff shocks into a common and an idiosyncratic component, ω and τ_i respectively:

$$\theta_i = \omega + \tau_i, \tag{16}$$

where ω and $\{\tau_i\}_{i \in N}$ are normally distributed and independent of each other.[2] It is now natural to allow information to reflect common and idiosyncratic components in different ways.

Multi-Dimensional Noisy Signals Our second information structure consists of *noisy multi-dimensional signals.* Each agent observes a separate noisy signal about all the idiosyncratic and the

[2] Given our assumption that the θ_i were normally distributed with mean 0, standard deviation σ_θ and correlation $\rho_{\theta\theta}$, this decomposition would have ω and the τ_i independently normally distributed with mean 0 and standard deviations $\sigma_\omega^2 = \rho_{\theta\theta}\sigma_\theta^2$ and $\sigma_\tau^2 = (1 - \rho_{\theta\theta})\sigma_\theta^2$ respectively. Observe that $\sigma_\theta^2 = \mathrm{var}(\omega + \frac{1}{N}\sum \tau_i) = \sigma_\omega^2 + \sigma_\tau^2/N$.

common components in the payoff state, and thus each agent i observes $N+1$ signals:

$$\forall i \in N, \qquad s_{ii} = \tau_i, \tag{17}$$

$$\forall j \neq i \in N, \qquad s_{ij} = \tau_j + \delta\varepsilon_{ij}, \tag{18}$$

$$\forall i \in N, \qquad s_{i\omega} = \omega + \gamma\varepsilon_{i\omega} \tag{19}$$

where all noise terms are again independent standard normal. Agents know their own idiosyncratic component for sure. They additionally have signals of others' idiosyncratic components, which we assume to be very accurate (i.e., $0 < \delta \ll 1$). The *multidimensional signals* are parametrized by a one-dimensional parameter: the standard deviation of noise on the common component $\gamma \in [0, \infty)$. For every γ, there is a unique linear Nash equilibrium.

In Figure 3 we plot the set of market power and price volatility that are achieved by multi-dimensional noisy signals for all $\gamma \in \mathbb{R}$ in a green curve (the red dashed curve is the set of outcomes under complete information). As before, point A corresponds to the outcome when $\gamma = 0$: an agent knows his own payoff shock but remains uncertain about the payoff shocks of the other agents. Initially, as γ increases, market power increases. The intuition is similar to the case of one-dimensional noisy signals; because agents have interdependent values an agent wants to increase the correlation between the quantity he buys and the quantity bought by other agents. But as $\gamma \to \infty$ the signals about the common shock become irrelevant, and so we are back to the case in which all the relevant sources of uncertainty are the idiosyncratic shocks. Therefore as $\gamma \to \infty$, market power is reduced back to the same level as $\gamma = 0$, but with lower volatility because the price does not reflect the common component.

The picture illustrates that the set of market power and price volatility under multi-dimensional signals "tracks" very closely the set of outcomes under complete information. The agents are effectively close to complete information as each agent i observes precise signals about $\{\tau_j\}_{j \in N}$ and ω. The market power is determined by agent i's relative uncertainty about τ_j and ω rather than by an absolute level of uncertainty. Thus even close to complete information, we can have large changes in the induced level of market power and price volatility. Point B in Figure 3 corresponds to a point in which both δ and γ are small, but γ is relatively larger than δ.[3] This degree of uncertainty about payoff shocks did not have a significant impact in the case of one-dimensional normal signals because relative uncertainty about common and idiosyncratic components was not present.

Market power is equal to 1 when agents have common values; this would happen if an agent observed perfectly the idiosyncratic shock of other agents (i.e. if the variance of the noise in (18) was 0 instead

[3] The parametrization is given by $\delta = 0.01$ and $\gamma = 0.53$. The variances of the payoff shocks are given by $\sigma_\tau = 1$ and $\sigma_\omega = 5/2$. Thus, both (18) and (19) are precise signals about the respective payoff shocks.

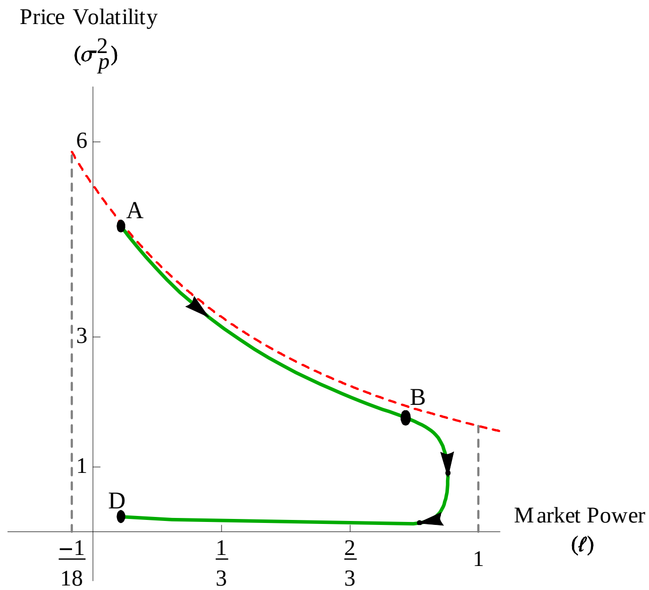

Figure 3: Set of equilibrium pairs (l, σ_p^2) of market power and price volatility under noisy one-dimensional signals.

of δ). In this case, the price perfectly reveals the expected value of ω conditional on all private signals. So an increase in the quantity bought by agent i leads to an equal increase in the quantity bought by all other agents. Although a small amount of incomplete information can generate a large market power, in our multi-dimensional signals example, market power is never above 1. However, Theorem 1 establishes that there is no upper bound on market power across all information structures. This is because it is possible to construct information structures in which an increase in the quantity bought by agent i leads to an even bigger increase in the quantity bought by all other agents, which in turns leads to a market power larger than 1.

Confounding Signals The third information structure consists of *confounding signals.* Each agent observes a weighted sum of the common and idiosyncratic components of his payoff state. Agent i observes the *confounding* signal:

$$s_i = \tau_i + \gamma\omega, \tag{20}$$

with $\gamma \in \mathbb{R}$ (note that we allow for $\gamma < 0$). Here there is no noise, but the one-dimensional signal may disproportionately reflect either the common component or the idiosyncratic component. The *confounding signals* are parametrized by a one-dimensional parameter: the confounding parameter $\gamma \in \mathbb{R}$. For every γ, there is a unique linear Nash equilibrium.

An alternative representation of the confounding signals is to note that θ_i can be written as

$$\theta_i = v(s_i, \omega),$$

where $v(\cdot, \cdot)$ is a linear function. Hence, the utility of agent i depends directly on the signal s_i that he observes and on the realization of a common value shock. In addition the signals are independently distributed across agents conditional on ω. Information structures similar to (20) have been used by Wilson (1977) and Reny and Perry (2006), among others.

In Figure 4 we plot the set of market power and price volatility that are achieved by one-dimensional noisy signals for all $\gamma \in \mathbb{R}$ in the blue curve (the red dashed curve is the set of outcomes under complete information). In this case, point A corresponds to the outcome when $\gamma = 1$. As γ decreases to 0, the market power and price volatility approaches point D. As γ diverges to ∞, the market power and price volatility approaches point C. The rest of the points are achieved by a negative γ.

It is clear to see that point C (achieved in the limit $\gamma \to \infty$) already achieves a higher price volatility and a lower market power than point A (a possibility that did not arise with the noisy multi-dimensional signals example). The reason is that a high signal of agent j is indicative of a low shock of agent i. Hence, agent i submits a more *elastic* demand, in order to decrease the correlation between the quantity he buys and the quantity bought by agent j. This in turn decreases the market power of agent j and increases the price volatility.

We note that it is even possible to achieve negative market power. This happens when a good signal for agent i is sufficiently negative information for agent j. In this case, when agent i increases the quantity he buys, this induces an even bigger decrease in the total quantity bought by the other agents. Hence, overall, when agent i buys a larger quantity the price decreases (due to the response of other agents).

We used three information structures to provide an intuition of how private information impacts market power and price volatility. Each of these information structures yield different comparative statics and can be used to understand how information determines the equilibrium outcome. The fact that it was necessary to study three "natural" information structures to account for the richness that come into play in Theorem 1 and Theorem 2 should also be a sign of caution; this illustrates how sensitive the set of possible equilibrium outcomes are to the exact specification of the information structure. An analyst, when assuming a specific information structure, may be inadvertently imposing severe restrictions on the set outcomes that are being considered. Thus, we seek predictions regarding the demand function competition outcomes that are robust to the specification of the information structure.

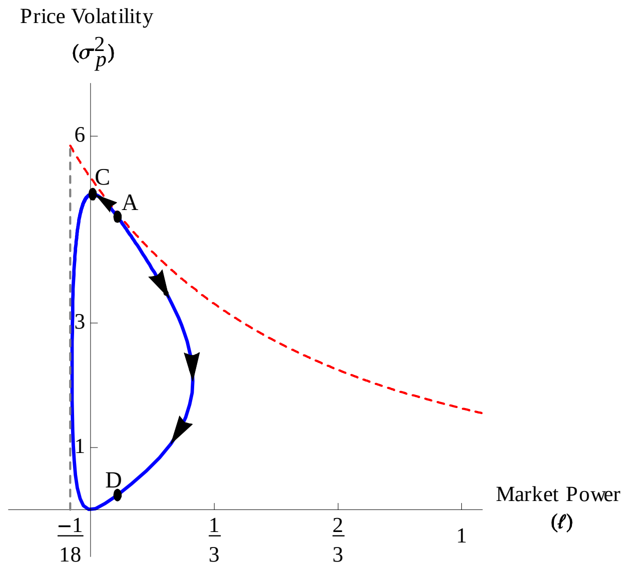

Figure 4: Set of equilibrium pairs $\left(l, \sigma_p^2\right)$ of market power and price volatility under confounding signals.

5 The Entire Set of Equilibrium Outcomes

In Section 4 we studied how the structure of private information can determine market power and price volatility. We showed that the set of outcomes under complete information completely captures the set of all possible equilibrium pairs of market power and price volatility that can be attained with any information structure. There are two natural questions that follow: (i) does the set of complete information equilibria also identify the set of all possible outcomes if we consider other relevant statistics of an equilibrium outcome; and (ii) can we characterize all moments of the equilibrium outcomes rather than a lower dimensional subset of statistics? We now answer both of these questions by characterizing the set of all possible equilibrium outcomes.

5.1 Distribution of Outcomes

We provide a description of the equilibrium outcomes from an ex ante perspective. We say that the joint distribution of variables $(\theta_1, ..., \theta_N, q_1, ..., q_N, p)$ is an *outcome distribution* of the demand function competition if the distribution is induced by an equilibrium outcome. The advantage of the description in terms of distributions of equilibrium outcomes is that it does not depend on the detailed description of the information structure. That is, two information structures may induce different beliefs and may induce different realizations over outcomes ex post, but as long as the distribution of outcomes ex ante is the same, these two information structures will be indistinguishable in terms of outcomes.

Since we focus on symmetric outcomes, we can simplify the description of the distribution. We define the common component of the payoff shock and the common component of the quantities:

$$\bar{\theta} \triangleq \frac{1}{N}\sum_{i\in N}\theta_i; \;\; \bar{q} \triangleq \frac{1}{N}\sum_{i\in N} q_i.$$

In symmetric environments, the joint distribution of variables $(\theta_1,...,\theta_N,q_1,...,q_N,p)$ is fully determined by the joint distribution of variables $(\theta_i,\bar{\theta},q_i,\bar{q},p)$. That is, we can focus attention on the joint distribution of the payoff shock and quantity of an individual agent with the corresponding averages. Finally, since the average quantity $\bar{q}$ is collinear with the price p due to the market clearing condition, we can omit the average quantity and we simply describe the joint distribution of variables $(\theta_i,\bar{\theta},q_i,p)$.

In the multivariate normal environment, the joint distribution is hence completely characterized by the first and second moments:

$$\begin{pmatrix}\theta_i\\ \bar{\theta}\\ q_i\\ p\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}\mu_\theta\\ \mu_\theta\\ \mu_q\\ \mu_p\end{pmatrix}, \begin{pmatrix}\sigma_\theta^2 & \rho_{\bar{\theta}\theta}\sigma_\theta\sigma_{\bar{\theta}} & \rho_{q\theta}\sigma_\theta\sigma_q & \rho_{p\theta}\sigma_\theta\sigma_p\\ \rho_{\theta\bar{\theta}}\sigma_\theta\sigma_{\bar{\theta}} & \sigma_{\bar{\theta}}^2 & \rho_{q\bar{\theta}}\sigma_{\bar{\theta}}\sigma_q & \rho_{p\bar{\theta}}\sigma_{\bar{\theta}}\sigma_p\\ \rho_{q\theta}\sigma_\theta\sigma_q & \rho_{q\bar{\theta}}\sigma_{\bar{\theta}}\sigma_q & \sigma_q^2 & \rho_{qp}\sigma_q\sigma_p\\ \rho_{p\theta}\sigma_\theta\sigma_p & \rho_{p\bar{\theta}}\sigma_{\bar{\theta}}\sigma_p & \rho_{qp}\sigma_q\sigma_p & \sigma_p^2\end{pmatrix}\right). \tag{21}$$

Some of the coefficients are part of the distribution of payoff shocks, and hence, they are exogenously determined: (i) the expected payoff shock of every agent (μ_θ), (ii) the expected average payoff shock $(\mu_{\bar{\theta}})$, (iii) the variance of the payoff shock of an agent σ_θ^2, (iv) the variance of the average payoff shock $\sigma_{\bar{\theta}}^2$, and (v) the correlation between the payoff shock of an agent and the average payoff shock $(\rho_{\theta\bar{\theta}})$. The rest of the coefficients are endogenously determined by the equilibrium outcome.

The joint distribution of outcomes thus contains nine endogenous variables: (i) the mean quantity bought by agent (μ_q) , (ii) the mean price (μ_p), (iii) the variance of the quantity bought by an agent (σ_q^2), (iv) the price volatility (σ_p^2), (v) the correlation between the price and the payoff shock of an agent $(\rho_{p\theta})$, (vi) the correlation between the price and the average payoff shock $(\rho_{p\bar{\theta}})$, (vii) the correlation between the quantity bought by an agent and the payoff shock of this agent $(\rho_{q\theta})$, $(viii)$ the correlation between the quantity bought by an agent and the average payoff shock $(\rho_{q\bar{\theta}})$, (ix) the correlation between the quantity bought by an agent and the price (ρ_{qp}).

To characterize the set of all possible feasible distributions it is useful to define the orthogonal components in the payoffs shocks and the demanded quantities:

$$\Delta\theta_i \triangleq \theta_i - \overline{\theta} \text{ and } \Delta q_i \triangleq q_i - \overline{q}.$$

The variable $\Delta\theta_i$ is the difference between the payoff shock of agent i and the average payoff shock (and analogously Δq_i). Hence, the variance $\sigma_{\Delta\theta_i}^2$ is the dispersion of the payoff shocks (and analogously $\sigma_{\Delta q_i}^2$).

The correlation $\rho_{\Delta q \Delta \theta}$ is an economically important quantity; it measures how efficiently the good is allocated across agents. In other words, the correlation $\rho_{\Delta q \Delta \theta}$ measures how much of the dispersion in the allocation across agents is caused by fundamental shocks and how much it is caused by noise. Note that $\Delta\theta_i$ is a linear combination of the variables θ_i and $\bar{\theta}$, while Δq_i is a linear combination of the variables q_i and p. Hence, the distribution (21) completely determines the correlation $\rho_{\Delta q_i, \Delta \theta_i}$.

5.2 Set of Feasible Distributions

We now provide a description of all equilibrium outcome distributions. For this it is useful to note that any distribution (21) completely determines the induced market power (as defined in (5)).

Theorem 3 (Set of Feasible Outcomes)
There exists an information structure that induces outcome distribution (21) if and only if the induced triple $(l, \rho_{p\bar{\theta}}, \rho_{\Delta q \Delta \theta})$ satisfies:

$$l \geq -\frac{1}{2\beta N} \text{ ; } \rho_{p\bar{\theta}} \in [0,1] \text{ ; } \rho_{\Delta q \Delta \theta} \in [0,1]. \tag{22}$$

The theorem characterizes the set of all outcome distributions that can be implemented as a Nash equilibrium of the demand function competition game for some information structure. The theorem provides two different results regarding the set of outcomes. First, it shows that an equilibrium outcome is fully determined by the triple $(l, \rho_{p\bar{\theta}}, \rho_{\Delta q \Delta \theta})$. Hence, any other moment of the distribution can be inferred simply from observing these three coefficients. Second, it establishes that there are few restrictions on the set of feasible triples $(l, \rho_{p\bar{\theta}}, \rho_{\Delta q \Delta \theta})$. More precisely, the only restrictions on these three coefficients are: (i) market power is bounded from below by $-1/2\beta N$, and (ii) the correlations are positive. Thus, the equilibrium conditions of demand function competition impose essentially no restrictions on these three coefficients — not even the distribution of payoff shocks (which is exogenous) imposes any restrictions on the triple $(l, \rho_{p\bar{\theta}}, \rho_{\Delta q \Delta \theta})$.

We now describe how the triple $(l, \rho_{p\bar{\theta}}, \rho_{\Delta q \Delta \theta})$ determines the distribution of outcomes. We distinguish between two types of restrictions (i) statistical restrictions that are independent of the equilibrium conditions (that is, they hold for any strategy profile of agents, not only the equilibrium ones), and (ii) restrictions imposed by the equilibrium conditions. Among the latter ones, we separate the restrictions imposed on the first and second moments of the distribution.

The equilibrium conditions are derived using the individual best response conditions. In particular, in any linear Nash equilibrium the quantity bought by agent i, the payoff shock of agent i, the price p

and the market power must satisfy the following condition:

$$q_i = \frac{\mathbb{E}[\theta_i|s_i, p] - p}{1 + \beta N(1 + l)}. \tag{23}$$

Heuristically, this relation can be derived from simply taking the first order condition of (1) with respect to q_i and replacing $\frac{\partial p}{\partial q_i}$ with $\beta N l$. The market clearing condition and (23) allow us to write an equation for the price

$$p^* = \frac{\beta}{1 + \beta N(1 + l)} \sum_{i \in N} \mathbb{E}[\theta_i|s_i, p^*], \tag{24}$$

which describes the equilibrium price in terms of the equilibrium beliefs of the agents. This is sufficient to characterize the mean price.

Lemma 1 (Equilibrium Mean Price)

The expected price of any linear Nash equilibrium must satisfy:

$$\mu_p = \frac{N\beta\mu_\theta}{1 + \beta N(1 + l)}, \tag{25}$$

where l is the equilibrium market power, and the expected equilibrium demand is

$$\mu_q = \frac{\mu_p}{\beta N}.$$

The expected equilibrium price is then determined only by the mean payoff shock and the equilibrium market power. In particular, there is a one-to-one relation between the mean price and the equilibrium market power. The relation between market power and the mean price is derived by taking the expectation of (24) and using the law of iterated expectations. The relation between the expected equilibrium demand and the expected price is implied by the market clearing condition.

We now show how (23) and (24) provide additional restrictions on the second moments of any equilibrium distribution.

Lemma 2 (Equilibrium Variance)

The second moments of any linear Nash equilibrium must satisfy:

$$\sigma_p^2 = \left(\frac{\rho_{p\bar{\theta}} \sigma_{\bar{\theta}} \beta N}{1 + \beta N(1 + l)} \right)^2, \tag{26}$$

$$\sigma_q^2 = \left(\frac{\rho_{p\bar{\theta}} \sigma_{\bar{\theta}}}{1 + \beta N(1 + l)} \right)^2 + \left(\frac{\rho_{\Delta q \Delta \theta}}{1 + l\beta N} \right)^2 (\sigma_\theta^2 - \sigma_{\bar{\theta}}^2), \tag{27}$$

and

$$\sigma_p = \rho_{qp} \sigma_q \beta N. \tag{28}$$

We can see that the variance of the price and the variance of the quantities bought by agents is determined by (i) the correlations $\rho_{p\bar{\theta}}$ and $\rho_{\Delta q \Delta \theta}$, and (ii) the market power. The market clearing condition then imposes the relationship between the variance of price and quantity. The price volatility is increasing with $\rho_{p\bar{\theta}}$ and thus the price volatility is driven by the average payoff shocks. Furthermore, the price volatility is decreasing with l as more market power means that the agents trade less. By multiplying (24) by p^*, taking expectations and using the law of iterated expectations we derive the following equation:

$$\mathbb{E}[(p^*)^2] = \frac{\beta}{1+\beta N(1+l)} \sum_{i \in N} \mathbb{E}[\theta_i p^*]. \tag{29}$$

The restriction (26) can now be derived directly from (25) and (29).

The variance of the aggregate quantity, σ_q^2, can be understood in a similar way. The expression (27) identifies two components that contribute to the quantity volatility. The first component depends only on the average quantity $\bar{q}$ traded by agents. As the average quantity purchased by the agents is collinear with the price, the intuition for this component is similar to the price volatility. In addition to the contribution of the average quantity $\bar{q}$, the variance of the quantity traded by agents is also determined by the orthogonal component of the quantity traded by the agents; this is the second component of (27). If the quantity traded by agent i is very volatile even when conditioning on $\bar{q}$, then this will contribute substantially to σ_q.

Finally, we can use the payoff environment and the symmetry condition to determine the remaining moments of the distribution.

Lemma 3 (Statistical Conditions on the Distributions)
Every distribution of outcomes must satisfy

$$\rho_{p\theta} = \rho_{p\bar{\theta}}\rho_{\theta\bar{\theta}}, \quad \rho_{q\bar{\theta}} = \rho_{p\bar{\theta}}\rho_{qp}; \tag{30}$$

and

$$\rho_{\Delta q \Delta \theta} = \frac{\rho_{q\theta} - \rho_{p\bar{\theta}}\rho_{pq}\rho_{\theta\bar{\theta}}}{\sqrt{(1-\rho_{qp}^2)(1-\rho_{\theta\bar{\theta}}^2)}}. \tag{31}$$

The previous lemma imposes several restrictions on the moments of a distribution of outcomes. The constraints (30) and (31) are consistency requirements that arise only from the fact that the distribution of quantities and payoff shocks is symmetric and that the price is collinear with the average quantity traded by agents.

5.3 How Private Information Determines the Moments

So far we approached the equilibrium outcomes of demand function competition by deliberately avoiding the use of specific information structure that give rise to the equilibrium behavior. This contrasts with the more conventional approach in the analysis of games with incomplete information. There, specific assumption are made regarding the true information structure and the analysts solves the equilibrium for a given information structure, or a class of parametrized information structures. In this subsection, we indicate how to relate these two approaches. In particular, we define some parametrized classes of information structures and describe the equilibrium outcomes they induce. For each information structure we compute the triple $(l, \rho_{p\bar{\theta}}, \rho_{\Delta q \Delta\theta})$; the market power, the correlation between the price and the average shocks and the correlation between the orthogonal components of quantities and payoff shocks. In doing so, we link the representation of the equilibrium outcomes in Theorem 3 with specific classes of parameterized information structures.

Recall that in Section 4.2, we studied three different information structures (one-dimensional noisy signals, multi-dimensional noisy signals and confounding signals) in order to get intuition for how market power and price volatility varied as we varied a one-dimensional parameter in natural information structures. In this section, we complement the analysis therein by studying two additional information structures: noise-free signals and canonical signals. The former allows us to decentralize the same outcomes as the set of equilibria under complete information while the latter allows decentralizing *all* equilibrium outcomes. Thus, they will give a complementary view of how the information structure determines the equilibrium outcome. We also use the one-dimensional signals (studied in Section 4.2) to help build intuition for how the canonical signals determine the equilibrium outcome.

Noise-Free Signals The class of noise-free signals decentralize all outcomes that arise under complete information as a unique equilibrium under incomplete information, and we used them to establish Theorem 1. Here, each agent i observes:

$$s_i = \theta_i + (\gamma - 1)\frac{1}{N}\sum_{j\in N}\theta_j, \tag{32}$$

where $\gamma \in \mathbb{R}$. In this case, the outcome of the demand function competition game is given by:

$$\rho_{p\bar{\theta}} = 1, \quad \rho_{\Delta\theta\Delta q} = 1, \quad l = L(\gamma),$$

where the function $L(\cdot)$ is defined as follows:

$$L(\gamma) \triangleq \frac{1}{2\beta N}\left(-N\beta\frac{(N-1)\gamma - 1}{(N-1)\gamma + 1} - 1 + \sqrt{\left(N\beta\frac{(N-1)\gamma - 1}{(N-1)\gamma + 1}\right)^2 + 2\beta N + 1}\right).$$

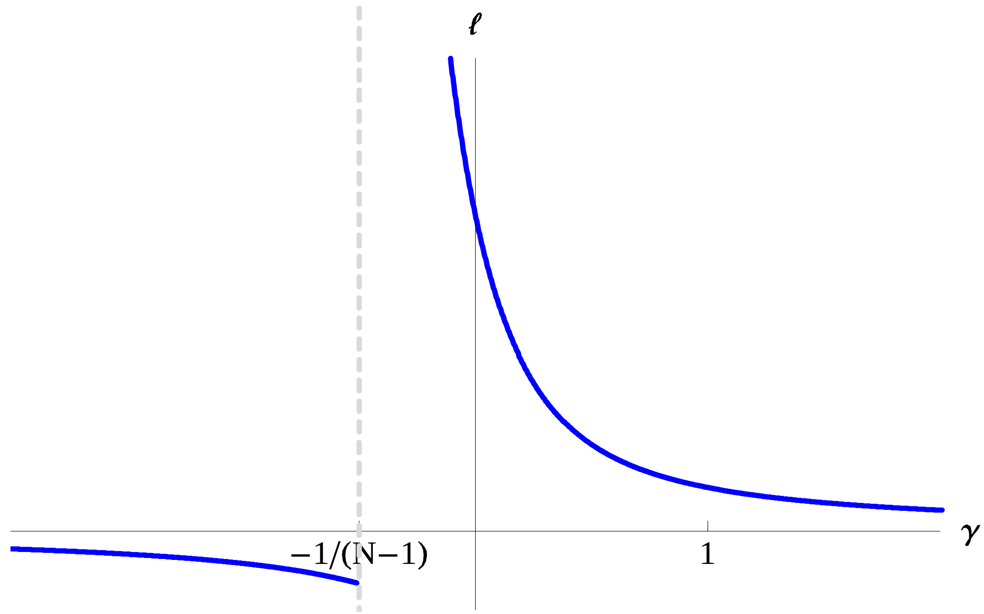

Figure 5: Set of equilibrium pairs $\left(l, \sigma_p^2\right)$ of market power and price volatility under noise-free signals.

Under the noise-free signals the correlations $(\rho_{p\bar{\theta}}, \rho_{\Delta\theta\Delta q})$ are equal to one. The noise free signals decentralize the outcomes of the complete information equilibria in which there is no extraneous noise in the outcomes — thus the correlations are equal to one. On the other hand the market power is determined by the confounding parameter γ using the function $L(\gamma)$, which we plot in Figure 5. We can see that the range of $L(\gamma)$ is $[-1/2, \infty)$, which implies that all market powers can be decentralized for some γ. As $\gamma \to -1/(N-1)$, $L(\gamma)$ approaches an asymptote; for values $\gamma < -1/(N-1)$ the function is negative; in the limit as $\gamma \to \infty$ the function $L(\gamma)$ approaches 0.

Finally, we remark that (32) is not the same signal as (20). If we decompose the payoff shock in terms of a common and an idiosyncratic component (as in (16)), then (32) would be written as follows:

$$s_i = \tau_i + \gamma \cdot \omega + (\gamma - 1)\frac{1}{N}\sum_{j \in N} \tau_j.$$

Even though this signal and (20) look similar, the former decentralizes all the points in the red dashed curve while the latter allows to decentralize the points in a blue curve in Figure 4. This serves as an additional illustration of how small changes in an information structure may lead to large changes in the equilibrium outcomes.

Canonical Signals The third class of signal combines elements of the noise-free signals and the one-dimensional noisy signals. We assume that agent i observes a one-dimensional signal s_i given by:

$$s_i = \theta_i + \varepsilon_i + (\gamma - 1)(\bar{\theta} + \bar{\varepsilon}). \tag{33}$$

The term ε_i is a noise term that is independent of all payoff shocks $\{\theta_i\}_{i \in N}$, has a variance of σ_ε^2, and a correlation $\rho_{\varepsilon\varepsilon}$ across signals.

We refer to (33) as canonical signals because they allow us to decentralize all feasible outcomes. In other words, for every distribution of outcomes that can be implemented by some information structure, it can also be implemented by a canonical information structure. The relevant moments of the equilibrium outcome are now given by:

$$\rho_{p\bar{\theta}}^2 = \frac{1}{1 + \frac{\rho_{\varepsilon\varepsilon} N + (1-\rho_{\varepsilon\varepsilon})}{\rho_{\theta\theta} N + (1-\rho_{\theta\theta})} \frac{\sigma_\varepsilon^2}{\sigma_\theta^2}}, \; \rho_{\Delta q \Delta\theta}^2 = \frac{1}{1 + \frac{1-\rho_{\varepsilon\varepsilon}}{1-\rho_{\theta\theta}} \frac{\sigma_\varepsilon^2}{\sigma_\theta^2}}, \;\; l = L(\gamma \frac{\rho_{\Delta\theta\Delta q}^2}{\rho_{p\bar{\theta}}^2}).$$

The correlation coefficient are determined by the variance of the noise term in an analogous way to the noisy one-dimensional signals. In contrast to the noisy one-dimensional signals, we now allow the noise terms to be correlated across agents, which is incorporated in the computation of the correlations. The market power combines the intuitions from the noise-free signals and the noisy one-dimensional signals; here the market power is determined by both the confounding parameter and the correlations. It is now easy to check that by varying $(\sigma_\varepsilon^2, \rho_{\varepsilon\varepsilon}, \gamma)$ we can span all values of $(\rho_{p\bar{\theta}}, \rho_{\Delta q \Delta\theta}, l)$. Hence, these canonical signals allow us to span all possible outcomes.

6 Cournot vs. Demand Function Competition

The competition in demand functions constitutes a market mechanism that balances demand and supply with a *uniform price* across traders. As the competition in demand function is only one of many mechanisms that match demand and supply on the basis of a uniform price, it is natural to compare the outcome under demand function competition with other uniform price market mechanisms. A natural candidate to consider is Cournot competition or competition in quantities.

We maintain the payoff environment as described in Section 2, but we now assume that the agents submit unconditional quantities $\{q_i\}_{i \in N}$. The market clearing price is given by:

$$p^* = \beta \sum_{i \in N} q_i.$$

The equilibrium trading behavior in quantity competition differs from demand function competition in two important respects. First, in demand function competition the agents can make their trade contingent on the equilibrium price, whereas in quantity competition the demand has to be stated unconditional. Second, in demand function competition the price impact of each agent depends on the submitted demand function of all other agents, whereas in quantity competition the price impact is constant and simply given by the supply conditions. We show how these two aspects induce important differences in the set of possible outcomes across these two forms of market mechanisms, even when we compare across all possible information structure.[4] In recent work, Lambert, Ostrovksy, and Panov (2018) consider how informationally sensitive the trading outcomes are in a hybrid model between demand function competition and Cournot competition. They study the informational efficiency of the Kyle (1985) model in a high-dimensional model and their analysis also relies on the multivariate normal structure of payoffs and signals.

We compare the set of feasible pairs of market power and price volatility for Cournot competition and demand function competition. Similarly to the analysis of the demand function competition, we can obtain a description of the equilibrium outcomes under all information structures. In this section we refine the definition of market power in terms of the ratio of the expectations:

$$l = \frac{1}{N}\frac{\mathbb{E}\left[\sum_{i\in N}\theta_i - q_i - p\right]}{\mathbb{E}\left[p\right]}.$$

With demand function competition, this always coincides with the expectation of the ratio. Here, we use this refined measure to convey most directly that under Cournot competition, the agents' market power is constant.

Theorem 4 (Cournot Equilibria Under All Information Structures)
There exists an information structure that induces a pair of market power and price volatility (l, σ_p^2) *if and only if:*

$$l = \frac{1}{N} \quad \textit{and} \quad \sigma_p^2 \leq \frac{1}{4}\left(\frac{\sqrt{1+\beta}\sigma_{\bar{\theta}} + \sqrt{(\beta+\beta N+1)\sigma_{\Delta\theta}^2 + (1+\beta)\sigma_{\bar{\theta}}^2}}{\sqrt{1+\beta}(\beta+\beta N+1)}\right)^2. \tag{34}$$

Moreover, all feasible pairs of market power and price volatility (l, σ_p^2) *are induced by a unique equilibrium for some information structure.*

In Cournot competition the first moment of the individual and aggregate demand is *independent* of the information structure. In particular, the market power is always equal to $l = 1/N$. By contrast, in

[4] In an early version of this paper, Bergemann, Heumann, and Morris (2015a), we provide a more exhaustive comparison of the equilibrium behavior across many uniform price mechanisms, including the Bertrand price mechanism, the Kyle trading mechanism and noisy price mechanism.

demand function competition, the set of feasible market powers is a one-dimensional object without an upper bound.

Yet, under Cournot competition the maximum price volatility can be larger than under demand function competition. From Theorem 2 we can infer that the maximum price volatility under demand function competition is given by:

$$\sigma_p^2 = \frac{(\beta N)^2}{(\frac{1}{2} + \beta N)^2}\sigma_{\bar{\theta}}^2,$$

and the maximum price volatility under quantity competition is displayed in (34). In contrast to demand function competition, the price volatility can grow even as the variance of the average payoff shock shrinks. As the inequality in (34) documents, the maximal price volatility under quantity competition grows if the contribution of the idiosyncratic component in the payoff shock increases, that is if $1 - \rho_{\theta\theta}$ increases.

The extra degree of freedom that demand function competition has on the first moment is a reflection of the fact that market power is endogenously determined. The extra degree of freedom that Cournot competition has in the second moments is reflective of the fact that the agents cannot condition the quantity bought on the equilibrium price. Hence, there is less information that disciplines the quantities bought by agents. In Cournot competition, the price volatility and the volatility in the quantity demanded by the agents are not determined separately (as σ_p^2 and σ_q^2 in (26) and (27)) but rather there is a single equation that jointly determines the volatility in the quantities demanded by agents. This implies that the price volatility can increase with the absolute level of uncertainty about payoff shocks, σ_θ^2, and not only with the uncertainty about the average payoff shock $\sigma_{\bar{\theta}}^2$. We illustrate the different behavior of the first and second moments across these market mechanism in Figure 6. Most importantly, we see that for Cournot competition the level of market power is constant across information structures, while with demand function competition the market power varies substantially with the information structure.

The lack of common conditioning device in quantity competition also leads to fewer restriction on the correlation coefficients that describe the entire matrix of second moments. With quantity competition the set of feasible second moments is a three dimensional object. In particular, for any $(\rho_{p\bar{\theta}}, \rho_{\Delta q \Delta \theta}, \rho_{qq}) \in [0,1]^3$, there exists an information structure that induces a distribution of outcomes under quantity competition with correlations $(\rho_{p\bar{\theta}}, \rho_{\Delta q \Delta \theta}, \rho_{qq})$. Yet, for a fixed first moment, the set of possible second moments in the demand function competition is a two dimensional object as stated in Theorem 3.

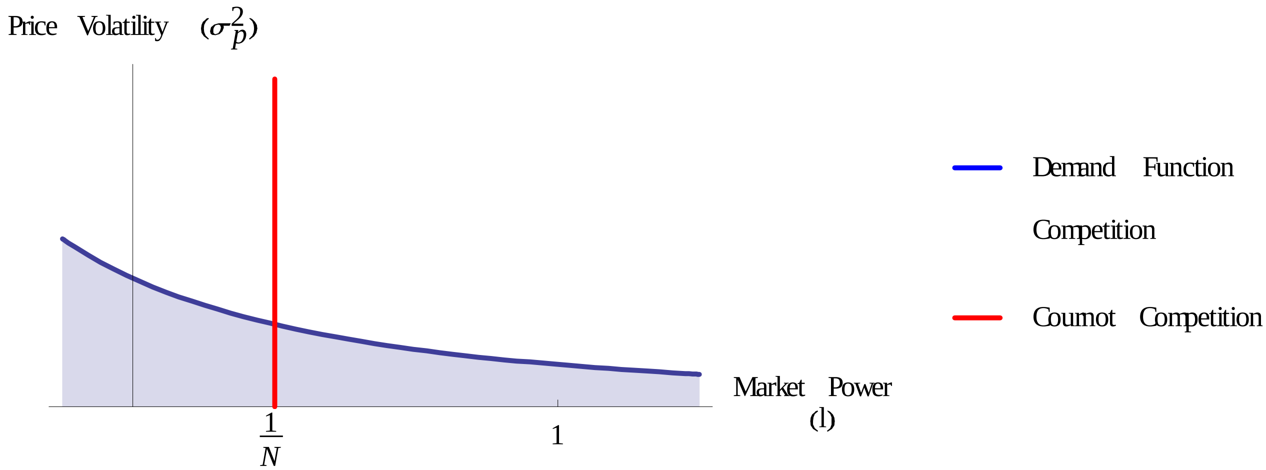

Figure 6: Comparison of the first and second equilibrium moments under demand function competition and quantity competition.

7 Conclusions

In this paper we study demand function competition. Our results provide positive and negative results regarding our ability to make predictions in this form of market microstructure. On the one hand, we showed that any market power is possible— from $-1/2$ to infinity. Considering small amounts of incomplete information does not allow us to provide any sharper predictions, unless one is able to make additional restrictive assumptions regarding the nature of the incomplete information. On the other hand, we showed that we can provide many substantive predictions on the outcome of demand function competition that are robust to the information structure.

The analysis in our paper provides a way of thinking about demand function competition in a more abstract way. In particular, we analyze directly quantities and payoff shocks, abstracting from the specific demands that are submitted in equilibrium. While this allows us to analyze demand function competition, it may also be helpful to analyze other forms of market microstructure, and perhaps more interestingly, to compare between them. We believe this may a fruitful direction for future work.

The comparison between demand function and Cournot competition indicates that distinct trading mechanisms for the same allocation problem may respond surprisingly different to small changes in the structure of private information.

8 Appendix

We first present three lemmas that are used to prove the results in the main text.

Lemma 4 (Characterization of Linear Nash Equilibrium)
The demand function $x(s_i, p) = c_0 + \sum_{j\in J} c_j s_{ij} - mp$ *is a linear Nash equilibrium if and only if:*

$$x(s_i, p) = c_0 + \sum_{j\in J} c_j s_{ij} - mp = \frac{\mathbb{E}[\theta_i|p, s_i] - p}{1+\lambda}, \tag{35}$$

where λ *is given by:*

$$\lambda = \frac{\beta}{1 + \beta m(N-1)}, \tag{36}$$

and it satisfies $\lambda \geq -1/2$*. The expectation* $\mathbb{E}[\theta_i|p, s_i]$ *is computed using the induced price distribution, which is given by:*

$$p = \frac{\beta(Nc_0 + \sum_{i\in N}\sum_{j\in J} c_j s_{ij})}{1 + m\beta N}. \tag{37}$$

Proof. We conjecture a symmetric linear Nash equilibrium in which agent i submits demand function:

$$x_i = c_0 + \sum_{j\in J} c_j s_{ij} - mp. \tag{38}$$

and show that this is a symmetric linear Nash equilibrium if and only if (35) and (36) are satisfied and the equilibrium price is determined by (37)

If all agents submit linear demand function as in (38), then market clearing implies that:

$$p^* = \beta \sum_{i\in N} x(s_i, p^*) = \beta(Nc_0 + \sum_{i\in N}\sum_{j\in J} c_j s_{ij}) - \beta N m p^*.$$

Solving for p^* we conclude that market clearing implies that:

$$p^* = \frac{\beta(Nc_0 + \sum_{i\in N}\sum_{j\in J} c_j s_{ij})}{1 + m\beta N}.$$

Thus (37) is satisfied.

We now examine agent i's maximization problem. Given the demands submitted by other agents $\{x_j(p)\}_{j\neq i}$, agent i maximizes:

$$\max_{x_i(p)\in\mathcal{C}(\mathbb{R})} \quad \mathbb{E}[\theta_i x_i(p^*) - p^* x_i(p^*) - \frac{x_i(p^*)^2}{2}] \tag{39}$$
$$\text{where } \beta \sum_{k\in N} x_k(p^*) = p^*.$$

A linear demand functions is a Nash equilibrium if and only if the demand function of agent i solves (39). An alternative way to write the market clearing condition is to write it in terms of agent i's residual supply. Agent i's residual supply is given by:

$$r_i(p) = \frac{p - \beta \sum_{k \neq i} x_k(p)}{\beta}. \tag{40}$$

If agent i submits a demand $x_i(p)$, then market clearing implies that $x_i(p^*) = r_i(p^*)$.

We first solve agent i's maximization problem assuming that he knows his residual supply. This corresponds to finding the quantity q_i that maximizes agent i's expected utility conditional on agent i's signals and agent i's residual supply. If agent i knows his residual supply, then he solves:

$$\max_{q_i \in \mathbb{R}} \mathbb{E}[\theta_i | r_i(p), s_i] q_i - r^{-1}(q_i) q_i - \frac{1}{2} q_i, \tag{41}$$

where $r_i^{-1}(\cdot)$ is the inverse function of r_i (defined in (40)). Note that the residual supply of agent i may contain information about θ_i so this is added as a conditioning variable. In other words, in a linear Nash equilibrium the intercept of the residual supply $r_i(p)$ is measurable with respect to:

$$\sum_{k \neq i} \sum_{j \in J} c_j s_{kj}.$$

Hence, agent i can use the intercept of $r_i(p)$ as additional information about θ_i. Note that in a linear Nash equilibrium the slope of $r_i(p)$ does not depend on the realization of the signals $\{s_{ij}\}_{i \in N, j \in J}$.

Taking the first order condition of (41) we obtain:

$$\mathbb{E}[\theta_i | r_i, s_i] - r^{-1}(q_i^*) - q_i^* \frac{\partial r^{-1}(q_i^*)}{\partial q_i^*} - q_i^* = 0.$$

The derivative of the inverse residual supply is given by:

$$\frac{\partial r^{-1}(q_i)}{\partial q_i} = \left(\frac{\partial r_i(p)}{\partial p} \right)^{-1} = \frac{\beta}{1 + \beta m (N-1)},$$

where the first equality is using the implicit function theorem and the second equality is taking the derivative of (40) with respect to p. Note that the derivative of the inverse residual supply is equal to λ (as defined in (36)):

$$\lambda = \left(\frac{\partial r_i(p)}{\partial p} \right)^{-1}.$$

The objective function of the maximization problem (41) is a quadratic function of q_i and the coefficient on the quadratic component is equal to $-(\lambda + 1/2)$. Thus, the second order conditions is satisfied if and only if $\lambda \geq -1/2$. It is clear that, if $\lambda < 1/2$ then the agent's objective function is strictly convex and hence (41) does not have a solution. Therefore, there is no equilibrium with $\lambda > 1/2$.

If agent i knows his residual demand, then the first order condition can be written as follows:

$$q_i^* = \frac{\mathbb{E}[\theta_i | r_i, s_i] - r^{-1}(q_i^*)}{1+\lambda}.$$

Note that $r^{-1}(q_i^*)$ is the equilibrium price:

$$p^* = r^{-1}(q_i^*).$$

Hence, we can write the first order condition of agent i as follows:

$$q_i^* = \frac{\mathbb{E}[\theta_i | p^*, s_i] - p^*}{1+\lambda}.$$

Note that the equilibrium price p^* is informationally equivalent to the intercept of the residual supply faced by agent i. This is because p^* is computed using r_i and the demand function submitted by agent i. Hence, for agent i, conditioning on the residual supply or the equilibrium price is informationally equivalent. Hence, we can replace it as a conditioning variables.

In demand function competition agent i does not know his residual supply but an agent submits a whole demand schedule. If agent i submits demand schedule:

$$x(p) = \frac{\mathbb{E}[\theta_i | p, s_i] - p}{1+\lambda}, \tag{42}$$

then he will buy the same quantity as if he knew his residual supply. Thus, for any set of linear demands submitted by the other agents $\{x_j(s_j)\}_{j \neq i}$, agent i's best response is given by (42). The expectation $\mathbb{E}[\theta_i | p, s_i]$ is computed the same way as if p was the equilibrium price. That is, for any residual supply $r_i(p)$, if agent i submits demand function (42), then p^* is chosen to satisfy $x(p^*) = r_i(p^*)$. Hence, agent i buys a quantity:

$$q_i^* = \frac{\mathbb{E}[\theta_i | r_i, s_i] - p}{1+\lambda},$$

which is the optimal quantity as if he knew his residual supply.

Hence, a linear Nash equilibrium is determined by constants $(c_0, ..., c_J, m)$ such that:

$$c_0 + \sum_{j \in J} c_j s_j - mp = \frac{\mathbb{E}[\theta_i | p, s_i] - p}{1+\lambda},$$

where λ is given by:

$$\lambda = \frac{\beta}{1 + \beta m (N-1)},$$

and where expectation $\mathbb{E}[\theta_i | p, s_i]$ is computed the same way as if p was the equilibrium price. ■

Lemma 5 (Relation between Price Impact and Market Power)

In every symmetric linear Nash equilibrium where agents' price impact is λ, the induced market power is $l = \lambda/(\beta N)$.

Proof. Lemma 4 shows that in every linear Nash equilibrium in which agents have price impact λ, they submit demands

$$x(s_i, p) = \frac{\mathbb{E}[\theta_i|p, s_i] - p}{1+\lambda}, \tag{43}$$

Rearranging terms, we obtain:

$$\lambda x(s_i, p) = \mathbb{E}[\theta_i|p, s_i] - x(s_i, p) - p. \tag{44}$$

Summing up over all agents and multiplying times β, we get:

$$\lambda\beta \sum_{i\in N} x(s_i, p) = \beta \sum_{i\in N} (\mathbb{E}[\theta_i|p, s_i] - x(s_i, p) - p). \tag{45}$$

Note that $x(s_i, p)$ is the quantity bought by agent i in equilibrium so the market clearing condition implies that $\beta \sum_{i\in N} x(s_i, p) = p$. Thus (45) can be written as follows:

$$\lambda p = \beta \sum_{i\in N} \mathbb{E}[\theta_i - q_i - p|p, s_i]. \tag{46}$$

Here we wrote q_i and p inside the expectation; this is possible because they are measurable with respect to the conditioning variables. Taking the expectation of the previous equation conditional on p (i.e. taking expectation $\mathbb{E}[\cdot|p]$) and using the law of iterated expectations:

$$\lambda p = \beta \sum_{i\in N} \mathbb{E}[\theta_i - q_i - p|p], \tag{47}$$

Rearranging terms, we have:

$$\lambda = (\beta N) \frac{1}{N} \sum_{i\in N} \mathbb{E}[\frac{\theta_i - q_i - p}{p}|p]. \tag{48}$$

Taking the expectation of the previous equation and using the law of iterated expectations

$$\frac{\lambda}{\beta N} = \frac{1}{N} \sum_{i\in N} \mathbb{E}[\frac{\theta_i - q_i - p}{p}] = l, \tag{49}$$

which establishes the result. ■

Lemma 6 (Continuum of Equilibria)

Under complete information, for every $\lambda \geq -1/2$, the following demand function is a symmetric Nash equilibrium:

$$x_i(p) = \frac{1}{1+\lambda}\left(\theta_i - (1 - \hat{\gamma}(\lambda))\bar{\theta}\right) - \frac{1}{N-1}(\frac{1}{\lambda} - \frac{1}{\beta})p, \tag{50}$$

where $\hat{\gamma}$ is defined as follows:

$$\hat{\gamma}(\lambda) \triangleq \frac{(\lambda+1)(\beta N - \lambda)}{\lambda(N-1)(\beta N + \lambda + 1)}. \tag{51}$$

Proof. We check that the demand function (50) satisfy (35) and (36). The equilibrium price satisfies

$$\beta \sum_{i \in N} x_i(p^*) = p^*.$$

When agents submit demand functions as in (50) the market clearing condition implies that:

$$\beta \sum_{i \in N} x_i(p^*) = \beta \left(\frac{N}{1+\lambda} \hat{\gamma} \bar{\theta} - \frac{N}{N-1} (\frac{1}{\lambda} - \frac{1}{\beta}) p^* \right) = p^*.$$

Rearranging terms, the equilibrium price can be written as follows:

$$p^* = \frac{\beta N \bar{\theta}}{1 + \lambda + \beta N}. \tag{52}$$

Using (52) we note that:

$$-\frac{(1-\hat{\gamma})\bar{\theta}}{1+\lambda} - \frac{1}{N-1}(\frac{1}{\lambda} - \frac{1}{\beta}) p^* + \frac{1}{1+\lambda} p^* = 0.$$

This equation can be verified by replacing p^* with the expression in (52). It follows that:

$$\frac{\mathbb{E}[\theta_i | \theta_i, p^*]}{1+\lambda} = \frac{\theta_i}{1+\lambda} - \frac{(1-\hat{\gamma})\bar{\theta}}{1+\lambda} - \frac{1}{N-1}(\frac{1}{\lambda} - \frac{1}{\beta}) p^* + \frac{1}{1+\lambda} p^*.$$

Here the equality follows from the fact that the last three terms cancel each other. Hence, we can write (50) as follows:

$$\begin{aligned} x_i(p) &= \frac{1}{1+\lambda} \left(\theta_i - (1-\hat{\gamma})\bar{\theta} \right) - \frac{1}{N-1}(\frac{1}{\lambda} - \frac{1}{\beta}) p \\ &= \frac{\theta_i - p}{1+\lambda} - \frac{(1-\hat{\gamma})\bar{\theta}}{1+\lambda} - \frac{1}{N-1}(\frac{1}{\lambda} - \frac{1}{\beta}) p + \frac{1}{1+\lambda} p \\ &= \frac{\mathbb{E}[\theta_i | s_i, p] - p}{1+\lambda}. \end{aligned} \tag{53}$$

Hence, (35) from Lemma 4 is satisfied. Additionally, note that, if agents submit demand functions as in (50), then

$$m = \frac{1}{N-1}(\frac{1}{\lambda} - \frac{1}{\beta}).$$

Inverting the function, we obtain:

$$\lambda = \frac{\beta}{1 + \beta m (N-1)}.$$

Hence, (36) is also satisfied. Using Lemma 4, this establishes the linear Nash equilibrium. ■

Proof of Proposition 1. In the proof of Lemma 6 we showed that in every symmetric linear Nash equilibrium in which agents have price impact λ, the equilibrium price is given by (see (52)):

$$p^* = \frac{\beta N \bar{\theta}}{1 + \lambda + \beta N}.$$

Moreover, we also proved that there exists an equilibrium in which agents have price impact λ for all $\lambda \geq -1/2$, which establishes the result. ■

Proof of Corollary 1. Lemma 5 states that in every symmetric linear Nash equilibrium in which agents' have price impact λ, the induce market power is $l = \lambda/(\beta N)$. In the proof of Lemma 6, we show that in every symmetric linear Nash equilibrium in which agents' have price impact λ, the equilibrium price is given by (see (52)):

$$p^* = \frac{\beta N \bar{\theta}}{1 + \lambda + \beta N}.$$

Thus, the price volatility is given by:

$$\sigma_p^2 = \left(\frac{\beta N}{1 + \lambda + \beta N} \right)^2 \mathrm{var}(\bar{\theta}),$$

which establishes the result. ■

Proof Theorem 1. We prove the result by decomposing the payoff shock, into two independent payoff shocks:

$$\theta_i \triangleq \eta_i + \phi_i. \tag{54}$$

We assume that the sets of payoff shocks $\{\eta_i\}_{i \in N}$ are independent of the shocks $\{\phi_i\}_{i \in N}$, the shocks are jointly normally distributed, and:

$$\mu_\eta = \mu_\phi = \frac{\mu_\theta}{2} \quad \text{and} \quad \mathrm{corr}(\eta_i, \eta_j) = \mathrm{corr}(\phi_i, \phi_j) = \mathrm{corr}(\theta_i, \theta_j). \tag{55}$$

Finally, we assume that the variance of the shocks $\{\phi_i\}_{i \in N}$ is equal to ε:

$$\mathrm{var}(\phi_i) = \varepsilon \quad \text{and} \quad \mathrm{var}(\eta_i) = (\sigma_\theta^2 - \varepsilon). \tag{56}$$

We remark that (55) and (56) guarantee that:

$$\mathrm{var}(\phi_i + \eta_i) = \sigma_\theta^2; \;\; \mathrm{cov}(\phi_i + \eta_i, \phi_j + \eta_j) = \mathrm{cov}(\theta_i, \theta_j),$$

and thus, the joint distribution of the random variables $\{\eta_i + \phi_i\}_{i \in N}$ is equal to the the joint distribution of the original payoff shocks $\{\theta_i\}_{i \in N}$.

We assume that every agent observes the realization of all shocks $\{\eta_i\}_{i \in N}$. In other words, each agent observes N signals, each signal being equal to one of the shocks η_i. Additionally, agent i observes a signal that is equal to a weighted difference between his shock ϕ_i and the average of all shocks $\{\phi_j\}_{j \in N}$

$$s_i = \phi_i - (1 - \gamma) \frac{1}{N} \sum_{j \in N} \phi_j. \tag{57}$$

Here $\gamma \in \mathbb{R}$ is any number in the real line. Throughout this proof s_i denotes only the one-dimensional signal (57) and not the whole vector of signals an agent observes. We remark that under this information structure:

$$\forall i, j \in N, \quad \text{var}(\theta_i | \eta_1, ..., \eta_N, s_j) = \text{var}(\phi_i | s_j) \leq \text{var}(\phi_i) = \varepsilon.$$

It follows that under this information structure (12) is satisfied.

In any linear Nash equilibrium, the equilibrium price must be a linear function of the shocks $\{\eta_i\}_{i \in N}$ and the signals $\{s_i\}_{i \in N}$. The symmetry of the conjectured equilibrium, implies that there exists constants $\hat{c}_0, \hat{c}_1, \hat{c}_2$ such that the equilibrium price satisfies:

$$p^* = \hat{c}_0 + \hat{c}_1 \bar{\phi} + \hat{c}_2 \bar{\eta}.$$[5]

Regardless of the values of $\hat{c}_0, \hat{c}_1, \hat{c}_2$ the following equation is satisfied:

$$\mathbb{E}[\theta_i | \{\eta_i\}_{i \in N}, s_i, p^*] = \theta_i.$$

That is, agent i can infer perfectly θ_i using the realization of the shocks $\{\eta_i\}_{i \in N}$, the signal s_i and the equilibrium price. This is because agent i can infer $\bar{\phi}$ from p^*, which in addition to s_i, allows agent i to perfectly infer ϕ_i (note that $\bar{\eta}$ is common knowledge).

Lemma 4 states that agent i submits demand function:

$$x_i(p) = \frac{\mathbb{E}[\theta_i | \{\eta_i\}_{i \in N}, s_i, p^*] - p}{1 + \lambda},$$

for some $\lambda \geq -1/2$. However, in equilibrium $\mathbb{E}[\theta_i | \{\eta_i\}_{i \in N}, s_i, p^*] = \theta_i$ so in equilibrium agent i buys a quantity equal to:

$$q_i^* = \frac{\theta_i - p^*}{1 + \lambda}, \tag{58}$$

for some $\lambda \geq -1/2$. The market clearing condition implies that $p^* = \beta \sum q_i^*$, and so the equilibrium price is given by:

$$p^* = \frac{\beta N \bar{\theta}}{1 + \lambda + \beta N}, \tag{59}$$

for some $\lambda \geq -1/2$. Hence, the equilibrium price is measurable with respect to $\bar{\theta}$. That is, the equilibrium price must satisfy that $\hat{c}_1 = \hat{c}_2$. It is important to clarify that the linearity and symmetry of the conjectured equilibrium guarantees that the price is an affine function of $\bar{\eta}$ and $\bar{\phi}$. Yet, since the equilibrium price plus the private signals observed by agent i allows agent i to infer θ_i, the quantity bought by agent i is measurable with respect to θ_i. Hence, using the linearity and the symmetry, the price must be a linear function of $\bar{\theta}$. Note that for a fixed γ, the quantity bough by agent i and the price

[5] Recall that according to the notation introduced in the main text $\bar{\eta} = \sum_{i \in N} \eta_i / N$ and $\bar{\phi} = \sum_{i \in N} \phi_i / N$.

are equal to (58) and (59) respectively. This is the same as the equilibrium under complete information when agents have price impact λ (compare with (53) and (52)). Thus, we are only left with showing that for a fixed γ there is a unique equilibrium and every price impact $\lambda \geq -1/2$ is spanned by some $\gamma \in \mathbb{R}$.

Given the equilibrium price in (59) (as a function of λ), we can find an expression for $\mathbb{E}[\theta_i|\{\eta_i\}_{i\in N}, s_i, p^*]$ (in terms of the conditioning variables). We first note that:

$$\left(\frac{p^*}{\beta N}(1+\lambda+\beta N) - \bar{\eta}\right) = \bar{\phi}.$$

Hence, the expectation can be written as follows:

$$\mathbb{E}[\theta_i|p^*, s_i, \{\eta_i\}_{i\in N}] = s_i + \eta_i + (1-\gamma)\left(\frac{p^*}{\beta N}(1+\lambda+\beta N) - \bar{\eta}\right) = \theta_i.$$

Recall that in equilibrium agent i submits demand function:

$$x_i(p) = \frac{\mathbb{E}[\theta_i|p^*, s_i, \{\eta_i\}_{i\in N}] - p}{1+\lambda}.$$

Hence, the slope of the demand submitted by agent i is equal to:

$$\begin{aligned} m &= -\frac{\partial x_i(p)}{\partial p} = \frac{-1}{1+\lambda}\left(\frac{\partial \mathbb{E}[\theta_i|p^*, s_i, \{\eta_i\}_{i\in N}]}{\partial p^*} - 1\right) \\ &= \frac{1-(1-\gamma)\frac{1}{\beta N}(1+\lambda+\beta N)}{1+\lambda}. \end{aligned}$$

The previous equation gives a relation between agent i's price impact (i.e. λ) and the slope of the demand function submitted by agent i (i.e. m). Equation (36) is a second equation that relates λ and m. Using these two equations we can find λ in terms of the confounding parameter γ:

$$\lambda = \frac{1}{2}\left(-1 - N\beta\frac{\gamma(N-1)-1}{\gamma(N-1)+1} \pm \sqrt{\left(N\beta\frac{\gamma(N-1)-1}{\gamma(N-1)+1}\right)^2 + 2N\beta + 1}\right). \quad (60)$$

Only the positive root is a valid solution as the negative root yields a λ less than $-1/2$ (which violates the condition in Lemma 4). Hence, there is a unique equilibrium in which the price impact is equal to the positive root of (60).

Finally, to show that the noise-free signals span the same outcomes as the outcomes under complete information we need to show that for all $\lambda \geq -1/2$, there exists a γ that satisfies (60) with the positive root. To check this note that inverting (60) (using the positive solution), we have that γ as a function of λ is given by (51). Hence, for any $\lambda \geq -1/2$, if γ is given by (51), there exists a unique linear Nash equilibrium in which the equilibrium outcome is the same as the equilibrium outcome under complete information when the price impact is λ. ■

Proof of Theorem 2. We prove necessity and sufficiency separately.

"If" Part. Let (l, σ_p^2) be such that (13) is satisfied. We show that there exists an information structure that induces this market power and price volatility as a unique equilibrium. Suppose the payoff shocks are decomposed as in (54) and (55). Additionally, assume that the variances are given by:

$$\mathrm{var}(\eta_i) = \sigma_p^2 \frac{N}{\rho_{\theta\theta}(N-1)+1} \frac{(1+\beta N(1+l))^2}{(\beta N)^2} \text{ and } \mathrm{var}(\phi_i) = (\sigma_\theta^2 - \mathrm{var}(\eta_i)). \tag{61}$$

Note that $\mathrm{var}(\phi_i)$ (as defined in (61)) is always positive because the theorem states that:

$$\sigma_p^2 \leq \frac{(\beta N)^2}{(1+\beta N(1+l))^2} \sigma_\theta^2$$

and so $\mathrm{var}(\eta_i)$ (as defined in (61)) is less or equal than $\mathrm{var}(\theta_i)$. We assume that the payoff shocks $\{\eta_i\}_{i\in N}$ are common knowledge (i.e. every agent observes all shocks $\{\eta_i\}_{i\in N}$) and agents have no information on the realization of the shocks $\{\phi_i\}_{i\in N}$. This model is isomorphic to a model in which agents have complete information and the only shocks are $\{\eta_i\}_{i\in N}$.

Corollary 1 states that in the complete information equilibrium with price impact is λ the induced market power is $l = \lambda/\beta \cdot N$ and the price volatility is:

$$\frac{(\beta N)^2}{(1+\beta N+\lambda)^2} \mathrm{var}(\frac{1}{N}\sum_{i\in N} \eta_i) = \frac{(\beta N)^2}{(1+\beta N+\lambda)^2} \frac{\rho_{\eta\eta}(N-1)+1}{N} \mathrm{var}(\eta_i).$$

Since $\mathrm{var}(\eta_i)$ is defined as in (61) and $\rho_{\eta\eta} = \rho_{\theta\theta}$, the previous equation implies that the price volatility is given by σ_p^2. Thus, there exists an equilibrium that induces (l, σ_p^2). In this equilibrium the shocks $\{\eta_i\}_{\in N}$ are common knowledge and agents have no information on $\{\phi_i\}_{\in N}$. Under this information structure the market power and price volatility (l, σ_p^2) are *not* induced as a unique equilibrium. However, the model is isomorphic to a model of complete information in which the only shocks are $\{\eta_i\}_{\in N}$. We can then use Theorem 1, which states that this market power and price volatility are induced as the unique symmetric linear Nash equilibrium for some information structure when agents have incomplete information. This concludes the first part of the proof.

"Only If" Part. Using Lemma 4, in any linear Nash equilibrium:

$$x_i(p^*) = \frac{\mathbb{E}[\theta_i|s_i, p^*] - p^*}{1+\lambda}, \tag{62}$$

where $\lambda \geq -1/2$ is agent i's price impact. Adding (62) over all agents and multiplying by β we get:

$$\beta \sum_{i\in N} x_i(p^*) = \beta \sum_{i\in N} \frac{\mathbb{E}[\theta_i|s_i, p^*] - p^*}{1+\lambda}.$$

Market clearing implies that $\beta \sum_{i\in N} x_i(p^*) = p^*$. It follows that

$$p^* = \beta \sum_{i\in N} \frac{\mathbb{E}[\theta_i|s_i, p^*] - p^*}{1+\lambda}.$$

Rearranging terms we obtain

$$p^* = \frac{\beta N}{1+\lambda+\beta N}\frac{1}{N}\sum_{i\in N}\mathbb{E}[\theta_i|s_i,p^*]. \tag{63}$$

Taking the expectation of the previous equation conditional on p^* (i.e., taking the expectation $\mathbb{E}[\cdot|p^*]$) and using the law of iterated expectations:

$$p^* = \frac{\beta N}{1+\lambda+\beta N}\frac{1}{N}\sum_{i\in N}\mathbb{E}[\theta_i|p^*] = \frac{\beta N}{1+\lambda+\beta N}\mathbb{E}[\frac{1}{N}\sum_{i\in N}\theta_i|p^*]. \tag{64}$$

It follows that:

$$\sigma_p^2 = \left(\frac{\beta N}{1+\lambda+\beta N}\right)\mathrm{cov}(p,\bar{\theta}).$$

Hence, we have that:

$$\sigma_p^2 = \left(\frac{\beta N}{1+\lambda+\beta N}\right)^2\rho_{p\bar{\theta}}^2\sigma_{\bar{\theta}}^2. \tag{65}$$

Since $\rho_{p\bar{\theta}}^2 \leq 1$, this proves the necessity part for the price volatility. Lemma 4 shows that in any linear Nash equilibrium agents' price impact is greater or equal than $-1/2$ (i.e. $\lambda \geq -1/2$). Lemma 5 shows that the induce market power is given by $l = \lambda/(\beta N)$. Thus, the equilibrium market power satisfies $l = -1/(2\beta N)$. This concludes the proof. ■

Proof of Theorem 3. We first note that $\mathrm{cov}(\Delta\theta_i,\bar{\theta}) = 0$, which can be verified as follows:

$$\begin{aligned}
\mathrm{cov}(\Delta\theta_i,\bar{\theta}) &= \mathrm{cov}(\sum_{i\in N}\theta_i-\bar{\theta},\bar{\theta}) && (66)\\
&= \sum_{i\in N}\mathrm{cov}(\theta_i-\bar{\theta},\bar{\theta}) && (67)\\
&= \sum_{i\in N}\sum_{j\in N}\mathrm{cov}(\theta_i,\theta_j)-\mathrm{cov}(\theta_i,\theta_j) = 0. && (68)
\end{aligned}$$

The explanations of the steps is as follows: (66) is by the definition of $\Delta\theta_i$, (67) is using the collinearity of the covariance, and (68) is using the definition of $\bar{\theta}$ and the collinearity of the covariance. In an analogous way it is easy to prove that $\mathrm{cov}(\Delta\theta_i,\bar{q}) = \mathrm{cov}(\Delta q_i,\bar{\theta}) = \mathrm{cov}(\Delta q_i,\bar{q}) = 0$.

Thus, we can write the joint distribution of random variables $(\Delta\theta_i,\bar{\theta},\Delta q_i,p)$ as follows:

$$\begin{pmatrix}\Delta\theta_i\\ \bar{\theta}\\ \Delta q_i\\ p\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}0\\ \mu_\theta\\ 0\\ \mu_p\end{pmatrix},\begin{pmatrix}\sigma_{\Delta\theta}^2 & 0 & \rho_{\Delta q\Delta\theta}\sigma_{\Delta\theta}\sigma_{\Delta q} & 0\\ 0 & \sigma_{\bar{\theta}}^2 & 0 & \rho_{p\bar{\theta}}\sigma_{\bar{\theta}}\sigma_p\\ \rho_{\Delta q\Delta\theta}\sigma_{\Delta\theta}\sigma_{\Delta q} & 0 & \sigma_{\Delta q}^2 & 0\\ 0 & \rho_{p\bar{\theta}}\sigma_{\bar{\theta}}\sigma_p & 0 & \sigma_p^2\end{pmatrix}\right). \tag{69}$$

Note that the variables $(\Delta\theta_i, \bar{\theta}, \Delta q_i, p)$ can be written as a linear combination of the variables $(\theta_i, \bar{\theta}, q_i, p)$ as follows:

$$\begin{pmatrix} \Delta\theta_i \\ \bar{\theta} \\ \Delta q_i \\ p \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/(\beta N) \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_i \\ \bar{\theta} \\ q_i \\ p \end{pmatrix}.$$

Thus by characterizing the joint distribution of variables $(\Delta\theta_i, \bar{\theta}, \Delta q_i, p)$ we are also characterizing the joint distribution variables of $(\theta_i, \theta, q_i, p)$.

"If" Part. We fix a triple $(\hat{l}, \hat{\rho}_{p\bar{\theta}}, \hat{\rho}_{\Delta q\Delta\theta}) \in [-1/2, \infty) \times [0,1] \times [0,1]$ that satisfies (22) and show there exists an information structure that induces this triple. We consider a set of N normally distributed noise terms $\{\varepsilon_i\}_{i\in N}$, which are jointly independent of $\{\theta_i\}_{i\in N}$ with variance and correlation across agents given by

$$\sigma_\varepsilon^2 = \frac{\hat{\rho}_{p\bar{\theta}}^2(-N\hat{\rho}_{\Delta q\Delta\theta}^2 + (N-1)(1-\rho_{\theta\theta})) + \hat{\rho}_{\Delta q\Delta\theta}^2((N-1)\rho_{\theta\theta}+1)}{N\hat{\rho}_{p\bar{\theta}}^2\hat{\rho}_{\Delta q\Delta\theta}^2}\sigma_\theta^2\ ;$$

$$\rho_{\varepsilon\varepsilon} = \frac{\hat{\rho}_{p\bar{\theta}}^2(-N\hat{\rho}_{\Delta q\Delta\theta}^2\rho_{\theta\theta} + \rho_{\theta\theta} - 1) + (N-1)\hat{\rho}_{\Delta q\Delta\theta}^2\rho_{\theta\theta} + \hat{\rho}_{\Delta q\Delta\theta}^2}{\hat{\rho}_{p\bar{\theta}}^2(-N(\hat{\rho}_{\Delta q\Delta\theta}^2 + \rho_{\theta\theta} - 1) + \rho_{\theta\theta} - 1) + (N-1)\hat{\rho}_{\Delta q\Delta\theta}^2\rho_{\theta\theta} + \hat{\rho}_{\Delta q\Delta\theta}^2}.$$

The variance and correlation of the noise terms satisfy the following relation with the correlations $(\hat{\rho}_{p\bar{\theta}}, \hat{\rho}_{\Delta q\Delta\theta})$:

$$\hat{\rho}_{p\bar{\theta}} = \sqrt{\frac{\sigma_{\bar{\theta}}^2}{\sigma_{\bar{\theta}}^2 + \sigma_{\bar{\varepsilon}}^2}} \quad \text{and} \quad \hat{\rho}_{\Delta q\Delta\theta} = \sqrt{\frac{\sigma_{\Delta\theta}^2}{\sigma_{\Delta\theta}^2 + \sigma_{\Delta\varepsilon}^2}}, \tag{70}$$

where $\bar{\varepsilon}$ and $\Delta\varepsilon_i$ are defined in an analogous way to $\bar{\theta}$ and $\Delta\theta_i$. The only relevant part of the construction of the noise terms is that (70) is satisfied; the specific definitions of σ_ε^2 and $\rho_{\varepsilon\varepsilon}$ are not used again throughout the proof. We assume agents observe a one-dimensional signal as follows:

$$s_i = (\Delta\theta_i + \Delta\varepsilon_i) + \gamma(\bar{\theta} + \bar{\varepsilon}), \tag{71}$$

were γ is given by:

$$\gamma = \frac{(\hat{l}\beta N + 1)(\beta N - \hat{l}\beta N)}{\hat{l}\beta N(N-1)(\beta N(1+\hat{l}) + 1)}\frac{\hat{\rho}_{p\bar{\theta}}^2}{\hat{\rho}_{\Delta q\Delta\theta}^2}. \tag{72}$$

The variance and correlation of the noise terms plus the definition of the signal (71) completely determines the information structure. We now show that the induced equilibrium triple is $(\hat{l}, \hat{\rho}_{p\bar{\theta}}, \hat{\rho}_{\Delta q\Delta\theta})$.

Consistent with the notation previously used we define:

$$\bar{s} \triangleq \gamma(\bar{\theta} + \bar{\varepsilon}) \text{ and } \Delta s_i \triangleq \Delta\theta_i + \Delta\varepsilon_i.$$

The random variables $(\bar{\theta}, \bar{\varepsilon}, \bar{s})$ are orthogonal to $(\Delta\theta_i, \Delta\varepsilon_i, \Delta s_i)$ (the proof is analogous to (66)-(68)) so the expectation of θ_i conditional on $(\Delta s_i, \bar{s})$ can be written as follows:

$$\mathbb{E}[\theta_i|s_i, \bar{s}] = \frac{\sigma^2_{\Delta\theta}}{\sigma^2_{\Delta\theta} + \sigma^2_{\Delta\varepsilon}} \Delta s_i + \frac{\sigma^2_{\bar{\theta}}}{\sigma^2_{\bar{\theta}} + \sigma^2_{\bar{\varepsilon}}} \frac{1}{\gamma} \bar{s}. \tag{73}$$

Finally, it is useful to define:

$$\tilde{\gamma} \triangleq \gamma \frac{\hat{\rho}^2_{\Delta q \Delta\theta}}{\hat{\rho}^2_{p\bar{\theta}}}. \tag{74}$$

The expectation (73) can be rewritten as follows:

$$\mathbb{E}[\theta_i|s_i, \bar{s}] = \frac{\sigma^2_{\Delta\theta}}{\sigma^2_{\Delta\theta} + \sigma^2_{\Delta\varepsilon}} s_i + (1 - \tilde{\gamma}) \left(\frac{\sigma_{\bar{\theta}}}{\sigma_{\bar{\theta}} + \sigma_{\bar{\varepsilon}}} \frac{1}{\gamma} \bar{s} \right). \tag{75}$$

In any linear Nash equilibrium, the equilibrium price must be a linear function of the signals $\{s_i\}_{i \in N}$. Using the symmetry of the conjectured equilibrium, we have that in any symmetric linear Nash equilibrium, there exists constants $\hat{c}_0, \hat{c}_1$ such that the equilibrium price satisfies:

$$p^* = \hat{c}_0 + \hat{c}_1 \left(\frac{1}{N} \sum_{i \in N} s_i \right).$$

We note that:

$$\mathbb{E}[\theta_i|s_i, p^*] = \mathbb{E}[\theta_i|s_i, \bar{s}].$$

Hence, agent i in equilibrium buys a quantity:

$$q_i^* = \frac{\mathbb{E}[\theta_i|s_i, \bar{s}] - p^*}{1 + \lambda}, \tag{76}$$

for some $\lambda \geq -1/2$. Replacing the expression for the expectation (73) in (76) and using the market clearing condition we obtain:

$$p^* = \beta \sum_{i \in N} q_i = N\beta \frac{\frac{\sigma^2_{\bar{\theta}}}{\sigma^2_{\bar{\theta}} + \sigma^2_{\bar{\varepsilon}}} \frac{1}{\gamma} \bar{s} - p^*}{1 + \lambda},$$

Note that $\sum_{i \in N} \Delta s_i = 0$ so the terms with Δs_i cancel out in the previous expression when we sum over all agents. Solving for p^* we obtain:

$$p^* = \frac{\beta N \left(\frac{\sigma^2_{\bar{\theta}}}{\sigma^2_{\bar{\theta}} + \sigma^2_{\bar{\varepsilon}}} \frac{1}{\gamma} \bar{s} \right)}{1 + \lambda + \beta N}. \tag{77}$$

Because p^* is collinear with $\bar{s}$, it is immediate that:

$$\text{corr}(p, \bar{\theta}) = \text{corr}(\bar{s}, \bar{\theta}) = \frac{\sigma^2_{\bar{\theta}}}{\sigma^2_{\bar{\theta}} + \sigma^2_{\bar{\varepsilon}}}.$$

Since σ_ε^2 and $\rho_{\varepsilon\varepsilon}$ were chosen so that (70) is satisfied, we have that the correlation between the price and the average shock is $\hat{\rho}_{p\bar{\theta}}$; as desired. Similarly, using (76) to compute Δq_i^* we obtain:

$$\Delta q_i^* = q_i^* - \bar{q}^* = \frac{\mathbb{E}[\Delta\theta_i|\Delta s_i]}{1+\lambda}. \tag{78}$$

Because Δq_i^* is collinear with Δs_i it is immediate that:

$$\text{corr}(\Delta q_i, \Delta\theta_i) = \text{corr}(\Delta s_i, \Delta\theta_i) = \frac{\sigma_{\Delta\theta}^2}{\sigma_{\Delta\theta}^2 + \sigma_{\Delta\varepsilon}^2}.$$

Since σ_ε^2 and $\rho_{\varepsilon\varepsilon}$ were fixed so that (70) is satisfied, we have that the correlation between the Δq_i and $\Delta\theta_i$ is $\hat{\rho}_{\Delta q\Delta\theta}$; as desired. Therefore, the induced correlations by the equilibrium are $(\hat{\rho}_{p\bar{\theta}}, \hat{\rho}_{\Delta q\Delta})$, which is the desired quantities. We now show that the equilibrium market power is $\hat{l}$.

Given the equilibrium price in (77), we can find an expression for the expected value of θ_i conditional on the private information of agent i and the equilibrium price:

$$\mathbb{E}[\theta_i|p^*, s_i] = \left(\frac{\sigma_{\Delta\theta}^2}{\sigma_{\Delta\theta}^2 + \sigma_{\Delta\varepsilon}^2} s_i + (1-\tilde{\gamma})\frac{p^*}{\beta N}(1+\lambda+\beta N)\right). \tag{79}$$

The previous equation is obtained by rewriting (75) in terms of p^* instead of $\bar{s}$. Recall that in equilibrium agent i submits demand function:

$$x_i(p) = \frac{\mathbb{E}[\theta_i|p^*, s_i] - p}{1+\lambda}.$$

Using the expression for the expectation (79) we write the slope of the demand submitted by agent i as follows:

$$m = \frac{1-(1-\tilde{\gamma})\frac{1}{\beta N}(1+\lambda+\beta N)}{1+\lambda}. \tag{80}$$

Recall that λ and m must also satisfy (36). Using (36) and (80) we can find the equilibrium price impact:

$$\lambda = \frac{1}{2}\left(-1 - N\beta\frac{\tilde{\gamma}(N-1)-1}{\tilde{\gamma}(N-1)+1} \pm \sqrt{\left(N\beta\frac{\tilde{\gamma}(N-1)-1}{\tilde{\gamma}(N-1)+1}\right)^2 + 2N\beta + 1}\right). \tag{81}$$

Only the positive root is a valid solution as the negative root yields a λ less than $-1/2$, which violates the condition in Lemma 4. Hence, for a fixed γ, there is a unique symmetric linear Nash equilibrium. Finally, we show that the definition of $\tilde{\gamma}$ implies that the induced market power is l (as conjectured).

Using the definitions of γ (see (72)), the variable $\tilde{\gamma}$ can be written as follows:

$$\tilde{\gamma} = \gamma\frac{\hat{\rho}_{\Delta q\Delta\theta}^2}{\hat{\rho}_{p\bar{\theta}}^2} = \frac{(\hat{l}\beta N+1)(\beta N - \hat{l}\beta N)}{\hat{l}\beta N(N-1)(\beta N(1+\hat{l})+1)}.$$

Replacing the previous expression of $\tilde{\gamma}$ into the expression for the price impact (i.e. (81)), we obtain that the price impact is given by:

$$\lambda = \hat{l}\beta N.$$

However, Lemma 5 shows that in this case the equilibrium market power is $\hat{l}$; which is the desired induced market power.

"Only If Part" Lemma 4 states that in any linear Nash equilibrium the price impact is greater or equal than -1/2 and Lemma 5 states that the price impact is equal to the market power divided by (βN). Thus in any linear Nash equilibrium the market power satisfies $l \geq -1/(2\beta N)$.

In the proof of Theorem 2 we established that (see (64)):

$$p^* = \frac{\beta N}{1+\lambda+\beta N}\mathbb{E}[\frac{1}{N}\sum_{i\in N}\theta_i|p^*] = \frac{\beta N}{1+\lambda+\beta N}\mathbb{E}[\bar{\theta}|p^*]. \tag{82}$$

Observe that p^* appears on the left-hand-side and as a conditioning variable on the right-hand-side, and thus, $\text{corr}(p^*, \bar{\theta}) \geq 0$. A statistical condition of a correlation is that $\text{corr}(p^*, \bar{\theta}) \leq 1$. Therefore, in any linear Nash equilibrium $\text{corr}(p^*, \bar{\theta}) \in [0, 1]$.

Using Lemma 4, in any linear Nash equilibrium:

$$x_i(p^*) = \frac{\mathbb{E}[\theta_i|s_i, p^*] - p^*}{1+\lambda}, \tag{83}$$

We now use that, $x_i(p^*) = q_i$ (i.e. $x_i(p^*)$ is the quantity bought by agent i in equilibrium) and market clearing implies $p^* = N\beta\bar{q}$, thus

$$\Delta q_i = q_i - \bar{q} = \frac{\mathbb{E}[\theta_i|s_i, p^*] - p^*}{1+\lambda} - \frac{p^*}{\beta N} = \frac{\beta N\mathbb{E}[\theta_i|s_i, p^*] - (1+\lambda+\beta N)p^*}{\beta N(1+\lambda)}, \tag{84}$$

We now observe that $(\Delta q_i, p^*)$ is measurable with respect to (s_i, p^*). Taking expectation of the previous equation conditional on $(\Delta q_i, p^*)$ (i.e., taking expectation $\mathbb{E}[\cdot|\Delta q_i, p^*]$) and using the law of iterated expectations we get:

$$\Delta q_i = \frac{\beta N\mathbb{E}[\theta_i|\Delta q_i, p^*] - (1+\lambda+\beta N)p^*}{\beta N(1+\lambda)}, \tag{85}$$

Using (64) we have that

$$p^* = \frac{\beta N}{1+\lambda+\beta N}\mathbb{E}[\frac{1}{N}\sum_{i\in N}\theta_i|p^*] = \frac{\beta N}{1+\lambda+\beta N}\mathbb{E}[\bar{\theta}|p^*].$$

Replacing p^* in (84) we get:

$$\Delta q_i = \frac{\mathbb{E}[\theta_i|\Delta q_i, p^*] - \mathbb{E}[\bar{\theta}|p^*]}{1+\lambda}. \tag{86}$$

Finally, we note that $cov(\bar{\theta}, \Delta q_i) = cov(\Delta\theta_i, p^*)$ and so $\mathbb{E}[\theta_i|\Delta q_i, p^*] = \mathbb{E}[\Delta\theta_i|\Delta q_i] + \mathbb{E}[\bar{\theta}|p^*]$. Therefore,

$$\Delta q_i = \frac{\mathbb{E}[\Delta\theta_i|\Delta q_i]}{1+\lambda}. \tag{87}$$

Observe that Δq_i appears on the left-hand-side and as a conditioning variable on the right-hand-side and so $\text{corr}(\Delta\theta_i, \Delta q_i) \geq 0$. A statistical condition of a correlation is that $\text{corr}(\Delta\theta_i, \Delta q_i) \leq 1$. Therefore, in any linear Nash equilibrium $\text{corr}(\Delta\theta_i, \Delta q_i) \in [0, 1]$.

"Uniqueness of Distribution". Finally, we prove that for every $(l, \rho_{p\bar{\theta}}, \rho_{\Delta\theta}) \in [-1/(\beta N2), \infty) \times [0,1] \times [0,1]$ there exists a unique distribution that is the outcome of a linear Nash equilibrium. For this we note that the only coefficients missing in distribution (69) are σ_p, $\sigma_{\Delta q}$ and μ_p. In Lemma 1 and Lemma 2 we show that these are uniquely determined by $(l, \rho_{p\bar{\theta}}, \rho_{\Delta\theta})$. ■

Proof of Lemma 1 . We rewrite (63) for the convenience of the reader:

$$p^* = \frac{\beta N}{1+\lambda+\beta N} \frac{1}{N} \sum_{i \in N} \mathbb{E}[\theta_i | s_i, p^*]. \tag{88}$$

Taking expectations of the previous equation and using the law of iterated expectations we get:

$$\mu_p = \frac{\beta N}{1+\lambda+\beta N} \mu_{\bar{\theta}}. \tag{89}$$

Lemma 5 states that $\lambda = l\beta N$. To prove the second equation note that the market clearing condition is given by:

$$p = \beta \sum_{i \in N} q_i.$$

Taking expectations of this equation:

$$\mathbb{E}[p] = \beta \sum_{i \in N} \mathbb{E}[q_i].$$

Symmetry implies that $\mathbb{E}[q_i] = \mathbb{E}[q_j]$, and so we get $\mu_p = \beta N \mu_q$. ■

Proof of Lemma 2. We first rewrite (65):

$$\sigma_p^2 = \left(\frac{\beta N}{1+\lambda+\beta N} \right)^2 \rho_{p\bar{\theta}}^2 \sigma_{\bar{\theta}}^2. \tag{90}$$

Lemma 5 shows that $\lambda = l\beta N$, which proves (26). We rewrite (87):

$$\Delta q_i = \frac{\mathbb{E}[\Delta\theta_i | \Delta q_i]}{1+\lambda}. \tag{91}$$

which implies that:

$$\sigma_{\Delta\theta_i}^2 = \frac{\rho_{\Delta\theta\Delta q}^2 \sigma_{\Delta\theta}^2}{(1+\lambda)^2}. \tag{92}$$

Since p^* is collinear in $\bar{q}$ (due to the market clearing condition), it is clear that

$$\sigma_{\bar{q}}^2 = \left(\frac{1}{1+\lambda+\beta N} \right)^2 \rho_{p\bar{\theta}}^2 \sigma_{\bar{\theta}}^2. \tag{93}$$

Since $\text{cov}(\Delta q_i, \bar{q}) = 0$, we have

$$\sigma_{q_i}^2 = \sigma_{\bar{q}}^2 + \sigma_{\Delta q_i}^2,$$

and thus:

$$\sigma_{q_i}^2 = \left(\frac{1}{1+\beta N(1+l)} \right)^2 \rho_{p\bar{\theta}}^2 \sigma_{\bar{\theta}}^2 + \frac{\rho_{\Delta\theta\Delta q}^2 \sigma_{\Delta\theta}^2}{(1+\beta N l)^2},$$

where we once again use Lemma 5 to write λ in terms of l. Finally, we note that $\sigma^2_{\Delta\theta} = \sigma^2_{\theta} - \sigma^2_{\bar{\theta}}$, and so we get the second equation.

To prove the last equation note that the price is collinear with the price, and hence:

$$\sigma_p = \beta N \sigma_{\bar{q}}.$$

We now use that $\sigma_{\bar{q}} = \rho_{q\bar{q}}\sigma_q = \rho_{qp}\sigma_q$ and conclude that $\sigma_p = \beta N \rho_{qp}\sigma_q$. ■

Proof of Lemma 3. Symmetry implies that $\text{cov}(\theta_i, p) = \text{cov}(\theta_j, p)$, and thus:

$$\text{cov}(\theta_i, p) = \frac{1}{N}\sum_{j\in N} \text{cov}(\theta_j, p) = \text{cov}(\bar{\theta}, p).$$

It follows that $\rho_{\theta p}\sigma_\theta\sigma_p = \rho_{\bar{\theta}p}\sigma_{\bar{\theta}}\sigma p$. We now note that $cov(\bar{\theta}, \bar{\theta}) = cov(\bar{\theta}, \theta_i)$, which implies that $\sigma_{\bar{\theta}} = \rho_{\theta\bar{\theta}}\sigma_\theta$. Therefore, we get $\rho_{\theta p} = \rho_{\bar{\theta}p}\rho_{\theta\bar{\theta}}$.

Symmetry implies that $\text{cov}(q_i, \bar{\theta}) = \text{cov}(q_j, \bar{\theta})$, and thus:

$$\text{cov}(q_i, \bar{\theta}) = \frac{1}{N}\sum_{j\in N} \text{cov}(q_j, \bar{\theta}) = \text{cov}(\bar{q}, \bar{\theta}).$$

We therefore get that $\rho_{q\bar{\theta}}\sigma_q\sigma_{\bar{\theta}} = \rho_{\bar{q}\bar{\theta}}\sigma_{\bar{q}}\sigma_{\bar{\theta}}$. We also have that $\sigma_{\bar{q}} = \rho_{q\bar{q}}\sigma_q$ (this can be proved the same way as we proved that $\sigma_{\bar{\theta}} = \rho_{\theta\bar{\theta}}\sigma_\theta$). Since $\bar{q}$ is collinear with p, $\rho_{\bar{q}\bar{\theta}} = \rho_{p\bar{\theta}}$ and $\rho_{q\bar{q}} = \rho_{qp}$. Hence, we get that $\rho_{q\bar{\theta}} = \rho_{p\bar{\theta}}\rho_{qp}$.

Finally, we have:

$$\begin{aligned}
\rho_{\Delta q \Delta\theta} &= \frac{\text{cov}(\Delta q_i, \Delta\theta_i)}{\sigma_{\Delta q}\sigma_{\Delta\theta}} && (94)\\
&= \frac{\text{cov}(q_i, \theta_i) - \text{cov}(\bar{q}, \bar{\theta})}{\sigma_{\Delta q}\sigma_{\Delta\theta}} && (95)\\
&= \frac{(\rho_{q\theta}\sigma_q\sigma_\theta - \rho_{p\bar{\theta}}\rho_{pq}\sigma_q\rho_{\theta\bar{\theta}}\sigma_\theta)}{\sigma_{\Delta q}\sigma_{\Delta\theta}} && (96)\\
&= \frac{(\rho_{q\theta}\sigma_q\sigma_\theta - \rho_{p\bar{\theta}}\rho_{pq}\sigma_q\rho_{\theta\bar{\theta}}\sigma_\theta)}{\sqrt{(1-\rho^2_{qp})\sigma^2_q(1-\rho^2_{\theta\bar{\theta}})\sigma^2_\theta}} && (97)\\
&= \frac{(\rho_{q\theta} - \rho_{p\bar{\theta}}\rho_{pq}\rho_{\theta\bar{\theta}})}{\sqrt{(1-\rho^2_{qp})(1-\rho^2_{\theta\bar{\theta}})}}. && (98)
\end{aligned}$$

The definition of the covariance is given by (94). (95) follows from:

$$\text{cov}(\Delta q_i, \Delta\theta_i) = \text{cov}(q_i - \bar{q}, \theta_i - \bar{\theta}) = \text{cov}(q_i, \theta_i) - \text{cov}(\bar{q}, \bar{\theta}),$$

where the symmetry of the distribution is used to show that $\text{cov}(\bar{q}, \bar{\theta}) = \text{cov}(q_i, \bar{\theta}) = \text{cov}(\bar{q}, \theta_i)$. The numerator of (96) is using the definition of the covariance and $\sigma^2_{\bar{\theta}} = \rho_{\bar{\theta}\theta}\sigma^2_\theta$ and $\sigma^2_{\bar{q}} = \rho_{pq}\sigma^2_q$ (note that p

is collinear with $\bar{q}$ and so $\rho_{pq} = \rho_{\bar{q}q}$). The denominator of (97) is found as follows:

$$\sigma^2_{\Delta\theta} = \text{cov}(\theta_i - \bar{\theta}, \theta_i - \bar{\theta}) = \sigma^2_{\theta} - \sigma^2_{\bar{\theta}} = (1 - \rho^2_{\theta\bar{\theta}})\sigma^2_{\theta},$$

where once again we use that $\sigma^2_{\bar{\theta}} = \rho_{\bar{\theta}\theta}\sigma^2_{\theta}$. The variance $\sigma^2_{\Delta q}$ is calculated in an analogous way. (98) is obtained after simplifying the variances. This proves (31). ■

Proof of Theorem 4. We first show that for all (l, σ^2_p) that satisfy (34) with equality, there exists an information structure that induced this market power and this price volatility. In a linear Nash equilibrium of the Cournot competition game an agent's best response is given by:

$$0 = \mathbb{E}[\theta_i - (1+\beta)q_i - \beta \sum_{j \in N} q_j | s_i].$$

The previous equation corresponds simply to the first order condition of (1) where we replace $p = \beta \sum_{j \in N} q_j$. Because q_i is measurable with respect to s_i, we can take q_i outside of the expectation. We can write the first order condition as follows:

$$\beta q_i = \mathbb{E}[\theta_i - q_i - \beta \sum_{j \in N} q_j | s_i].$$

Summing up the previous equation over all agents we obtain:

$$\beta \sum_{i \in N} q_i = p = \sum_{i \in N} \mathbb{E}[\theta_i - q_i - \beta \sum_{j \in N} q_j | s_i].$$

Here we used that market clearing implies that $\beta \sum_{i \in N} q_i$. Taking expectations of the previous equation and using the law of iterated expectations, we obtain:

$$\mathbb{E}[p] = \sum_{i \in N} \mathbb{E}[\theta_i - q_i - \beta \sum_{j \in N} q_j].$$

Thus, we obtain that:

$$\frac{\frac{1}{N} \sum_{i \in N} \mathbb{E}[\theta_i - q_i - \beta \sum_{j \in N} q_j]}{\mathbb{E}[p]} = \frac{1}{N}.$$

Therefore, the market power is constant and equal to $1/N$. To prove that the price volatility is less or equal than the expression in (34) we consider the following noise-free signals:

$$s_i = \theta_i - (1 - \lambda)\bar{\theta}.$$

By using a guess-and-verify method, we find that the unique linear Nash equilibrium is given by:

$$q_i(s_i) = \left(\frac{\lambda \sigma^2_{\bar{\theta}} + \sigma^2_{\Delta\theta}}{\beta \lambda^2 N \sigma^2_{\bar{\theta}} + (\beta + 1)\left(\lambda^2 \sigma^2_{\bar{\theta}} + \sigma^2_{\Delta\theta}\right)} \right) s_i.$$

The equilibrium price is given by $\beta \sum_{i\in N} q_i$ and so we can find the λ that maximizes the price volatility. We define:

$$(\sigma_p^*)^2 \triangleq \max_{\lambda \in \mathbb{R}} \ \beta \sum_{i\in N} \mathrm{var}\left(\left(\frac{\lambda\sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta}^2}{\beta\lambda^2 N\sigma_{\bar{\theta}}^2 + (\beta+1)\left(\lambda^2\sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta}^2\right)}\right)s_i\right),$$

and obtain that:

$$(\sigma_p^*)^2 = \frac{1}{4}\left(\frac{\sqrt{1+\beta}\sigma_{\bar{\theta}} + \sqrt{(\beta+\beta N+1)\sigma_{\Delta\theta}^2 + (1+\beta)\sigma_{\bar{\theta}}^2}}{\sqrt{1+\beta}(\beta+\beta N+1)}\right)^2.$$

This is the upper bound found in Theorem 4. By using the same arguments as in Bergemann, Heumann, and Morris (2015b) it is possible to show that all information structures yield a weakly lower price volatility, which establishes the result. Moreover, decomposing the payoff shock in an analogous way to that in the proof of Theorem 1 and Theorem 2, it is easy to check that all price volatilities can be achieved by some information structure. ■

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