FINANCIAL BUBBLE IMPLOSION

By

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Financial Bubble Implosion*

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Abstract

Expansion and collapse are two key features of a financial asset bubble. Bubble expansion may be modeled using a mildly explosive process. Bubble implosion may take several different forms depending on the nature of the collapse and therefore requires some flexibility in modeling. This paper develops analytics and studies the performance characteristics of the real time bubble monitoring strategy proposed in Phillips, Shi and Yu (2014b,c, PSY) under alternative forms of bubble implosion that can be represented in terms of mildly integrated processes which capture various return paths to market normalcy. We propose a new reverse sample use of the PSY procedure for detecting crises and estimating the date of market recovery. Consistency of the dating estimators is established and the limit theory addresses new complications arising from the alternative forms of bubble implosion and the endogeneity effects present in the reverse regression. Simulations explore the finite sample performance of the strategy for dating maket recovery and an illustration to the Nasdaq stock market is provided. A real-time version of the strategy is provided that is suited for practical implementation.

Keywords: Bubble implosion, Dating algorithm, Limit theory, Market recovery, Nasdaq.
JEL classification: C15, C22

1 Introduction

Following the GFC there has been wide recognition of the harm that financial bubbles can inflict on real economies. The slow recovery from the great recession in the USA and the continuing debt crisis in Europe has alerted central bankers and regulators to the need for greater understanding of the mechanisms by which financial bubbles form, the dynamics of their evolution and collapse,

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and the process of contagion through which other markets and the real economy are affected. Few national economies have been unaffected by the fallout from the GFC. The hazards have therefore become a matter of considerable concern to central banks and policy makers.[1],[2]

While the potentially damaging impact of financial bubbles on the real economy is widely acknowledged, policy makers face major difficulties in designing corrective measures and timing their implementation. Recent econometric work has assisted the design and timing of policy measures by providing empirical techniques that detect mildly explosive bubble-like behaviour in asset prices. While some economists think it is impossible to see bubbles in their inflationary phase (Cooper, 2008), recent developments in the econometric bubble literature deliver real-time monitoring strategies, such as recursive right-sided unit root testing procedures (Phillips, Wu and Yu, 2011, PWY; Phillips, Shi and Yu, 2013a; PSY), CUSUM monitoring techniques (Homm and Breitung, 2012; HB), and double-recursive algorithms (Phillips, Shi and Yu, 2013b,c) that enable bubble detection and consistent estimation of bubble origination and collapse dates.

The present paper focuses on the real-time monitoring procedure of PSY, which is an extended version of the PWY recursive testing approach. The PSY algorithm has been applied to a wide range of markets, including foreign exchange, real estate, commodities and financial assets, and has attracted attention from policy makers and the financial press.[3] The algorithm has been shown (PSY, 2013a,b; HM, 2012) to have superior real time monitoring and detection performance than other methods but nonetheless suffers from delay bias in detection.

The present paper strengthens the foundation of the PSY approach to bubble monitoring by exploring its asymptotics and behavioral characteristics under alternative collapse scenarios which enable more flexible modeling of bubble implosion. The paper also proposes an alternative implementation strategy for detecting financial crises and estimating the dates of crisis collapse origination and market recovery. The new algorithm assists in reducing the delay bias in change point detection.

The PSY double recursion strategy is particularly designed for detection purposes when there are periodically collapsing bubbles (Blanchard, 1979) in the data. Its asymptotic and finite sample performance has been studied under several different bubble generating processes and performance measures. In particular, PSY (2014c) demonstrate consistency of the strategy in estimating the bubble origination and termination dates when either single or multiple bubbles appear in the sample. The data generating processes considered in those exercises are based on the model proposed in PWY where asset prices follow a random walk in normal market conditions, switch to a mildly explosive process under bubble expansion, and return to martingale dynamics when the bubble implodes. In that model bubbles collapse abruptly within one sample period, an assumption made largely for analytic convenience but lacking empirical realism. Casual inspection of the trajectories of financial asset bubbles typically reveal a more complex process of market

[1] *Federal Reserve policymakers should deepen their understanding about how to combat speculative bubbles to reduce the chances of another financial crisis.* –Donald Kohn, Former Vice Chairman of the Federal Reserve Board, March 2010.

[2] *How do we know when irrational exuberance has unduly escalated asset values?* – Alan Greenspan, Formal Chairman of Federal Reserve, December 1996 .

[3] See, for example, Bohl et al. (2013), Etienne et al. (2013), Chen and Funke (2013), Meyer (2013), Gutierrez (2013), and Yiu et al (2012).

Figure 1: The monthly price-dividend ratio of the NASDAQ composite index for the sample period from January 1991 to December 2005, showing the bubble period detected by the PWY algorithm.

correction and reversion. For instance, Figure 1 shows monthly Nasdaq price-dividend ratio from January 1991 to December 2005 with origination and termination dates as determined in PWY. The observed trajectory shows that the collapse process of the Nasdaq in the early 2000s is a complex one that is neither immediate nor monotonic. Implosion of the famous dot-com bubble[4] does not conclude within a single time period but involves many months of realignment.

Historical episodes of collapse have been classified in the literature into 'sudden', 'disturbing', or 'smooth' crisis events (Rosser, 2000; Huang et al., 2010). 'Sudden crises' characterize precipitate declines and correspond to the PWY model of abrupt declines in prices. In 'smooth crises' (a somewhat oxymoronic description used in the literature), prices fall smoothly with a moderate but persistent decline. 'Disturbing crises' are considered to be intermediate in form between these two extremes.

One aim of the present paper is to examine the limit behavior and finite sample performance of the PSY strategy under a more realistic bubble generating process that allows for various forms of implosion that fall into the above categories. The process considered here, given in (1) below, is an extension of the model proposed in PWY (2011) and used recently in Harvey, Leybourne, and Sollis (2012; HLS). The model involves normal market and bubble exuberance dynamics similar to the PWY model, whereas bubble implosion is modeled by a (stationary) mildly integrated process that is intended to capture elements of the mean reversion process that occur as prices collapse to normal market levels concordant with past and present fundamentals. This extension of the bubble collapse mechanism to allow for transitional dynamics was envisaged in the original formulation of the PWY model[5] but was not pursued in that work. Figure 2 displays a typical realization of the PWY process and several realizations of the new process (1) with different collapse speeds and durations. As is evident in these graphs, the new bubble model is flexible and can produce richer dynamic trajectories in the collapse period.

[4]The presence of speculative bubble behavior in the dot-com market around this sample period has been documented in PWY.

[5]See the discussion following equation (14) of PWY.

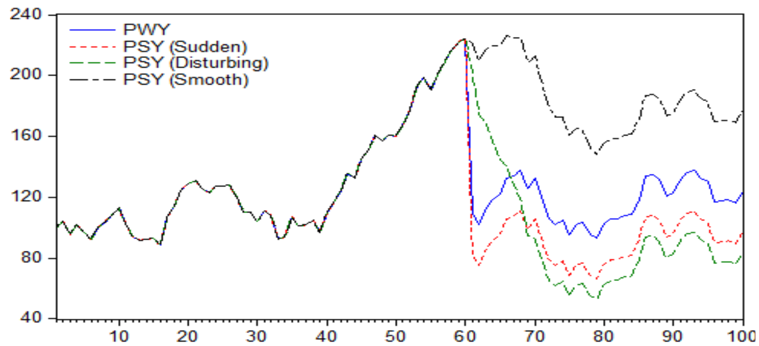

Figure 2: Typical bubble collapse patterns generated by a mildly integrated process, giving sudden, disturbing, and smooth correction trajectories. Model parameters in (2) are set as $T = 100$, $\eta = 1$, $\alpha = 0.7$, $d_{BT} = 0.2$, $\beta = 0.1, d_{CT} = \lfloor 0.01T \rceil$ for sudden collapse, $\beta = 0.5, d_{CT} = \lfloor 0.10T \rceil$ for disturbing crash, and $\beta = 0.9, d_{CT} = \lfloor 0.20T \rceil$ for smooth collapse.

An important feature of the new model is the embodiment of a recovery date break point in the process. Market recovery is defined as the date asset prices return to their normal martingale path, effectively the switch point from the mildly integrated collapse process to the martingale path. A second aim of the paper is to address the econometric issues associated with consistent estimation of the recovery date. For the simplified PWY bubble process, market recovery coincides with bubble implosion because immediate market correction ensures that the asset price returns abruptly to its martingale path up to a term of $O_p(1)$. In the new model, the date of bubble implosion (the switch point from exuberant behavior to market correction behavior) differs from the bubble implosion date. By construction, the market recovery date of the new process is the conclusion point of the mildly integrated collapsing process (see Figure 3). This feature of the model leads to new practical issues such as the existence and nature of the collapsing regime and the econometric estimation of the crisis recovery date.

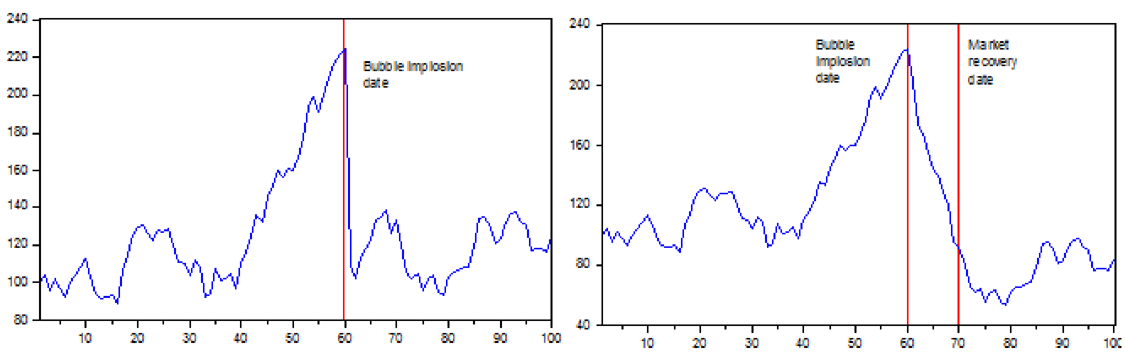

Figure 3: Turning points of different bubble generating processes showing abrupt correction (PWY) and mildly integrated correction with a separate market recovery date.

Econometric detection of the crisis regime is possible because the collapse process, modeled here by a mildly integrated process, is embedded in a long sample period that includes multiple regimes of martingale and explosive behavior. Doubly-recursive tests like those in PSY are well suited to deal with such break analysis. In an important related literature on testing for stationarity Leybourne, Taylor (2007) suggested doubly-recursive tests for unit roots against stationary alternatives. In their work on bubble break point testing HLS (2013) perform sequential procedures based on PWY tests, HB (2012) Chow tests, and a union of rejections strategy (combining PWY and HB tests) to identify bubble behavior. That procedure starts by estimating the regime switching points and is followed by the use of unit root tests (against different alternatives) within each regime. A challenging aspect with this approach is feasibility. Often bubble expansions and contractions are short-lived, producing insufficient observations for unit root tests to have good performance. In addition, the HLS model uses a simple collapse process like that of PWY - either an abrupt collapse (like PWY) or an immediate transition to normal market behavior (a unit root process initialized at the last period of the explosive regime). Further, their procedure is designed for cases where the number of bubbles in the sample period is known to the practitioner. In work that is much closer in spirit to that of the present paper and the extension suggested in PWY, HLS (2012) use a 4-regime model that incorporates an intermediate stationary regime to model a collapsing bubble. Their approach involves *ex post* dating of the regime change points rather than recursive procedures, so it is not intended for real time bubble identification or date stamping.

The present paper proposes a new method for detecting crises and estimating their associated multiple turning points. Specifically, we recommend applying the recursive (or doubly-recursive) PSY test to a data series which is arranged in reverse order of the original data series. This new strategy is suitable for cases with either a single crash or multiple (known or unknown number of) crashes. In contrast to PSY and PWY (2011) but similar to HLS (2012), the procedure is primarily designed for *ex post* analysis rather than real-time monitoring. We demonstrate that this procedure consistently estimates the origination, termination, and recovery dates of crises. It has good finite sample performance in simulations and helps to reduce bias in some of the date estimates. A real-time version of the strategy is provided that is suited for practical implementation and ongoing policy analysis. In view of the technical complications arising from multiple collapse processes, from the presence of a unknown recovery date, and from the endogeneity introduced by reverse regression, the limit theory of the date stamping procedures involves substantial extension of earlier work in PWY (2011) and PSY (2014c).

We illustrate the use of the new strategy in both bubble monitoring and crisis detection settings with an application to the Nasdaq stock market over 1973:M1-2013:M8. The tests suggest that the dot-com bubble originated in August 1995. Implosion occured in Februrary 2000 and the market recovered from November 2000, with a further correction in 2004M01-M05. In this long historical series, the testing procedures also identify several other bubble/crisis incidents: the 1973 and 1976 stock market crashes, the 1983 bubble episode, and the subsequent 1980s bubble leading up to the famous 'black Monday' crash of 1987.

The rest of the paper is organized as follows. Section 2 introduces the PSY procedure for bubble monitoring and crisis detection. Section 3 derives the limit theory for the PSY strategy in both settings under the new bubble generating process that allows for flexibility in the collapse mechanism. Finite sample performance is studied in Section 4. An empirical application to long historical Nasdaq series is conducted in Section 5. Section 6 concludes. Two appendices contain supporting lemmas and derivations for the limit theory presented in the paper. Complete details of the derivations and supporting lemmas are given in a technical supplement to the paper (Phillips and Shi, 2014).

2 Econometric Methods

The following development uses models in which a single bubble occurs. Extension of the methods to cases where there are multiple bubbles may be established using PSY (2014b, 2014c) and for brevity these are not provided here.

We denote the bubble origination and collapse dates by T_e and T_c, so that $B = [T_e, T_c]$ is the bubble period and $N_0 = [1, T_e)$ and $N_1 = (T_c, T]$ represent normal periods before and after the bubble episode. In this change point framework, the PWY bubble model has the form

$$X_t = \begin{cases} X_{t-1} + \varepsilon_t, & t \in N_0 \\ \delta_T X_{t-1} + \varepsilon_t, & t \in B \\ X^*_{T_c} + \sum_{i=T_c+1}^{t} \varepsilon_i, & t \in N_1 \end{cases} , \varepsilon_t \sim iid\left(0, \sigma^2\right) \tag{1}$$

where $t = 1, 2, \ldots, T$, $\delta_T = 1 + c_1 T^{-\alpha}$ with $c_1 > 0$ and $\alpha \in [0, 1)$, $X^*_{T_c} = X_{T_e} + X^*$ with $X^* = O_p(1)$, and $X_0 = o_p(1)$. Asset prices are assumed to be a pure random walk during normal periods. During market exuberance, asset prices follow a mildly explosive process. An abrupt collapse occurs at T_c, which brings the asset price back to the level when the bubble originated (i.e. X_{T_e}) plus a random perturbation X^*. The asset price then continues its martingale path towards the end of the sample period.

The new generating process considered here differs from the PWY model in three respects. First, it includes an asymptotically negligible drift in the martingale path during normal periods. Second, the collapse process is modeled directly as a transient mildly integrated process (Phillips and Magdalinos, 2007) that covers an explicit period of market collapse. Third, a market recovery date is introduced to capture the return to normal market behavior. The model has the following specification

$$X_t = \begin{cases} dT^{-\eta} + X_{t-1} + \varepsilon_t, & t \in N_0 \cup N_1 \\ \delta_T X_{t-1} + \varepsilon_t, & t \in B \\ \gamma_T X_{t-1} + \varepsilon_t, & t \in C \end{cases} , \tag{2}$$

where $B = [T_e, T_c]$ is the bubble episode as before, $C = (T_c, T_r]$ is the collapse period, T_r is the date of market recovery, and $N_0 \cup N_1 = [1, T_e) \cup (T_r, T]$ are the normal market periods. Following PSY (2014a), the asset price process during $N_0 \cup N_1$ involves an asymptotically negligible deterministic trend ($d\frac{t}{T^\eta}$ with constant d and some $\eta > 1/2$) which adds a small drift to the normal martingale path. Both autoregressive coefficients $\delta_T = 1 + c_1 T^{-\alpha}$ (with $c_1 > 0$ and $\alpha \in [0, 1)$) and $\gamma_T = 1 - c_2 T^{-\beta}$ (with $c_2 > 0$ and $\beta \in [0, 1)$) involve mild deviations from unity in the sense of Phillips and Magdalinos (2007), one (δ_T) in the explosive direction and the other (γ_T) in the stationary direction. For given $c_2 > 0$, the speed of collapse is controlled by the parameter β. The smaller is β, the faster is the implosion rate during the collapse period C. [6]

The specification $\gamma_T = 1 - c_2 T^{-\beta}$ for the autoregressive coefficient during the collapse period allows for flexibility in possible collapse trajectories while retaining a vicinity of unity or near martingale flavor. In particular, the data generating process (2) is capable of generating abrupt collapses like the PWY process (1) if the value of β is small and the collapse duration (i.e. $T_r - T_c$) is short. But the model can also generate smooth, slow or turbulent trajectories of correction for which the market collapse duration lasts longer, corresponding to quite different collapse processes in practice. Figure 2 shows a realization of the PWY process against some typical realizations of the mechanism (2) for various values of the parameters that indicate some of these possibilities.

2.1 The PSY strategy for bubble origination and collapse dates

The null hypothesis of the PSY (2014a) test is a unit root process with an asymptotically negligible drift, namely

$$X_t = kT^{-\gamma} + X_{t-1} + \varepsilon_t, \text{ with constant } k \text{ and } \gamma > 1/2. \tag{3}$$

The PSY strategy conducts a backward sup Dickey Fuller (BSDF) test for each observation of interest, which we briefly explain here for completeness. Let f_1 and f_2 be the (fractional) starting and ending points of the DF regression. The regression model includes an intercept but no time trend[7] such that

$$\Delta X_t = \alpha + \beta X_{t-1} + \varepsilon_t, \varepsilon_t \overset{i.i.d}{\sim} \left(0, \sigma^2\right), \tag{4}$$

where $t = \lfloor f_1 T \rfloor, \ldots, \lfloor f_2 T \rfloor$. The corresponding DF statistic sequence is $DF_{f_1}^{f_2}$.

Suppose interest focuses on the properties of the generating process at observation $t := \lfloor fT \rfloor$ where f is the sample fraction corresponding to t. We are particularly interested in whether there is a unit or explosive root in the process at this observation, therefore focusing attention on the upper tail (right side) of the distribution. The backward sup DF test (denoted $BSDF$) therefore calculates the sup of the DF statistics computed recursively over a sample sequence whose end point (expressed in fractional form) f_2 is fixed at f and whose start point f_1 runs from backwards from $f - f_0$ to 0, where f_0 is the smallest window size in these regressions. Specifically, we define

$$BSDF_f\left(f_0\right) = \sup_{f_1 \in [0, f - f_0], f_2 = f} \left\{DF_{f_1}^{f_2}\right\}, \text{ with } f \in \left[f_0, 1\right],$$

giving the statistic at t (or sample fraction f) using a minimum window of size f_0.

To identify a mildly explosive bubble episode, the BSDF statistic is compared to its corresponding right-tail critical value. Let $scv\left(\beta_T\right)$ be the $\left(1 - \beta_T\right) 100\%$ critical value of the $BSDF_f$ statistic and assume that $scv\left(\beta_T\right) \to \infty$ as $\beta_T \to 0$. The origination (termination) date of bubble expansion is then calculated as the first chronological observation whose BSDF statistic exceeds

[6] The process of Harvey, Leybourne and Sollis (2013) is similar but without the drift term in the random walk process. In addition, the AR coefficients in their model deviate discretely from unity so that in their model $\delta_T = 1 + d_1$ and $\gamma_T = 1 - d_2$, where $d_1 > 0$ and $d_2 > 0$ are constant parameters.

[7] See Phillips, Shi and Yu (2014a) for a detailed discussion of regression model specification for right-tailed unit root tests.

(falls below) its corresponding critical value. The (fractional) dates of bubble emergence (origination) and collapse are denoted by f_e and f_c with corresponding estimates $\hat{f}_e$ and $\hat{f}_c$ which are defined in terms of first crossing times. Specifically,

$$\hat{f}_e = \inf_{f \in [f_0, 1]} \left\{ f : BSDF_f \left(f_0 \right) > scv \left(\beta_T \right) \right\}, \tag{5}$$

$$\hat{f}_c = \inf_{f \in \left[\hat{f}_e + L_T, 1 \right]} \left\{ f : BSDF_f \left(f_0 \right) < scv \left(\beta_T \right) \right\}. \tag{6}$$

It is often useful to define a bubble as having a required minimum duration to eliminate potentially misleading information from short term blips in the data. PSY (2013a) used a minimum duration based on a slowly varying function such as $\log T$ to distinguish bubbles. In sample fraction form we can set $L_T = \delta \log (T) / T$ where δ is a sample-frequency dependent parameter (so that δ is greater for monthly data than quarterly or annual data). The sample fraction L_T is used in defining the collapse date $\hat{f}_c$ in (2) as the first crossing time following the minimum duration period of the bubble, $\hat{f}_e + L_T$, when the test sequence recursion $BSDF_f(f_0)$ falls below the critical value $scv(\beta_T)$.

2.2 Reverse recursion tests for crisis origination and termination

For identifying crisis episodes, we suggest applying the BSDF test to data X_t^* arranged in reverse order to the original series X_t, so that $X_t^* = X_{T+1-t}$, for $t = 1, 2, \ldots, T$. If asset prices follow model (2), the reversed series X_t^* satisfies the following dynamics

$$X_t^* = \begin{cases} -dT^{-\eta} + X_{t-1}^* + v_t, & t \in N_0 \cup N_1 \\ \delta_T^{-1} X_{t-1}^* + \delta_T^{-1} v_t, & t \in B \\ \gamma_T^{-1} X_{t-1}^* + \gamma_T^{-1} v_t, & t \in C \end{cases}. \tag{7}$$

with $v_t = -\varepsilon_{T-(t-2).}$ In (7) the original mildly integrated collapse process transforms to a mildly explosive process and vice versa. Hence, detecting crisis episodes in X_t is equivalent to testing for mildly explosive behavior in X_t^*.

The BSDF statistic for crisis episode detection is defined as

$$BSDF_g^* (g_0) \text{ with } g \in [g_0, 1] \text{ and } g = 1 - f,$$

where $BSDF_g^*(g_0)$ is the BSDF statistic for observation (fraction) g of X_t^* where the recursion (in reverse direction) initiates with a minimum window size g_0. The market recovery date (f_r) and crisis origination date (f_c), both expressed in fractions of the original series sequence are then calculated as follows:

$$\hat{f}_r = 1 - \hat{g}_e, \text{ where } \hat{g}_e = \inf_{g \in [g_0, 1]} \left\{ g : BSDF_g^* (g_0) > scv^* (\beta_T) \right\} \tag{8}$$

$$\hat{f}_c = 1 - \hat{g}_c, \text{ where } \hat{g}_c = \inf_{g \in [\hat{g}_e, 1]} \left\{ g : BSDF_g^* (g_0) < scv^* (\beta_T) \right\}, \tag{9}$$

where $scv^*(\beta_T)$ is the $(1-\beta_T)\,100\%$ critical value of the $BSDF_g^*(g_0)$ statistic, with $scv^*(\beta_T) \to \infty$ as $\beta_T \to 0$. According to these crossing times, market recovery $(\hat{f}_r)$ following a crash begins when normal market behavior changes to exuberance in the reverse series $(\hat{g}_e)$. Similarly, market collapse in the original series begins when exuberance in the reverse series shifts to collapse at $(\hat{g}_c)$. In many applications restrictions on crisis duration via the presence of a slowly varying function L_T in these crossing time expressions will not be needed, especially when there is interest in the detection of abrupt crisis movements in the data.

Notice that the information set for calculating the $BSDF_g^*(g_0)$ statistic in the original series is $I_f^R = \{T_f, T_f+1, \ldots, T\}$ with $T - T_f \geq \lfloor Tg_0 \rfloor$, the minimum window size used in the reverse recursion. In the reverse time series X_t^* this data corresponds to $I_g^R = \{1, 2, \ldots, T_g = \lfloor Tg \rfloor\}$ where $g = 1 - f$. Clearly, at any point in the sample $t < T$ the information set $\{t, t+1, \ldots, T\}$ contains future observations up to the sample end point T. Accordingly, we can regard this crisis detection strategy as an *ex post* identification tool. Nevertheless, there is a real time detector version of this algorithm that may be implemented in practical work as we now explain.

Specifically, the algorithm may be implemented on subsets of the data from any end point $K < T$. For example, suppose in the original series a bubble has been detected in the expansionary phase in real time so that $K > \lfloor fT \rfloor$. Then a question of major importance to all market participants and regulators is the timing of a market correction. To test for correction the above procedure may be implemented in reverse order from any sample point t to assess evidence of a correction. In particular, suppose the current observation is $t = \lfloor \kappa T \rfloor = K$ for some $\kappa > 0$. Reversing the series and writing $X_s^* = X_{K+1-s}$ for $s = 1, ..., \lfloor gK \rfloor$ with $g \in [0,1]$, the recursive statistics $BSDF_g^*(g_0)$ may be calculated from $\{X_s^*\}_{s=1}^{\lfloor gK \rfloor}$ starting from some minimal window size g_0. In real time applications, g_0 will need to be small (in sample observation terms perhaps $\lfloor g_0 T \rfloor \geq 6$) so that evidence for possible market correction is collected as early as possible. The main advantage of this approach (rather than testing for correction in the original series) is that right-sided unit root tests are typically much more sensitive to departures from the null than left-sided tests. In other words, the hypothesis of market correction is the existence of a mildly explosive process in the reverse series.

3 Asymptotics

From PSY (2014c), the asymptotic distribution of the $BSDF_f(f_0)$ statistic under the null hypothesis (3) has the form

$$F_f(W, f_0) := \sup_{f_1 \in [0, f - f_0]} \left\{ \frac{f_w \left[\int_{f_1}^{f} W(s)\, ds - \frac{1}{2} f_w \right] - \int_{f_1}^{f} W(s)\, ds. \int_{f_1}^{f} dW}{f_w^{1/2} \left\{ f_w \int_{f_1}^{f} W(s)^2 ds - \left[\int_{f_1}^{f} W(s)\, ds \right]^2 \right\}^{1/2}} \right\},$$

where W is a standard Wiener process. The reverse regression asymptotics are given in the following form.

Theorem 1 *When the regression model includes an intercept and the null hypothesis is (3), the limit distribution of the $BSDF_g^*(g_0)$ statistic is:*

$$F_g(W, g_0) := \sup_{g_1 \in [0, g-g_0]} \left\{ \frac{g_w \left[\int_{1-g}^{1-g_1} W(s)\, dW + \frac{1}{2} g_w \right] + \int_{1-g}^{1-g_1} W(s)\, ds . \int_{1-g}^{1-g_1} dW}{g_w^{1/2} \left\{ g_w \int_{1-g}^{1-g_1} W(s)^2\, ds - \left[\int_{1-g}^{1-g_1} W(s)\, ds \right]^2 \right\}^{1/2}} \right\}.$$

See Appendix A and the Supplement for the proof. Notice that the $BSDF_g^*(g_0)$ statistic has a noncentral asymptotic distribution. The noncentrality arises from the endogeneity induced by the non-martingale implications of reverse regression, viz., the error component $\sum_{j=2}^{T} X_{j-1}^* v_j$ which, by virtue of the construction of the series X_t^* and v_t, equals $-\sum_{j=2}^{T} X_{T-j+2}\varepsilon_{T-j+2}$ for which $\mathbb{E}(X_{T-j+2}\varepsilon_{T-j+2}) \neq 0$.

3.1 The $BSDF_f(f_0)$ statistic

The asymptotic properties of the BSDF statistic under the PWY bubble model are given in PSY (2014c). Here we derive the limit theory for the BSDF statistic under the more realistic bubble process (2) allowing for various forms of financial contraction captured by the parameterization within the collapse process. The derivations involve a non-trivial extension of the limit theory of PSY (2014c) to account for the additional regime, the drift in the normal martingale process, and the new bubble collapse process.

Theorem 2 (BSDF detector) *Under the alternative hypothesis of mildly explosive behavior in model (2), the limit forms of the $BSDF_f(f_0)$ statistic are as follows:*

$$BSDF_f(f_0) \sim \begin{cases} F_f(W, f_0) & \text{if } f \in N_0 \\ O_p\left(T^{1-\alpha/2}\right) \to +\infty & \text{if } f \in B \\ O_p\left(T^{\omega(\alpha,\beta)}\right) = \begin{cases} O_p\left(T^{\alpha/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{(1-\alpha+\beta)/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{(1-\beta+\alpha)/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to +\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} & \text{if } f \in C \end{cases}$$

Theorem 2 shows that the $BSDF$ statistic diverges to infinity at rate $O_p\left(T^{1-\alpha/2}\right)$ when $f \in B$ and is $O_p\left(T^{\omega(\alpha,\beta)}\right)$ when $f \in C$ where the order $\omega(\alpha, \beta)$ depends on the values of the rate parameters (α, β). These results differ from those under the PWY model analyzed in PSY (2014c) where the BSDF statistic diverges to infinity at rate $O_p\left(T^{1-\alpha/2}\right)$ when $f \in B$, as above, but diverges to negative infinity at rate $O_p\left(T^{(1-\alpha)/2}\right)$ when $f \in C$ in contrast to the rate $O_p\left(T^{\omega(\alpha,\beta)}\right)$ above, which depends on the relative strengths (α, β) of the bubble and collapse processes. In particular, when the collapse regime follows a mildly integrated process (rather than an abrupt collapse), for $f \in C$ the limit form of the BSDF statistic may diverge to positive or negative infinity depending on the relative speeds of the bubble expansion and collapse, which are controlled by the rate parameters α and β. These parameters then play a major role in the conditions for consistent estimation of the bubble origination and termination dates, as shown in the following result.

Theorem 3 (BSDF detector) *Suppose $\hat{f}_e$ and $\hat{f}_c$ are the date estimates obtained from the backward sup DF statistic crossing times (5). Under the alternative hypothesis of mildly explosive behavior in model (2), if the following conditions hold*

$$\begin{cases} \frac{T^{\alpha/2}}{scv^{\beta}T} + \frac{scv^{\beta}T}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \frac{T^{(1-\alpha+\beta)/2}}{scv^{\beta}T} + \frac{scv^{\beta}T}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \frac{T^{(1-\beta+\alpha)/2}}{scv^{\beta}T} + \frac{scv^{\beta}T}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \frac{T^{\beta/2}}{scv^{\beta}T} + \frac{scv^{\beta}T}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

we have $\hat{f}_e \xrightarrow{p} f_e$ and $\hat{f}_c \xrightarrow{p} f_c$ as $T \to \infty$.

Theorem (3) shows that consistent estimation of the dates of bubble origination and collapse requires certain conditions on the expansion rate of the test critical value $scv\,(\beta_T)$. In particular, depending on the values of the rate parameters (α, β), the critical value $scv\,(\beta_T)$ needs to lie respectively in the intervals $\left(T^{\alpha/2}, T^{1-\alpha/2}\right)$, $\left(T^{(1-\alpha+\beta)/2}, T^{1-\alpha/2}\right)$, $\left(T^{(1-\beta+\alpha)/2}, T^{1-\alpha/2}\right)$, and $\left(T^{\beta/2}, T^{1-\alpha/2}\right)$ according as $\{\alpha > \beta$ and $1+\beta < 2\alpha\}$, $\{\alpha > \beta$ and $1+\beta > 2\alpha\}$, $\{\alpha < \beta$ and $1+\alpha > 2\beta,\}$ and $\{\alpha < \beta$ and $1+\alpha < 2\beta\}$. These conditions are more restrictive than the simple condition that applies in the PWY bubble model, where the expansion rate of $scv\,(\beta_T)$ is only required to be lower than $T^{1-\alpha/2}$. Importantly, theorem (3) reveals that the conditions become increasingly restrictive as the rate of bubble collapse becomes slower with larger values of the collapse rate parameter β. In this case, as might be expected, when the collapse is slow rather than rapid, large values of β make it harder for the algorithm to distinguish the explosive bubble regime from the collapse regime.

3.2 The $BSDF_g^*\,(g_0)$ statistic

We first derive the limit properties of the $BSDF_g^*\,(g_0)$ under the data generating process (7). Note that the volatility of X_t^* differs in regimes B and C, as is clear from the reverse model specification (7), and there is a switch in the interpretation of the regimes since the autoregressive coefficients for the explosive and stationary regimes are now $\gamma_T^{-1} \sim 1+c_2T^{-\beta}$ and $\delta_T^{-1} \sim 1-c_1T^{-\alpha}$, respectively. The mildly explosive rate is now governed by the parameter β and the collapse rate is controlled by α.

Theorem 4 (The $BSDF_g^*\,(g_0)$ statistic) *Under the alternative hypothesis of mildly explosive* *behavior in model (7), the limit behavior of the $BSDF_g^*(g_0)$ statistic is as follows:*

$$BSDF_g^*(g_0) \sim_a \begin{cases} F_g(W, g_0) & \text{if } g \in N_1 \\ \begin{cases} O_p\left(T^{1/2}\right) \to +\infty & \text{if } \alpha > \beta \\ O_p\left(T^{1-\beta/2}\right) \to +\infty & \text{if } \alpha < \beta \end{cases} & \text{if } g \in C \\ O_p\left(T^{\omega^*(\alpha,\beta)}\right) = \begin{cases} O_p\left(T^{\alpha/2}\right) \to +\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{(1-\alpha+\beta)/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{(1-\beta+\alpha)/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} & \text{if } g \in B \end{cases};$$

Like the BSDF statistic, the $BSDF_g^*(g_0)$ statistic diverges to positive infinity when X^* is in the explosive regime (i.e. when $g \in C$). The rate of divergence is faster when $\alpha < \beta$ (i.e. $T^{1-\beta/2}$) than it is when $\alpha > \beta$ (i.e. $T^{1/2}$). In this case, the divergence rate $O_p\left(T^{1-\beta/2}\right)$ of the statistic also increases as β decreases. Intuitively, as β and α decrease both the collapse rate and bubble expansion rate increase, making detection of the collapse easier. In regime B, the limiting form of the $BSDF^*$ statistic has magnitude $O_p\left(T^{\omega^*(\alpha,\beta)}\right)$, which is the same as that for the BSDF statistic in regime C, so in this case the reverse and forward regressions are balanced.

Theorem 5 (The $BSDF_g^*$ detector) *Suppose $\hat{f}_c$ and $\hat{f}_r$ are the date estimates obtained from the $BSDF_g^*(g_0)$ statistic crossing times rules (9). Under the alternative hypothesis of mildly explosive behavior in model (2), if*

$$\begin{cases} \frac{T^{\alpha/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \frac{T^{(1-\alpha+\beta)/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \frac{T^{(1-\beta+\alpha)/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1-\beta/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \frac{T^{\beta/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1-\beta/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

we have $\hat{f}_r \xrightarrow{p} f_r$ and $\hat{f}_c \xrightarrow{p} f_c$ as $T \to \infty$.

To obtain consistent estimators of the crisis origination and termination dates, the expansion rate of $scv^*(\beta_T)$ is required to fall respectively in the intervals $\left(T^{\alpha/2}, T^{1/2}\right)$, $\left(T^{(1-\alpha+\beta)/2}, T^{1/2}\right)$, $\left(T^{(1-\beta+\alpha)/2}, T^{1-\beta/2}\right)$, and $\left(T^{\beta/2}, T^{1-\beta/2}\right)$ for the cases $\{\alpha > \beta \text{ and } 1+\beta < 2\alpha\}$, $\{\alpha > \beta \text{ and } 1+\beta > 2\alpha\}$, $\{\alpha < \beta \text{ and } 1+\alpha > 2\beta\}$, and $\{\alpha < \beta \text{ and } 1+\alpha < 2\beta\}$. Despite their apparent complexity, these conditions generally accord with intuition because they tend to be less restrictive in the following cases: (i) as the value of α decreases (that is, as the collapse rate for X_t^* becomes faster, or the mildly explosive rate for X_t increases); and (ii) as the value of β decreases (that is, as the explosive rate for X_t^* increases, or the collapse rate of X_t increases).

4 Simulation Evidence

Extensive simulation studies were conducted in PSY (2014a,b) showing that the PSY dating strategy performs well under the PWY data generating process compared with other strategies.

These simulations are extended here to explore finite sample performance under the new bubble generating process (2) that allows for multiple forms of collapse regime. We consider bubble detection performance as well as bubble origination and collapse date determination. These findings complement those of PSY for abrupt collapse conditions. We also investigate performance characteristics of the $BSDF^*$ test for crisis detection under the new bubble generating process.

The base parameter settings used here are the same as those in PSY (2014b), namely $y_0 = 100$, $\sigma = 6.79$, and $c = c_1 = c_2 = 1$, where y_0 and σ are selected to match the initial value and sample standard deviation of the normalized S&P 500 price-dividend ratio examined in PSY. Bubbles were identified using respective finite sample 95% quantiles obtained from simulations with 2,000 replications. The minimum window size has 24 observations and lag order is set to zero. The simulations explore the effects of various settings for the mildly explosive and collapse regime rate parameters (α, β) that enter the new process (2).

4.1 The BSDF test for bubble origination and collapse dates

As in PSY (2014c) we examine the proportion of successful bubble detection, along with the empirical mean and standard deviation (given in parentheses in the following tables) of the estimated origination and collapse dates. Successful detection of a bubble is defined as an outcome where the estimated origination date is greater than or equal to the true origination date and smaller than the true collapse date of that particular bubble (i.e. $f_{ie} \leq \hat{f}_{ie} < f_{ic}$). So the narrower is the interval (f_{ie}, f_{ic}) the more challenging is the requirement.

Table 1: Detection rate and estimation of the origination and collapse dates (for different drift values). Parameters are set to: $y_0 = 100, \sigma = 6.79, c = c_1 = c_2 = 1, \alpha = 0.60, \beta = 0.1, d_{BT} = \lfloor 0.20T \rfloor, d_{CT} = \lfloor 0.01T \rfloor, f_e = 0.4, T = 100$. Figures in parentheses are standard deviations.

	DGP (PWY)	DGP (PSY)		
		$\eta = 0.6$	$\eta = 1$	$\eta = 2$
Detection Rate (B)	0.90	0.91	0.90	0.90
$\hat{f}_e - f_e$	0.07 (0.04)	0.07 (0.04)	0.07 (0.04)	0.07 (0.04)
$\hat{f}_c - f_c$	-0.00 (0.01)	-0.00 (0.01)	-0.00 (0.01)	-0.00 (0.01)

Note: Calculations are based on 2,000 replications. The minimum window has 24 observations.

The first step is to investigate the impact of drift in the unit root process on bubble detection accuracy. The PSY estimate of the bubble origination date depends solely on past information. So when there is a single bubble in the sample period, the later mildly integrated collapse regime should have no impact on the accuracy of the bubble origination estimate. But the presence of drift, even asymptotically negligible drift, in the unit root process that characterizes normal periods may well impact bubble origination detection.

In the simulation, we fix β at 0.1 and the duration of bubble collapse d_{CT} at $\lfloor 0.01T \rceil$. This setting delivers an instantaneous one-period collapse when the sample size $T = 100$, which resembles the abrupt collapse pattern of PWY. We let η take the values $\{0.6, 1, 2\}$, which give corresponding drift values of $\{0.063, 0.010, 0.000\}$. For each parameter constellation, 2,000 replications were employed. As evident in Table 1, the parameter η has no material impact on bubble detection when $\eta > 0.5$ as here, which is concordant with asymptotic theory. Thus, the PSY strategy continues to deliver consistent estimates despite the inclusion of this type of drift. However, under these parameter settings, there is an seven-observation delay in detecting the bubble, indicating some upward bias in the date estimates. On the other hand, just as in PSY, due to the sudden one-period collapse, the termination date of the bubble expansion is estimated with great accuracy.

The next step is to explore the influence of a mildly integrated collapse regime. The model specification of this regime allows for processes with different collapse speeds and durations. We consider three collapse patterns here: sudden collapse (i.e $\beta = 0.1$ and $d_{CT} = \lfloor 0.01T \rceil$), disturbing collapse (i.e. $\beta = 0.5$ and $d_{CT} = \lfloor 0.10T \rceil$), and smooth collapse (i.e. $\beta = 0.9$ and $d_{CT} = \lfloor 0.20T \rceil$). We let $\eta = 1$, $\alpha = 0.6$, $d_{BT} = 0.2$. Figure 2 plots one typical realization of these three collapse processes.

Table 2 displays the estimation results of the PSY strategy under different data generating processes. We observe an increasing delay in bubble collapse date estimates (viz., f_c) as the collapse process becomes smoother. For instance, the bias ($\hat{f}_c - f_c$) is one observation in a disturbing collapse regime but increases to nine observations for a smooth collapse. The bubble origination date is mainly unaffected by the collapse regime dynamics, as expected and indicated above.

Table 2: Detection rate and estimation of the origination and collapse dates (under different collapse patterns). Parameters are set to: $y_0 = 100, \sigma = 6.79, c = c_1 = c_2 = 1, \alpha = 0.70, d_{BT} = \lfloor 0.20T \rfloor, f_e = 0.4, T = 100$, $\beta = 0.1, d_{CT} = \lfloor 0.01T \rceil$ for sudden collapse, $\beta = 0.5, d_{CT} = \lfloor 0.10T \rceil$ for disturbing crash, and $\beta = 0.9, d_{CT} = \lfloor 0.20T \rceil$ for smooth collapse. Figures in parentheses are standard deviations.

	PWY	Sudden	Disturbing	Smooth
Detection Rate (B)	0.90	0.90	0.91	0.91
$\hat{f}_e - f_e$	0.07 (0.04)	0.07 (0.04)	0.07 (0.04)	0.07 (0.04)
$\hat{f}_c - f_c$	-0.00 (0.01)	-0.00 (0.01)	0.02 (0.01)	0.09 (0.04)

Note: Calculations are based on 2,000 replications. The minimum window has 24 observations.

4.2 The $BSDF^*$ test for crisis origination and market recovery dates

In the case of a bubble-led crisis such as model (2), the origination date of the crisis coincides with the date of collapse. Successful detection of a bubble-led crisis may then be defined as an outcome where the estimated collapse date is greater than or equal to the true bubble origination date and smaller than the true recovery date of that particular crisis (i.e. $f_{ie} \leq \hat{f}_{ic} < f_{ir}$). As earlier, we report the detection rate of a crisis, along with the empirical mean and standard deviation (shown in parentheses) of the estimated crisis origination and termination dates.

Table 3 reports the performance characteristics of the $BSDF^*$ test under sudden, disturbing, and smooth collapse processes with different bubble expansion rates. As is evident in the Table,

Table 3: Detection rate and estimation of the collapse and recovery dates (with various expansion and collapse patterns). Parameters are set to: $y_0 = 100, \sigma = 6.79, c = c_1 = c_2 = 1, d_{BT} = \lfloor 0.20T \rfloor, f_e = 0.4, T = 100$, $\beta = 0.1, d_{CT} = \lfloor 0.01T \rceil$ for sudden collapse, $\beta = 0.5, d_{CT} = \lfloor 0.10T \rceil$ for disturbing crash, and $\beta = 0.9, d_{CT} = \lfloor 0.20T \rceil$ for smooth collapse. Figures in parentheses are standard deviations.

	Sudden	Disturbing	Smooth
$\alpha = 0.6$			
Detection Rate (C)	0.33	0.96	0.66
$\hat{f}_c - f_c$	-0.00 (0.01)	-0.03 (0.02)	0.00 (0.05)
$\hat{f}_r - f_r$	0.00 (0.02)	-0.03 (0.03)	-0.12 (0.06)
$\alpha = 0.7$			
Detection Rate (C)	0.37	0.92	0.64
$\hat{f}_c - f_c$	-0.01 (0.02)	-0.04 (0.02)	-0.01 (0.07)
$\hat{f}_r - f_r$	0.00 (0.02)	-0.04 (0.03)	-0.15 (0.08)
$\alpha = 0.8$			
Detection Rate (C)	0.35	0.88	0.58
$\hat{f}_c - f_c$	-0.01 (0.03)	-0.04 (0.03)	-0.02 (0.08)
$\hat{f}_r - f_r$	0.00 (0.02)	-0.05 (0.03)	-0.17 (0.09)

Note: Calculations are based on 2,000 replications. The minimum window has 24 observations.

the crisis detection rate is highest for disturbing collapses (96% when $\alpha = 0.6$) and lowest for sudden collapses (41% when $\alpha = 0.6$). These findings are explained by the fact that sudden crises have short duration and collapse date estimates come late, whereas in a disturbing crisis there is a delay before recovery which aids crisis detection. The crisis detection rate increases with the rate of bubble expansion. As a case in point, when α decreases from 0.8 to 0.6, the detection rate for disturbing collapses rises from 88% to 96%.

In addition, we see that the estimated crisis origination date is generally bias downward. Moreover, the bias (i.e. $\hat{f}_c - f_c$) is larger for disturbing collapses than sudden and smooth collapses. Interestingly, the estimation accuracy of the crisis origination date also depends on the bubble expansion rate. Namely, the bias is marginal larger when the bubble expansion rate is slower. This is consistent with our earlier finding for the BSDF test where the estimated crisis origination (or bubble collapse) date is affected by the collapse pattern.

Estimates of the recovery date evidently highly accurate for sudden collapses, whereas there is substantial downward bias for smooth collapses. For instance, when $\alpha = 0.6$, there is a twelve-period bias (earlier than the true date) in the estimates of f_r under a smooth collapse, which accounts for 60% of the collapse duration. The bias for disturbing collapses is much smaller and is close to the near zero bias for sudden collapses. For both disturbing and smooth collapses, the bias increases as the bubble expansion rate becomes slower (with larger α).

Table 4: Detection rate and collapse and recovery date estimation (for different collapse rates and collapse durations). Parameters are set to: $y_0 = 100, \sigma = 6.79, c = c_1 = c_2 = 1, \alpha = 0.7, d_{BT} = \lfloor 0.20T \rfloor, r_o = 0.4, T = 100$, $\beta = 0.1, d_{CT} = \lfloor 0.01T \rceil$ for sudden collapse, $\beta = 0.5, d_{CT} = \lfloor 0.10T \rceil$ for disturbing crash, and $\beta = 0.9, d_{CT} = \lfloor 0.20T \rceil$ for smooth collapse. Figures in parentheses are standard deviations.

	$d_{CT} = \lfloor 0.05T \rfloor$	$d_{CT} = \lfloor 0.10T \rfloor$	$d_{CT} = \lfloor 0.15T \rfloor$
$\beta = 0.3$			
Detection Rate (C)	0.96	0.97	0.96
$\hat{r}_c - r_c$	-0.04 (0.02)	-0.04 (0.02)	-0.04 (0.02)
$\hat{r}_r - r_r$	-0.02 (0.02)	-0.04 (0.02)	-0.08 (0.02)
$\beta = 0.5$			
Detection Rate (C)	0.91	0.96	0.93
$\hat{r}_c - r_c$	-0.01 (0.02)	-0.03 (0.02)	-0.03 (0.02)
$\hat{r}_r - r_r$	-0.02 (0.03)	-0.03 (0.03)	-0.05 (0.03)
$\beta = 0.7$			
Detection Rate (C)	0.71	0.90	0.91
$\hat{r}_c - r_c$	-0.01 (0.04)	-0.01 (0.03)	-0.01 (0.03)
$\hat{r}_r - r_r$	-0.03 (0.04)	-0.05 (0.04)	-0.06 (0.05)

Note: Calculations are based on 2,000 replications. The minimum window has 24 observations.

For further investigation, we extend the parameter specification in the case of disturbing collapses by varying β from 0.3 to 0.7 and d_{CT} from 5% of the total sample to 15%. Consistent with expectations, the $BSDF^*$ strategy provides higher crisis detection rates when bubbles collapse faster (Table 4). For instance, with a collapse duration of $\lfloor 0.05T \rfloor$, the detection rate increases from 71% to 96% when the value of β declines from 0.7 to 0.3 (corresponding to faster collapse in the mildly integrated process over this period). Moreover, the crisis termination date is more accurately estimated when there is shorter collapse duration. For example, assuming $\beta = 0.3$, the estimated recovery date is two observations earlier than the true recovery date when $d_{CT} = \lfloor 0.05T \rfloor$. However, it increases to eight observations when the duration increases to 15% of the total sample. The bias direction in these estimates is consistent with the reverse regression scenario underlying the $BSDF^*$ detection strategy.

4.3 Real Time Monitoring of Market Correction

The goal of this use of reverse regression is to aid the ongoing detection of market recovery dates. We propose to implement the reverse procedure repeatedly for each observation starting from the date of the bubble collapse, that is T_c in chronological time. The dating rules in (8) and (9) can always be rewritten as

$$\hat{f}_c = \inf_{f \in [0, 1-g_0]} \left\{ f : BSDF_f^{**}(g_0) > scv^{**}(\beta_T) \right\} \tag{10}$$

$$\hat{f}_r = \inf_{f \in [\hat{f}_c, 1-g_0]} \left\{ f : BSDF_f^{**}(g_0) > scv^{**}(\beta_T) \right\} \tag{11}$$

where $BSDF_f^{**}(g_0)$ and $scv^{**}(\beta_T)$ are the reverse series of $BSDF_g^{*}(g_0)$ and $scv^{*}(\beta_T)$. Suppose we conduct the reverse PSY test on a sample period running from some initial (reverse) observation 1 through to K (which includes observation T_c in chronological time) and identify one occurrence of market correction, namely $BSDF_{t'-1}^{**} > scv^{**}$ with $BSDF_{t'}^{**} < scv^{**}$ and $t' \geq T_c$ for some given critical value scv^{**} in the reverse regression. Suppose, in addition, that no market correction is detected in samples from 1 to s, with $s < K$. Then, we conclude that t' is the date of market recovery (i.e. $\hat{f}_r = t'/T$) and K is the date at which the correction is detected in the data.

Table 5 reports the average delay of the estimated market recovery date ($\hat{f}_r - f_r$) and the average delay in detecting this correction ($DD = K/T - \hat{f}_r$). By construction, the delay in detecting the market correction is bounded below by the minimum window size f_0. For early detection, one would need to consider setting f_0 to be a small value. On the other hand, choosing too small a value for f_0 may result in inaccurate estimation of model parameters and lead to corresponding distortions in the market recovery date estimates. Hence, the choice of f_0 is important for performance of the monitoring procedure. In simulations we allowed the minimum window size to vary from half a year to two years (i.e f_0 varies from 0.06 to 0.24) to assess sensitivies to choice of f_0. The number of replications was 2,000.

Table 5: The estimated dates of market recovery and the delays in detecting market correction. Parameters are set to: $y_0 = 100, \sigma = 6.79, c = c_1 = c_2 = 1, \alpha = 0.6, d_{BT} = \lfloor 0.20T \rfloor, f_e = 0.4, T = 100$. Figures in parentheses are standard deviations.

(1) Disturbing Crises				
	$f_0 = 0.06$	$f_0 = 0.08$	$f_0 = 0.12$	$f_0 = 0.24$
Detection Rate	0.99	0.98	0.98	0.97
$\hat{f}_r - f_r$	-0.23 (0.28)	-0.19 (0.28)	-0.06 (0.14)	-0.06 (0.13)
DD	0.21 (0.08)	0.20 (0.10)	0.12 (0.10)	0.22 (0.15)
(2) Smooth Crises				
	$f_0 = 0.06$	$f_0 = 0.08$	$f_0 = 0.12$	$f_0 = 0.24$
Detection Rate	0.91	0.90	0.90	0.86
$\hat{f}_r - f_r$	-0.19 (0.28)	-0.22 (0.23)	-0.27 (0.27)	-0.28 (0.31)
DD	0.20 (0.10)	0.09 (0.24)	0.17 (0.24)	0.28 (0.32)

Note: Calculations are based on 2,000 replications.

As expected, the procedure does not perform well in detecting market recovery dates for smooth crises. Nevertheless, for disturbing crises (perhaps the most frequently observed type in practical work), best performance is achieved with 12 observations in the minimum window, which corresponds to $f_0 = 0.12$. As is evident in the table, with this setting the average bias of the market recovery date estimates is -0.06 (or six observations early) and the average delay in detecting the correction is the minimum window size (12 observations).

5 Empirical Applications

We examine NASDAQ stocks as in PWY (2011), extending the sample period to August 2013. The NASDAQ composite index and NASDAQ dividend yield are obtained in monthly frequency from Datastream International, starting from January 1973. The dividend index at the current period (t), which is included as a proxy for stock market fundamentals, is calculated by multiplying the price index and dividend yield at period $t+1$.

We apply BSADF and BSADF* tests to the NASDAQ price-dividend ratio. The smallest window size has 24 observations (2 years). The lag order in the ADF regression is selected using BIC with a maximum lag order of 4. Bubble duration is restricted to be no less than one year.

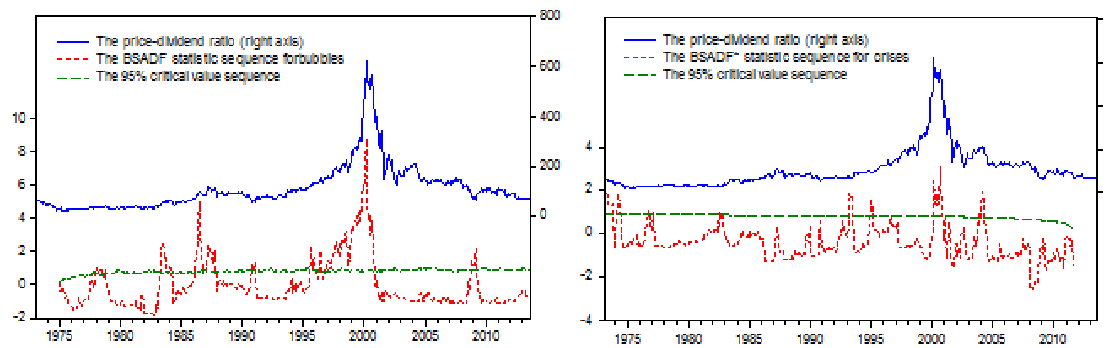

Figure 4: The NASDAQ stock market

Figure 4 plots the recursive BSADF and BSADF* statistics against their corresponding 95% critical value sequences separately in panels (a) and (b). The test procedures identify several bubble and crisis episodes. There are two bubble episodes with abrupt collapses: 1983M04-1984M03 and 1986M02-1987M11. Enlarged plots for these two episodes are displayed in Figure 5, the first panel showing the bubble episode in 1983 and the second the 'black Monday' episode in 1987. The BSADF* procedure does not detect collapses for these two episodes, which is perhaps unsurprising becasue simulations show that the crisis detection rate is lower when the collapse is rapid with short duration as in these two case.

The most interesting episode identified by the tests is the famous dot-com bubble in the late 1990s, displayed in enlarge form in Figure 6. The origination of the bubble episode is detected (using BSADF) as August 1995, close to the original estimate obtained in PWY of July 1995. The estimated collapse dates obtained from the BSADF and BSADF* procedures are 2000M11 and 2000M2, respectively. Notice that the peak of the dot-com bubble is 2000M03, so the BSADF* test anticipates the crash (but does so using the *ex post* subsequent crash data, of course). The collapse process for the dot-com bubble episode is much smoother than the previous two episodes,

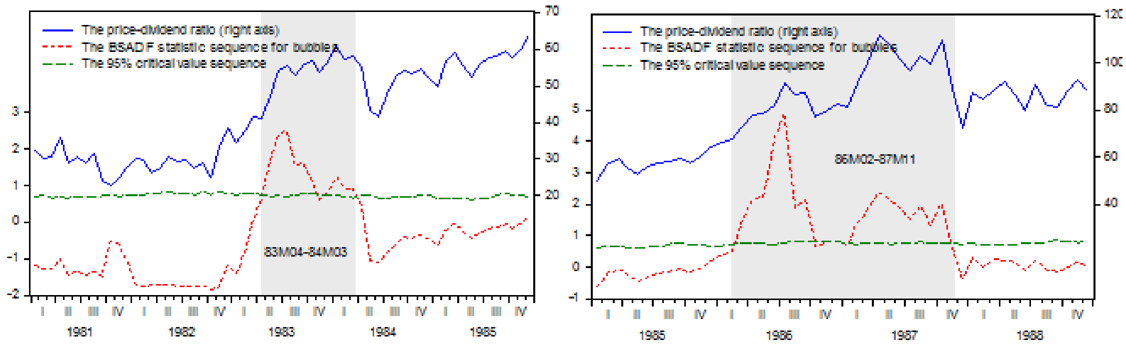

Figure 5: The 1983 and 1986 bubble episodes

which partly explains the detective capability of the BSADF* test since the date of the bubble collapse clearly differs from the date of later market recovery. According to the BSADF* test, the crash lasts from March to November 2000, followed by a further correction in January 2004 and full return to normal market conditions in May 2004.

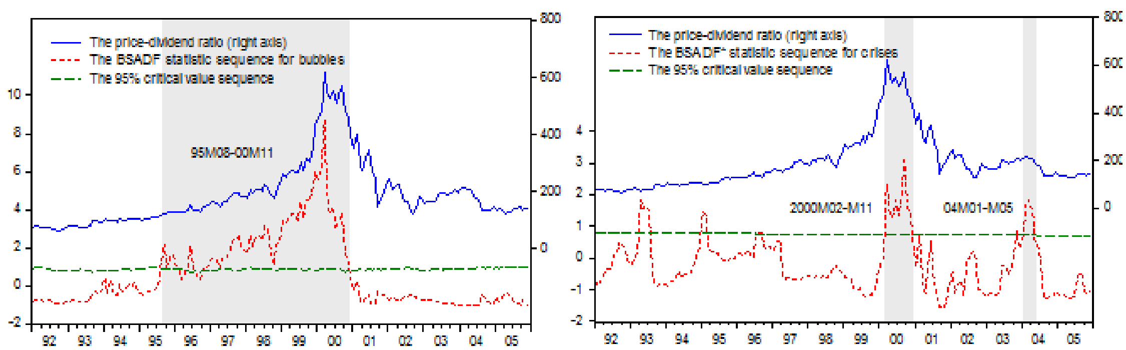

Figure 6: The dot-com bubble episode

Enlarged details of the 1973/74 and 1976 stock market downturn episodes are shown in Figure 7. According to the BSADF* test, the 1973/74 episode lasts for 17 months (starting from February 1973) and the 1976 episode lasts from September to November 1976. The procedure also identifies three minor downturns in late 1982, mid of 1993M04 and 1995M02.

Finally, we conduct a pseudo-real-time monitoring exercise to assess market collapse and recovery for the dot-com bubble episode. Specifically, we start implementing the reverse procedure repeatedly for each observation between March 2000 (the peak of the episode) and December 2006. Specifically, we conduct the reserve dating technique first for the window running from January 1973 to March 2000 to see whether any market correction has occurred. If affirmative, we calculate the date of market recovery and record March 2000 as the data detecting this correction. Otherwise, we expand the sample one observation forward and apply the same technique to the expanded sample period until detecting the occurence of market correction. One can either stop the investigation upon detection of market correction or continue the procedure for possible futher

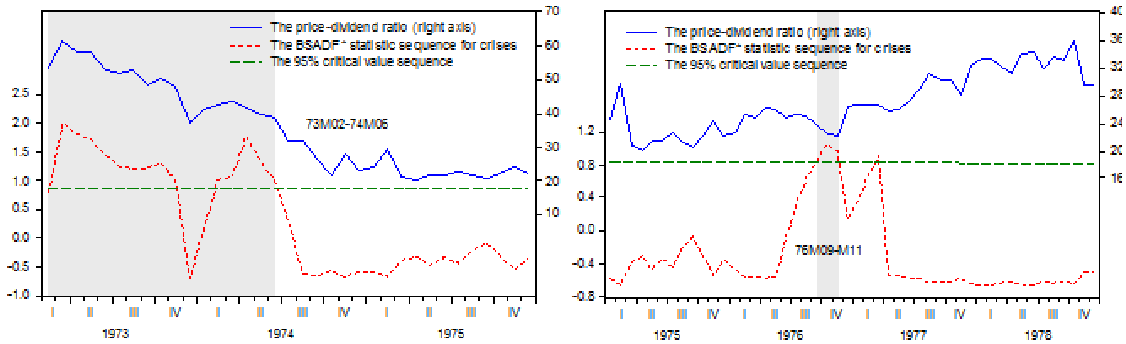

Figure 7: The 1973 and 1976 stock market crashes

correction. Here, we choose the latter and stop the pseudo-real-time investigation in December 2006. The minimum window size has 12 observations and the lag length is fixed at one. Similar to the *ex-post* identification strategy, we identify two episodes of market correction within this sample period. The first correction is in October 2000 (detected by the procedure in November 2002) and the other one is in May 2004 (detected in October 2005).

6 Conclusion

Financial bubbles are typically characterized by mildly explosive expansions and subsequent contractions that can be abrupt, extended, or various combinations of the two. In this work we have modeled financial contractions using a mildly integrated process that can capture a variety of forms of reversion to normal martingale behavior in a financial market. The resulting model has multiple break points corresponding to the bubble origination and peak, the implosion of the bubble, and the reversion to normality. This framework is intended to be more realistic than simpler models that assume an abrupt collapse of a bubble to normality and is therefore suited to a wider range of practical applications. The limit theory shows that the bubble dating strategy of Phillips, Shi and Yu (2013a,b) delivers consistent date estimates within this more realistic bubble generating framework and simulations corroborate its advantages in finite samples.

The new reverse regression implementation of the PSY strategy developed here helps to detect crises and estimate their associated turning points. The strategy is suited to cases of a single crash or multiple crashes and provides consistent estimates of the origination, termination, and recovery dates of a crisis. Crisis detection methodology of this type can be applied when crises occur in seemingly normal periods that are not prefaced by an expansionary bubble phase. In applying these methods to the Nasdaq stock market over 1973-2013, the tests identify several crisis incidents, including the 1973 and 1976 stock market crashes and the famous ‘black Monday’ crash of 1987, in addition to the 1990s Nasdaq bubble episode.

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Appendix: Limit Theory and Proofs of Main Results

This Appendix has two main sections (A and B). The first provides limit theory for the BSADF* statistic under the null based on the reverse regression model formulation (7). The second gives limit theory for the BSADF and BSADF* statistics under the alternative. Supporting lemmas are stated here. Complete details of the derivations and the supporting lemmas are given in Phillips and Shi (2014, PS) which provides a full technical supplement to the paper. The general framework of the derivations follows PSY (2014b&c) but important differences arise because of the treatment of the reverse regression statistic, the endogeneity involved in that regression, and the presence of an asymptotically negligible drift in the time series.

A The limit behaviour of the BSADF* statistic under the null

Lemmas A1 and A2 below provide some standard partial sum asymptotics that hold under the following assumption, where the input process ε_t is assumed to be *iid* for convenience but may be extended to martingale differences with appropriate changes to the limit theory. These results mirror those given in PSY (2014b).

Assumption (EC) *Let $u_t = \psi(L)\varepsilon_t = \Sigma_{j=0}^{\infty}\psi_j\varepsilon_{t-j}$, where $\Sigma_{j=0}^{\infty} j\left|\psi_j\right| < \infty$ and $\{\varepsilon_t\}$ is an i.i.d sequence with mean zero, variance σ^2 and finite fourth moment.*

Lemma A.1 *Suppose u_t satisfies error condition EC. Define $M_T(g) = 1/T\sum_{s=1}^{[Tg]} u_s$ with $r \in [g_0, 1]$ and $\xi_t = \sum_{s=1}^{t} u_s$. Let $g_2, g_w \in [g_0, 1]$ and $g_1 = g_2 - g_w$. The following hold:*

(1) $\sum_{s=1}^{t} u_s = \psi(1)\sum_{s=1}^{t}\varepsilon_s + \eta_t - \eta_0$, *where* $\eta_t = \sum_{j=0}^{\infty}\alpha_j\varepsilon_{t-j}$, $\eta_0 = \sum_{j=0}^{\infty}\alpha_j\varepsilon_{-j}$ *and* $\alpha_j = -\sum_{i=1}^{\infty}\psi_{j+i}$, *which is absolutely summable.*

(2) $\frac{1}{T}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\varepsilon_t^2 \xrightarrow{p} \sigma^2 g_w$.

(3) $T^{-1/2}\sum_{t=1}^{[Tg]}\varepsilon_t \xrightarrow{L} \sigma W(g)$.

(4) $T^{-1}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\sum_{s=1}^{t-1}\varepsilon_s\varepsilon_t \xrightarrow{L} \sigma^2\left[\int_{g_1}^{g_2} W(s)\,dW - \frac{1}{2}g_w\right]$.

(5) $T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\varepsilon_t t \xrightarrow{L} \sigma\left[g_2 W(g_2) - g_1 W(g_1) - \int_{g_1}^{g_2} W(s)\,ds\right]$.

(6) $T^{-1}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\left(\eta_{t-1} - \eta_0\right)\varepsilon_t \xrightarrow{p} 0$.

(7) $T^{-1/2}\left(\eta_{[Tg]} - \eta_0\right) \xrightarrow{p} 0$.

(8) $\sqrt{T}M_T(g) \xrightarrow{L} \psi(1)\sigma W(g)$.

(9) $T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\xi_{t-1} \xrightarrow{L} \psi(1)\sigma\int_{g_1}^{g_2} W(s)\,ds$.

(10) $T^{-5/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\xi_{t-1}t \xrightarrow{L} \psi(1)\sigma\int_{g_1}^{g_2} W(s)\,s ds$.

(11) $T^{-2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\xi_{t-1}^2 \xrightarrow{L} \sigma^2\psi(1)^2\int_{g_1}^{g_2} W(s)^2\,ds$.

(12) $T^{-3/2}\sum_{t=\lfloor Tg_1\rfloor}^{\lfloor Tg_2\rfloor}\xi_t\varepsilon_{t-j} \xrightarrow{p} 0,\ \forall j = 0, \pm 1, \pm 2, ...$

Lemma A.2 *Define* $X_t^* = -\alpha_T \psi(1) t + \sum_{s=1}^{t} \omega_s$, *where* $\alpha_T = cT^{-\eta}$ *with* $\eta > 1/2$ *and* $\omega_t = -u_{T+2-t} = \psi(L) v_t$. *Let* u_t *satisfy condition EC. Then*

$$(a)\ T^{-1} \sum_{t=\lfloor Tg_1 \rfloor}^{\lfloor Tg_2 \rfloor} X_{t-1}^* v_t \xrightarrow{L} \psi(1) \sigma^2 \left[\int_{1-g_2}^{1-g_1} W(s) dW + \frac{1}{2} g_w \right].$$

$$(b)\ T^{-3/2} \sum_{t=\lfloor Tg_1 \rfloor}^{\lfloor Tg_2 \rfloor} X_{t-1}^* \xrightarrow{L} \psi(1) \sigma \int_{1-g_2}^{1-g_1} W(s) ds.$$

$$(c)\ T^{-2} \sum_{t=\lfloor Tg_1 \rfloor}^{\lfloor Tg_2 \rfloor} X_{t-1}^{*2} \xrightarrow{L} \sigma^2 \psi(1)^2 \int_{1-g_2}^{1-g_1} W(s)^2 ds.$$

$$(d)\ T^{-3/2} \sum_{t=\lfloor Tg_1 \rfloor}^{\lfloor Tg_2 \rfloor} X_{t-1}^* v_{t-j} \xrightarrow{p} 0, \quad j = 0, 1, \cdots.$$

$$(e)\ T^{-1/2} \sum_{t=\lfloor Tg_1 \rfloor}^{\lfloor Tg_2 \rfloor} v_t \xrightarrow{L} -\sigma \int_{1-g_2}^{1-g_1} dW(s).$$

With these results and using standard weak convergence methods, we can derive the asymptotic distribution of the $BSADF_g^*(g_0)$ statistic given in Theorem 3.1. A complete proof of Theorem 3.1 is provided in the technical supplement (PS, 2014).

B The limit behaviour of the BSDF statistic under the Alternative

B.1 Notation

- The bubble period $B = [T_e, T_c]$, where $T_e = \lfloor Tf_e \rfloor$ and $T_c = \lfloor Tf_c \rfloor$.
- The crisis period $C = (T_c, T_r]$, where $T_r = \lfloor Tf_r \rfloor$.
- The normal market periods $N_0 = [1, T_e)$ and $N_1 = [T_r + 1, T]$, where T is the last observation of the sample.
- The data generating process is specified as

$$X_t = \begin{cases} cT^{-\eta} + X_{t-1} + \varepsilon_t, \text{ constant } c,\ \eta > 1/2, & t \in N_0 \cup N_1 \\ \delta_T X_{t-1} + \varepsilon_t, & t \in B \\ \gamma_T X_{t-1} + \varepsilon_t, & t \in C \end{cases}, \tag{12}$$

  where $\varepsilon_t \sim N(0, \sigma^2)$, $X_0 = O_p(1)$, $\delta_T = 1 + c_1 T^{-\alpha}$ and $\gamma_T = 1 - c_2 T^{-\beta}$ with $c_1, c_2 > 0$ and $\alpha, \beta \in [0, 1)$. If $\alpha > \beta$, the rate of bubble collapse is faster than that of the bubble expansion. If $\alpha < \beta$, the rate of bubble collapse is slower than that of the bubble expansion.

- Let $X_t^* = X_{T+1-t}$. The dynamics of X_t^* are

$$X_t^* = \begin{cases} -cT^{-\eta} + X_{t-1}^* + v_t, \text{ constant } c,\, \eta > 1/2, & t \in N_0 \cup N_1 \\ \gamma_T^{-1} X_{t-1}^* + \gamma_T^{-1} v_t, & t \in C \\ \delta_T^{-1}\left[X_{t-1}^* + v_t\right], & t \in B \end{cases}, \tag{13}$$

  where $v_t = -\varepsilon_{T+2-t} \sim N\left(0, \sigma^2\right)$ and $X_0^* = X_{T+1}$.

- Let $\tau_1 = \lfloor Tg_1 \rfloor$ and $\tau_2 = \lfloor Tg_2 \rfloor$ be the start and end points of the regression. We have $T_1 = T + 1 - \tau_2$, $T_2 = T + 1 - \tau_1$ and $\tau_w = \lfloor Tg_w \rfloor$ be the regression window size.

- Let $\tau_e = \lfloor Tg_e \rfloor$, $\tau_r = \lfloor Tg_r \rfloor$, and $\tau_c = \lfloor Tg_c \rfloor$, where $g_e = 1 - f_r$, $g_c = 1 - f_c$, $g_r = 1 - f_e$. This suggests that $N_1 = [1, \tau_e)$, $C = [\tau_e, \tau_c)$, $B = [\tau_c, \tau_r]$, $N_0 = (\tau_r, T]$.

B.2 Dating Bubble Expansion

Lemma B.1 *Under the data generating process (12):*
(1) *For* $t \in N_0$, $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p)$.
(2) *For* $t \in B$, $X_{t=\lfloor Tp \rfloor} = \delta_T^{t-T_e} X_{T_e} \{1 + o_p(1)\} \sim_a T^{1/2} \delta_T^{t-T_e} B(f_e)$.
(3) *For* $t \in C$,

$$X_{t=\lfloor Tp \rfloor} = \gamma_T^{t-T_c} X_{T_c} + \sum_{j=0}^{t-T_c-1} \gamma_T^j \varepsilon_{t-j} \sim_a T^{1/2} \delta_T^{T_c-T_e} \gamma_T^{t-T_c} B(f_e) + T^{\beta/2} X_{c_2}.$$

(4) *For* $t \in N_1$,

$$X_{t=\lfloor Tp \rfloor} = \begin{cases} \sum_{j=0}^{t-T_r-1} \varepsilon_{t-j} \{1 + o_p(1)\} \sim_a T^{1/2} \left[B(p) - B(f_r)\right] & \text{if } \alpha > \beta \\ \gamma_T^{T_r-T_c} X_{T_c} \sim_a T^{1/2} \gamma_T^{T_r-T_c} \delta_T^{T_c-T_e} B(f_e) & \text{if } \alpha < \beta \end{cases}.$$

Lemma B.2 *Under the data generating process (12):*
(1) *For* $T_1 \in N_0$ *and* $T_2 \in B$,

$$\frac{1}{T_w} \sum_{j=T_1}^{T_2} X_j = \frac{T^\alpha \delta_T^{T_2-T_e}}{T_w c_1} X_{T_e} \{1 + o_p(1)\} \sim_a T^{\alpha-1/2} \delta_T^{T_2-T_e} \frac{1}{f_w c_1} B(f_e).$$

(2) *For* $T_1 \in N_0$ *and* $T_2 \in C$,

$$\frac{1}{T_w} \sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \{1 + o_p(1)\} \sim_a T^{\alpha-1/2} \delta_T^{T_c-T_e} \frac{1}{f_w c_1} B(f_e) & \text{if } \alpha > \beta \\ X_{T_c} \frac{T^\beta}{T_w c_2} \{1 + o_p(1)\} \sim_a T^{\beta-1/2} \delta_T^{T_c-T_e} \frac{B(f_e)}{f_w c_2} & \text{if } \alpha < \beta \end{cases}.$$

(3) *For* $T_1 \in N_0$ *and* $T_2 \in N_1$,

$$\frac{1}{T_w} \sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{T^\alpha \delta_T^{T_c-T_e}}{T_w c_1} X_{T_e} \{1 + o_p(1)\} \sim_a T^{\alpha-1/2} \delta_T^{T_c-T_e} \frac{1}{f_w c_1} B(f_e) & \text{if } \alpha > \beta \\ \frac{T^\beta}{T_w c_2} X_{T_c} \{1 + o_p(1)\} \sim_a T^{\beta-1/2} \delta_T^{T_c-T_e} \frac{1}{f_w c_2} B(f_e) & \text{if } \alpha < \beta \end{cases}.$$

(4) *For* $T_1 \in B$ *and* $T_2 \in C$,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1} X_{T_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B\left(f_e\right) & \text{if } \alpha > \beta \\ X_{T_c}\frac{T^{\beta}}{T_w c_2}\left\{1+o_p(1)\right\} \sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B\left(f_e\right) & \text{if } \alpha < \beta \end{cases}.$$

(5) *For* $T_1 \in B$ *and* $T_2 \in N_1$,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{T^{\alpha}\delta_T^{T_c-T_1}}{T_w c_1} X_{T_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_1}B\left(f_e\right) & \text{if } \alpha > \beta \\ X_{T_c}\frac{T^{\beta}}{T_w c_2}\left\{1+o_p(1)\right\} \sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\frac{1}{f_w c_2}B\left(f_e\right) & \text{if } \alpha < \beta \end{cases}.$$

(6) *For* $T_1 \in C$ *and* $T_2 \in N_1$,

$$\frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j = \begin{cases} \frac{1}{T_w}\sum_{j=T_r+1}^{T_2}\sum_{i=0}^{j-T_r-1}\varepsilon_{j-i}\left\{1+o_p(1)\right\} \sim_a T^{1/2}\frac{f_2-f_r}{f_w}\int_{f_r}^{f_2}\left[B(s)-B\left(f_r\right)\right]ds & \text{if } \alpha > \beta \\ X_{T_c}\frac{\gamma_T^{T_1-T_c}T^{\beta}}{T_w c_2}\left\{1+o_p(1)\right\} \sim_a T^{\beta-1/2}\delta_T^{T_c-T_e}\gamma_T^{T_1-T_c}\frac{1}{c_2 f_w}B\left(f_e\right) & \text{if } \alpha < \beta \end{cases}.$$

Lemma B.3 *Define the centered quantity* $\tilde{X}_t = X_t - T_w^{-1}\sum_{j=T_1}^{T_2} X_j$.

(1) *For* $T_1 \in N_0$ *and* $T_2 \in B$,

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{T_2-T_e}}{T_w c_1}X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^{\alpha}\delta_T^{T_2-T_e}}{T_w c_1}\right]X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in B \end{cases}.$$

(2) *For* $T_1 \in N_0$ *and* $T_2 \in C$, *if* $\alpha > \beta$

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}\right]X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c}X_{T_c} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\right]\left\{1+o_p(1)\right\} & \text{if } t \in C \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} -\frac{T^{\beta}}{T_w c_2}X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e}X_{T_e} - \frac{T^{\beta}}{T_w c_2}X_{T_c}\right]\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^{\beta}}{T_w c_2}\right]X_{T_c}\left\{1+o_p(1)\right\} & \text{if } t \in C \end{cases}.$$

(3) *For* $T_1 \in N_0$ *and* $T_2 \in N_1$, *if* $\alpha > \beta$

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}\right]X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c}X_{T_c} - \frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\right]\left\{1+o_p(1)\right\} & \text{if } t \in C \\ -\frac{T^{\alpha}\delta_T^{T_c-T_e}}{T_w c_1}X_{T_e}\left\{1+o_p(1)\right\} & \text{if } t \in N_1 \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\beta}{T_w c_2} X_{T_c} \{1 + o_p(1)\} & \text{if } t \in N_0 \\ \left[\delta_T^{t-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right] \{1 + o_p(1)\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c} \{1 + o_p(1)\} & \text{if } t \in C \\ -\frac{T^\beta}{T_w c_2} X_{T_c} \{1 + o_p(1)\} & \text{if } t \in N_1 \end{cases}.$$

(4) *For* $T_1 \in B$ *and* $T_2 \in C$, *if* $\alpha > \beta$

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1}\right] X_{T_e} \{1 + o_p(1)\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e}\right] \{1 + o_p(1)\} & \text{if } t \in C \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-T_e} X_{T_e} - X_{T_c} \frac{T^\beta}{T_w c_2}\right] \{1 + o_p(1)\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c} \{1 + o_p(1)\} & \text{if } t \in C \end{cases}$$

(5) *For* $T_1 \in B$ *and* $T_2 \in N_1$, *if* $\alpha > \beta$

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-T_e} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1}\right] X_{T_e} \{1 + o_p(1)\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} X_{T_c} - \frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e}\right] \{1 + o_p(1)\} & \text{if } t \in C \\ -\frac{T^\alpha \delta_T^{T_c - T_e}}{T_w c_1} X_{T_e} \{1 + o_p(1)\} & \text{if } t \in N_1 \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-T_e} X_{T_e} - \frac{T^\beta}{T_w c_2} X_{T_c}\right] \{1 + o_p(1)\} & \text{if } t \in B \\ \left[\gamma_T^{t-T_c} - \frac{T^\beta}{T_w c_2}\right] X_{T_c} \{1 + o_p(1)\} & \text{if } t \in C \\ -\frac{T^\beta}{T_w c_2} X_{T_c} \{1 + o_p(1)\} & \text{if } t \in N_1 \end{cases}$$

(6) *For* $T_1 \in C$ *and* $T_2 \in N_1$, *if* $\alpha > \beta$,

$$\tilde{X}_t = \begin{cases} \left[\gamma_T^{t-T_c} X_{T_c} - \frac{1}{T_w} \sum_{j=T_r+1}^{T_2} \sum_{i=0}^{j-T_r-1} \varepsilon_{j-i}\right] \{1 + o_p(1)\} & \text{if } t \in C \\ \left[\sum_{j=0}^{t-T_r-1} \varepsilon_{t-j} - \frac{1}{T_w} \sum_{j=T_r+1}^{T_2} \sum_{i=0}^{j-T_r-1} \varepsilon_{j-i}\right] \{1 + o_p(1)\} & \text{if } t \in N_1 \end{cases}$$

and if $\alpha < \beta$,

$$\tilde{X}_t = \begin{cases} \left[\gamma_T^{t-T_c} - \frac{\gamma_T^{T_1 - T_c} T^\beta}{T_w c_2}\right] X_{T_c} \{1 + o_p(1)\} & \text{if } t \in C \\ -\frac{\gamma_T^{T_1 - T_c} T^\beta}{T_w c_2} X_{T_c} \{1 + o_p(1)\} & \text{if } t \in N_1 \end{cases}.$$

Lemma B.4 *Quadratic terms in $\tilde{X}_t$ behave as follows.*

(1) *For $T_1 \in N_0$ and $T_2 \in B$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a T^{1+\alpha} \frac{\delta_T^{2(T_2-T_e)}}{2c_1} B(f_e)^2 .$$

(2) *For $T_1 \in N_0$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha}\delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha}\delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha \le \beta \end{cases} .$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha}\delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^{1+\alpha}\delta_T^{2(T_c-T_e)} \frac{1}{2c_1} B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} .$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2 \sim_a \begin{cases} T^2 (f_2 - f_r) \left\{ \int_{f_r}^{f_2} [B(s) - B(f_r)]^2 ds - \frac{f_2 - f_r}{f_w} \left[\int_{f_r}^{f_2} [B(s) - B(f_r)] ds \right]^2 \right\} & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-T_c)} \frac{1}{2c_2} B(f_e)^2 & \text{if } \alpha < \beta \end{cases} . \tag{14}$$

Lemma B.5 *Cross-product terms involving $\tilde{X}_t$ and ε_t behave as follows.*

(1) *For $T_1 \in N_0$ and $T_2 \in B$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2} \delta_T^{T_2-T_e} X_{c_1} B(f_e) .$$

(2) *For $T_1 \in N_0$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2} \delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases} .$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_1} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} \begin{array}{l} T\left\{\frac{1}{2}\left[B(f_2)-B(f_r)\right]^2-\frac{1}{2}(f_2-f_r)\sigma^2 \right. \\ \left. -2\frac{f_2-f_r}{f_w}\left[B(f_2)-B(f_r)\right]\int_{f_r}^{f_2}\left[B(s)-B(f_r)\right]ds\right\} \end{array} & \text{if } \alpha > \beta \\ T^{(1+\beta)/2}\gamma_T^{T_1-T_c-1}\delta_T^{T_c-T_e} B(f_e) X_{c_2} & \text{if } \alpha < \beta \end{cases}. \tag{15}$$

Lemma B.6 *Cross-product terms involving $\tilde{X}_{j-1}$ and $\tilde{X}_j - \delta_T\tilde{X}_{j-1}$ behave as follows.*

(1) *For $T_1 \in N_0$ and $T_2 \in B$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \sim_a -T^{\alpha}\delta_T^{2(T_2-T_e)}\frac{f_e-f_1}{f_w}B(f_e)^2.$$

(2) *For $T_1 \in N_0$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j-\delta_T\tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_r-f_c}{f_w c_1^2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B(f_e)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)}\frac{c_1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2-\beta} c_2 \frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2} \left[\int_{f_r}^{f_2} \left[B\left(s\right) - B\left(f_r\right)\right] ds\right]^2 & \text{if } \alpha > \beta \\ -T^{1+\beta-\alpha}\delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-T_c)} \frac{c_1}{2c_2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

Lemma B.7 *The sums of cross-product of $\tilde{X}_{j-1}$ and $\tilde{X}_j - \gamma_T \tilde{X}_{j-1}$ behave as follows.*

(1) *For $T_1 \in N_0$ and $T_2 \in B$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_2-T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_2-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

(2) *For $T_1 \in N_0$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(3) *For $T_1 \in N_0$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \sim_a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_e}{f_w c_2^2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(4) *For $T_1 \in B$ and $T_2 \in C$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_1}{f_w c_2^2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(5) *For* $T_1 \in B$ *and* $T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right) \sim^a \begin{cases} T^{1+\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{c_2}{2c_1}B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ T\delta_T^{2(T_c-T_e)}\frac{1}{2}B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(6) *For* $T_1 \in C$ *and* $T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right)$$

$$\sim_a \begin{cases} T^{2-\beta}c_2\left(f_2-f_r\right)\left\{\int_{f_r}^{f_2}\left[B\left(s\right)-B\left(f_r\right)\right]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}\left[B\left(s\right)-B\left(f_r\right)\right]ds\right]^2\right\} & \text{if } \alpha > \beta \\ T^{\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{f_2-f_r}{f_w c_2}B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

Lemma B.8 *The sums of cross-product of* $\tilde{X}_{j-1}$ *and* $\tilde{X}_j - \tilde{X}_{j-1}$ *behave as follows.*

(1) *For* $T_1 \in N_0$ *and* $T_2 \in B$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a T\delta_T^{2(T_2-T_e)}\frac{1}{2}B\left(f_e\right)^2.$$

(2) *For* $T_1 \in N_0$ *and* $T_2 \in C$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_e-f_1}{f_w c_2^2}B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(3) *For* $T_1 \in N_0$ *and* $T_2 \in N_1$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_w c_2^2}B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(4) *For* $T_1 \in B$ *and* $T_2 \in C$,

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_w c_1^2}B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_w c_2^2}B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(5) *For $T_1 \in B$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim^a \begin{cases} -T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)} c_1 \frac{f_c-f_1}{f_w c_2^2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(6) *For $T_1 \in C$ and $T_2 \in N_1$,*

$$\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right) \sim_a \begin{cases} -T^{2-\beta} c_2 \frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2} \left[\int_{f_r}^{f_2} \left[B\left(s\right) - B\left(f_r\right)\right] ds\right]^2 & \text{if } \alpha > \beta \\ -T\delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-T_c)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

B.2.1 Test asymptotics

The fitted regression model for the recursive unit root tests is

$$X_t = \hat{\alpha}_{f_1,f_2} + \hat{\rho}_{f_1,f_2} X_{t-1} + \hat{\varepsilon}_t,$$

where the intercept $\hat{\alpha}_{f_1,f_2}$ and slope coefficient $\hat{\rho}_{f_1,f_2}$ are obtained using data over the subperiod $[f_1, f_2]$.

Remark 1 *Based on Lemma B.4 and Lemma B.6, we can obtain limit forms of $\hat{\rho}_{f_1,f_2} - \delta_T$ using*

$$\hat{\rho}_{f_1,f_2} - \delta_T = \frac{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \delta_T \tilde{X}_{j-1}\right)}{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2}.$$

When $T_1 \in N_0$ and $T_2 \in B$

$$\hat{\rho}_{f_1,f_2} - \delta_T \sim_a -\frac{1}{T} 2c_1 \frac{f_e - f_1}{f_w}.$$

When $T_1 \in C$ and $T_2 \in N_1$,

$$\hat{\rho}_{f_1,f_2} - \delta_T \sim_a \begin{cases} -T^{-\beta} \frac{c_2 \frac{(f_r-f_1)(f_2-f_r)}{f_w^2} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds\right]^2}{\left\{\int_{f_r}^{f_2} [B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds\right]^2\right\}} & \text{if } \alpha > \beta \\ -T^{-\alpha} c_1 & \text{if } \alpha < \beta \end{cases}.$$

and for all other cases

$$\hat{\rho}_{f_1,f_2} - \delta_T \sim_a \begin{cases} -T^{\alpha-\beta-1} 2c_2 \frac{f_2-f_c}{f_w c_1} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -T^{-\alpha} c_1 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{-\alpha} c_1 & \text{if } \alpha < \beta \end{cases}$$

Remark 2 *Based on Lemma B.4 and Lemma B.7, we can obtain limit forms of $\hat{\rho}_{f_1,f_2} - \gamma_T$ using*

$$\hat{\rho}_{f_1,f_2} - \gamma_T = \frac{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \gamma_T \tilde{X}_{j-1}\right)}{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2}.$$

When $T_1 \in N_0$ and $T_2 \in B$

$$\hat{\rho}_{f_1,f_2} - \gamma_T \sim_a \begin{cases} T^{-\beta}c_2 & \text{if } \alpha > \beta \\ T^{-\alpha}c_1 & \text{if } \alpha < \beta \end{cases}.$$

When $T_1 \in C$ and $T_2 \in N_1$,

$$\hat{\rho}_{f_1,f_2} - \gamma_T \sim_a \begin{cases} T^{-\beta}c_2 \frac{\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\}}{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2} & \text{if } \alpha > \beta \\ 2\frac{1}{T}\frac{f_2-f_r}{f_w} & \text{if } \alpha < \beta \end{cases}.$$

and for all other cases

$$\hat{\rho}_{f_1,f_2} - \gamma_T \sim_a \begin{cases} T^{-\beta}c_2 & \text{if } \alpha > \beta \\ T^{-\beta}c_2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{\beta-\alpha-1}2c_1\frac{f_c-f_e}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

Remark 3 *Based on Lemma B.4 and Lemma B.8, we can obtain limit forms of $\hat{\rho}_{f_1,f_2} - 1$ using*

$$\hat{\rho}_{f_1,f_2} - 1 = \frac{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}\left(\tilde{X}_j - \tilde{X}_{j-1}\right)}{\sum_{j=T_1}^{T_2} \tilde{X}_{j-1}^2}.$$

When $T_1 \in N_0$ and $T_2 \in B$

$$\hat{\rho}_{f_1,f_2} - 1 \sim_a \frac{c_1}{T^\alpha}.$$

When $T_1 \in C$ and $T_2 \in N_1$,

$$\hat{\rho}_{f_1,f_2} - 1 \sim_a \begin{cases} -T^{-\beta}c_2 \frac{\frac{(f_r-f_1)(f_2-f_r)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2}{\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2\right\}} & \text{if } \alpha > \beta \\ -T^{-\beta}c_2 & \text{if } \alpha < \beta \end{cases}.$$

When $T_1 \in N_0$ and $T_2 \in C$

$$\hat{\rho}_{f_1,f_2} - 1 \sim_a \begin{cases} -T^{\alpha-\beta-1}2c_2\frac{f_2-f_c}{f_w c_1} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -c_1 T^{-\alpha} & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -c_2 T^{-\beta} & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{\beta-\alpha-1}2c_1\frac{f_e-f_1}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

And for all other cases

$$\hat{\rho}_{f_1,f_2} - 1 \sim_a \begin{cases} -T^{\alpha-\beta-1} 2c_2 \frac{f_2-f_c}{f_w c_1} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T^{-\alpha}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T^{-\beta}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ T^{\beta-\alpha-1} 2c_1 \frac{f_e-f_1}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

Based on the above three remarks, one can see that the quantity $\hat{\rho}_{f_1,f_2} - \delta_T$ diverges to negative infinity and the quantity $\hat{\rho}_{f_1,f_2} - \gamma_T$ diverges to positive infinity. In other words, the estimated value of $\hat{\rho}_{f_1,f_2}$ is bounded by δ_T and γ_T. Furthermore, the quantity $\hat{\rho}_{f_1,f_2} - 1$ diverges to positive infinity when $T_1 \in N_0$ and $T_2 \in B$ and negative infinity when $T_1 \in C$ and $T_2 \in N_1$.

Lemma B.9 *To obtain the asymptotic behavior of the Dickey-Fuller t-statistic, we first obtain the equation standard error of the regression over* $[T_1, T_2]$ *is*

$$\hat{\sigma}_{f_1 f_2} = \left\{ T_w^{-1} \sum_{j=T_1}^{T_2} \left(\tilde{X}_j - \hat{\rho}_{f_1,f_2} \tilde{X}_{j-1} \right)^2 \right\}^{1/2}.$$

(1) *When* $T_1 \in N_0$ *and* $T_2 \in B$,

$$\hat{\sigma}^2_{f_1 f_2} = O_p\left(T^{-1} \delta_T^{2(T_2-T_e)} \right)$$

(2) *When* $T_1 \in N_0$ *and* $T_2 \in C$,

$$\hat{\sigma}^2_{f_1 f_2} = \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

(3) *When* $T_1 \in N_0$ *and* $T_2 \in N_1$,

$$\hat{\sigma}^2_{f_1 f_2} \sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(4) *When* $T_1 \in B$ *and* $T_2 \in C$,

$$\hat{\sigma}^2_{f_1 f_2} \sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

(5) *When* $T_1 \in B$ *and* $T_2 \in N_1$,

$$\hat{\sigma}^2_{f_1 f_2} \sim_a \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

(6) *When* $T_1 \in C$ *and* $T_2 \in N_1$,

$$\hat{\sigma}^2_{f_1 f_2} \sim_a \begin{cases} O_p\left(T^{1-2\beta}\right) & \text{if } \alpha > \beta \\ O_p\left(T^{\beta-2\alpha}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right) & \text{if } \alpha < \beta \end{cases}.$$

The asymptotic distribution of the Dickey-Fuller t statistic can be calculated as follows

$$DF^t_{f_1,f_2} = \left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}^2_{j-1}}{\hat{\sigma}^2_{f_1 f_2}}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right).$$

Notice that the sign of the DF statistics depend on that of $\hat{\rho}_{f_1,f_2}-1$.

Remark 4 (1) *When* $T_1 \in N_0$ *and* $T_2 \in B$,

$$DF^t_{f_1,f_2} = \left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}^2_{j-1}}{\hat{\sigma}^2_{f_1 f_2}}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right) = O_p\left(T^{1-\alpha/2}\right) \to +\infty.$$

When $T_1 \in C$ *and* $T_2 \in N_1$

$$DF^t_{f_1,f_2} = \left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}^2_{j-1}}{\hat{\sigma}^2_{f_1 f_2}}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right) = \begin{cases} O_p\left(T^{1/2}\right) \to -\infty & \text{if } \alpha > \beta \\ O_p\left(T^{1/2+\alpha-\beta}\right) \to -\infty & \text{if } \alpha < \beta \end{cases}.$$

When $T_1 \in N_0$ *and* $T_2 \in C$

$$DF^t_{f_1,f_2} = \left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}^2_{j-1}}{\hat{\sigma}^2_{f_1 f_2}}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right) = \begin{cases} O_p\left(T^{\alpha/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{(1-\alpha+\beta)/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{(1-\beta+\alpha)/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to +\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

For all other cases

$$DF^t_{f_1,f_2} = \left(\frac{\sum_{j=T_1}^{T_2}\tilde{X}^2_{j-1}}{\hat{\sigma}^2_{f_1 f_2}}\right)^{1/2}\left(\hat{\rho}_{f_1,f_2}-1\right) = \begin{cases} O_p\left(T^{\alpha/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T^{(1-\alpha+\beta)/2}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T^{(1-\beta+\alpha)/2}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to +\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Given that $f_2 = f$ and $f_1 \in [0, f - f_0]$, the asymptotic behavior of the backward sup DF statistic under the alternative hypothesis are:

$$BSDF_f(f_0) \sim \begin{cases} F_f(W, f_0) & \text{if } f \in N_0 \\ O_p\left(T^{1-\alpha/2}\right) \to +\infty & \text{if } f \in B \\ O_p\left(T^{\omega(\alpha,\beta)}\right) = \begin{cases} O_p\left(T^{\alpha/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{(1-\alpha+\beta)/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{(1-\beta+\alpha)/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to +\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} & \text{if } f \in C \end{cases}$$

This proves Theorem 3.2. Following the standard probability arguments (see PSY), we deduce that $\Pr\left\{|\hat{f}_e - f_e| > \eta\right\} \to 0$ and $\Pr\left\{|\hat{f}_c - f_c| > \gamma\right\} \to 0$ for any $\eta, \gamma > 0$ as $T \to \infty$, provided that

$$\begin{cases} \frac{T^{\alpha/2}}{scv^{\beta}T} + \frac{scv^{\beta}T}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \frac{T^{(1-\alpha+\beta)/2}}{scv^{\beta}T} + \frac{scv^{\beta}T}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \frac{T^{(1-\beta+\alpha)/2}}{scv^{\beta}T} + \frac{scv^{\beta}T}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \frac{T^{\beta/2}}{scv^{\beta}T} + \frac{scv^{\beta}T}{T^{1-\alpha/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Therefore, $\hat{f}_e$ and $\hat{f}_c$ are consistent estimators of f_e and f_c. This proves Theorem 3.3.

B.3 Dating Bubble Contractions

Define the demeaned quantity as $\tilde{X}_t^* \equiv X_t^* - \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j^*$. Since $\tau_w = T_w$ and $\sum_{j=\tau_1}^{\tau_2} X_j^* = \sum_{i=T_1}^{T_2} X_i$, we have

$$\tilde{X}_t^* = X_t^* - \frac{1}{T_w}\sum_{j=T_1}^{T_2} X_j^* = X_{T+1-t} - \frac{1}{T_w}\sum_{i=T_1}^{T_2} X_i = \tilde{X}_{T+1-t}.$$

Based on this linkage, we derive the next two lemmas.

Lemma B.10 *Quadratic terms in $\tilde{X}_t^*$ behave as follows.*

(1) *For $\tau_1 \in B$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2} = \sum_{j=T_1}^{T_2} \tilde{X}_{j+1}^2 \sim_a T^{1+\alpha}\delta_T^{2(T_2-T_e)}\frac{1}{2c_1}B(f_e)^2 \ .$$

(2) *For $\tau_1 \in C$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2} = \sum_{j=T_1}^{T_2} \tilde{X}_{j+1}^2 \sim_a \begin{cases} T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(3) *For $\tau_1 \in N_1$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2} = \sum_{j=T_1}^{T_2} \tilde{X}_{j+1}^{2} \sim_a \begin{cases} T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha \leq \beta \end{cases}.$$

(4) *For $\tau_1 \in C$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2} = \sum_{j=T_1}^{T_2} \tilde{X}_{j+1}^{2} \sim_a \begin{cases} T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(5) *For $\tau_1 \in N_1$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2} = \sum_{j=T_1}^{T_2} \tilde{X}_{j+1}^{2} \sim_a \begin{cases} T^{1+\alpha}\delta_T^{2(T_c-T_e)}\frac{1}{2c_1}B(f_e)^2 & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}.$$

(6) *For $\tau_1 \in N_1$ and $\tau_2 \in C$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2} = \sum_{j=T_1}^{T_2} \tilde{X}_{j+1}^{2} \sim_a \begin{cases} T^2(f_2-f_r)\left\{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]\,ds\right]^2\right\} & \text{if } \alpha > \beta \\ T^{1+\beta}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2c_2}B(f_e)^2 & \text{if } \alpha < \beta \end{cases}$$

Lemma B.11 *Cross-product terms involving $\tilde{X}_t^*$ and v_t behave as follows.*

(1) *For $\tau_1 \in B$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a -T^{(1+\alpha)/2}\delta_T^{T_2-T_e}X_{c_1}B(f_e).$$

(2) *For $\tau_1 \in C$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(3) *For $\tau_1 \in N_1$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} v_j \sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(4) *For $\tau_1 \in C$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a \begin{cases} -T^{(1+\alpha)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2}\delta_T^{T_c-T_e}B(f_e)X_{c_2} & \text{if } \alpha < \beta \end{cases}.$$

(5) *For $\tau_1 \in C$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a \begin{cases} -T^{(1+\alpha)/2} \delta_T^{T_c - T_e} B\left(f_e\right) X_{c_1} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \delta_T^{T_c - T_e} B\left(f_e\right) X_{c_2} & \text{if } \alpha < \beta \end{cases} .$$

(6) *For $\tau_1 \in C$ and $\tau_2 \in N_1$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} v_j \sim_a \begin{cases} \begin{array}{l} -T \left\{ \frac{1}{2} \left[B\left(f_2\right) - B\left(f_r\right) \right]^2 + \frac{1}{2} \left(f_2 - f_r\right) \sigma^2 \right. \\ \left. -\frac{f_2 - f_r}{f_w} \left[B\left(f_2\right) - 2B\left(f_r\right) + B\left(f_1\right) \right] \int_{f_r}^{f_2} \left[B\left(s\right) - B\left(f_r\right) \right] ds \right\} \end{array} & \text{if } \alpha > \beta \\ -T^{(1+\beta)/2} \gamma_T^{T_1 - T_c} \delta_T^{T_c - T_e} B\left(f_e\right) X_{c_2} & \text{if } \alpha < \beta \end{cases}$$

Lemma B.12 *Cross-product terms involving $\tilde{X}_{j-1}^{*}$ and $\tilde{X}_j^{*} - \gamma_T^{-1} \tilde{X}_{j-1}^{*}$ behave as follows.*

(1) *For $\tau_1 \in B$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} \left(\tilde{X}_j^{*} - \gamma_T^{-1} \tilde{X}_{j-1}^{*} \right) \sim_a \begin{cases} -T^{1+\alpha-\beta} \delta_T^{2(T_2 - T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ -T \delta_T^{2(T_2 - T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases} .$$

(2) *For $\tau_1 \in C$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} \left(\tilde{X}_j^{*} - \gamma_T^{-1} \tilde{X}_{j-1}^{*} \right) \sim_a \begin{cases} -T^{1+\alpha-\beta} \delta_T^{2(T_c - T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ -T \delta_T^{2(T_c - T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c - T_e)} c_1 \frac{f_c - f_e}{f_w c_2^2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} .$$

(3) *For $\tau_1 \in N_1$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} \left(\tilde{X}_j^{*} - \gamma_T^{-1} \tilde{X}_{j-1}^{*} \right) \sim_a \begin{cases} -T^{1+\alpha-\beta} \delta_T^{2(T_c - T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ -T \delta_T^{2(T_c - T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c - T_e)} c_1 \frac{f_c - f_e}{f_w c_2^2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} .$$

(4) *For $\tau_1 \in C$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} \left(\tilde{X}_j^{*} - \gamma_T^{-1} \tilde{X}_{j-1}^{*} \right) \sim_a \begin{cases} -T^{1+\alpha-\beta} \delta_T^{2(T_c - T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ -T \delta_T^{2(T_c - T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c - T_e)} c_1 \frac{f_c - f_1}{f_w c_2^2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} .$$

(5) *For $\tau_1 \in N_1$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*} \left(\tilde{X}_j^{*} - \gamma_T^{-1} \tilde{X}_{j-1}^{*} \right) \sim_a \begin{cases} -T^{1+\alpha-\beta} \delta_T^{2(T_c - T_e)} \frac{c_2}{2c_1} B\left(f_e\right)^2 & \text{if } \alpha > \beta \\ -T \delta_T^{2(T_c - T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{2\beta-\alpha} \delta_T^{2(T_c - T_e)} c_1 \frac{f_c - f_1}{f_w c_2^2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} .$$

(6) *For $\tau_1 \in N_1$ and $\tau_2 \in C$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1} \tilde{X}_{j-1}^*\right)$$

$$\sim_a \begin{cases} -T^{2-\beta} c_2 \left(f_2 - f_r\right) \left\{ \int_{f_r}^{f_2} \left[B\left(s\right) - B\left(f_r\right)\right]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2} \left[\int_{f_r}^{f_2} \left[B\left(s\right) - B\left(f_r\right)\right] ds\right]^2 \right\} & \text{if } \alpha > \beta \\ -T^{\beta} \delta_T^{2(T_c-T_e)} \gamma_T^{2(T_1-T_c)} \frac{f_2-f_r}{f_w c_2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

Lemma B.13 *The sums of cross-product of $\tilde{X}_{j-1}^*$ and $\tilde{X}_j^* - \delta_T^{-1}\tilde{X}_{j-1}^*$ behave as follows.*

(1) *For $\tau_1 \in B$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1} \tilde{X}_{j-1}^*\right) \sim_a T^{\alpha} \delta_T^{2(T_2-T_e)} \frac{f_e - f_1}{f_w} B\left(f_e\right)^2.$$

(2) *For $\tau_1 \in C$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1} \tilde{X}_{j-1}^*\right) \sim_a \begin{cases} T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T \delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T^{1+\beta-\alpha} \delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

(3) *For $\tau_1 \in N_1$ and $\tau_2 \in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1} \tilde{X}_{j-1}^*\right) \sim_a \begin{cases} T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T \delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T^{1+\beta-\alpha} \delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

(4) *For $\tau_1 \in C$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1} \tilde{X}_{j-1}^*\right) \sim_a \begin{cases} T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_2-f_c}{f_w c_1^2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T \delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T^{1+\beta-\alpha} \delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

(5) *For $\tau_1 \in N_1$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1} \tilde{X}_{j-1}^*\right) \sim_a \begin{cases} T^{2\alpha-\beta} \delta_T^{2(T_c-T_e)} c_2 \frac{f_r-f_c}{f_w c_1^2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ T \delta_T^{2(T_c-T_e)} \frac{1}{2} B\left(f_e\right)^2 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ T^{1+\beta-\alpha} \delta_T^{2(T_c-T_e)} \frac{c_1}{2c_2} B\left(f_e\right)^2 & \text{if } \alpha < \beta \end{cases}.$$

(6) *For $\tau_1 \in N_1$ and $\tau_2 \in C$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \delta_T^{-1} \tilde{X}_{j-1}^*\right)$$

$$\sim_a \begin{cases} T^{2-\alpha}c_1\left(f_2-f_r\right)\left\{\int_{f_r}^{f_2}\left[B\left(s\right)-B\left(f_r\right)\right]^2ds+\frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2}\left[\int_{f_r}^{f_2}\left[B\left(s\right)-B\left(f_r\right)\right]ds\right]^2\right\} & \text{if } \alpha>\beta \\ T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}c_1\frac{f_2-f_r}{f_wc_2^2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \end{cases}.$$

Lemma B.14 *The sums of cross-product of $\tilde{X}_{j-1}^*$ and $\tilde{X}_j^*-\tilde{X}_{j-1}^*$ behave as follows.*

(1) *For $\tau_1\in B$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\sim_a T\delta_T^{2(T_2-T_e)}\frac{1}{2}B\left(f_e\right)^2.$$

(2) *For $\tau_1\in C$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_wc_1^2}B\left(f_e\right)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & f\ \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_wc_2^2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(3) *For $\tau_1\in N_1$ and $\tau_2\in N_0$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_wc_1^2}B\left(f_e\right)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & f\ \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_wc_2^2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(4) *For $\tau_1\in C$ and $\tau_2\in B$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}c_2\frac{f_2-f_c}{f_wc_1^2}B\left(f_e\right)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & f\ \alpha>\beta \text{ and } 1+\beta>2\alpha \\ o_p\left(T\delta_T^{2(T_c-T_e)}\right) & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_e}{f_wc_2^2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}.$$

(5) *For $\tau_1\in N_1$ and $\tau_2\in B$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2\alpha-\beta}\delta_T^{2(T_c-T_e)}\frac{f_2-f_r}{f_wc_2}B\left(f_e\right)^2 & \text{if } \alpha>\beta \text{ and } 1+\beta<2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B\left(f_e\right)^2 & f\ \alpha>\beta \text{ and } 1+\beta>2\alpha \\ -T\delta_T^{2(T_c-T_e)}\frac{1}{2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha>2\beta \\ -T^{2\beta-\alpha}\delta_T^{2(T_c-T_e)}c_1\frac{f_c-f_1}{f_wc_2^2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \text{ and } 1+\alpha<2\beta \end{cases}$$

(6) *For $\tau_1\in N_1$ and $\tau_2\in C$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^*\left(\tilde{X}_j^*-\tilde{X}_{j-1}^*\right)\sim_a \begin{cases} T^{2-\beta}c_2\frac{(f_r-f_1)(f_2-f_r)^2}{f_w^2}\left[\int_{f_r}^{f_2}\left[B\left(s\right)-B\left(f_r\right)\right]ds\right]^2 & \text{if } \alpha>\beta \\ T\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\frac{1}{2}B\left(f_e\right)^2 & \text{if } \alpha<\beta \end{cases}.$$

B.3.1 Test asymptotics

The fitted regression model for the recursive unit root tests is

$$X_t^* = \hat{\alpha} + \hat{\rho}_{g_1,g_2} X_{t-1}^* + \hat{\upsilon}_t,$$

where the intercept $\hat{\alpha}$ and slope coefficient $\hat{\rho}_{g_1,g_2}$ are obtained using data over the subperiod $[g_1, g_2]$.

Remark 5 *Based on Lemma B.10 and Lemma B.12, we can obtain limit forms of $\hat{\gamma}_T^{-1} - \gamma_T^{-1}$ using*

$$\hat{\rho}_{g_1,g_2} - \gamma_T^{-1} = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(\tilde{X}_j^* - \gamma_T^{-1} \tilde{X}_{j-1}^* \right)}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2}}.$$

When $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$\hat{\rho}_{g_1,g_2} - \gamma_T^{-1} \sim_a \begin{cases} -c_2 T^{-\beta} & \text{if } \alpha > \beta \\ -c_1 T^{-\alpha} & \text{if } \alpha < \beta \end{cases}.$$

when $\tau_1 \in N_1$ and $\tau_2 \in C$,

$$\hat{\rho}_{g_1,g_2} - \gamma_T^{-1} \sim_a \begin{cases} -T^{-\beta} c_2 \frac{\left\{ \int_{f_r}^{f_2} [B(s)-B(f_r)]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds \right]^2 \right\}}{\left\{ \int_{f_r}^{f_2} [B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds \right]^2 \right\}} & \text{if } \alpha > \beta \\ -2T^{-1} \frac{f_2-f_r}{f_w} & \text{if } \alpha < \beta \end{cases}$$

for all other cases, we have

$$\hat{\rho}_{g_1,g_2} - \gamma_T^{-1} \sim_a \begin{cases} -T^{-\beta} c_2 & \text{if } \alpha > \beta \\ -T^{-\beta} c_2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{\beta-\alpha-1} 2c_1 \frac{f_c-f_e}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Remark 6 *Based on Lemma B.10 and Lemma B.13, we can obtain limit forms of $\hat{\rho}_{g_1,g_2} - \delta_T^{-1}$ using*

$$\hat{\rho}_{g_1,g_2} - \delta_T^{-1} = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(X_j^* - \delta_T^{-1} X_{j-1}^* \right)}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2}}.$$

When $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$\hat{\rho}_{g_1,g_2} - \delta_T^{-1} \sim_a \frac{1}{T} 2c_1 \frac{f_e - f_1}{f_w};$$

When $\tau_1 \in C$ and $\tau_2 \in N_0$,

$$\hat{\rho}_{g_1,g_2} - \delta_T^{-1} \sim_a \begin{cases} T^{-\alpha} \frac{c_1 \left\{ \int_{f_r}^{f_2} [B(s)-B(f_r)]^2 ds + \frac{(f_2-f_r)(f_2-f_r-2f_w)}{f_w^2} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds \right]^2 \right\}}{\int_{f_r}^{f_2} [B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w} \left[\int_{f_r}^{f_2} [B(s)-B(f_r)] ds \right]^2} & \text{if } \alpha > \beta \\ 2T^{\beta-\alpha-1} c_1 \frac{f_2-f_r}{f_w c_2} & \text{if } \alpha < \beta \end{cases};$$

For all other cases

$$\hat{\rho}_{g_1,g_2} - \delta_T^{-1} \sim_a \begin{cases} T^{\alpha-\beta-1}K & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ c_1 T^{-\alpha} & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ c_1 T^{-\alpha} & \text{if } \alpha < \beta \end{cases};$$

where K is a constant which equals $2c_1c_2\frac{f_r-f_c}{f_w c_1^2}$ when $\tau_1 \in N_1$ and $\tau_2 \in N_0$ and when $\tau_1 \in N_1$ and $\tau_2 \in B$ and equals $2c_1c_2\frac{f_2-f_c}{f_w c_1^2}$ when $\tau_1 \in C$ and $\tau_2 \in B$ and when $\tau_1 \in C$ and $\tau_2 \in N_0$.

Remark 7 *Based on Lemma B.10 and Lemma B.13, we can obtain limit forms of $\hat{\rho}_{g_1,g_2} - 1$ using*

$$\hat{\rho}_{g_1,g_2} - 1 = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^* \left(X_j^* - X_{j-1}^*\right)}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^{*2}}.$$

When $\tau_1 \in B$ and $\tau_2 \in N_0$,

$$\hat{\rho}_{g_1,g_2} - 1 \sim_a \frac{c_1}{T^\alpha}.$$

when $\tau_1 \in N_1$ and $\tau_2 \in C$,

$$\hat{\rho}_{g_1,g_2} - 1 \tilde{} \begin{cases} T^{-\beta} c_2 \dfrac{\frac{(f_r-f_1)(f_2-f_r)}{f_w^2}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2}{\int_{f_r}^{f_2}[B(s)-B(f_r)]^2 ds - \frac{f_2-f_r}{f_w}\left[\int_{f_r}^{f_2}[B(s)-B(f_r)]ds\right]^2} & \text{if } \alpha > \beta \\ T^{-\beta} c_2 & \text{if } \alpha < \beta \end{cases}$$

when $\tau_1 \in N_1$ and $\tau_2 \in B$,

$$\hat{\rho}_{g_1,g_2} - 1 \sim_a \begin{cases} T^{\alpha-\beta-1} 2c_1 \frac{f_2-f_r}{f_w c_2} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ -c_1 T^{-\alpha} & f\ \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ -T^{-\beta} c_2 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{\beta-\alpha-1} 2c_1 \frac{f_c-f_1}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

for all other cases, we have

$$\hat{\rho}_{g_1,g_2} - 1 \sim_a \begin{cases} T^{\alpha-\beta-1} 2c_1 \frac{f_2-f_r}{f_w c_2} & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T^{-\alpha}\right) & f\ \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T^{-\beta}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ -T^{\beta-\alpha-1} 2c_1 \frac{f_c-f_1}{f_w c_2} & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}$$

Based on the above three remarks, one can see that the quantity $\hat{\rho}_{g_1,g_2} - \gamma_T^{-1}$ diverges to negative infinity and the quantity $\hat{\rho}_{g_1,g_2} - \delta_T^{-1}$ diverges to positive infinity. In other words, the estimated value of $\hat{\rho}_{g_1,g_2}$ is bounded by γ_T^{-1} and δ_T^{-1}. Furthermore, the quantity $\hat{\rho}_{g_1,g_2} - 1$ diverges to positive infinity when $\tau_1 \in B$ and $\tau_2 \in N_0$ and when $\tau_1 \in N_1$ and $\tau_2 \in C$. For all other cases, the quantity $\hat{\rho}_{g_1,g_2} - 1$ diverges to positive infinity when bubble collapsing speed is much faster than expansion rate (i.e. $1+\beta < 2\alpha$) and to negative infinity otherwise.

Lemma B.15 *To obtain the asymptotic behavior of the Dickey-Fuller t-statistic, we first obtain the equation standard error of the regression over* $[T_1, T_2]$ *is*

$$\hat{\sigma}_{g_1 g_2} = \left\{ \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j^* - \hat{\rho}_{g_1,g_2} \tilde{X}_{j-1}^* \right)^2 \right\}^{1/2}.$$

(1) *When* $\tau_1 \in B$ *and* $\tau_2 \in N_0$,

$$\hat{\sigma}_{g_1 g_2}^2 = O_p\left(T^{-1} \delta_T^{2(T_2 - T_e)} \right).$$

(2) *When* $\tau_1 \in C$ *and* $\tau_2 \in N_0$,

$$\hat{\sigma}_{g_1,g_2}^2 = \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha > \beta\ and\ 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & f\ \alpha > \beta\ and\ 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha < \beta\ and\ 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha < \beta\ and\ 1+\alpha < 2\beta \end{cases}$$

(3) *When* $\tau_1 \in N_1$ *and* $\tau_2 \in N_0$,

$$\hat{\sigma}_{g_1,g_2}^2 = \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha > \beta\ and\ 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & f\ \alpha > \beta\ and\ 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha < \beta\ and\ 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha < \beta\ and\ 1+\alpha < 2\beta \end{cases}$$

(4) *When* $\tau_1 \in C$ *and* $\tau_2 \in B$,

$$\hat{\sigma}_{g_1,g_2}^2 = \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha > \beta\ and\ 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & f\ \alpha > \beta\ and\ 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha < \beta\ and\ 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha < \beta\ and\ 1+\alpha < 2\beta \end{cases}$$

(5) *When* $\tau_1 \in N_1$ *and* $\tau_2 \in B$,

$$\hat{\sigma}_{g_1,g_2}^2 = \begin{cases} O_p\left(T^{2\alpha-2\beta-1}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha > \beta\ and\ 1+\beta < 2\alpha \\ O_p\left(T^{-\beta}\delta_T^{2(T_c-T_e)}\right) & f\ \alpha > \beta\ and\ 1+\beta > 2\alpha \\ O_p\left(T^{-\alpha}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha < \beta\ and\ 1+\alpha > 2\beta \\ O_p\left(T^{2\beta-2\alpha-1}\delta_T^{2(T_c-T_e)}\right) & if\ \alpha < \beta\ and\ 1+\alpha < 2\beta \end{cases}$$

(6) *When $\tau_1 \in N_1$ and $\tau_2 \in C$,*

$$\hat{\sigma}^2_{g_1,g_2} = \begin{cases} O_p\left(T^{1-2\beta}\right) & \text{if } \alpha > \beta \\ O_p\left(T^{-1}\delta_T^{2(T_c-T_e)}\gamma_T^{2(T_1-T_c)}\right) & \text{if } \alpha < \beta \end{cases}$$

The asymptotic distribution of the Dickey-Fuller t statistic

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2}\tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2}\right)^{1/2}\left(\hat{\rho}_{g_1,g_2}-1\right)$$

can be calculated as follows. Notice that the sign of the DF statistic is determined by the quantity $\hat{\rho}_{g_1,g_2}-1$.

Remark 8 *When $\tau_1 \in B$ and $\tau_2 \in N_0$,*

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2}\tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2}\right)^{1/2}\left(\hat{\rho}_{g_1,g_2}-1\right) = O_p\left(T^{1-\alpha/2}\right) \to +\infty$$

when $\tau_1 \in N_1$ and $\tau_2 \in C$,

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2}\tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2}\right)^{1/2}\left(\hat{\rho}_{g_1,g_2}-1\right) = \begin{cases} O_p\left(T^{1/2}\right) \to +\infty & \text{if } \alpha > \beta \\ O_p\left(T^{1-\beta/2}\right) \to +\infty & \text{if } \alpha < \beta \end{cases}.$$

when $\tau_1 \in N_1$ and $\tau_2 \in B$,

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2}\tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2}\right)^{1/2}\left(\hat{\rho}_{g_1,g_2}-1\right) \sim_a \begin{cases} O_p\left(T^{\alpha/2}\right) \to +\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{(1-\alpha+\beta)/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{(1-\beta+\alpha)/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

for all other cases

$$DF^t_{g_1,g_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2}\tilde{X}^{*2}_{j-1}}{\hat{\sigma}^2}\right)^{1/2}\left(\hat{\rho}_{g_1,g_2}-1\right) \sim_a \begin{cases} O_p\left(T^{\alpha/2}\right) \to +\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ o_p\left(T^{(1-\alpha+\beta)/2}\right) & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ o_p\left(T^{(1-\beta+\alpha)/2}\right) & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Given that $g_2 = g$ and $g_1 \in [0, g-g_0]$, the asymptotic behavior of the backward sup DF statistic under the alternative hypothesis are:

$$BSDF^*_g(g_0) \sim_a \begin{cases} F_g(W, g_0) & \text{if } g \in N_1 \\ \begin{cases} O_p\left(T^{1/2}\right) \to +\infty & \text{if } \alpha > \beta \\ O_p\left(T^{1-\beta/2}\right) \to +\infty & \text{if } \alpha < \beta \end{cases} & \text{if } g \in C \\ O_p\left(T^{\omega^*(\alpha,\beta)}\right) = \begin{cases} O_p\left(T^{\alpha/2}\right) \to +\infty & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ O_p\left(T^{(1-\alpha+\beta)/2}\right) \to -\infty & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ O_p\left(T^{(1-\beta+\alpha)/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ O_p\left(T^{\beta/2}\right) \to -\infty & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases} & \text{if } g \in B \end{cases};$$

This proves Theorem 3.4. Following the standard probability arguments (see PSY), we deduce that $\Pr\{|\hat{g}_e - g_e| > \eta\} \to 0$ and $\Pr\{|\hat{g}_c - g_c| > \gamma\} \to 0$ for any $\eta, \gamma > 0$ as $T \to \infty$, provided that

$$\begin{cases} \frac{T^{\alpha/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta < 2\alpha \\ \frac{T^{(1-\alpha+\beta)/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1/2}} \to 0 & \text{if } \alpha > \beta \text{ and } 1+\beta > 2\alpha \\ \frac{T^{(1-\beta+\alpha)/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1-\beta/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha > 2\beta \\ \frac{T^{\beta/2}}{scv^*(\beta_T)} + \frac{scv^*(\beta_T)}{T^{1-\beta/2}} \to 0 & \text{if } \alpha < \beta \text{ and } 1+\alpha < 2\beta \end{cases}.$$

Therefore, $\hat{f}_r = 1 - \hat{g}_e$ and $\hat{f}_c = 1 - \hat{g}_c$ are consistent estimators of f_r and f_c. This proves Theorem 3.5.