INFORMATION AND VOLATILITY

By

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Abstract

In an economy of interacting agents with both idiosyncratic and aggregate shocks, we examine how the information structure determines aggregate volatility. We show that the maximal aggregate volatility is attained in a noise free information structure in which the agents confound idiosyncratic and common components of the payoff state, and display excess response to the common component, as in Lucas (1972). The upper bound on aggregate volatility is linearly increasing in the variance of idiosyncratic shocks, for any given variance of aggregate shocks. Our results hold in a setting of symmetric agents with linear best responses and normal uncertainty. We show our results by providing a characterization of the set of all joint distributions over actions and states that can arise in equilibrium under any information structure. This tractable characterization, extending results in Bergemann and Morris (2013b), can be used to address a wide variety of questions.

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1   Introduction

Consider an economy of interacting agents, each of whom picks an action. Agents are subject to idiosyncratic and aggregate shocks. A classical economic question in this environment is to ask how aggregate and idiosyncratic shocks map into "aggregate volatility" - the variance of the average action. Versions of this question arises in many different economic contexts. In particular, a central question in macroeconomics is how aggregate and individual productivity shocks translate into variation in GDP. Another classical question is when and how asymmetric information can influence this mapping, and in particular exacerbate aggregate volatility. A difficulty addressing this question is that the answer depends on the nature of the asymmetric information, something that is not easily observed. Thus results may be sensitive to the exact information structure assumed.

This paper considers a very simple stylized economy where we can completely characterize what can happen for all information structures. In particular, we consider a setting with a continuum of agents with linear best responses that depend on average actions of others and idiosyncratic and aggregate shocks. We assume that shocks, actions and signals are symmetrically normally distributed across agents, maintaining symmetry and normality of the information structure. Our sharp clean characterization of what can happen across all information structures in this symmetric normal class can be used to address many economic questions of interest. In particular, we can study the two classical questions described above, providing an upper bound on aggregate volatility as a function of fundamentals and identifying the critical information structures that give rise to maximal volatility.

The information structure that maximizes aggregate volatility turns out to be "noise free:" each agent observes a one dimensional signal which is a deterministic function of his idiosyncratic and the aggregate shock. While there is no noise in such signals, they are imperfect because they leave the agent uncertain about the size of aggregate and idiosyncratic shocks. The shocks are confounded in the agent's signal. Aggregate volatility is highest when signals overweight the common shock relative to the idiosyncratic shock. In this case, agents who want to tailor their actions to their idiosyncratic shocks have no choice but to overweight the common shock, generating aggregate volatility. We show how maximum aggregate volatility increases linearly in the variance of idiosyncratic shocks even if the variance of aggregate shocks is held constant. The critical noise free signal generating the maximal aggregate volatility puts proportionately more weight on the aggregate shock of constant variance as the variance of the idiosyncratic shock becomes larger.

These noise free information structures are also critical for many questions of interest, including for dispersion (the variance of individual actions around the mean action) and individual volatility (the variance of individual actions, which is equal to the sum of aggregate volatility and dispersion). But different noise free information structures maximize these variables. Thus dispersion is highest when signals overweight the idiosyncratic shock relative to the common shock. In this case, agents who want to tailor their actions to their common shocks have no choice but to overweight the idiosyncratic shock, generating dispersion.

The fact that confounding shocks can lead to overreaction has been long recognized, notably by Lucas (1972) and more recently by Hellwig and Venkateswaran (2009) and Venkateswaran (2013). Our contribution is to highlight that, in this setting with idiosyncratic and aggregate shocks, noise free confounding information structures are extremal and provide global bounds on how much volatility can arise via the information structure. The intuition for the bounding result is simple and comes in two parts. First, suppose that agents observed a one dimensional signal that was linear function of the idiosyncratic shock, the aggregate shock and a noise term which may be correlated across agents. Equilibrium actions must be linear in the signal. The impact of the noise in the signal must be to dampen the response of agents to the signal and thus to both the idiosyncratic and aggregate shocks. Thus among one dimensional symmetric information structures, noise free information structures generate the most volatility. Second, imagine any other, perhaps multidimensional and perhaps noisy, symmetric information structure. By symmetry, each agent's equilibrium action choice (assumed to be one dimensional) can be expressed as a linear function of the idiosyncratic shock, the aggregate shock and a noise term which may be correlated across agents. Now we can replace the original information structure by the one dimensional one where each agent observes a signal which is linear in the equilibrium action he would have chosen under the old information structure. Equilibrium in this new information structure will now generate the same outcomes as the equilibrium with the richer information structure. Thus it is enough to study one dimensional information structures where signals are linear function of the idiosyncratic shock, the aggregate shock and a noise term which may be correlated across agents. We provide a direct characterization of all symmetric joint distributions of individual actions, mean action, idiosyncratic shocks and aggregate shocks can arise for a given distribution of shocks in equilibrium for some information structure; these are called Bayes correlated equilibria. Every Bayes correlated equilibrium can be written as the (Bayes Nash) equilibrium for some one dimensional information structure.

While noise free information structures generate maximal aggregate volatility with *both* idiosyncratic and aggregate shocks, this is not longer true if there are only idiosyncratic shocks. In this case, if each agent responded to his idiosyncratic shock only, there would be no aggregate volatility by the law of large numbers. On the other hand, if each agent had no information about his idiosyncratic shock, his action would be constant and there would again be no aggregate volatility. The information structure which maximizes aggregate volatility would be one where each agent observed his idiosyncratic shock with an intermediate level of noise, and where the noise in agents' signals was perfectly correlated. Angeletos and La'O (2013) have analyzed the role of such common shocks to beliefs about purely idiosyncratic uncertainty in a macroeconomic model, describing them as "sentiment" shocks. How can we relate this finding that - with only idiosyncratic shocks - adding noise maximizes aggregate volatility to our finding - adding common shocks - that noise can only decrease aggregate volatility? We can reconcile the results by considering what happens if we let the variance of common shocks decline towards zero in our model. In this case, our results show that the information structure that maximizes aggregate volatility is a noise free information structure where the signal puts a larger and larger weight on the common shock and a smaller and smaller weight on the idiosyncratic shock. The agent - in order to respond to the idiosyncratic shock - must put a larger and larger weight on the signal. The total sensitivity to the common shock (multiplying the weight on the common shock in the signal with the equilibrium weight on the signal) converges to a constant as the common shock disappears, so that the dependence on the common shock becomes dependence on a common payoff irrelevant noise term, i.e., the sentiment shock. Thus this paper highlights a tight connection between noise free confounding information structures and sentiment shocks.

Similarly, in environments with purely common shocks, such as Morris and Shin (2002), Woodford (2003) and Angeletos and Pavan (2007), there is no idiosyncratic shock, and hence there can be no confusion between the idiosyncratic shock and the common shock. The only noise free information structure is the complete information structure. Now by contrast to the pure idiosyncratic shock environment, the complete information, and hence noise free information structure generates the largest aggregate volatility. But symmetrically to the common "sentiment" shocks in Angeletos and La'O (2013), an idiosyncratic "sentiment" shock may be needed to generate the largest individual volatility in the aggregate shock environment as shown in Bergemann and Morris (2013b).

We maintain the assumption that agent's best responses are linear in their expectation of the average action of others. The results described thus far hold independent of whether the weight on

the average action, r, is negative (the strategic substitutes case), zero (the purely decision theoretic case) or positive (the strategic complementarities case). A striking property of our characterization of Bayes correlated equilibria - i.e., what can happen in all symmetric information structures - is that the set of feasible correlations between individual and average actions and individual and common shocks is independent of r and determined only by statistical constraints. There are three degrees of freedom in describing the correlation structure: the correlation between each agent's action and his private state (the sum of the aggregate shock and his idiosyncratic shock), the correlation between any two agents' actions, and the correlation between any agent's action and any other agent's private state. On the other hand, once one pins down the correlation structure, the mean and variance of individual actions are pinned down. This follows from the simple observation that any Bayes correlated equilibrium distribution could have arisen from one dimensional signals. In equilibrium, strategies will be linear in the one dimensional signals. So the correlation structure is pinned down by the information structure. The best response parameter r describes how to translate correlation structures into first and second moments. Thus there is a three dimensional class of Bayes correlated equilibria which are extremely tractable.

While we can restrict attention to one dimensional information structures in deriving bounds on volatility, we may want to assume that agents have access to particular multidimensional signals. In general, this will impose restrictions on what can happen so that upper bounds on volatility cannot be obtained. For example, in the work of Angeletos and La'O (2013), it is assumed that agents know their own payoff shock but are uncertain only about others' payoff shocks. In this case, we would want to assume that agents know at least the sum of their idiosyncratic and aggregate shock, but may also know more. We characterize the subset of Bayes correlated equilibria that could arise from information structures of this form, and also other information structures. In this case, there is interaction between the best response parameter r and the set of correlation structures that can arise. Angeletos and Pavan (2009) studied a rich three dimensional class of normal signals, where agents observe a noisy signal of the idiosyncratic shock, and private (conditionally independent) signal of the aggregate shock and a public signal of the aggregate shock. Although the set of Bayes correlated equilibria is also three dimensional, the Angeletos and Pavan (2009) information structures will generally not give rise to maximal volatility. There is a lower upper bound on volatility within this class, because the information structure does not allow overweighting of the common shock via confounding.

This paper is an application of a general approach to analyzing equilibrium behavior of agents

for a given description of the fundamentals for *all* possible information structures. In Bergemann and Morris (2013a), we considered this problem in an abstract game theoretic setting. We show that a general version of Bayes correlated equilibrium characterizes the set of outcomes that could arise in any Bayes Nash equilibrium of an incomplete information game where agents may or may not have access to more information beyond the given common prior over the fundamentals. In Bergemann and Morris (2013b) we pursue this argument in detail and characterize the set of Bayes correlated equilibria in the class of games with quadratic payoffs and normally distributed uncertainty, but there we restricted our attention to the special environment with only aggregate shocks, or *pure common values*. In the present contribution, we generalize the environment to interdependent values (and thus more general information structures) and analyze the interaction between the heterogeneity in the fundamentals and the information structure of the agents. A key finding of Bergemann and Morris (2013b) was that - with only aggregate shocks - it was without loss of generality to restrict attention to a standard two dimensional information structure where agents observe a private and a public signal of the aggregate shock. By contrast, in the richer environment of this paper, there is not a standard class of information structures that generates all Bayes correlated equilibria.

One question addressed in Bergemann and Morris (2013b) was what was the optimal information structure from the point of view of agents. A large classical literature on information sharing in oligopoly, pioneered by Novshek and Sonnenschein (1982), Clarke (1983) and Vives (1984), was conducted in the context of linear normal models with strategic substitutes and thus fits the framework of analysis of Bergemann and Morris (2013b) and this paper. In Bergemann and Morris (2013b) - with only aggregate shocks - we argued that agents would sometimes prefer to have an information structure where they observed the common shock with conditionally independent noise. This minimizes the correlation of agents' actions (which is desirable because of strategic substitutes) for any given level of correlation of an agent's actions with the true aggregate state. Producer surplus turns out to be equivalent to the variance of individual volatility, and thus the results in this paper can be applied to generalize these welfare results to models with idiosyncratic shocks, and we briefly describe the relationship to the present analysis in the concluding section.

By giving a complete analysis of the impact of information in a stripped down model, we can connect with richer macroeconomic and other applied models that work with parameterized classes of information structures. As noted above, the work of Lucas (1972) and Venkateswaran (2013) have highlighted the importance of confounding shocks in macroeconomics and Angeletos and La'O

(2013) have highlighted the importance of "sentiment shocks" - common noise in agents' signals about idiosyncratic shocks - in models without aggregate shocks, and our approach suggests a deep connection between the two kinds of information structures. Angeletos and La'O (2013) impose the additional restriction that agents know their own payoff (in their case, productivity) shocks and are uncertain only about others' payoff shocks. But - as they emphasize - with no aggregate shocks and known private shocks, there can be no aggregate volatility under the maintained assumption of this paper that others actions matter only through their aggregate, i.e., average action. It is a natural extension of the approach in this paper to allow asymmetric strategic interaction, i.e., so that each agent cares about the actions of a subset of agents, not the population mean. In Angeletos and La'O (2013), agents interact in pairs on "islands" and we briefly discuss how our results can be extended to agents caring about a random partner's action in order to relate our work to Angeletos and La'O (2013) and other work in the macro literature.

The remainder of the paper is organized as follows. Section 2 introduces the model and the equilibrium concept. Section 3 introduces the noise free information structures and analyzes the Bayes Nash equilibrium behavior in this class of information structures. Section 4 discusses the determination of maximal volatility and dispersion in a benchmark model without strategic interaction and identifies the associated information structures, and thus establishes the link between information and volatility. Section 5 examines how the nature of strategic interaction impacts these results. Section 6 introduces the solution concept of Bayes correlated equilibrium. We establish an equivalence between the set of Bayes correlated equilibria and the set of Bayes Nash equilibria under any information structure. Section 7 considers a number of specific information structures that have appeared widely in the literature and we describe the subtle restrictions that they impose on the equilibrium behavior. Section 8 discusses the relationship of our results to recent contribution in macroeconomics on the source and scope of volatility and concludes. Section 9 constitutes the appendix and contains most of the proofs.

2 Model

We consider a continuum of agents, with mass normalized to 1. Agent $i \in [0, 1]$ chooses an action $a_i \in \mathbb{R}$ and is assumed to have a quadratic payoff function:

$$u_i : \mathbb{R}^3 \to \mathbb{R},$$

which is function of his action a_i, the mean action taken by all agents, A:

$$A \triangleq \int_j a_j dj,$$

and the individual payoff state, $\theta_i \in \mathbb{R}$, thus $u_i(a_i, A, \theta_i)$. Given the quadratic property of the payoff function, each agent i has a linear best response function:

$$a_i = r\mathbb{E}[A|\mathcal{I}_i] + \mathbb{E}[\theta_i|\mathcal{I}_i], \tag{1}$$

where $\mathbb{E}[\cdot|\mathcal{I}_i]$ is the expectation conditional on the information $\mathcal{I}_i$ agent i has prior to taking an action a_i. The parameter $r \in \mathbb{R}$ of the best response function represents the strategic interaction among the agents. If $r < 0$, then we have a game of *strategic substitutes*, if $r > 0$, then we have a game of *strategic complements*. We shall assume that the interaction parameter r is bounded above, or $r \in (-\infty, 1)$.

We assume that the individual payoff state θ_i is given by the linear combination of a *aggregate shock* $\bar{\theta}$ and an *idiosyncratic shock* $\Delta\theta_i$:

$$\theta_i = \bar{\theta} + \Delta\theta_i.$$

Throughout the text, we shall also refer to the *aggregate shock* and the *idiosyncratic shock* as the common value and the idiosyncratic (or private) value *component* of the payoff state θ_i. Each component of the payoff state θ_i is assumed to be normally distributed. While $\bar{\theta}$ is common to all agents, the idiosyncratic component $\Delta\theta_i$ is identically distributed across agents, independent of the common component. The payoff uncertainty is thus completely described by the pair $(\bar{\theta}, \Delta\theta_i)$ of random variables:

$$\begin{pmatrix} \bar{\theta} \\ \Delta\theta_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{\bar{\theta}} \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\bar{\theta}}^2 & 0 \\ 0 & \sigma_{\Delta\theta_i}^2 \end{pmatrix} \right). \tag{2}$$

It follows that the sample average of the idiosyncratic component across all agents always equals zero. We denote the sample average across the entire population, that is across all i, as $\mathbb{E}_i[\cdot]$, and

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so $\mathbb{E}_i[\Delta\theta_i] = 0$. The common component can be interpreted as the sample mean or average payoff state, and so $\bar{\theta} = \mathbb{E}_i[\theta_i]$.

Given the independence and the symmetry of the idiosyncratic component $\Delta\theta_i$ across agents, the above joint distribution can be expressed in terms of the variance of the individual state:

$$\sigma_\theta^2 \triangleq \sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2,$$

and the correlation (coefficient) $\rho_{\theta\theta}$ between any two states of any two agents i and j, θ_i and θ_j. After all, by construction the covariance of θ_i and θ_j is equal to the covariance between θ_i and $\bar{\theta}$, and in turn also represents the variance of the common component, that is $\sigma_{\bar{\theta}}^2 = \rho_{\theta\theta}\sigma_\theta^2$:

$$\begin{pmatrix} \theta_i \\ \bar{\theta} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{\bar{\theta}} \\ \mu_{\bar{\theta}} \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 \\ \rho_{\theta\theta}\sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 \end{pmatrix} \right). \tag{3}$$

The joint normal distribution given by (3) is the commonly known *common prior*. We shall almost exclusively use the representation (3) of the payoff uncertainty and only occasionally the former representation, see (2).

We often take the variance (and the mean) of the individual state θ_i as given by σ_θ^2, and describe the changes in the equilibrium behavior as we change $\rho_{\theta\theta}$ between 0 and 1. As we keep the variance σ_θ^2 of the individual state constant, by changing $\rho_{\theta\theta}$ we implicitly change the relative contribution of the idiosyncratic and the common variance as

$$\rho_{\theta\theta} = \frac{\sigma_{\bar{\theta}}^2}{\sigma_\theta^2} = \frac{\sigma_{\bar{\theta}}^2}{\sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2}.$$

We refer to the special cases of $\rho_{\theta\theta} = 0$ and $\rho_{\theta\theta} = 1$ as the case of *pure private* and *pure common* values.

The present model of a continuum of players with quadratic payoffs and normally distributed values, encompassing both private and common value environments, was first proposed by Vives (1990) to analyze information sharing among agents with private, but noisy information about the fundamentals. The focus of the present paper is rather different, but we shall briefly indicate how our approach to determine critical information structures also yields new insights to the large literature on information sharing in the conclusion.

3 Noise Free Bayes Nash Equilibrium

We begin the analysis by considering a class of noise free information structures and then derive the Bayes Nash equilibrium behavior under these one-dimensional information structures. We consider the following *one-dimensional* class of signals:

$$s_i \triangleq \lambda \Delta \theta_i + (1 - \lambda) \bar{\theta}, \tag{4}$$

where the linear composition of the signal s_i is determined by the parameter $\lambda \in [0, 1]$. We restrict attention to symmetric information structures (across agents), and hence all agents have the same parameter λ, and we also refer to it as the noise free information structure λ. In the present section, we consider the case of interdependent values, thus $\rho_{\theta\theta} \in (0, 1)$, and we discuss the limit cases of pure private and pure common values, thus $\rho_{\theta\theta} \in \{0, 1\}$, in the next section.

The information structure λ is *noise free* in the sense that every signal s_i is a linear combination of the components of the payoff state, $\Delta\theta_i$ and $\bar{\theta}$, and no extraneous noise or error enters the signal of each agent. Nonetheless, since the signal s_i combines the idiosyncratic and the common component of the payoff state, each signal s_i leaves agent i with residual uncertainty about the components of the payoff state. Moreover, unless the weight λ in the information structure exactly mirrors the composition of the payoff state $\theta_i = \bar{\theta} + \Delta\theta_i$, and hence exactly equals $1/2$, agent i still faces residual uncertainty about his payoff state θ_i. Thus, the signal *confounds* the two sources of fundamental uncertainty.

Given the information structure λ, we can compute the conditional expectation of agent i given the signal s_i about the idiosyncratic component $\Delta\theta_i$:[1]

$$\mathbb{E}\left[\Delta\theta_i \,|\, s_i\right] = \frac{\text{cov}\left(\Delta\theta_i, s_i\right)}{\text{var}\left(s_i\right)} = \frac{(1 - \rho_{\theta\theta})\,\lambda}{\rho_{\theta\theta}\,(1 - \lambda)^2 + (1 - \rho_{\theta\theta})\,\lambda^2}\,s_i, \tag{5}$$

the common component $\bar{\theta}$:

$$\mathbb{E}\left[\bar{\theta} \,|\, s_i\right] = \frac{\text{cov}\left(\bar{\theta}, s_i\right)}{\text{var}\left(s_i\right)} = \frac{\rho_{\theta\theta}\,(1 - \lambda)}{\rho_{\theta\theta}\,(1 - \lambda)^2 + (1 - \rho_{\theta\theta})\,\lambda^2}\,s_i, \tag{6}$$

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[1]For the remainder of the paper, we report all of the conditional expectations under the normalization of $\mu_\theta = 0$. With $\mu_\theta \neq 0$, the conditional expectations, such as (5)-(7) below, are given by a convex combination of the signal s_i and prior mean μ_θ. By normalizing $\mu_\theta = 0$, the statistical expressions become easier to read with minor loss of generality. By contrast, the description of the equilibrium in terms of mean and variance, as in Proposition 2 will always be stated for $\mu_\theta \in \mathbb{R}$.

and the payoff state θ_i of agent i:

$$\mathbb{E}\left[\theta_i \left| s_i \right.\right] = \frac{\mathrm{cov}\left(\theta_i, s_i\right)}{\mathrm{var}\left(s_i\right)} = \frac{\rho_{\theta\theta}\left(1 - \lambda\right) + \left(1 - \rho_{\theta\theta}\right)\lambda}{\rho_{\theta\theta}\left(1 - \lambda\right)^2 + \left(1 - \rho_{\theta\theta}\right)\lambda^2} s_i. \tag{7}$$

A few noise free information structures are of particular interest. If $\lambda = 1/2$, then each agent knows his own payoff state θ_i with certainty, as (7) reduces to:

$$\mathbb{E}\left[\theta_i \left| s_i \right.\right] = 2s_i = \Delta\theta_i + \bar{\theta},$$

but remains uncertain about the exact value of the idiosyncratic and common component. Similarly, if $\lambda = 0$, then the common component is known with certainty by each agent, as $\mathbb{E}\left[\bar{\theta} \left| s_i \right.\right] = s_i = \bar{\theta}$, but there remains residual uncertainty about the idiosyncratic component $\Delta\theta_i$ and a fortiori about the payoff state θ_i. Likewise, if $\lambda = 1$, then the idiosyncratic component is known with certainty, as $\mathbb{E}\left[\Delta\theta_i \left| s_i \right.\right] = s_i = \Delta\theta_i$, but there remains residual uncertainty about the common component $\bar{\theta}$ and a fortiori about the payoff state θ_i.

We define Bayes Nash equilibrium as the solution concept given the noise free information structures in (4).

Definition 1 (Bayes Nash Equilibrium)

Given an information structure λ the strategy profile

$$a^* : \mathbb{R} \to \mathbb{R},$$

forms a pure strategy symmetric Bayes Nash equilibrium if and only if:

$$a^*\left(s_i\right) = \mathbb{E}[\theta_i + rA | s_i], \quad \forall s_i \in \mathbb{R}.$$

The construction of the linear equilibrium strategy in the multivariate normal environment is by now standard, see Vives (1999) and Veldkamp (2011). Given the information structure λ, we denote the responsiveness, and in the linear strategy, the slope of the strategy in the signal s_i, by $w(\lambda)$.

Proposition 1 (Noise Free BNE)

For every noise free information structure λ, there is a unique Bayes Nash equilibrium and the strategy of each agent i is linear in the signal s_i:

$$a_i^*\left(s_i\right) = w\left(\lambda\right) s_i, \tag{8}$$

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with weight $w(\lambda)$:

$$w(\lambda) = \frac{\rho_{\theta\theta}(1-\lambda) + (1-\rho_{\theta\theta})\lambda}{(1-r)\rho_{\theta\theta}(1-\lambda)^2 + \lambda^2(1-\rho_{\theta\theta})}. \tag{9}$$

The responsiveness of the individual strategy is in general affected by the interaction parameter r, but in the special case of $r = 0$, each agent solves an pure statistical prediction problem and the optimal weight corresponds to the Bayesian updating rule given by (7). If $r > 0$, then the agents are in a game with strategic complements and respond stronger to the signal than Bayesian updating would suggest because of the inherent coordination motive with respect to the common component in the state, represented by the weight $\rho_{\theta\theta}(1-\lambda)^2$.

Given the information structure λ and the linearity of the unique Bayes Nash equilibrium, we can immediately derive the properties of the joint distribution of the equilibrium variables.

Proposition 2 (Moments of the Noise Free BNE)

For every noise free information structure λ:

1. *the mean of the individual action is $\mathbb{E}[a_i] = \mu_\theta / (1-r)$;*

2. *the variance of the individual action is:*

$$\mathrm{var}\,(a_i) \triangleq \sigma_a^2 = w(\lambda)^2 (\rho_{\theta\theta}(1-\lambda)^2 + (1-\rho_{\theta\theta})\lambda^2)\sigma_\theta^2; \tag{10}$$

3. *the covariances are given by:*

$$\mathrm{cov}(a_i, a_j) \triangleq \rho_{aa}\sigma_a^2 = w(\lambda)^2 \rho_{\theta\theta}(1-\lambda)^2 \sigma_\theta^2,$$

and

$$\mathrm{cov}(a_i, \theta_i) \triangleq \rho_{a\theta}\sigma_a\sigma_\theta = w(\lambda)\left(\rho_{\theta\theta}(1-\lambda) + (1-\rho_{\theta\theta})\lambda\right)\sigma_\theta^2.$$

We observe that the mean of the individual action is only a function of the mean μ_θ of the payoff state θ_i and the interaction parameter r, and thus is invariant with respect to the interdependence $\rho_{\theta\theta}$ in the payoff states and to the information structure λ. By contrast, the second moments, respond to the interdependence $\rho_{\theta\theta}$ in the payoff states and to the information structure λ. Given the normal distribution of the payoff states, and the linearity of the strategy, the variance and covariance terms are naturally the products of the weights, λ and $w(\lambda)$, and the variance σ_θ^2 of the fundamental uncertainty.

We are interested in the correlation between any pair of actions by agents i and j, a_i and a_j, and the correlation between the action a_i and the state θ_i of agent i. We denote the correlation coefficients by ρ_{aa} and $\rho_{a\theta}$, respectively. From the variance/covariance terms in the above proposition, we obtain the correlation coefficients explicitly:

$$\rho_{aa} = \frac{\rho_{\theta\theta}(1-\lambda)^2}{\rho_{\theta\theta}(1-\lambda)^2 + (1-\rho_{\theta\theta})\lambda^2}, \tag{11}$$

and

$$\rho_{a\theta} = \frac{\rho_{\theta\theta}(1-\lambda) + (1-\rho_{\theta\theta})\lambda}{\sqrt{\rho_{\theta\theta}(1-\lambda)^2 + (1-\rho_{\theta\theta})\lambda^2}}. \tag{12}$$

For completeness, we note that the covariance between the action a_i of agent i and the payoff state θ_j of agent j, is given by:

$$\text{cov}(a_i, \theta_j) = \text{cov}(a_i, \bar{\theta}) \triangleq \rho_{a\phi}\sigma_a\sigma_\theta = w(\lambda)(1-\lambda)\rho_{\theta\theta}\sigma_\theta^2. \tag{13}$$

As correlation *coefficients* only reflect the direction (and the non-linearity) in the relationship of two random variables, the interaction parameter r which only affects the slope of the equilibrium strategy, but not the composition of the signals, does not appear in the correlation coefficients.

Proposition 3 (Characterization of Noise-Free BNE)

For all $\rho_{\theta\theta} \in (0,1)$, the set of noise free BNE in the space of correlation coefficients $(\rho_{aa}, \rho_{a\theta})$ is given by:

$$\{(\rho_{aa}, \rho_{a\theta}) \in [0,1]^2 : \rho_{a\theta} = \sqrt{\rho_{aa}\rho_{\theta\theta}} + \sqrt{(1-\rho_{\theta\theta})(1-\rho_{aa})}\}. \tag{14}$$

To visualize the noise free equilibria in the space of correlation coefficients we plot them in Figure 1 for different coefficients $\rho_{\theta\theta}$ of interdependence. The left panel represents the case of $\rho_{\theta\theta} = 1/2$, whereas the right panel represents $\rho_{\theta\theta} = 1/4$ and $\rho_{\theta\theta} = 3/4$. In each case, the set of noise free equilibria is described as the positive solution to a quadratic equation (14) which identifies $\rho_{a\theta}$ as a function of ρ_{aa} for a given $\rho_{\theta\theta}$. Accordingly, the correlation coefficient $\rho_{a\theta}$ between action a_i and state θ_i of agent i is the sum of two roots. For a given correlation ρ_{aa} of the agents' actions, the correlation $\rho_{a\theta}$ is the sum of the products that arise from correlation between agents' actions and agents' states, $\rho_{aa}\rho_{\theta\theta}$, which is induced by the common component in the signal, and the anti-correlation between agents' actions and agents' states, $(1-\rho_{aa})(1-\rho_{\theta\theta})$, which is induced by the idiosyncratic component in the signal.

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We recall that the noise free information structure λ generates signal s_i by:

$$s_i \triangleq \lambda \Delta \theta_i + (1 - \lambda) \bar{\theta},$$

and λ varies between 0 and 1. For $\lambda = 1$, all the weight is on the idiosyncratic component, and the resulting equilibrium correlation coefficient ρ_{aa} of the actions of the agents is zero. As λ decreases starting from 1, the signal s_i begins to contain more information about the common component, and hence the correlation ρ_{aa} increases to eventually reach the maximum of 1 for $\lambda = 0$. In between 0 and 1, the highest correlation $\rho_{a\theta} = 1$ is always achieved at the information structure $\lambda = 1/2$ at which the signal exactly informs the agent about his payoff state θ_i. In consequence, at the maximum $\rho_{a\theta} = 1$, the correlation coefficient ρ_{aa} of the actions mirrors exactly the correlation coefficient of the states, or $\rho_{aa} = \rho_{\theta\theta}$. The right panel indicates how the interdependence in payoff states, $\rho_{\theta\theta}$, shapes the relationship between ρ_{aa} and $\rho_{a\theta}$.
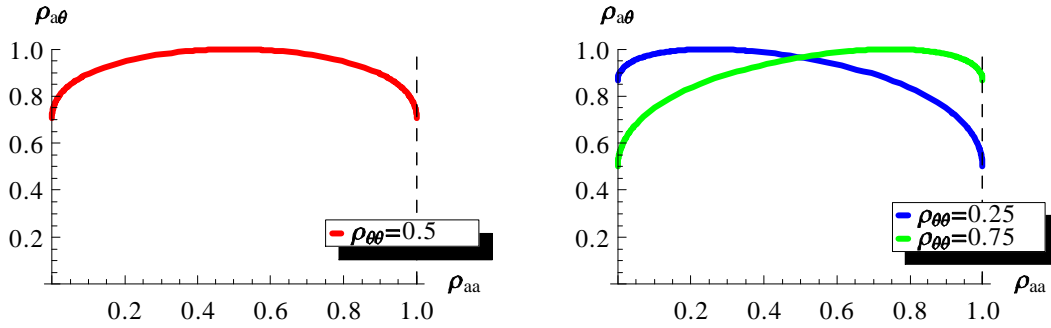

Figure 1: The set of noise-free BNE

At this point, it might be useful to compare the Bayes Nash equilibrium under the noise free information structure with the Nash equilibrium under complete information. The noise free information structure gives each agent access to a *one-dimensional* signal s_i that combines, and hence confounds, the components of the payoff state, $\bar{\theta}$ and $\Delta \theta_i$. By contrast, under complete information, each agent observes the private and common component separately, and hence each player receives a noise free *two-dimensional* signal $t_i = (\bar{\theta}, \Delta \theta_i)$. Given the linear structure of the best response, the Bayes Nash equilibrium under complete information is easily established to be:

$$a_i(t_i) \triangleq \Delta \theta_i + \frac{\bar{\theta}}{1 - r}.$$

In the complete information equilibrium, the strategy remains linear in the components and each agent assigns weight 1 to the idiosyncratic component $\Delta \theta_i$, and weight $1/(1 - r)$ to the common

component $\bar{\theta}$. Given the linearity, we may expect that a *one-dimensional* noise free information structure may still be able to replicate the outcome under the two-dimensional complete information structure. Indeed, we verify that the one-dimensional information structure $\widehat{\lambda}$ defined by:

$$\widehat{\lambda} \triangleq \frac{1-r}{2-r},$$

is the unique information structure $\lambda \in [0,1]$ such that the corresponding slope of the equilibrium strategy:

$$w\left(\widehat{\lambda}\right) = \frac{1}{\widehat{\lambda}} = \frac{2-r}{1-r},$$

yields the complete information equilibrium action for every realized component pair $\left(\Delta\theta_i, \bar{\theta}\right)$:

$$a_i^*\left(s_i\right) = \Delta\theta_i + \frac{\bar{\theta}}{1-r}.$$

This critical value $\widehat{\lambda}$ of a noise free information structure will be an important benchmark as we analyze how the information structure changes the responsiveness of the agents relative to the complete information equilibrium. We observe that $\widehat{\lambda}$ is the unique information structure among all $\lambda \in [0,1]$ such that the slope $w(\lambda)$ is invariant with the correlation structure $\rho_{\theta\theta}$. It always reproduces the complete information outcome, independent of the correlation structure. By comparison with the information structure $\lambda = 1/2$ at which each agent is guaranteed to learn his payoff state θ_i, we find that

$$\widehat{\lambda} < \frac{1}{2} \Leftrightarrow r > 0.$$

Thus in a game of strategic complementarities, the agent wishes to put less weight on the idiosyncratic component than on the common component, and conversely for strategic complements. The critical property of the information structure $\widehat{\lambda}$ is that the signal s_i is a sufficient statistic with respect to the equilibrium action. We can furthermore evaluate the equilibrium moments of Proposition 2 at $\widehat{\lambda}$ and recover the equilibrium moments of the complete information Nash equilibrium. In particular, we find that the correlation coefficient $\widehat{\rho}_{aa}$ of the actions is given by

$$\widehat{\rho}_{aa} \triangleq \frac{\rho_{\theta\theta}}{\left(1 - \rho_{\theta\theta}\right)\left(1-r\right)^2 + \rho_{\theta\theta}},$$

and as the above relationship of the critical value $\widehat{\lambda}$ with the balanced information structure of $\lambda = 1/2$ suggest, we have that

$$\widehat{\rho}_{aa} > \rho_{\theta\theta} \Leftrightarrow r > 0.$$

Thus the equilibrium correlation $\widehat{\rho}_{aa}$ under complete information is larger than under the balanced information structure $\lambda = 1/2$, where the equilibrium correlation is given by $\rho_{aa} = \rho_{\theta\theta}$, if and only if there are strategic complements.

4   Individual Decisions and Aggregate Volatility

We first consider aggregate volatility in the absence of any strategic interaction, and thus we are setting the strategic parameter r equal to zero. While we focus on aggregate volatility, we will also report results for individual volatility and dispersion.

With $r = 0$, the best response of each agent simply reflects a statistical prediction problem, namely to predict the payoff state θ_i given the signal s_i:

$$a_i = \mathbb{E}\left[\theta_i \,|\, s_i\right] = \frac{(1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})}{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})} s_i. \tag{15}$$

The individual prediction problem is more responsive to the signal s_i if and only if the signal contains more information about the fundamental payoff state θ_i. As we observed earlier, the information structure $\lambda = 1/2$ allows each agent to perfectly predict the payoff state, and given $r = 0$, the complete information benchmark is indeed $\widehat{\lambda} = 1/2$. Thus the responsiveness, and hence the variance of the *individual action* σ_a^2 is maximized at $\lambda = 1/2$, at which:

$$\sigma_a^2 = \sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2.$$

Now, to the extent that the individual payoff states θ_i and θ_j are correlated, we find that even though each agent i only solves an individual prediction problem, their actions are correlated by means of the underlying correlation in terms of the payoff state. Under the information structure $\lambda = 1/2$, the aggregate volatility is given by:

$$\sigma_A^2 = \rho_{aa}\sigma_a^2 = \sigma_{\bar{\theta}}^2.$$

Now we can ask, whether the aggregate volatility may reach higher levels under information structures different from $\lambda = 1/2$. As the information structure departs from $\lambda = 1/2$, it necessarily introduces a bias in the signal s_i towards one of the two components of the payoff state θ_i. Clearly, the signal s_i is losing informational quality with respect to the payoff state θ_i as λ moves away from $1/2$. Thus the individual prediction problem (15) is becoming noisier, and in consequence the response of the individual agent to the signal s_i is attenuated. But a larger weight, $1 - \lambda$, on

the common component $\bar{\theta}$, may support correlation in the actions across agents, and thus support aggregate volatility. At the same time, the response of the agent is likely to be attenuated, and thus a trade-off appears between bias and loss of information. We can then ask what is the maximal aggregate volatility that can be sustained across all noise free information structures.

Proposition 4 (Maximal Aggregate Volatility)

The maximal aggregate volatility:

$$\max_{\lambda} \{ \text{var} \, (A) \} = \frac{\left(\sigma_{\bar{\theta}} + \sqrt{\sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2} \right)^2}{4}, \tag{16}$$

is achieved by the information structure λ^* :

$$\lambda^* \triangleq \arg\max_{\lambda} \{ \text{var} \, (A) \} = \frac{\sigma_{\bar{\theta}}}{2\sigma_{\bar{\theta}} + \sqrt{\sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2}} < \frac{1}{2}. \tag{17}$$

Thus, the aggregate volatility is indeed maximized by an information structure which biases the signal towards the common component of the payoff state, as stated by (17). We recall that we defined the variance of the payoff state θ_i as $\sigma_\theta^2 = \sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2$, and hence can express the variance of the components also in terms of the variance of the state θ_i and the correlation coefficient $\rho_{\theta\theta}$, or $\sigma_{\bar{\theta}}^2 = \rho_{\theta\theta} \sigma_\theta^2$ and $\sigma_{\Delta\theta_i}^2 = (1 - \rho_{\theta\theta}) \sigma_\theta^2$. It follows that the information structure that maximizes the aggregate volatility, given by (17), can also expressed in terms of the correlation coefficient $\rho_{\theta\theta}$:

$$\arg\max_{\lambda} \{ \text{var} \, (A) \} = \frac{\sqrt{\rho_{\theta\theta}}}{1 + 2\sqrt{\rho_{\theta\theta}}},$$

and the maximal volatility given by (16) can be expressed as:

$$\max_{\lambda} \{ \text{var} \, (A) \} = \frac{1}{4} \left(1 + \sqrt{\rho_{\theta\theta}} \right)^2 \sigma_\theta^2.$$

Thus, as we approach the pure *common* value environment with $\rho_{\theta\theta} \to 1$ (or equivalently as the contribution from the idiosyncratic component vanishes with $\sigma_{\Delta\theta_i}^2 \to 0$), the maximal aggregate volatility of the actions coincides with the variance of the common component. This is achieved by the complete information equilibrium in which the action of each agent matches the realization of the payoff state θ_i. In the pure common value environment, all the variance in the payoff state θ_i, comes from the common component, and hence the aggregate variance, the variance of the average action, exactly attains the variance of the individual action.

More surprisingly, as we approach the pure *private* value environment, and hence consider an environment with purely idiosyncratic payoff uncertainty, the maximal aggregate volatility does not converge to zero, rather it is bounded away from 0, and given by $\sigma^2_{\Delta\theta_i}/4$, as stated in following corollary of Proposition 4.

Corollary 1 (Maximal Volatility with Pure Common and Pure PrivateValues)
In the limit to pure common *values:*

$$\lim_{\sigma^2_{\Delta\theta_i}\to 0}\max_\lambda\left\{\text{var}(A)\right\}=\sigma^2_{\bar\theta},$$

and to pure private *values:*

$$\lim_{\sigma^2_{\bar\theta}\to 0}\max_\lambda\left\{\text{var}(A)\right\}=\frac{\sigma^2_{\Delta\theta_i}}{4}. \tag{18}$$

As the payoff environment approaches the pure private value, the information structure puts more and more weight on the common component which itself has diminishing variance. The volatility maximizing information structure amplifies the response to the small common shock and hence maintains a substantial correlation in the signals (and actions) across the agents, even though the payoff states are almost idiosyncratic and thus almost independent. In fact, if we form the ratio of the standard deviations of the *weighted* idiosyncratic and common component to the signal s_i of the information structure (17), then we find that:

$$\frac{(1-\lambda^*)\sigma_{\bar\theta}}{\lambda^*\sigma_{\Delta\theta_i}}=\frac{1+\sqrt{\rho_{\theta\theta}}}{\sqrt{1-\rho_{\theta\theta}}},$$

and in the limit as $\rho_{\theta\theta}\to 0$ (or equivalently as $\sigma_{\bar\theta}\to 0$):

$$\lim_{\sigma_{\bar\theta}\to 0}\left\{\frac{(1-\lambda^*)\sigma_{\bar\theta}}{\lambda^*\sigma_{\Delta\theta_i}}\right\}=1. \tag{19}$$

Thus, the economy can maintain a large aggregate volatility even in the presence of vanishing aggregate uncertainty by confounding the payoff relevant information about the private component with the (in the limit) payoff irrelevant information about the common component.

In Section 4 we analyzed the equilibrium behavior in the noise free information structures assuming interdependent values, and hence confining the analysis to $\rho_{\theta\theta}\in(0,1)$. The above limit argument towards the pure private value environment suggests that as long as there is some *arbitrarily small* variation in the payoff state, the signal can always amplify the informational importance of the shock much beyond its payoff importance. But, if the variance in either the idiosyncratic

or the common component completely ceases to exists, then of course no amplification is possible. Nonetheless, the limits stated in Corollary 1 can still be attained with zero variance in either one of the components, but now require noise in the signal that is *payoff irrelevant*. We illustrate this for the case of pure private values, or idiosyncratic uncertainty.

In an environment with pure private values, consider an information structure in which each agent i observes a signal that contains an error ε common to all agents with mean 0 and variance σ_ε^2:

$$s_i = \Delta\theta_i + \varepsilon. \tag{20}$$

Given the signal s_i, the best response of agent i will be:

$$a_i = \mathbb{E}\left(\Delta\theta_i | s_i = \Delta\theta_i + \varepsilon\right) = \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_\varepsilon^2} s_i, \tag{21}$$

and it follows that the realized average action A is:

$$A = \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_\varepsilon^2} \varepsilon.$$

The resulting variance of the average action, the aggregate volatility is:

$$\sigma_A^2 = \left(\frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_\varepsilon^2}\right)^2 \sigma_\varepsilon^2,$$

and the aggregate volatility is maximized by setting the variance of the error term equal to the variance of the idiosyncratic component:

$$\sigma_\varepsilon^2 = \sigma_{\Delta\theta_i}^2. \tag{22}$$

This results in a positive level of aggregate volatility σ_A^2 driven by purely idiosyncratic uncertainty $\sigma_{\Delta\theta_i}^2$, that is:

$$\sigma_A^2 = \frac{1}{4}\sigma_{\Delta\theta_i}^2.$$

The noisy information structure (20) thus achieves the limit (18) of Corollary 1 with the noise to signal ratio of 1 implied by (22), which we derived earlier in (19) as the limiting ratio.

The maximal aggregate volatility is therefore achieved by an information structure that finds an optimal trade-off between biasing the information towards the common shock, and here simply common error, and maintaining responsiveness of agent i towards the signal s_i as given by the best response condition (21). Specifically, an increase in the variance σ_ε^2 of the error leads to larger

aggregate volatility only to the extent that the response of each agent to the signal does not become too attenuated. As the slope $\sigma^2_{\Delta\theta_i} / \left(\sigma^2_{\Delta\theta_i} + \sigma^2_\varepsilon \right)$ of the best response is decreasing in the variance σ^2_ε of the error term, the idiosyncratic payoff shock can only absorb a finite variance of the error term, namely $\sigma^2_\varepsilon = \sigma^2_{\Delta\theta_i}$, before the response to the signal becomes too weak to generate additional aggregate volatility.

In the special cases of pure private or pure common values, $\rho_{\theta\theta} = 0$ and $\rho_{\theta\theta} = 1$, respectively, the payoff uncertainty is described completely by either $\Delta\theta_i$ or $\bar{\theta}$, and reduces from a two dimensional to a one dimensional space of uncertainty. In either case, across all $\lambda \in [0,1]$, there are only *two* possible noise free equilibrium outcomes. Namely, either players respond perfectly to the state of the world (complete information) or players do not respond at all (zero information). For example, with purely private values, that is $\rho_{\theta\theta} = 0$, we have $\sigma^2_{\Delta\theta_i} = \sigma^2_\theta$, and $\sigma^2_{\bar{\theta}} = 0$. Then, the signal s_i is perfectly informative for all $\lambda \neq 0$ about the idiosyncratic component, and we are effectively in a complete information setting. By contrast, if $\lambda = 0$, then the signal s_i is completely uninformative, and each agent makes a deterministic choice given the expected value $\mathbb{E}\left[\Delta\theta_i\right] = 0$ of the state. Correspondingly, for purely common values, the critical and sole value under which the information structure is completely uninformative is $\lambda = 1$. Therefore there is a discontinuity at $\rho_{\theta\theta} \in \{0,1\}$ in the set of *noise free* Bayes Nash equilibria, but as the construction of the noisy information structure (20) suggests there is no discontinuity in the set of outcomes. The reason is simple and stems from the fact that as $\rho_{\theta\theta}$ approaches zero or one, one of the dimensions of the uncertainty about fundamentals vanishes. Yet we should emphasize, that even as the payoff states approach the case of pure common or pure private values, the part of the fundamental that becomes small can be arbitrarily amplified by the weight λ. For example, as $\rho_{\theta\theta} \to 1$, the environment becomes arbitrarily close to pure common values, yet the shock $\Delta\theta_i$ still can be amplified by letting $\lambda \to 1$ in the construction of signal (4) above. Thus, the component $\Delta\theta_i$ acts similarly to a purely idiosyncratic noise in an environment with pure common values. After all, the component $\Delta\theta_i$ only affects the payoffs in a negligible way, but with a large enough weight, it has a non-negligible effect on the actions that the players take. This suggests that for the case in which the correlation of states approaches the case of pure common or pure private values, there is no longer a sharp distinction between what is noise and what is fundamentals.

The dispersion of the individual action,

$$\Delta a_i \triangleq a_i - A,$$

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is defined as the volatility of the individual action beyond the aggregate volatility:

$$\text{var}(\Delta a_i) = (1 - \rho_{aa}) \sigma_a^2.$$

The analysis of dispersion is symmetric to that for aggregate volatility. We exactly replicate the analysis of aggregate volatility after we redefine the relevant variables in the obvious way:

$$\tilde{\lambda} \triangleq (1 - \lambda), \ \tilde{\rho}_{\theta\theta} \triangleq (1 - \rho_{\theta\theta}).$$

The result below then follows directly from Proposition 4.

Corollary 2 (Maximal Dispersion)
The maximal dispersion:

$$\max_{\lambda} \{\text{var}(\Delta a_i)\} = \frac{1}{4} \left(1 + \sqrt{1 - \rho_{\theta\theta}}\right)^2 \sigma_\theta^2;$$

is achieved by the information structure λ *:*

$$\arg\max_{\lambda} \{\text{var}(\Delta a_i)\} = \frac{1 + \sqrt{1 - \rho_{\theta\theta}}}{1 + 2\sqrt{1 - \rho_{\theta\theta}}} > \frac{1}{2}.$$

5 Interactive Decisions and Aggregate Volatility

Next, we extend the analysis from individual to interactive decisions, that is strategic environments with $r \neq 0$. The responsiveness of the action to the signal now depends on the nature of the signal but also on the nature of the interaction. In Section 3, we established that for every level r of strategic interaction, there is a noise free information structure $\widehat{\lambda}$ under which the agents' equilibrium behavior exactly mimics the complete information Nash equilibrium. This benchmark information structure $\widehat{\lambda}$ is useful to establish how the information structure λ affects the equilibrium behavior, and in particular the responsiveness to the payoff fundamentals, relative to the complete information outcome.

The action of each agent can be decomposed in terms of the responsiveness to the components of the payoff state θ_i, namely the idiosyncratic component $\Delta\theta_i$ and the common component $\bar{\theta}$. Given the multivariate normal distribution, the responsiveness of the agent to the components of his payoff state is directly expressed by the covariance:

$$\frac{\partial \mathbb{E}[a_i | \Delta\theta_i]}{\partial \Delta\theta_i} = \frac{\text{cov}(a_i, \Delta\theta_i)}{\sigma_{\Delta\theta}^2} = \lambda w(\lambda), \quad \frac{\partial \mathbb{E}[a_i | \bar{\theta}]}{\partial \bar{\theta}} = \frac{\text{cov}(a_i, \bar{\theta})}{\sigma_{\bar{\theta}}^2} = (1 - \lambda) w(\lambda). \tag{23}$$

Proposition 5 (Responsiveness to Fundamentals)

In the noise free BNE with information structure λ:

1. $\lambda \in (\widehat{\lambda}, 1) \Leftrightarrow \frac{\text{cov}(a_i, \Delta\theta_i)}{\sigma^2_{\Delta\theta}} > 1$;

2. $\lambda \in (0, \widehat{\lambda}) \Leftrightarrow \frac{\text{cov}(a_i, \bar{\theta})}{\sigma^2_{\bar{\theta}}} > \frac{1}{1-r}$.

Thus, the responsiveness of the action to each component of the payoff state is determined by the weight λ that the signal assigns relative to complete information benchmark $\hat{\lambda}$. Importantly for any given information structure λ, the responsiveness is typically stronger than in the complete information environment for exactly one of the components. Importantly, with interdependent values, the maximal responsiveness of the individual action to either the common or the idiosyncratic component is achieved with uncertainty about the payoff state, and not under complete information. We recall that with pure common values, any residual uncertainty about the payoff state inevitably reduced the responsiveness of the individual agent to the common state, and ultimately also reduced the aggregate responsiveness. Similarly, with pure private values, for each individual agent the residual uncertainty attenuated the responsiveness to his payoff state θ_i. By contrast, with interdependent values, the interaction between the idiosyncratic and the common component in the payoff state can correlate the responsiveness of the agents without attenuating the individual response, thus leading to a greater responsiveness than could be achieved under complete information.

In Figure 2 we plot the responsiveness to the two components of the fundamental state for the case of $\rho_{\theta\theta} = 0.5$ and different interaction parameters r. The threshold values $\widehat{\lambda}_r$ simply corresponds to the critical value $\widehat{\lambda}$ for each of the considered interaction parameters $r \in \left\{-\frac{3}{4}, 0, +\frac{3}{4}\right\}$. The horizontal black lines represent the responsiveness to the common component $\bar{\theta}$ in the complete information equilibrium which is equal to $1/(1-r)$, and the responsiveness to the idiosyncratic part, which is always equal to 1. By contrast, the red curves represent the responsiveness to the common component along the noise free equilibrium, and the blue curves represent the responsiveness to the idiosyncratic component. Thus if $\lambda < \widehat{\lambda}$, then the responsiveness to the common component $\bar{\theta}$ is larger than in the complete information equilibrium, and conversely for $\Delta\theta_i$. Moreover, we observe that the maximum responsiveness to the common component is never attained under either the complete information equilibrium or at the boundary values of λ, at 0 or 1. This immediately implies that the responsiveness is not monotonic in the informational content. We now provide some general comparative static results with respect to the strategic environment represented by r.
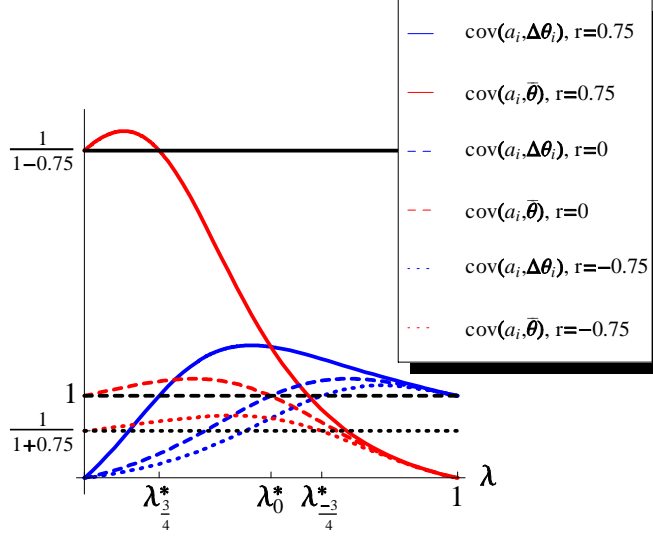
Figure 2: Responsiveness to Fundamentals for $\rho_{\theta\theta} = 1/2$

Proposition 6 (Informational Weight and Maximal Volatility)

For all $\rho_{\theta\theta} \in (0, 1)$:

1. *the informational weights λ that maximize the second moments satisfy:*

$$\text{argmax}_\lambda \ \left\{ (1 - \rho_{aa})\sigma_a^2 \right\} > \text{argmax}_\lambda \ \left\{ \sigma_a^2 \right\} > \text{argmax}_\lambda \ \left\{ \rho_{aa}\sigma_a^2 \right\};$$

2. *the informational weights λ that maximize the second moments: $\text{argmax}_\lambda \sigma_a^2$, $\text{argmax}_\lambda \rho_{aa}\sigma_a^2$, $\text{argmax}_\lambda(1 - \rho_{aa})\sigma_a^2$ are strictly decreasing in r;*

3. *the maximal second moments: $\max_\lambda \sigma_a^2$, $\max_\lambda \rho_{aa}\sigma_a^2$, $\max_\lambda(1 - \rho_{aa})\sigma_a^2$ are strictly increasing in r.*

Thus, the maximal volatility, both individual and aggregate, is increasing in the level of complementarity r. Even the maximal dispersion is increasing in r. In the equilibrium with maximum dispersion, the agents confound the idiosyncratic and aggregate component of the payoff state and overreact to the idiosyncratic part, this effect increases with r. This implies that the responsiveness to the common component $\overline{\theta}$ increases, and hence the overreaction to the idiosyncratic component $\Delta\theta_i$ increases as well. Moreover, the optimal weight on the aggregate component increases in r for all of the second moments.

We can contrast the behavior of aggregate volatility with pure common values and with interdependent values even more dramatically.

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Proposition 7 (Aggregate Volatility)

The maximal aggregate volatility is given by:

$$\max_{\lambda} \left\{ \rho_{aa} \sigma_a^2 \right\} = \frac{\sigma_{\Delta\theta_i}^4}{4 \left(\sqrt{\sigma_{\bar{\theta}}^2 + (1-r)\sigma_{\Delta\theta_i}^2} - \sigma_{\bar{\theta}} \right)^2} \tag{24}$$

and is strictly increasing and without bound in the idiosyncratic uncertainty $\sigma_{\Delta\theta_i}^2$ for all $r \in (-\infty, 1)$.

In other words, as we move away from the model of pure common values, that is $\sigma_{\Delta\theta_i}^2 \neq 0$, the aggregate volatility is largest with some amount of incomplete information. In consequence, the maximum aggregate volatility is not bounded by the aggregate volatility under the complete information equilibrium as it is the case with common values. In fact, the aggregate volatility is increasing without bounds in the variance of the idiosyncratic component even in the absence of variance of the common component $\bar{\theta}$. The latter result is in stark contrast to the complete information equilibrium in which the aggregate volatility is unaffected by the variance of the idiosyncratic component. This illustrates in a simple way that the aggregate volatility may result from uncertainty about *either* the aggregate fundamental *or* the idiosyncratic fundamental. As in the case of individual decision-making, we can consider the limits of the aggregate volatility as we approach a model of pure common or pure private values.

Corollary 3 (Maximal Volatility with Pure Common and Pure PrivateValues)

In the limit to pure common *values:*

$$\lim_{\sigma_{\Delta\theta_i}^2 \to 0} \max_{\lambda} \left\{ \text{var}(A) \right\} = \sigma_{\bar{\theta}}^2 / (1-r)^2,$$

and to pure private *values:*

$$\lim_{\sigma_{\bar{\theta}}^2 \to 0} \max_{\lambda} \left\{ \text{var}(A) \right\} = \sigma_{\Delta\theta_i}^2 / (4(1-r)).$$

Earlier, we suggested that the impact of the confounding information on the equilibrium behavior is distinct in the interdependent value environment relative to either the pure private and pure common value environment. We can make this now precise by evaluating the impact the introduction of a public component has in a world of pure idiosyncratic uncertainty. By evaluating the aggregate volatility and ask how much can it be increased by adding a common payoff shock with arbitrarily small variance, we find from (24) that:

$$\frac{\partial \max_{\lambda} \left\{ \text{var}(A) \right\}}{\partial \sigma_{\bar{\theta}}} \Big|_{\sigma_{\bar{\theta}} = 0} = \frac{\sigma_{\theta}^2}{2(1-r)^{3/2}}.$$

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More generally, there is positive interaction between the variance of the idiosyncratic and the common component with respect to the aggregate volatility in equilibrium as the cross-derivatives are:

$$\frac{\partial^2 \max_\lambda \{\text{var}(A)\}}{\partial \sigma_{\Delta\theta_i} \partial \sigma_{\bar{\theta}}} = \frac{\sigma_{\Delta\theta_i}^3}{2 \left(\sigma_{\bar{\theta}}^2 + (1-r)\sigma_{\Delta\theta_i}^2\right)^{3/2}} > 0.$$

Interestingly, for a given variance of the common component, the positive interaction effect as measured by the cross derivatives occurs at finite values of the variance of the idiosyncratic component.

Confounding Information   The idea that confounding shocks can lead to overreaction goes back at least to Lucas (1972). In a seminal contribution, he shows how monetary shocks can have a real effect in the economy, even when under complete information monetary shocks would have no real effect. As agents observe just a one dimensional signal that confounds two shocks, namely the labor market conditions and the monetary supply shock, however they respond to the signal, they respond to the two shocks in the same way. By contrast, under complete information they would condition their labor decision only on the real market conditions, yet as the one dimensional signal does not allow them to disentangle both shocks, in equilibrium they respond to both shocks. Thus, this can be seen as an overreaction to monetary shocks due to "informational frictions". The idea has been present also in more resent papers, for example, Venkateswaran (2013) uses a similar idea to show how firms can have an excess reaction to aggregate shocks when these are confounded with idiosyncratic shocks. In Mackowiak and Wiederholt (2009), confounding informational conditions are derived from a model of rational inattention where the amount of noise on each signal is chosen optimally by the agents, subject to a global informational constraint motivated by limited attention.

In a recent contribution, Angeletos and La'O (2013) show that an economy without any kind of aggregate uncertainty might still have aggregate fluctuations. In Angeletos and La'O (2013) the fundamental uncertainty is purely idiosyncratic and agents *know* the realization of their own payoff state. But each agent is only assumed to interact with a specific trading partner rather than the aggregate market. Now, even though the payoff uncertainty is purely idiosyncratic, and each agent knows his own payoff state, the pairwise interaction leaves each agent uncertain about the action of his trading partner. It is uncertainty that can be affected by a common noise term, and hence generate aggregate volatility across the agents. They interpret this common noise term as *sentiments*, which generate aggregate fluctuations.

The current analysis allow us to extract some very simple intuitions on when informational

frictions can have a big effect on aggregate outcomes. We saw that as we approach a world of idiosyncratic uncertainty ($\rho_{\theta\theta} \to 0$), the maximum aggregate volatility is bounded away from 0 and it is a achieved by a noise free equilibria. The information structure amplified the payoff relevant common shock that had small variance and lead to a big response by the agents provided that the informational weight on the common shock was sufficiently large. Conversely, a payoff shock that has a very high variance generally leads to a response that is similar to the response they would have in a complete information environment. This is simply because a shock that has a large variance and a non-vanishing informational weight in the signal will allow the agents to approximately learn the true value of the shock. Thus, we find that in a model with idiosyncratic uncertainty such as Angeletos and La'O (2013), any arbitrarily small aggregate shock can have a huge effect, that is can be amplified if it receives a sufficiently large weight in the signal of the agents.

It is perhaps worth emphasizing that the idea that agents can play as if they had complete information, but with rather limited information has also been explored in the literature. In what may at first appear to be a result conflicting with the discussion here, Hellwig and Venkateswaran (2009) show that agents may take actions which resemble the complete information equilibria, even when they only receive a one dimensional signal that leaves them uncertain about the state of the world. Nevertheless, as the signal they receive may be a good statistic of what their *optimal action* should be, they can still take actions that resemble what their complete information actions would be. This idea is used to show that the effect of informational frictions on monetary rigidities can be dampened by the fact that the signal that the agents receive may have an adequate weight on the common and the idiosyncratic shock. Thus, even if agents have little information on the aggregate shocks, they can respond as if they knew, because their signals are a precise statistic of their complete information action.

6   Bayes Correlated Equilibrium

So far, we analyzed the outcomes under the Bayes Nash equilibrium for a very special class of information structures. Now, we establish that these special, noise free information structures indeed form the boundaries of the equilibrium behavior for a very large class of information structures. At first glance, the argument may appear to be an indirect route, but we hope to convince the reader that it is arguably the most expeditious route to take. We first define a solution concept, the Bayes correlated equilibrium, that describes the behavior (and outcomes) independent of the

specific information structure that the agents may have access to. We then establish that the set of outcomes under the Bayes correlated equilibrium is equivalent to the set of outcomes that may arise under Bayes Nash equilibrium for all possible information structures. The set of Bayes correlated equilibria has the advantage that it can be completely described by a small set of parameters and inequalities that restrict the second moments of the equilibrium distribution. We then show that the equilibria which form the boundary of the equilibrium set in terms of the inequalities can be compactly described as noise free Bayes correlated equilibria. In turn, these equilibria, as the name suggest, are shown to be equivalent to the set of noise free Bayes Nash equilibria we described earlier. Finally, we show that a large class of "welfare functions", including those that represent the individual or the aggregate volatility, are maximized by choosing equilibria and hence information structures that are on the boundary of the equilibrium set. Thus, we conclude that the special class of noise free information structures are indeed the relevant information structures as we wish to analyze the maximal volatility or dispersion that can arise in equilibrium.

6.1   Definition of Bayes Correlated Equilibrium

We first define the new solution concept.

Definition 2 (Bayes Correlated Equilibrium)
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a symmetric and normally distributed Bayes correlated equilibrium (BCE) if their joint distribution is given by a multivariate normal distribution and for all i and a_i :

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \tag{25}$$

The Bayes correlated equilibrium requires that the joint distribution of states and actions satisfies for every action a_i in the support of the joint distribution the best response condition (25). As before, we shall restrict our attention to symmetric and normally distributed equilibrium outcomes.

We emphasize that the equilibrium notion does not at all refer to any information structure or signals, and thus is defined without reference to any specific information structure. As the equilibrium object is the joint distribution $(\theta_i, \bar{\theta}, a_i, A)$, the only informational restrictions that are imposed by the equilibrium notion are that: (i) the marginal distribution over the payoff relevant states $(\theta_i, \bar{\theta})$ coincides with common prior, and (ii) each agent conditions on the information contained in joint distribution and hence the conditional expectation given a_i when choosing action a_i. The equilibrium notion thus differs notably from the Bayes Nash equilibrium under a given information structure.

We denote the variance-covariance matrix of the joint distribution of $(\theta_i, \bar{\theta}, a_i, A)$ by $\mathbb{V}$. Next, we identify necessary and sufficient conditions such that the random variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes correlated equilibrium. These conditions can be separated into two distinct sets of requirements: the first set consists of conditions such that the variance-covariance matrix $\mathbb{V}$ of the joint multivariate distribution constitutes a valid variance-covariance matrix, namely that it is positive-semidefinite; and a second set of conditions that guarantee that the best response conditions (25) hold. The first set of conditions are purely statistical requirements. The second set of conditions are necessary for any BCE, and these later conditions merely rely on the linearity of the best response. Importantly, both set of conditions are necessary *independent* of the assumption of normal distributed payoff uncertainty. The normality assumption will simply ensure that the equilibrium distributions are completely determined by the first and second moment. Thus, the normality assumptions allows us to describe the set of BCE in terms of restrictions that are necessary *and* sufficient.

6.2   Characterization of Bayes Correlated Equilibria

We begin the analysis of the Bayes correlated equilibrium by reducing the dimensionality of the variance-covariance matrix. We appeal to the symmetry condition to express the aggregate variance in terms of the individual variance and the correlation between individual terms.   Just as we described above the variance $\sigma_{\bar{\theta}}^2$ of the common component $\bar{\theta}$ in terms of the covariance between any two individual payoff states in (3), or $\sigma_{\bar{\theta}}^2 = \rho_{\theta\theta}\sigma_\theta^2$, we can describe the variance of aggregate action σ_A^2 in terms of the covariance of any two individual actions, or $\sigma_A^2 = \rho_{aa}\sigma_a^2$.   Earlier, see Proposition 2, we denoted the correlation coefficient between action a_i and payoff state θ_i of player i by $\rho_{a\theta}$:

$$\operatorname{cov}(a_i, \theta_i) \triangleq \rho_{a\theta}\sigma_a\sigma_\theta,$$

and the correlation coefficient between the action a_i of agent i and the payoff state θ_j of a different agent j as $\rho_{a\phi}$:

$$\operatorname{cov}(a_i, \theta_j) \triangleq \rho_{a\phi}\sigma_a\sigma_\theta.$$

These three correlation coefficients, $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$, parameterize the entire variance-covariance matrix. To see why, observe that the covariance between a purely idiosyncratic random variable and a common random variable is always 0. This implies that both the covariance between the aggregate action A and the payoff state θ_j of player j and the covariance between the agent i's action, a_i, and the common component of the payoff state, $\bar{\theta}$, are the same as the covariance between the

action of player i and the payoff state θ_j of player j, or $\rho_{a\phi}\sigma_a\sigma_\theta$. Thus we can reduce the number of variance terms, and in particular the number of correlation coefficients needed to describe the variance-covariance matrix $\mathbb{V}$ without loss of generality.

Lemma 1 (Symmetric Bayes Correlated Equilibrium)

The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a symmetric and normally distributed Bayes correlated equilibrium (BCE) if and only if there exist parameters of the first and second moments, $(\mu_a, \sigma_a, \rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$, such that the joint distribution is given by:

$$\begin{pmatrix} \theta_i \\ \bar{\theta} \\ a_i \\ A \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_\theta \\ \mu_\theta \\ \mu_a \\ \mu_a \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 & \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta \\ \rho_{\theta\theta}\sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta \\ \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta & \sigma_a^2 & \rho_{aa}\sigma_a^2 \\ \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{aa}\sigma_a^2 & \rho_{aa}\sigma_a^2 \end{pmatrix} \right), \tag{26}$$

and for all i and a_i :

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \tag{27}$$

With a multivariate normal distribution the conditional expectations have the familiar linear form and we can write the best response condition (27) in terms of the first and second moments:

$$a_i = r(\mu_a + \rho_{aa}(a_i - \mu_a)) + (\mu_\theta + \rho_{a\theta}\frac{\sigma_\theta}{\sigma_a}(a_i - \mu_a)). \tag{28}$$

By taking expectation we determine the mean μ_a of the individual action:

$$\mu_a = r\mu_a + \mu_\theta \Leftrightarrow \mu_a = \frac{\mu_\theta}{1-r}. \tag{29}$$

Taking the derivative of (28) with respect to a_i we determine the variance σ_a of the individual action:

$$1 = r\rho_{aa} + \rho_{a\theta}\frac{\sigma_\theta}{\sigma_a} \Leftrightarrow \sigma_a = \frac{\rho_{a\theta}\sigma_\theta}{1-r\rho_{aa}}. \tag{30}$$

We thus have a complete determination of the individual mean and variance. Now, for $\mathbb{V}$ to be a valid variance-covariance matrix, it has to be positive semi-definite, and this imposes restrictions on the remaining covariance terms.

Proposition 8 (Characterization of BCE)

A multivariate normal distribution of $(\theta_i, \bar{\theta}, a_i, A)$ is a symmetric Bayes correlated equilibrium if and only if :

1. *the mean of the individual action is:*

$$\mu_a = \frac{\mu_\theta}{1 - r}; \tag{31}$$

2. *the standard deviation of the individual action is:*

$$\sigma_a = \frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}} \geq 0; \tag{32}$$

3. *the correlation coefficients $\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}$ satisfy $\rho_{aa}, \rho_{a\theta} \geq 0$ and the inequalities:*

$$(i) \quad (\rho_{a\phi})^2 \leq \rho_{\theta\theta}\rho_{aa}, \quad (ii) \quad (1 - \rho_{aa})(1 - \rho_{\theta\theta}) \geq (\rho_{a\theta} - \rho_{a\phi})^2. \tag{33}$$

There are two aspects of Proposition 8 that we should highlight. First, the mean μ_a of the individual action (and a fortiori the mean of the aggregate action A) is completely pinned down by the payoff fundamentals. This implies that any differences across Bayes correlated equilibria must manifest themselves in the second moments only. Second, the restrictions on the equilibrium correlation coefficients do not at all depend on the interaction parameter r. The restrictions on the set of equilibrium correlations are purely statistical and stem from the condition that the variance-covariance matrix $\mathbb{V}$ forms a positive semi-definite matrix. By contrast, the mean μ_a and the variance σ_a^2 of the individual actions do depend on the interaction parameter r, as they are determined by the best response condition (27). We will show in Section 7 that the disentanglement of the set of feasible correlations and the interaction parameter is possible only if we allow for all possible information structures, i.e. when we do not impose any restrictions on the private information that agents may have.

In the special case of pure private (or pure common) values the set of outcomes in terms of the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ reduces to a two dimensional set. The reduction in dimensionality arises as the correlation coefficient $\rho_{a\phi}$ of the individual action and the common state is either zero (as in the pure private value case) or equal to the correlation coefficient between the individual action and individual state (as in the pure common value case), and thus redundant in either case. Thus, in the case of pure common values, $\rho_{\theta\theta} = 1$, the conditions (33) reduce to $\rho_{a\phi} = \rho_{a\theta}$, and $\rho_{a\theta}^2 \leq \rho_{aa}$ as established earlier in Bergemann and Morris (2013b).

We should note that the conditions in Proposition 8 remain necessary conditions for any symmetric Bayes correlated equilibrium in the absence of any distributional assumptions on the distribution of payoff states and actions. The conditions on the first and second moments, (31) and (32), are obtained by using the law of iterated expectations given the linearity of the best response. The conditions on the correlation coefficients, (33), arise from the requirement that the variance/covariance matrix of the joint distribution must be positive semi-definite, which has to hold for any multivariate distribution. Thus the assumption of normality of the joint distribution merely turns these necessary conditions into sufficient conditions.

6.3 Equivalence between BNE and BCE

Next we describe the relationship between the joint distributions $\left(\theta_i, \bar{\theta}, a_i, A\right)$ that can arise as Bayes correlated equilibria and the distributions that may arise as a Bayes Nash equilibrium for some information structure $\mathcal{I} = \{S_i\}_{i \in [0,1]}$. In contrast to the restriction to one-dimensional noise free information structure made in Section 3, we now (implicitly) allow for a much larger class of information structures, including noisy and multi-dimensional information structures. In fact, for the present purpose, it is sufficient to merely require that the associated symmetric equilibrium strategy $\{a_i\}_{i \in [0,1]}$: $a_i : S_i \to \mathbb{R}$ forms a multivariate normal distribution.

Definition 3 (Bayes Nash Equilibrium)
The random variables $\{a_i\}_{i \in [0,1]}$ form a normally distributed Bayes Nash Equilibrium under information structure $\mathcal{I} = \{S_i\}_{i \in [0,1]}$ if and only if the random variables $\{a_i\}_{i \in [0,1]}$ are normally distributed and

$$a_i = \mathbb{E}[\theta_i + rA | s_i], \quad \forall i, \ \forall s_i.$$

In Section 7, we explicitly analyze certain classes of multi-dimensional normally distributed information structures, many of which have already appeared in the literature. As we will see then, and as it has been established in the literature, the normality of the signals together with the linearity of the best response function leads to the normality of the outcome distribution. We postpone the discussion until the next section, as for the moment the relevant condition is the normality of the outcome distribution itself.

Proposition 9 (Equivalence Between BCE and BNE)
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a (normal) Bayes correlated equilibrium if and only if there exists some information structure $\mathcal{I}$ under which the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes Nash equilibrium.

The important insight of the equivalence is that the set of outcomes that can be achieved as a BNE for *some* information structure can equivalently be described as a BCE. Thus, the solution concept of BCE allows us to study the set of outcomes that can be achieved as a BNE, importantly without the need to specify a specific information structure. In Bergemann and Morris (2013a), we establish the equivalence between Bayes correlated equilibrium and Bayes Nash equilibrium for canonical finite games and arbitrary information structures (see Theorem 1 there). The above proposition specializes the proof to an environment with linear best responses and normally distributed payoff states and actions.

We will discuss specific information structures and their associated equilibrium behavior in Section 7. Here, we describe a one-dimensional class of signals that is already sufficiently rich to decentralize the entire set of Bayes correlated equilibria as Bayes Nash equilibria. For this, we modify the set of noise free structure studied in Section 3 by allowing the weights on $\Delta\theta_i$ and $\bar{\theta}$ to have different signs, and adding noise. This generates a signal s_i :[2]

$$s_i = \lambda\Delta\theta_i + (1 - |\lambda|)\bar{\theta} + \varepsilon_i, \tag{34}$$

where $\lambda \in [-1, 1]$ and ε_i is normally distributed with mean zero and variance σ_ε^2. Similar to the definition of the payoff relevant fundamentals, the individual error term ε_i can have a common component:

$$\bar{\varepsilon} \triangleq \mathbb{E}_i[\varepsilon_i],$$

and an idiosyncratic component:

$$\Delta\varepsilon_i \triangleq \varepsilon_i - \bar{\varepsilon},$$

while being independent of the fundamental component. Thus, the joint distribution of the states and signals is given by:

$$
\begin{pmatrix} \Delta\theta_i \\ \bar{\theta} \\ \Delta\varepsilon_i \\ \bar{\varepsilon} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \mu_\theta \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (1 - \rho_{\theta\theta})\sigma_\theta^2 & 0 & 0 & 0 \\ 0 & \rho_{\theta\theta}\sigma_\theta^2 & 0 & 0 \\ 0 & 0 & (1 - \rho_{\varepsilon\varepsilon})\sigma_\varepsilon^2 & 0 \\ 0 & 0 & 0 & \rho_{\varepsilon\varepsilon}\sigma_\varepsilon^2 \end{pmatrix} \right), \tag{35}
$$

and the standard deviation $\sigma_\varepsilon > 0$ and the correlation coefficient $\rho_{\varepsilon\varepsilon} \in [0, 1]$ are the parameters of the fully specified information structure $\mathcal{I} = \{\mathcal{I}_i\}_{i \in [0,1]}$, together with the confounding parameter

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[2]Note that in contrast to the noise free information structure λ studied in Section 3, we now allow for $\lambda < 0$. This allows for $\Delta\theta_i$ and $\bar{\theta}$ to have different sign on the signal. The importance will become clear in what follows.

λ. We observe that the dimensionality of information structure $\mathcal{I}$, given by (34) and (35), and thus parametrized by $(\lambda, \sigma_\varepsilon, \rho_{\varepsilon\varepsilon})$, matches the dimensionality of the Bayes correlated equilibrium in terms of the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$. Relative to the boundary of the BCE in the $(\rho_{aa}, \rho_{a\theta})$ space, intuitively, the variance σ_ε^2 of the noise term controls how far the equilibrium correlation $\rho_{a\theta}$ falls vertically below the boundary, whereas the correlation $\rho_{\varepsilon\varepsilon}$ in the errors across agents controls the correlation in the agents actions as the noise in the observation increases.

Proposition 10 (Decentralization of BCE)

The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a (normal) Bayes correlated equilibrium if and only if there exist some information structure $(\lambda, \sigma_\varepsilon^2, \rho_{\varepsilon\varepsilon})$ under which the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes Nash equilibrium.

6.4 The Boundary of Bayes Correlated Equilibria

In Proposition 8, we characterized the entire set of Bayes correlated equilibria. The mean and the variance of the individual action were determined by the equalities (31) and (32), whereas the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ were restricted by the two inequalities given by (33). We will now provide a interpretation of these inequalities, and a characterization of when they hold with equality.

As we observed earlier, we can decompose the action of each agent in terms of his responsiveness to the components of his payoff state θ_i, namely the idiosyncratic component $\Delta\theta_i$ and the common component $\bar{\theta}$, and any residual responsiveness has to be attributed to noise, see (23). The action a_i itself also has an idiosyncratic and a common component as $a_i = A + \Delta a_i$. The conditional variance of these components of a_i can be expressed in terms of the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}, \rho_{\theta\theta})$, which are subject to the restrictions of Proposition 8. By using the familiar property of the multivariate normal distribution for the *conditional variance*, we obtain a diagonal matrix:

$$\text{var}\left[\begin{array}{c|c} \Delta a_i & \Delta\theta_i \\ A & \bar{\theta} \end{array}\right] = \sigma_a^2 \left(\begin{array}{cc} (1-\rho_{aa}) - \frac{(\rho_{a\theta}-\rho_{a\phi})^2}{1-\rho_{\theta\theta}} & 0 \\ 0 & \rho_{aa} - \frac{\rho_{a\phi}^2}{\rho_{\theta\theta}} \end{array}\right). \tag{36}$$

If the components A and Δa_i of the agent's action are completely explained by the components of the payoff state, $\bar{\theta}$ and $\Delta\theta_i$, then the conditional variance of the action components, and a fortiori of the action itself, is equal to zero. This suggests the following definition.

Definition 4 (Noise Free BCE)

A BCE is noise free *if a_i has zero variance, conditional on $\bar{\theta}$ and $\Delta\theta_i$.*

We observe that the above matrix of conditional variances is only well-defined for interdependent values, that is for $\rho_{\theta\theta} \in (0,1)$. For the case of pure private or pure common values, $\rho_{\theta\theta} = 0$ or $\rho_{\theta\theta} = 1$, only one of the off diagonal terms is meaningful, as the other conditioning terms, $\bar{\theta}$ or $\Delta\theta_i$, have zero variance by definition.

Now, by comparing the conditional variances above with the conditions of Proposition 8, it is easy to see that the conditional variances are equal to zero if and only if the conditions of Proposition 8 are satisfied as equalities. Moreover, in any BCE, the conditional variance of action a_i can be equal to zero if and only if the conditional variances of the components, A and Δa_i are each equal to zero. We can now provide a characterization of the noise free Bayes correlated equilibria.

Corollary 4 (Characterization of Noise Free Bayes Correlated Equilibria)

For all $\rho_{\theta\theta} \in (0,1)$, the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ form a noise free Bayes correlated equilibria if and only if the two inequalities given by (33) are satisfied with equality. Moreover, the set of noise free equilibria is given by:

$$\{(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}) \in [0,1]^2 \times [-1,1] : \rho_{a\theta} = |\sqrt{\rho_{aa}\rho_{\theta\theta}} \pm \sqrt{(1-\rho_{\theta\theta})(1-\rho_{aa})}|, \ \rho_{a\phi} = \pm\sqrt{\rho_{aa}\rho_{\theta\theta}} \}. \ (37)$$

It is not difficult to see, that the set of noise free ayes correlated equilibria can be decentralized by signals of the form (34) and imposing $\sigma_\varepsilon = 0$. In Section 3 we studied a subset of these class of signals, namely signals of the form (34) in which $\sigma_\varepsilon = 0$ *and* $\lambda \in [0,1]$. Although a priori, this may seem like an arbitrary restriction, it turns out the signals with positive values of λ (or noise free structures) are the most interesting ones for many applications.

By looking at the characterization of the set of noise free Bayes correlated equilibria, we can see that there were two possible signs in the characterization of $\rho_{a\theta}$. The positive root will correspond to the boundary of the set of feasible correlations in the $(\rho_{a\theta}, \rho_{aa})$ space. We will refer to these set of equilibria as simply the *boundary* of the Bayes correlated equilibria, which formally is defined as follows:

$$\{(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}) \in [0,1]^2 \times [-1,1] : \rho_{a\theta} = \sqrt{\rho_{aa}\rho_{\theta\theta}} + \sqrt{(1-\rho_{\theta\theta})(1-\rho_{aa})}, \ \rho_{a\phi} = \sqrt{\rho_{aa}\rho_{\theta\theta}} \}. \quad (38)$$

We now provide an equivalence between the boundary of the Bayes correlated equilibria and the set of Bayes Nash equilibria that can be decentralized by a noise free structure λ.

Corollary 5 (Equivalence of Noise Free BNE and BCE)

For all $\rho_{\theta\theta} \in (0,1)$, the coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ are a boundary Bayes correlated equilibria if and only if they form a BNE for some information structure λ.

Therefore, the boundary Bayes correlated equilibria corresponds to the set of equilibria already studied in Section 3. The definition of the boundary suggests that monotone functions of these correlation coefficients might achieve their maximum across all equilibria somewhere on the boundary described above by the noise free property.

Proposition 11 (Maximal Volatility and Dispersion)

Among all Bayes correlated equilibria, individual volatility, aggregate volatility, and dispersion are all maximized by a boundary Bayes correlated equilibrium.

Thus we can already conclude that information structures that maximize either individual or aggregate volatility, or dispersion, are precisely the noise free information structures that we analyzed in Section 3. Moreover, the proof of Proposition 11 indicates that these noise free information structures would remain the critical ones if we were to conduct a more comprehensive welfare analysis beyond the analysis of the second moments. Notably, an auxiliary result, Lemma 2, establishes that any continuous function that is strictly increasing in $\rho_{a\theta}$ and weakly increasing in $\rho_{a\phi}$, achieves its maximum in the set of all feasible BCE by a noise free BCE. As the conditions of Lemma 2 are silent about the correlation coefficient ρ_{aa}, we can accommodate strategic environments (and payoffs and associated objective functions) with either strategic substitutes or complements.

In the discussion following Proposition 8, we argued that the moment restrictions remain necessary conditions even in the absence of any distributional assumptions of normality. Therefore, Proposition 11 can actually be stated in a stronger version. Suppose we maintain the assumption of normality in the payoff states, but neither do we require the normality in actions nor the joint normality in actions and states. Then, we would still have the result that the volatility is maximized by the noise free and normally distributed equilibria of Proposition 11, as the necessary boundary conditions of the BCE given by Proposition 8 are indeed attained by the linear combinations of the common and idiosyncratic components of the payoff states.[3]

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[3] We conjecture that Proposition 11 remains to hold more generally in environments without normally distributed payoff states. But in the absence of normally distributed payoff states, the associated noise free information structure is likely to be a nonlinear, rather than linear, function of the components of the payoff state.

7 Information Structures and Equilibrium Behavior

We began our analysis in Section 3 with a class of specific, namely noise free, information structures, and analyzed the resulting Bayes Nash equilibrium behavior with respect to volatility and dispersion. We established in Proposition 8 and 9 that the noise free information structures indeed form the boundary of equilibrium behavior with respect to the relevant correlation coefficients across all possibly symmetric and normally distributed information structures. As the literature has frequently restricted attention to specific information structures, we can now ask how restrictive or permissive commonly used classes of information structures are with respect to the entire set of feasible equilibrium behavior. An important insight that emerges here is that the dimensionality of the information structures by itself is not sufficient statistic for the dimensionality of the supported equilibrium behavior, and that information structures limit equilibrium behavior in subtle ways. The specific information structures that we study are given by, or form a subset of, the following three-dimensional information structures:

$$\mathbf{s}_i \triangleq \{s_i^1 = \theta_i + \varepsilon_i^1 \ , \ s_i^2 = \bar{\theta} + \varepsilon_i^2 \ , \ s_i^3 = \bar{\theta} + \bar{\varepsilon}^3\}, \tag{39}$$

where $\varepsilon_i^1, \varepsilon_i^2$ are idiosyncratic noise terms and $\bar{\varepsilon}^3$ is a common noise term, all normally distributed, independent and with zero mean. This class of information structures appears in the analysis of Angeletos and Pavan (2009) and is parameterized by three variables, namely the variances $(\sigma_{\varepsilon^1}^2, \sigma_{\varepsilon^2}^2, \sigma_{\varepsilon^3}^2)$ of the noise terms. We begin by characterizing the set of feasible correlations when agents only observe a noisy and idiosyncratic signal of their payoff state θ_i:[4]

$$s_i^1 = \theta_i + \varepsilon_i^1,$$

and thus we set $\sigma_{\varepsilon^2}^2 = \sigma_{\varepsilon^3}^2 = \infty$. This class of signals is frequently used in the literature on information sharing, see Vives (1990) and Raith (1996).

Proposition 12 (Noisy Signal of Payoff State)
A set of correlations $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ can be achieved as a Bayes Nash equilibrium with an information structure $\{s_i^1\}_{i\in[0,1]}$, if and only if:

$$\rho_{a\theta} = \sqrt{\frac{\rho_{aa}}{\rho_{\theta\theta}}}; \quad \rho_{a\phi} = \rho_{a\theta}\rho_{\theta\theta}; \quad \rho_{aa} \in [0, \rho_{\theta\theta}]. \tag{40}$$

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[4]We emphasize the fact that conditions (31) and (32) must hold for any signal structure. Thus, by characterizing the set of feasible correlations, we are characterizing the set of feasible outcomes.
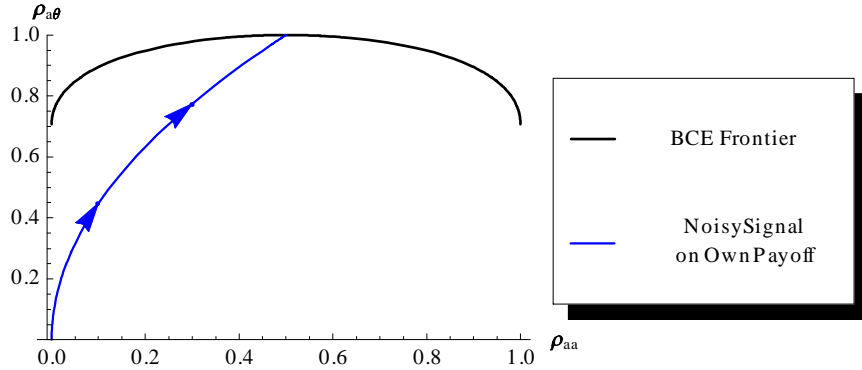
Figure 3: Feasible BNE correlation coefficients with noisy signals about payoff state θ_i

We observe that the set of feasible correlations when the agents receive only a one-dimensional signal of the form s_i^1 does not depend on the interaction parameter r. This, at first perhaps surprising result is actually true for any class of one dimensional signals. In fact, earlier we showed that a larger set of one dimensional signals allows us to span the entire set of feasible outcomes (that is, all Bayes correlated equilibria). And indeed, for the entire set of feasible outcomes, the set of feasible correlations is independent of r. We illustrate in Figure 3, the locus of attainable correlations with information structures $\{s_i^1\}_{i \in [0,1]}$ for $\rho_{\theta\theta} = 1/2$. The arrows point in the direction of greater precision (i.e., lower variance) of the error term. Notably, all the attainable equilibrium coefficients are below the frontier given by the Bayes correlated equilibria, except for a single point that is identified by zero noise, or $\sigma_{\varepsilon^1}^2 = 0$.

A second interesting information structures to study is the case in which, besides signal $\{s_i^1\}_{i \in [0,1]}$, the players also know the average payoff state $\bar{\theta}$, and thus we set $\sigma_{\varepsilon^2}^2 = \sigma_{\varepsilon^3}^2 = 0$, and each agent observes $\bar{\theta}$ and s_i^1. Although a priori this may not seem like an information structure that might arise exogenously, it is the information structure that arises when agents receive endogenous information on the average action taken by other players. For example, in a rational expectation equilibrium with a continuum of sellers as studied by Vives (2014), each seller chooses a supply function, given a signal about his private cost θ_i. The resulting equilibrium price p can be shown to be a linear function of the average supply, and in turn a linear function of the average cost $\bar{\theta}$. Similarly, in recent work by Benhabib, Wang, and Wen (2012), (2013) and Bergemann, Heumann, and Morris (2014) the equilibrium condition of the rational expectations equilibrium has a linear structure in which each agent conditions his decision on the common component $\bar{\theta}$.
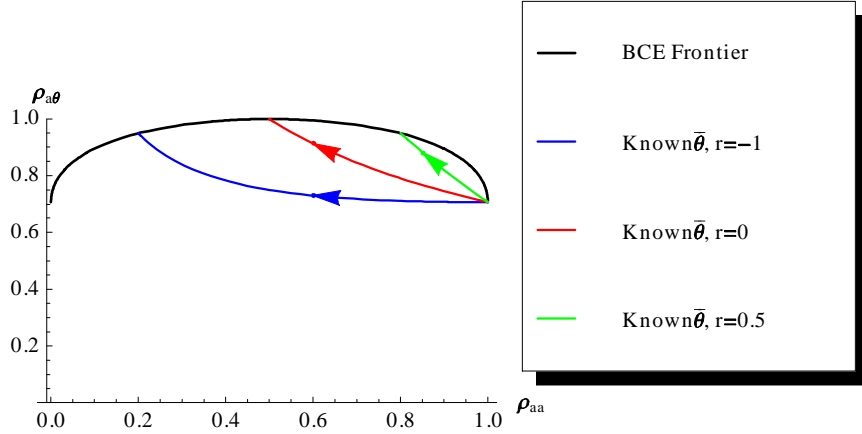
Figure 4: Set of feasible correlations for subset of information structures ($\rho_{\theta\theta} = 1/2$)

Proposition 13 (Noisy Signal of Payoff State / Noiseless Signal of Common Component)
A set of correlations $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ can be achieved as a Bayes Nash equilibrium with an information structure $\{s_i^1, \overline{\theta}\}_{i\in[0,1]}$ if and only if:

$$\rho_{a\theta} = \sqrt{\frac{\rho_{\theta\theta}}{\rho_{aa}}} \frac{1 - r\rho_{aa}}{1 - r}; \ \ \rho_{a\phi} = \sqrt{\rho_{\theta\theta}\rho_{aa}}; \ \ \rho_{aa} \in [\widehat{\rho}_{aa}, 1]. \tag{41}$$

Now, the set of feasible correlations indeed depends on r, and the information structure $\left(\{s_i^1\}_{i\in[0,1]}, \overline{\theta}\right)$ generates a two-dimensional signal for every agent i. We also find that the correlation coefficient ρ_{aa} has a lower bound, which is the correlation coefficient $\widehat{\rho}_{aa}$ achieved in the complete information equilibrium. In Figure 4 we illustrate the set of correlations that can be achieved when each agent receives a noisy signal s_i^1 of his payoff state θ_i, and *in addition* knows the average payoff state $\overline{\theta}$. We observe that the class of two dimensional information structures $\{s_i^1, \overline{\theta}\}_{i\in[0,1]}$ induce a one dimensional subspace of $(\rho_{a\theta}, \rho_{aa})$ that does depend on the nature of the interaction. Indeed Proposition 13 establishes that these signals maintain a one-dimensional subspace even with respect to the full three dimensional space of correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$. Importantly, the feasible correlation coefficients remained bounded away from the frontier, except for the two points of $\sigma_{\varepsilon^1}^2 = 0$ and $\sigma_{\varepsilon^1}^2 = \infty$, where the later leads to perfect correlation in the actions across agents: $\rho_{aa} = 1$, at the expense of low correlation with the payoff state: $\rho_{a\theta} = \sqrt{\rho_{\theta\theta}}$.

We now characterize the set of feasible correlations when agents know their own payoff state, and thus we set $\sigma_{\varepsilon^1}^2 = 0$. That is, all possible outcomes that are consistent with agents knowing *at least* θ_i. Since each agent knows his own payoff state, the residual uncertainty is with respect

to the actions taken by other players. This informational assumption of knowing the own payoff state θ_i commonly appears in the macroeconomics literature. For example, Angeletos and La'O (2009), Angeletos, Iovino, and La'O (2011) and Angeletos and La'O (2012) consider models with interdependent values and imperfect information, but assume that each agent know his own payoff state θ_i. In a model with idiosyncratic rather than aggregate interaction, Angeletos and La'O (2013) analyze the impact of informational friction on aggregate fluctuations. Again, they assume that each agent knows his own payoff state θ_i, but is uncertain about the payoff state θ_j of the trading partner j. Similarly, Lorenzoni (2010) investigates the optimal monetary policy with dispersed information. He also considers a form of individual matching rather than aggregate interaction. The common informational assumption in all of these models is that every agent i knows his own payoff state θ_i, and thus all uncertainty is purely strategic.

The characterization is achieved in two steps. First, we describe the set of feasible action correlations ρ_{aa}. If each agent only knows his own payoff state θ_i, then the correlation ρ_{aa} is equal to $\rho_{\theta\theta}$ as the actions of any two agents can only be correlated to the extent that their payoff states are correlated. By contrast, if the agents have complete information, then the correlation is given, as established earlier, by $\rho_{aa} = \widehat{\rho}_{aa}$. We find that the set of feasible action correlations is always between these two quantities, providing the lower and upper bound respectively. If $r > 0$, then the complete information bound is the upper bound, if $r < 0$, it is the lower bound. For $r = 0$ they coincide as θ_i is a sufficient statistic of the action taken by each agent under complete information.

Second, we describe the set of feasible correlations between action and state, $\rho_{a\theta}$, for any feasible ρ_{aa}. The set of feasible $\rho_{a\theta}$ is determined by two functions of ρ_{aa}, which provide the lower and upper bound for the feasible $\rho_{a\theta}$. We denote these functions by $\rho_{a\theta}^i(\rho_{aa})$ and $\rho_{a\theta}^c(\rho_{aa})$ as these bounds are achieved by information structures in which each agent knows his own payoff state and receives a second signal, either an idiosyncratic signal of the common component $\bar{\theta} : s_i^2 \triangleq \bar{\theta} + \varepsilon_i^2$ or a common signal of $\bar{\theta} : s_i^3 \triangleq \bar{\theta} + \bar{\varepsilon}^3$.

Proposition 14 (Known Payoff State θ_i)

A set of correlations $(\rho_{aa}, \rho_{a\theta})$ can be induced by a linear Bayes Nash equilibrium in which each agent knows his payoff state θ_i if and only if

$$\rho_{aa} \in [\min\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}, \max\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}]; \tag{42}$$

and for any ρ_{aa} satisfying (42):

$$\rho_{a\theta} \in [\min\{\rho_{a\theta}^c(\rho_{aa}), \ \rho_{a\theta}^i(\rho_{aa})\}, \max\{\rho_{a\theta}^c(\rho_{aa}), \ \rho_{a\theta}^i(\rho_{aa})\}].$$

In Figure 5, we illustrate the Bayes Nash equilibrium set for different values of r for a given correlation $\rho_{\theta\theta} = 1/2$. Each interaction value r is represented by a differently colored pair of lower and upper bounds. For each value of r, the entire set of BNE is given by the area enclosed by the lower and upper bound. Notably, the bounds $\rho_{a\theta}^c(\rho_{aa})$ and $\rho_{a\theta}^i(\rho_{aa})$ intersect in two points, corresponding to each agent knowing his payoff state θ_i only (at $\rho_{aa} = \rho_{\theta\theta} = 1/2$) and to complete information, at the low or high end of ρ_{aa} depending on the nature of the interaction, respectively. In fact these, and only these, two points, are also noise free equilibria of the unrestricted set of BCE. When $r \geq 0$, the upper bound is given by a signal with an idiosyncratic error term, s_i^2, while the lower bound is given by a signal with a common error term, s_i^3, and conversely for $r \leq 0$. With $r > 0$, if the additional signal contains an idiosyncratic error, then it forces the agent to stay closely to his known payoff state, as this is where the desired correlation with the other agents arises, and only slowly incorporate the information about the common component $\overline{\theta}$ of the state, thus overall tracking as closely as possible his own payoff state θ_i, and achieving the upper bound. The argument for the lower bound follows a similar logic.

Thus, if each agent knows at least his own payoff state, then we observe a dramatic reduction in the set of feasible BNE. Notably, every element, with the exception of the information structures mentioned in the above paragraph, are in the interior of the unrestricted set of BCE. Moreover, the nature of the interaction has a profound impact on the shape of the correlation that can arise in equilibrium, both in terms of the size as well as the location of the set in the unit square of $(\rho_{aa}, \rho_{a\theta})$.[5]

If each agent i is assumed to know his payoff state θ_i, then we can restate the best response condition (1) with respect to a_i after the following change of variables in terms of deviations from the payoff state:

$$\tilde{a}_i \triangleq a_i - \theta_i, \quad \widetilde{A} \triangleq A - \overline{\theta}. \tag{43}$$

The best response condition (1) in terms of a_i reduces to the following best response condition in terms of $\tilde{a}_i$:

$$\tilde{a}_i = r\mathbb{E}\left[\overline{\theta}\,|\theta_i, \mathcal{I}_i\right] + r\mathbb{E}\left[\widetilde{A}\,|\theta_i, \mathcal{I}_i\right], \tag{44}$$

---

[5]In the case of pure common or pure private values, the set of Bayes correlated equilibria where each agent knows his own payoff state is degenerate. Under either pure common or pure private values, if each agent knows his own payoff state there is no uncertainty left, and thus the only possible outcome corresponds to the complete information outcome.
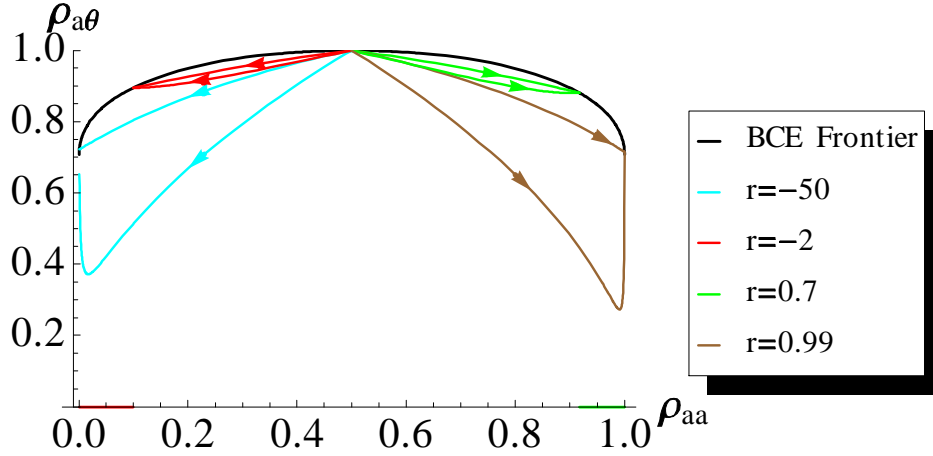
Figure 5: Boundary of the set of feasible correlations when agents know own payoff ($\rho_{\theta\theta} = 1/2$)

where $\mathcal{I}_i$ is any information agent i gets beyond knowing θ_i. The resulting best response condition is now isomorphic to one where the payoff environment is a pure common value environment, *and* where the payoff state and average action receive the same weight in the best response condition of the individual agent. This provides a distinct intuition on the strong restrictions on behavior that arise from imposing that agents know their own payoff state as stated in Proposition 14.

So far, we confined attention to strict subsets of the three-dimensional information structures as summarized by Table 1:

	Noisy Idiosyncratic Information	Known Common Component	Known Own Payoff State
$\sigma_{\varepsilon 1}$	$\in [0, \infty)$	$\in [0, \infty)$	0
$\sigma_{\varepsilon 2}$	∞	0	$\in [0, \infty)$
$\sigma_{\varepsilon 3}$	∞	0	$\in [0, \infty)$
Proposition	12	13	14

Table 1: Information as Restriction on Equilibrium Behavior

Now that we understand how the observability of the individual state or components of the individual state affect the equilibrium correlation, we ask what is the entire set of equilibrium correlations that can be achieved with the three dimensional signal structure. As the dimensionality of the information structure defined by (39) coincides with the dimensionality of the set of equilibrium correlations, one might expect that the set of information structures is sufficiently rich to span the

equilibrium set defined by the BCE. In fact, in the case of pure common values Bergemann and Morris (2013b) show that any BCE can be decentralized by considering a pair of noisy signals, a private and a public signal of the payoff state, namely $s_i^1 = \theta_i + \varepsilon_i^1$ and $s_i^3 = \bar{\theta} + \bar{\varepsilon}^3$ in terms of the current language. By contrast, in the present general environment neither the class of binary nor the extended class of tertiary information structures can decentralize the entire sets of BCE. We characterize the set of feasible correlations in $(\rho_{aa}, \rho_{a\theta})$ coefficient space. In order to describe the set of feasible correlations in the $(\rho_{aa}, \rho_{a\theta})$ space, we define the upper bound $\bar{\rho}_{a\theta}(\rho_{aa})$ that can be attained by a correlation coefficient ρ_{aa} in a Bayes Nash equilibrium by an information structure given by (39).

Proposition 15 (Boundary of Correlations under s_i)

The boundary of feasible correlations $\bar{\rho}_{a\theta}(\rho_{aa})$ for Bayes Nash equilibria with $\mathbf{s}_i$ is given by:

1. *If $\rho_{aa} \in [0, \min\{\hat{\rho}_{aa}, \rho_{\theta\theta}\}]$, then $\bar{\rho}_{a\theta}(\rho_{aa})$ is attained by an information structure in which each agent gets a noisy signal on his own payoff state: $\sigma_{\varepsilon^1} \in [0, \infty)$, $\sigma_{\varepsilon^2} = \sigma_{\varepsilon^3} = \infty$ with $\bar{\rho}_{a\theta}(\rho_{aa}) = \sqrt{\rho_{aa}/\rho_{\theta\theta}}$.*

2. *If $\rho_{aa} \in [\min\{\hat{\rho}_{aa}, \rho_{\theta\theta}\}, \max\{\hat{\rho}_{aa}, \rho_{\theta\theta}\}]$, then $\bar{\rho}_{a\theta}(\rho_{aa})$ is attained by an information structure in which each agents knows his own payoff state: $\sigma_{\varepsilon^1} = 0, \sigma_{\varepsilon^2}, \sigma_{\varepsilon^3} \in [0, \infty)$ with $\bar{\rho}_{a\theta}(\rho_{aa}) = \max\{\rho_{a\theta}^c, \rho_{a\theta}^i\}$.*

3. *If $\rho_{aa} \in [\max\{\hat{\rho}_{aa}, \rho_{\theta\theta}\}, 1]$, then $\bar{\rho}_{a\theta}(\rho_{aa})$ is attained by an information structure in which agent know the common component: $\sigma_{\varepsilon^1} \in [0, \infty)$, $\sigma_{\varepsilon^2} = \sigma_{\varepsilon^3} = 0$ with:*

$$\bar{\rho}_{a\theta}(\rho_{aa}) = \sqrt{\frac{\rho_{\theta\theta}}{\rho_{aa}}} \frac{1 - r\rho_{aa}}{1 - r}.$$

The upper bound $\bar{\rho}_{a\theta}(\rho_{aa})$ can therefore be constructed from the union of the information structures that we considered in Proposition 12-14. In Figure 6, the left panel illustrates the behavior that can be achieved for $r = 0$. In this special case, the correlation coefficient of the complete information equilibrium $\hat{\rho}_{aa}$ coincides with the correlation coefficient of the payoff states $\rho_{\theta\theta}$, and hence $\min\{\hat{\rho}_{aa}, \rho_{\theta\theta}\} = \max\{\hat{\rho}_{aa}, \rho_{\theta\theta}\}$. Thus the entire boundary can be achieved by noisy and idiosyncratic signals of the payoff state θ_i with either zero or complete information about the common component $\bar{\theta}$, appealing to Proposition 15.1 and 15.3 respectively. The right panel illustrates the Proposition for $r \neq 0$, and complements the description of the boundary with Proposition 15.2. As stated earlier in Proposition 14, the intermediate segment of the boundary is obtained with a noisy
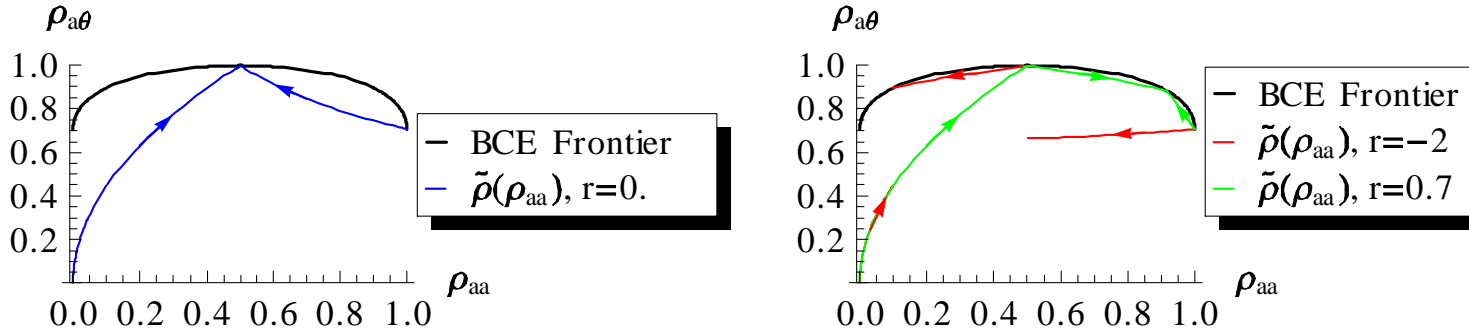
Figure 6: Boundary of the set of feasible correlations for information structures $\mathbf{s}_i$ ($\rho_{\theta\theta} = 1/2$)

signal of the common components that comes with a common error in the case of strategic substitutes: $\sigma_{\varepsilon^1} = \sigma_{\varepsilon^2} = \infty, \sigma_{\varepsilon^3} \in [0, \infty)$; or an idiosyncratic error in the case of strategic complements: $\sigma_{\varepsilon^1} = \sigma_{\varepsilon^3} = \infty, \sigma_{\varepsilon^2} \in [0, \infty)$. Using positive variances of the error terms, it is easy to show that for all $(\rho_{aa}, \rho_{a\theta})$ such that $\rho_{a\theta} \in [0, \overline{\rho}_{a\theta}(\rho_{aa})]$, there exists an information structure of the form $\mathbf{s}_i$ that achieves theses correlations as a Bayes Nash equilibrium.

A common feature across all strategic environments is the property that the boundary $\overline{\rho}_{a\theta}$ is strictly below the frontier of all Bayes correlated equilibrium, which implies that the set of three-dimensional information structures given by (39) is indeed restrictive. We find that the boundary $\overline{\rho}_{a\theta}$ attains the frontier of the BCE at exactly three points: (i) the complete information equilibrium, (ii) the equilibrium in which each agent knows θ_i and (iii) the equilibrium in which each agent only knows $\bar{\theta}$. It is perhaps worth highlighting that the boundary $\overline{\rho}_{a\theta}$ is discontinuous for strategic substitutes, that is $r < 0$. This result emphasizes the subtle role that the information structure has on equilibrium outcomes.

8   Discussion

We conclude by discussing the relevance of the current analysis to environments with heterogeneous rather than aggregate interaction. We end by relating our analysis to the large literature on information sharing among firms and suggest how the current tools might yield new results there as well.

Beyond Aggregate Interaction   We deliberately restricted our analysis to an environment with aggregate interaction. Every agent formed a best response against the average of the population.

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Yet, within the linear quadratic framework, it appears feasible to extend the analysis to much richer interaction structures, such as pairwise interaction or even general network interaction structures. In the macroeconomic literature, models of pairwise matching have appeared prominently in the literature, see for example Lorenzoni (2010) and Angeletos and La'O (2013). Notably, these models of pairwise interaction assume that each agent knows his own payoff state θ_i but is still uncertain about the payoff state of his matched partner, θ_j, the class of information structures that we investigated in Proposition 15. As each agent i is assumed to know his own payoff state θ_i, there is no *fundamental uncertainty* (about his own payoffs) anymore and so the residual uncertainty is all about the *strategic uncertainty*, namely the expected action of the other agent.[6]

Interestingly, even if we were interested in strategic uncertainty in the absence of fundamental uncertainty, the noise free information structures remain of central importance for the aggregate behavior. To see this, consider a simple model of pairwise interaction as in Angeletos and La'O (2013). We assume there is a pairwise matching between i and j and that agents interact with their partner as well as with the aggregate population. Thus, the first order condition of agent i who is matched with j is given by:

$$a_i = \mathbb{E}\left[\theta_i \,|\, \theta_i, \mathcal{I}_i\right] + r_a \mathbb{E}\left[a_j \,|\, \theta_i, \mathcal{I}_i\right] + r_A \mathbb{E}\left[A \,|\, \theta_i, \mathcal{I}_i\right].$$

If we make the same change of variables as earlier in (43), so that we express the choice variables in terms of their deviation from the payoff state: $\tilde{a}_i \triangleq a_i - \theta_i, \quad \widetilde{A} \triangleq A - \overline{\theta}$, then the associated first order conditions are given by:

$$\tilde{a}_i = r_a \mathbb{E}\left[\Delta\theta_j \,|\, \theta_i, \mathcal{I}_i\right] + r_A \mathbb{E}\left[\overline{\theta} \,|\, \theta_i, \mathcal{I}_i\right] + r_a \mathbb{E}\left[\tilde{a}_j \,|\, \theta_i, \mathcal{I}_i\right] + r_A \mathbb{E}\left[\widetilde{A} \,|\, \theta_i, \mathcal{I}_i\right]. \tag{45}$$

Thus, we have a similar model as the one we have been studying so far, but with some differences. First, agents have some prior information on $\overline{\theta}$ which comes from knowing θ_i. Second, the size of the shocks $\Delta\theta_j$ and $\overline{\theta}$ are scaled by r_a and r_A in the first order conditions. Besides these differences, a model with heterogeneous interaction in which each agent knows his own payoff state is almost identical to our original model. Namely, each agent's uncertainty is still two-dimensional, with a common and an idiosyncratic component (equal to $\overline{\theta}$ and $\Delta\theta_j$ respectively). Thus, we see that even if we were interested in strategic uncertainty in the absence of fundamental uncertainty, the same basic intuitions and ideas still apply. A key factor to consider in the information of agents would

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[6]We thank our discussant, Marios Angeletos, for emphasizing the importance of the distinct contribution of each source of uncertainty to the aggregate volatility.

be to see how the signal of each agents leads to confusion between $\Delta\theta_j$ and $\bar{\theta}$. And as before, the confusion would lead to overreaction and underreaction to some of these fundamentals, respectively.

Information Sharing   We described the impact that the private information structure has on the second moments of the economy, in particular the volatility of the aggregate outcome. Naturally, we could expand the analysis to functions of the (second) moments of the economy. In the large literature on information sharing among firms, pioneered in work by Novshek and Sonnenschein (1982), Clarke (1983) and Vives (1984), the expected profit function of the individual firm is a function both of the individual and the aggregate volatility of the outcome. In this literature, which is presented in a very general framework in Raith (1996) and surveyed in Vives (1999), each firm receives a private signal about a source of uncertainty, say a demand or cost shock. The central question is under which conditions the firms have an incentive to commit ex-ante to an agreement to share information in some form. The present analysis of the impact of information structures on the set feasible correlation structure suggest additional and novel insights into the nature of optimal information sharing policy.

We briefly illustrate this within a competitive equilibrium with a continuum of producers, each one of them with a quadratic cost of production $c(a_i) = a_i^2/2$, and facing a linear inverse demand function dependent on the common state of demand θ and the aggregate supply A: $p(\bar{\theta}, A) = \bar{\theta} + rA$, so that the resulting best response function is again given by (1).[7] Now, in the space of the correlation coefficients $(\rho_{aa}, \rho_{a\theta})$, we can depict the iso-profit curve $\bar{\pi}$ defined implicitly by a constant expected profit $\bar{\pi}$ of the representative firm: $\bar{\pi} \triangleq \mathbb{E}\left[a_i p - \frac{1}{2}a_i^2\right]$, which can be shown to be linear in ρ_{aa} (indicated by the red dashed line) and the slope is determined by the responsiveness r of the price to supply.

The maximal correlation $\rho_{a\theta}$ that is achievable with disclosure of a common signal, denoted earlier in Proposition 14 by $\rho_{a\theta}^c(\rho_{aa})$, is convex in ρ_{aa}, whereas the maximal correlation achievable with disclosure of an idiosyncratic signal is given by $\rho_{a\theta}^i(\rho_{aa}) \triangleq \sqrt{\rho_{aa}}$, is concave in ρ_{aa}. In fact with pure common values, the idiosyncratic signals $s_i^1 = \theta_i + \varepsilon_i$ trace out the entire boundary of the BCE coefficients, as illustrated in Figure 7. We therefore can conclude that the optimal disclosure policy with a *public signal* is either zero or complete disclosure, which was a central finding in

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[7]The restriction to the pure common value environment, $\theta_i = \bar{\theta}$, allows us to directly use arguments in Bergemann and Morris (2013b), in particular Proposition 8, but the insights naturally extend to the interdependent payoff environment.
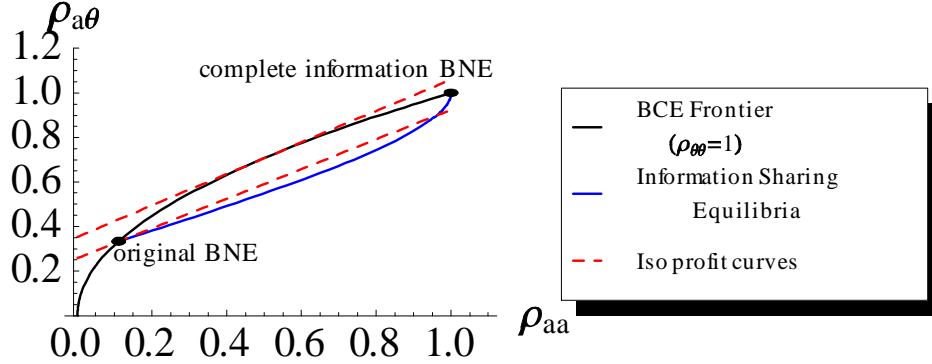
Figure 7: Information Sharing under Public and Private Disclosure Rules

Kirby (1988), Vives (1990) and Raith (1996).[8] By contrast, the optimal disclosure policy of a *private signal* depends on r and can be noisy. The iso-profit curve generates a linear trade-off in the correlation of the individual supply decision a_i with the state of demand θ, and the supply of the other firms. A better match with the level of aggregate demand increases profit, but a better match with the supply of the other firms decreases the profit. With *public* disclosure of a noisy signal s_i, the convexity in the trade-off suggests *either* zero disclosure *or* complete disclosure of the aggregate information. With *private* disclosure, the trade-off is resolved in favor of a better match with the state of demand, without an undue increase in the correlation of the supply decisions. Thus, we find that the industry wide preferred disclosure policy is frequently a partial disclosure of information, but one which is noisy and idiosyncratic, as opposed to a bang-bang like solution that was previously obtained in the literature under the restriction to public disclosure policies. Thus we find that a common and hence perfectly correlated disclosure policy is (always) weakly and (sometimes) strictly dominated by a private and hence imperfectly correlated disclosure policy. The analysis in Section 7 suggests that the above results for the pure common value environment extend to the interdependent value environment and general information policies. We leave a more comprehensive analysis for future research.

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[8]In Section 8.4 of Vives (1999), the design of the optimal information sharing policy in a large market with a continuum of agents is posed as the problem of a mediator who elicits and then transmits the collected information to the agents. The analysis is thus close to the present perspective of the Bayes correlated equilibrium, but also restricts the transmission policy to public signals, and hence leads to the same conclusion as the above mentioned literature.

9 Appendix

The appendix collects the omitted proofs from the main body of the text.

Proof of Proposition 1. Since the actions of players must be measurable with respect to s_i, in any linear strategy the actions of players must be given by $a_i = w(\lambda)s_i + \nu$, where ν and $w(\lambda)$ are constants. Thus $A = w(\lambda)((1 - \lambda)\bar{\theta}) + \nu$. Thus, we must have that:

$$a_i = w(\lambda)s_i + \nu = \mathbb{E}[r(w(\lambda)((1 - \lambda)\bar{\theta}) + \nu) + \theta_i|s_i].$$

By taking expectations and using the law of iterated expectations, we get:

$$w(\lambda)(1 - \lambda)\mu_\theta + \nu = rw(\lambda)((1 - \lambda)\mu_\theta + \nu) + \mu_\theta.$$

Using that $\mu_\theta = 0$, we get that $\nu = 0$. Thus, we know that $a_i = w(\lambda)((1 - \lambda)\bar{\theta} + \lambda\Delta\theta_i)$ and $A = w(\lambda)(1 - \lambda)\bar{\theta}$. Multiplying by a_i we get: $a_i^2 = \mathbb{E}[rAa_i + \theta_i a_i|s_i]$, and appealing to the law of iterated expectations we get:

$$w(\lambda)((1 - r)(1 - \lambda)^2\rho_{\theta\theta} + \lambda^2(1 - \rho_{\theta\theta})) = ((1 - \lambda)\rho_{\theta\theta} + \lambda(1 - \rho_{\theta\theta})),$$

and solving for $w(\lambda)$ yields the expression in (9). The uniqueness of the Bayes Nash equilibrium is established in Ui and Yoshizawa (2012). ∎

Proof of Proposition 2. By using the law of iterated expectations we obtain $\mu_a = \mu_\theta/(1 - r)$. We can compute the variance and covariances by using (8) and (9). It is easy to see that:

$$\sigma_a^2 = \text{var}(a_i) = w(\lambda)^2 \text{var}(s_i) = w(\lambda)^2((1 - \lambda)^2\rho_{\theta\theta} + \lambda^2(1 - \rho_{\theta\theta}))\sigma_\theta^2,$$

thus we get (10). Similarly, we obtain:

$$\rho_{aa}\sigma_a^2 = \text{cov}(a_i, a_j) = w(\lambda)^2\mathbb{E}[((1 - \lambda)\bar{\theta} + \lambda\Delta\theta_i)((1 - \lambda)\bar{\theta} + \lambda\Delta\theta_j)] = w(\lambda)^2(1 - \lambda)^2\rho_{\theta\theta}\sigma_\theta^2,$$

and

$$\rho_{a\theta}\sigma_a\sigma_\theta = \text{cov}(a_i, \theta_i) = w(\lambda)\mathbb{E}[((1 - \lambda)\bar{\theta} + \lambda\Delta\theta_i)\theta_i] = w(\lambda)((1 - |\lambda|)\rho_{\theta\theta} + \lambda(1 - \rho_{\theta\theta}))\sigma_\theta^2,$$

which establishes the result. ∎

Proof of Proposition 3. By solving for λ in (11) and restrict the root $\lambda \in [0, 1]$, we get:

$$\lambda = \frac{(\rho_{aa} - 1)\rho_{\theta\theta} + \sqrt{(\rho_{aa} - 1)\rho_{aa}(\rho_{\theta\theta} - 1)\rho_{\theta\theta}}}{\rho_{aa} - \rho_{\theta\theta}}.$$

By inserting the above expression in (12), we obtain the result. ∎

Proof of Proposition 4. The result follows directly from solving the following maximization problem:

$$\max_{\lambda} \left(\frac{(1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})}{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})} \right)^2 (1-\lambda)^2\rho_{\theta\theta}.$$

The solution (16) follows from Proposition 7. ∎

Proof of Proposition 5. Given a noise free equilibrium parametrized by λ we have that:

$$\text{cov}(a_i, \bar{\theta}) = w(\lambda)(1-\lambda)\bar{\theta} = \frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}(1-\lambda)\bar{\theta},$$

$$\text{cov}(a_i, \Delta\theta_i) = w(\lambda)\lambda\Delta\theta_i = \frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\lambda\Delta\theta_i.$$

But, note that if $\lambda < \widehat{\lambda}$, then $\frac{\lambda}{(1-r)} < (1-\lambda)$, but then

$$
\begin{aligned}
\text{cov}(a_i, \Delta\theta_i) &= \frac{((1-\lambda)\rho_{\theta\theta}\lambda + \lambda^2(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\Delta\theta_i \\
&\geq \frac{((1-\lambda)^2(1-r)\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} = 1,
\end{aligned}
$$

with strict inequality if $\lambda > \widehat{\lambda}$. Thus, the response to the idiosyncratic component is greater than in the complete information equilibrium if $\lambda \in (\widehat{\lambda}, 1)$. For the second part we repeat the same argument. Note that if $\lambda < \widehat{\lambda}$, then $\lambda < (1-\lambda)(1-r)$, but then:

$$
\begin{aligned}
\text{cov}(a_i, \bar{\theta}) &= \frac{((1-\lambda)^2\rho_{\theta\theta} + (1-\lambda)\lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\bar{\theta} \\
&\geq \frac{1}{1-r}\frac{((1-\lambda)^2(1-r)\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} = \frac{1}{1-r},
\end{aligned}
$$

with strict inequality if $\lambda < \widehat{\lambda}$. ∎

Proof of Proposition 6. The comparative statics with respect to the argmax are shown by proving that the quantities have a unique maximum, which is interior, and then use the sign of the cross derivatives (the derivative with respect to λ and r). The ordering of the information structures that maximizes the different second moments is proved by comparing the derivatives.

(**2.**) We begin by rewriting the individual variance, and using (10) we can write it in terms of λ:

$$
\begin{aligned}
\sigma_a^2 &= (\frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))})^2((1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))\sigma_\theta^2 \\
&= \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}(1+x^2)\sigma_\theta^2,
\end{aligned}
$$

where

$$x \triangleq \frac{\sqrt{(1 - \rho_{\theta\theta})}\lambda}{\sqrt{\rho_{\theta\theta}}(1 - \lambda)}, \quad y \triangleq \frac{\sqrt{1 - \rho_{\theta\theta}}}{\sqrt{\rho_{\theta\theta}}}. \tag{46}$$

Note that x is strictly increasing in λ, and if $\lambda \in [0, 1]$ then $x \in [0, \infty]$, and thus maximizing with respect to $x \in [0, \infty]$ is equivalent to maximizing with respect to $\lambda \in [0, 1]$. Finding the derivative we get:

$$\frac{\partial \sigma_a^2}{\partial x} = -\frac{2(xy + 1)\left(x^3 + (2r - 1)yx^2 + (r + 1)x - (1 - r)y\right)}{(x^2 + 1 - r)^3}\sigma_\theta^2.$$

It is easy to see that $\frac{d\sigma_a^2}{dx}$ is positive at $x = 0$ and negative if we take a x large enough, and thus the maximum must be in $x \in (0, \infty)$. We would like to show that the polynomial:

$$\left(x^3 + (2r - 1)yx^2 + (r + 1)x - (1 - r)y\right),$$

has a unique root in $x \in (0, \infty)$. If $r < -1$, then the function is increasing in x and has a negative value at $x = 0$, thus it has a unique root. If $x > 1/2$, then the function is negative and decreasing at x. Since it is a cubic polynomial and the term next to x^3 is positive, it must have a unique positive root. For $r \in [-1, 1/2]$ we define the determinant of the cubic equation:

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2.$$

We know that if $\Delta < 0$ then the polynomial has a unique root. Replacing by the respective values of the cubic polynomial we get:

$$\Delta = 4y^4(2r - 1)^3(1 - r) + y^2((2r - 1)^2(1 + r)^2 - 18(1 - r^2)(2r - 1) - 27(1 - r)^2)) - 4(1 + r)^3,$$

using the fact that for $r \in [-1, 1/2]$ we have that $(2r - 1) \leq 0$ and $1 + r \geq 0$, we know that the term with y^4 and without y are negative. We just need to check the term with y^2, but this is also negative for $r \in [-1, 1/2]$. Thus, $\Delta < 0$, and thus for $r \in [-1, 1/2]$ the polynomial has a unique root.

Thus, we have that there exists a unique λ that maximizes σ_a^2. Finally, we have that:

$$\frac{\partial \sigma_a^2}{\partial r} = 2\frac{(1 - \lambda)^2 \rho_{\theta\theta}}{((1 - r)(1 - \lambda)^2 \rho_{\theta\theta} + \lambda^2(1 - \rho_{\theta\theta}))}\sigma_a^2.$$

Note that

$$\frac{\partial}{\partial \lambda}\frac{(1 - \lambda)^2 \rho_{\theta\theta}}{((1 - r)(1 - \lambda)^2 \rho_{\theta\theta} + \lambda^2(1 - \rho_{\theta\theta}))} < 0,$$

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and thus at the maximum:

$$\frac{\partial^2 \sigma_a^2}{\partial r \partial \lambda} = 2\sigma_a^2 \frac{\partial}{\partial \lambda} \frac{(1-\lambda)^2 \rho_{\theta\theta}}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2 (1-\rho_{\theta\theta}))} < 0,$$

and thus $\text{argmax}_\lambda \sigma_a^2$ is decreasing in r.

Next, we consider the aggregate variance $\rho_{aa}\sigma_a^2$, and write it in terms of λ:

$$\rho_{aa}\sigma_a^2 = \left(\frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\right)^2 (1-\lambda)^2 \rho_{\theta\theta}\sigma_\theta^2 = \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}\sigma_\theta^2, \qquad (47)$$

where x and y are defined as in (46). Maximizing with respect to $x \in [0, \infty]$ is equivalent to maximizing with respect to $\lambda \in [0, 1]$. Finding the derivative we get:

$$\frac{\partial \rho_{aa}\sigma_a^2}{\partial x} = -\frac{2(xy+1)\left(2x + (x^2 + r - 1)y\right)}{(x^2 + 1 - r)^3}\sigma_\theta^2. \qquad (48)$$

Again, we have that $\left(2x + (x^2 + r - 1)y\right)$ has a unique root in $(0, \infty)$ Thus, we have that there exists a unique λ that maximizes $\rho_{aa}\sigma_a^2$. Finally, we have that:

$$\frac{\partial \rho_{aa}\sigma_a^2}{\partial r} = 2\frac{(1-\lambda)^2 \rho_{\theta\theta}}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\rho_{aa}\sigma_a^2.$$

Note that,

$$\frac{\partial}{\partial \lambda}\frac{(1-\lambda)^2 \rho_{\theta\theta}}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} < 0,$$

and thus at the maximum $\frac{\partial^2 \sigma_a^2}{\partial r \partial \lambda} < 0$, and thus $\text{argmax}_\lambda \rho_{aa}\sigma_a^2$ is decreasing in r.

Finally, we consider the dispersion, $(1 - \rho_{aa})\sigma_a^2$, expressed in terms of λ:

$$\begin{aligned}(1 - \rho_{aa})\sigma_a^2 &= \left(\frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\right)^2 \lambda^2 (1-\rho_{\theta\theta})\sigma_\theta^2 \\ &= \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}x^2\sigma_\theta^2,\end{aligned}$$

where x and y are defined in (46). As before, maximizing with respect to $x \in [0, \infty]$ is equivalent to maximizing with respect to $\lambda \in [0, 1]$. Finding the derivative we get:

$$\frac{\partial (1 - \rho_{aa})\sigma_a^2}{\partial x} = -\frac{2x(xy+1)\left(x^2 + 2(r-1)yx + r - 1\right)}{(x^2 + 1 - r)^3}\sigma_\theta^2.$$

Again, we have that $\left(x^2 + 2(r-1)yx + r - 1\right)$ has a unique root in $(0, \infty)$ Thus, there exists a unique λ that maximizes $(1 - \rho_{aa})\sigma_a^2$. Finally, we have that:

$$\frac{\partial (1 - \rho_{aa})\sigma_a^2}{\partial r} = 2\frac{(1-\lambda)^2 \rho_{\theta\theta}}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}(1 - \rho_{aa})\sigma_a^2.$$

Note that

$$\frac{\partial}{\partial\lambda}\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta}+\lambda^2(1-\rho_{\theta\theta}))} < 0,$$

and thus at the maximum $\frac{\partial^2\sigma_a^2}{\partial r\partial\lambda} < 0$, and thus $\mathrm{argmax}_\lambda\,(1-\rho_{aa})\sigma_a^2$ is decreasing in r.

(1.) Finally, we want to show that $\mathrm{argmax}_\lambda\,(1-\rho_{aa})\sigma_a^2 > \mathrm{argmax}_\lambda\,\sigma_a^2 > \mathrm{argmax}_\lambda\,\rho_{aa}\sigma_a^2$. These inequalities follows from comparing the derivatives of $(1-\rho_{aa})\sigma_a^2$, σ_a^2 and $\rho_{aa}\sigma_a^2$ with respect to λ (or equivalently x). It is easy to see that:

$$\frac{\partial\log(1-\rho_{aa})\sigma_a^2}{\partial x} < \frac{\partial\log\sigma_a^2}{\partial x} < \frac{\partial\log\rho_{aa}\sigma_a^2}{\partial x}.$$

Since the derivatives satisfy the previous inequalities, and the quantities have a unique maximum, the argument of the maximum must also satisfy the same inequalities.

(3.) The comparative static results with respect to the maximum follow directly from the envelope theorem. ∎

Proof of Proposition 7. We first solve for $\max_\lambda\{\rho_{aa}\sigma_a^2\}$. By setting (48) equal to 0, we have that the aggregate volatility is maximized at,

$$x = \frac{\sqrt{1+y^2(1-r)}-1}{y}.$$

In terms of the original variables this can be written as follows:

$$\lambda = \frac{\rho_{\theta\theta}\left(\sqrt{\frac{1-r}{\rho_{\theta\theta}}+r}+r-2\right)}{(r-4)\rho_{\theta\theta}+1}.$$

Substituting the solution in (47) and using the definitions of x and y we get that the maximum volatility is equal to:

$$\frac{\sigma_\theta^2(1-\rho_{\theta\theta})^2}{4(\sqrt{\rho_{\theta\theta}}-\sqrt{\rho_{\theta\theta}+(1-r)(1-\rho_{\theta\theta})})^2}.$$

Using the definition of $\sigma_{\bar\theta}$ and σ_θ^2 we get (24). Note that by imposing $r=0$ we also get (16) and (17). ∎

Proof of Corollary 3. It follows directly from (24) that:

$$\lim_{\sigma_\theta^2\to0}\max_\lambda\{\rho_{aa}\sigma_a^2\} = \frac{\sigma_\theta^2}{4(1-r)}.$$

Thus, we are only left with proving that,

$$\lim_{\sigma_{\Delta\theta_i}^2\to0}\max_\lambda\{\rho_{aa}\sigma_a^2\} = \sigma_{\bar\theta}^2/(1-r)^2.$$

The limit can be easily calculated using L'Hopital's rule. That is, just note that as $\sigma^2_{\Delta\theta_i} \to 0$ we have that:

$$4(\sigma_{\bar\theta} - \sqrt{\sigma^2_{\bar\theta} + (1-r)\sigma^2_{\Delta\theta_i}})^2 \approx \sigma^4_{\Delta\theta_i}(1-r)^2/\sigma^2_{\bar\theta} + o(\sigma^6_{\Delta\theta_i}),$$

and hence we get the result. ∎

Proof of Lemma 1. We need to prove that given the assumption of symmetry, the parameters $(\mu_a, \rho_{aa}, \rho_{a\theta}, \rho_{a\phi}, \sigma_a)$ are sufficient to characterize the distribution of the random variables $(\theta_i, \bar\theta, a_i, A)$. Clearly, we have that $\mu_a = \mu_A$, as it follows from the law of iterated expectations. By the previous definition (and decomposition) of the idiosyncratic state θ_i, we observe that the expectations of the following products all agree: $\mathbb{E}_i[a_i\bar\theta] = \mathbb{E}_i[A\theta\theta_i] = \mathbb{E}_i[A\bar\theta]$. This can be easily seen as follows:

$$\mathbb{E}[\theta_i A] = \mathbb{E}[\bar\theta A] + \mathbb{E}[\Delta\theta_i A] = \mathbb{E}[\bar\theta A] + \mathbb{E}[A \cdot \underbrace{\mathbb{E}[\Delta\theta_i|A]}_{=0}] = \mathbb{E}[\bar\theta A],$$

where we just use the law of iterated expectations and the fact that the expected value of a idiosyncratic variable conditioned on an aggregate variable must be 0. Thus:

$$\mathrm{cov}(a_i, \bar\theta) = \mathrm{cov}(A, \theta_i) = \mathrm{cov}(A, \bar\theta) = \mathrm{cov}(a_i, \theta_j) = \mathbb{E}[a_i\theta_j] - \mu_a\mu_\theta = \rho_{a\phi}\sigma_\theta\sigma_a.$$

Similarly, since we consider a symmetric Bayes correlated equilibrium, the covariance of the actions of any two individuals, a_i and a_j, which is denoted by $\rho_{aa}\sigma^2_a$, is equal to the aggregate variance. Once again, this can be easily seen as follows,

$$\mathbb{E}[a_ia_j] = \mathbb{E}[A^2] + \mathbb{E}[A\Delta a_j] + \mathbb{E}[\Delta a_i A] + \mathbb{E}[\Delta a_i \Delta a_j] = \mathbb{E}[A^2],$$

where in this case we need to use that the equilibrium is symmetric and thus $\mathbb{E}[\Delta a_i \Delta a_j] = 0$. Thus, we have $\sigma^2_A = \mathrm{cov}(a_i, a_j) = \mathrm{cov}(A, a_i) = \rho_{aa}\sigma^2_a$. ∎

Proof of Proposition 8. The moment equalities (1) and (2) were established in (29) and (30). Thus we proceed to verify that the inequality constraints (3) are necessary and sufficient to guarantee that the matrix $\mathbb{V}$ is positive semi-definite.

Here we express the equilibrium conditions, by a change of variables, in terms of different variables, which facilitates the calculation. Let:

$$M \triangleq \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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Thus, we have that:

$$\begin{pmatrix} \Delta\theta_i \\ \bar{\theta} \\ \Delta a_i \\ A \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \mu_\theta \\ 0 \\ \mu_a \end{pmatrix}, M\mathbb{V}M' \right),$$

where

$$\mathbb{V}_\perp \triangleq M\mathbb{V}M' = \begin{pmatrix} (1-\rho_{\theta\theta})\sigma_\theta^2 & 0 & (\rho_{a\theta}-\rho_{a\phi})\sigma_a\sigma_\theta & 0 \\ 0 & \rho_{\theta\theta}\sigma_\theta^2 & 0 & \rho_{a\phi}\sigma_a\sigma_\theta \\ (\rho_{a\theta}-\rho_{a\phi})\sigma_a\sigma_\theta & 0 & (1-\rho_{aa})\sigma_a^2 & 0 \\ 0 & \rho_{a\phi}\sigma_a\sigma_\theta & 0 & \rho_{aa}\sigma_a^2 \end{pmatrix}. \tag{49}$$

We use $\mathbb{V}_\perp$ to denote the variance/covariance matrix expressed in terms of $(\Delta\theta_i, \bar{\theta}, \Delta a_i, A)$. It is easy to verify that $\mathbb{V}_\perp$ is positive semi-definite if and only if the inequality conditions (3) are satisfied. To check this it is sufficient to note that the leading principal minors are positive if and only if these conditions are satisfied, and thus $\mathbb{V}_\perp$ is positive semi-definite if and only if these conditions are satisfied. ∎

Proof of Proposition 9. ($\Leftarrow$) We first prove that if the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a BNE for some information structure $\mathcal{I}_i$ (and associated signals), then the variables $(\theta_i, \bar{\theta}, a_i, A)$ also form a BCE. Consider the case in which agents receive normally distributed signals through the information structure $\mathcal{I}_i$, which by minor abuse of notation also serves as conditioning event. Then in any BNE of the game, we have that the actions of the agents are given by:

$$a_i = r\mathbb{E}[A|\mathcal{I}_i] + \mathbb{E}[\theta_i|\mathcal{I}_i], \quad \forall i, \forall \mathcal{I}_i, \tag{50}$$

and since the information is normally distributed, the variables $(\theta_i, \bar{\theta}, a_i, A)$ are jointly normal as well. By taking the expectation of (50) conditional on the information set $\mathcal{I}_i' = \{\mathcal{I}_i, a_i\}$ we get:

$$\begin{aligned} \mathbb{E}[a_i|\mathcal{I}_i, a_i] = a_i &= r\mathbb{E}[\mathbb{E}[A|\mathcal{I}_i]|\mathcal{I}_i, a_i] + \mathbb{E}[\mathbb{E}[\theta_i|\mathcal{I}_i]|\mathcal{I}_i, a_i] \\ &= r\mathbb{E}[A|\mathcal{I}_i, a_i] + \mathbb{E}[\theta_i|\mathcal{I}_i, a_i]. \end{aligned} \tag{51}$$

In other words, agents know the recommended action they are supposed to take, and thus, we can assume that the agents condition on their own actions. By taking expectations of (51) conditional on $\{a_i\}$ we get:

$$\begin{aligned} \mathbb{E}[a_i|a_i] = a_i &= r\mathbb{E}[\mathbb{E}[A|\mathcal{I}_i, a_i]|a_i] + \mathbb{E}[\mathbb{E}[\theta_i|\mathcal{I}_i, a_i]|a_i] \\ &= r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i], \end{aligned} \tag{52}$$

where we used the law of iterated expectations. In other words, the information contained in $\{a_i\}$ is a sufficient statistic for agents to compute their best response, and thus the agents compute the same best response if they know $\{\mathcal{I}_i, a_i\}$ or if they just know $\{a_i\}$. Yet, looking at (52), by definition $(\theta_i, \bar{\theta}, a_i, A)$ form a BCE.

($\Rightarrow$) We now prove that if $(\theta_i, \bar{\theta}, a_i, A)$ form a BCE, then there exists an information structure $\mathcal{I}_i$ such that the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a BNE when agents receive this information structure. We consider the case in which the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a BCE, and thus the variables are jointly normal and

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \tag{53}$$

Since the variables are jointly normal we can always find $w \in \mathbb{R}$ and $\lambda \in [-1, 1]$, such that:

$$a_i = w(\lambda \Delta\theta_i + (1 - |\lambda|)\bar{\theta} + \varepsilon_i).$$

The variables (λ, w) and the random variable ε are defined by the following equations of the BCE equilibrium distribution:

$$w\lambda = \frac{\text{cov}(a_i, \Delta\theta_i)}{\sigma^2_{\Delta\theta_i}}, \qquad w(1 - |\lambda|) = \frac{\text{cov}(a_i, \bar{\theta})}{\sigma^2_{\bar{\theta}}},$$

and

$$\varepsilon = \frac{a_i - \frac{\text{cov}(a_i, \Delta\theta_i)\Delta\theta_i}{\sigma^2_{\Delta\theta_i}} - \frac{\text{cov}(a_i, \bar{\theta})}{\sigma^2_{\bar{\theta}}})}{w}.$$

Now consider the case in which agents receive a one-dimensional signal

$$s_i \triangleq \frac{a_i}{w} = (\lambda \Delta\theta_i + (1 - \lambda)\bar{\theta} + \varepsilon_i). \tag{54}$$

Then, by definition, we have that:

$$a_i = ws_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i] = r\mathbb{E}[A|s_i] + \mathbb{E}[\theta_i|s_i],$$

where we use the fact that conditioning on a_i is equivalent to conditioning on s_i. Thus, when agent i receives information structure (and associated signal s_i): $\mathcal{I}_i = \{s_i\}$, then agent i taking action $a_i = ws_i$ constitutes a Bayes Nash equilibrium, as it complies with the best response condition. Thus, the distribution $(\theta_i, \bar{\theta}, a_i, A)$ forms a BNE when agents receive signals $\mathcal{I}_i = \{s_i\}$. ∎

Proof of Proposition 10. Note that (54) has the form stated in the Proposition, and thus this was implicitly established by the proof of Proposition 9. ∎

Proof of Corollary 4. The first part follows directly from (36), which we rewrite for the ease of the reader:

$$
\text{var} \left[\begin{array}{c|c} \Delta a_i & \Delta \theta_i \\ A & \bar{\theta} \end{array} \right] = \sigma_a^2 \left(\begin{array}{cc} (1 - \rho_{aa}) - \frac{(\rho_{a\theta} - \rho_{a\phi})^2}{1 - \rho_{\theta\theta}} & 0 \\ 0 & \rho_{aa} - \frac{\rho_{a\phi}^2}{\rho_{\theta\theta}} \end{array} \right).
$$

It is clear that both inequalities in (33) are satisfied with equality, if and only if,

$$
\text{var} \left[\begin{array}{c|c} \Delta a_i & \Delta \theta_i \\ A & \bar{\theta} \end{array} \right] = \binom{0}{0}.
$$

Thus, Δa_i and A can be written as linear functions of $\Delta \theta_i$ and $\bar{\theta}$ respectively. This implies that $a_i = A + \Delta a_i$ is a linear function of $\Delta \theta_i$ and $\bar{\theta}$, and hence noise free. Moreover, it is easy to see that these inequalities will be satisfied with equality if and only if the inequalities of restriction (33) are satisfied with equality.

The second part comes from imposing that inequalities (33) are satisfied with equality and solving for $\rho_{a\theta}$ and $\rho_{a\phi}$ in terms of ρ_{aa}. As we have the additional constraints that $\rho_{a\theta}, \rho_{aa} \geq 0$, we keep the roots the satisfy these conditions. ∎

To establish Proposition 11, we use the following lemma that is of independent interest. Consider an arbitrary continuous function:

$$
\psi : [0, 1] \times [0, 1] \times [-1, 1] \to \mathbb{R},
$$

whose domain is given by the triple of correlation coefficients: $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$.

Lemma 2

If $\psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ is a continuous function, strictly increasing in $\rho_{a\theta}$ and weakly increasing in $\rho_{a\phi}$, then the BCE that maximizes ψ is an noise free BCE.

Proof. By rewriting the constraints (33) of Proposition 8 we obtain:

1. $\rho_{\theta\theta} \rho_{aa} - (\rho_{a\phi})^2 \geq 0$;

2. $(1 - \rho_{aa})(1 - \rho_{\theta\theta}) - (\rho_{a\theta} - \rho_{a\phi})^2 \geq 0$.

If $\psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ is strictly increasing, then in the optimum the above inequality (2) must bind. Moreover, if the constraint (1) does not bind, then we can just increase $\rho_{a\theta}$ and $\rho_{a\phi}$ in equal amounts, without violating (2) and increasing the value of ψ. Thus, in the maximum of ψ we must have that

both bind. Moreover, since $\psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ is strictly increasing in $\rho_{a\theta}$ and weakly increasing in $\rho_{a\phi}$, it is clear that the maximum will be achieved positive root of (14). ∎

Proof of Proposition 11. From (32) the individual volatility, aggregate volatility and dispersion can be written as follows:

$$\frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}}, \ \rho_{aa}\frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}}, \ (1 - \rho_{aa})\frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}},$$

and the result follows directly.∎

Proof of Proposition 12. The action of agent i is given by $a_i = \nu s_i^1 = \nu(\theta_i + \varepsilon_i)$, for some $\nu \in \mathbb{R}$. Thus, we can calculate $\rho_{a\theta}$ and ρ_{aa} in terms of σ_ε. We find that:

$$\rho_{a\theta} = \frac{\text{cov}(a_i, \theta_i)}{\sigma_a \sigma_\theta} = \frac{\nu\sigma_\theta^2}{\sigma_\theta \nu \sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}} = \frac{\sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}},$$

$$\rho_{aa} = \frac{\text{cov}(a_i, a_j)}{\sigma_a^2} = \frac{\nu^2\sigma_\theta^2}{\nu^2(\sigma_\theta^2 + \sigma_\varepsilon^2)} = \frac{\rho_{\theta\theta}\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2},$$

$$\rho_{a\phi} = \frac{\text{cov}(a_i, \bar{\theta})}{\sigma_a \sigma_\theta} = \frac{\nu\sigma_{\bar{\theta}}^2}{\sigma_\theta \nu \sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}} = \frac{\rho_{\theta\theta}\sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}.$$

From the above equalities it follows directly that (40) is satisfied. ∎

Proof of Proposition 13. In equilibrium, the best response of players is given by:

$$a_i = \frac{\bar{\theta}}{1 - r} + \mathbb{E}[\Delta\theta_i | \Delta s_i^1] = \frac{\bar{\theta}}{1 - r} + \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_{\varepsilon^1}^2}(s_i^1 - \bar{\theta}) = \frac{\bar{\theta}}{1 - r} + b(s_i^1 - \bar{\theta}),$$

where

$$\Delta s_i^1 \triangleq s_i^1 - \bar{\theta} = \Delta\theta_i + \varepsilon_i^1 \text{ and } b \triangleq \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_{\varepsilon^1}^2}.$$

We can now calculate the correlations $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ in terms of σ_ε. We get:

$$\rho_{a\theta} = \frac{\text{cov}(a_i, \theta_i)}{\sigma_a \sigma_\theta} = \frac{\frac{\sigma_{\bar{\theta}}^2}{1-r} + b\sigma_{\Delta\theta_i}^2}{\sigma_\theta \sqrt{\sigma_{\bar{\theta}}^2/(1-r)^2 + b\sigma_{\Delta\theta_i}^2}};$$

$$\rho_{aa} = \frac{\text{cov}(a_i, a_j)}{\sigma_a^2} = \frac{\frac{\sigma_{\bar{\theta}}^2}{(1-r)^2}}{\sigma_{\bar{\theta}}^2/(1-r)^2 + b\sigma_{\Delta\theta_i}^2};$$

$$\rho_{a\phi} = \frac{\mathrm{cov}(a_i, \bar{\theta})}{\sigma_a \sigma_\theta} = \frac{\frac{\sigma_\theta^2}{1-r}}{\sigma_\theta \sqrt{\sigma_\theta^2/(1-r)^2 + b\sigma_{\Delta\theta_i}^2}}.$$

Note that by definition $b \in [0, 1]$, and thus we must have that $\rho_{aa} \in [\widehat{\rho}_{aa}, 1]$. By solving for $\rho_{a\theta}$ and $\rho_{a\phi}$ we obtain (41). $\blacksquare$

Proof of Proposition 14. From the best response conditions, we have that, $a_i = \theta_i + r\mathbb{E}[A|\mathcal{I}_i]$, and multiplying by θ_i and taking expectations (note that because θ_i is in $\mathcal{I}_i$, we have that $\theta_i \mathbb{E}[A|\mathcal{I}_i] = \mathbb{E}[\theta_i A|\mathcal{I}_i]$), we find that $\rho_{a\theta}\sigma_a = \sigma_\theta + r\rho_{a\phi}\sigma_a$. We use the fact that $\sigma_a = \rho_{a\theta}\sigma_\theta + r\rho_{aa}\sigma_a$, and hence inserting into the former we obtain:

$$\rho_{a\phi} = \frac{1}{r}(\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}). \tag{55}$$

Thus, the inequalities in (33) can be written as follows:

$$(1 - \rho_{aa})(1 - \rho_{\theta\theta}) \geq \frac{1}{r^2}((1-r)\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}})^2, \tag{56}$$

$$\rho_{aa}\rho_{\theta\theta} \geq \frac{1}{r^2}(\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}})^2. \tag{57}$$

For both of the previous inequalities the right hand side is a convex function of $\rho_{a\theta}$. Thus, for a fixed ρ_{aa}, inequalities (56) and (57) independently constrain the set of feasible $\rho_{a\theta}$ to be in a convex interval with non-empty interior for all $\rho_{aa} \in [0, 1]$. Thus, if we impose both inequalities jointly, we find the intersection of both intervals which is also a convex interval. We first find the set of feasible ρ_{aa} and prove that it is always case that one of the inequalities provides the lower bound and the other inequality provides the upper bound on the set of feasible $\rho_{a\theta}$ for each feasible ρ_{aa}. For this we make several observations.

First, when agents know their own payoff state there are only two noise free equilibria, one in which agents know only their state and the complete information equilibria. This implies that there are only two pairs of values for $(\rho_{aa}, \rho_{a\theta})$ such that inequalities (56) and (57) both hold with equality. Second, the previous point implies that there are only two pairs of values for $(\rho_{aa}, \rho_{a\theta})$ such that the bound of the intervals imposed by inequalities (56) and (57) are the same. Third, the previous two points imply that there are only two possible ρ_{aa} such that the set of feasible $\rho_{a\theta}$ is a singleton. These values are $\rho_{aa} \in \{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}$, which corresponds to the ρ_{aa} of the complete information equilibria and the equilibria in which agents only know their state. Fourth, it is clear that there are no feasible BCE with $\rho_{aa} \in \{0, 1\}$. Fifth, the upper and lower bound on the feasible

$\rho_{a\theta}$ that are imposed by inequalities (56) and (57) move smoothly with ρ_{aa}. Sixth, this implies that the set of feasible ρ_{aa} is bounded by the values of ρ_{aa} in which the set of feasible $\rho_{a\theta}$ is a singleton. Thus, the set of feasible ρ_{aa} is in $[\min\{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}, \max\{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}]$. Moreover, it is easy to see that for all ρ_{aa} in the interior of this interval one of the inequalities provides the upper bound for $\rho_{a\theta}$ while the other inequality will provide the lower bound.

We now provide the explicit functional forms for the upper and lower bounds. To check which of the inequalities provides the upper and lower bound respectively we can just look at the equilibria in which agents know only their own state, and thus $\rho_{aa} = \rho_{\theta\theta}$. In this case we have the following inequalities for $\rho_{a\theta}$:

$$(1 - \rho_{\theta\theta})^2 \geq \frac{1}{r^2}((1-r)\rho_{a\theta} - \frac{1 - r\rho_{\theta\theta}}{\rho_{a\theta}})^2, \quad \rho_{\theta\theta}^2 \geq \frac{1}{r^2}(\rho_{a\theta} - \frac{1 - r\rho_{\theta\theta}}{\rho_{a\theta}})^2.$$

It is easy to check that $\rho_{a\theta} = 1$ satisfies both inequalities. Moreover, it is easy to see that if $r > 0$ then $\rho_{a\theta} = 1$ provides a lower bound on the set of $\rho_{a\theta}$ that satisfies the first inequality while $\rho_{a\theta} = 1$ provides an upper bound on the set of $\rho_{a\theta}$ that satisfies the second inequality. If $r < 0$ we get the opposite, $\rho_{a\theta} = 1$ provides a upper bound on the set of $\rho_{a\theta}$ that satisfies the first inequality while $\rho_{a\theta} = 1$ provides a lower bound on the set of $\rho_{a\theta}$ that satisfies the second inequality. Thus, if $r > 0$, then the inequality (56) provides the lower bound for $\rho_{a\theta}$ and the inequality (57) provides an upper bound on the set of feasible $\rho_{a\theta}$ for all ρ_{aa}. If $r < 0$ we get the opposite result. Note that the conclusions about the bounds when $\rho_{aa} = \rho_{\theta\theta}$ can be extended for all feasible ρ_{aa} because we know that the bounds of the intervals are different for all $\rho_{aa} \notin \{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}$, and they move continuously, thus the relative order is preserved. Finally, we define implicitly the functions $\rho_{a\theta}^i$ and $\rho_{a\theta}^c$: For a given ρ_{aa}, $\rho_{a\theta}^c$ and $\rho_{a\theta}^i$ represent the solutions of the following equations:

$$\frac{1}{r}(-(1-r)(\rho_{a\theta}^c)^2 + 1 - r\rho_{aa}) - \rho_{a\theta}^c \sqrt{(1 - \rho_{aa})(1 - \rho_{\theta\theta})} = 0, \tag{58}$$

and

$$\frac{1}{r}((\rho_{a\theta}^i)^2 - (1 - r\rho_{aa})) - \rho_{a\theta}^i \sqrt{\rho_{aa}\rho_{\theta\theta}} = 0. \tag{59}$$

Therefore, for all feasible BCE in which agents know their own state $\rho_{aa} \in [\min\{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}, \max\{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}]$, while the set of $\rho_{a\theta}$ is bounded by the functions $\rho_{a\theta}^i$ and $\rho_{a\theta}^c$. If $r > 0$ then function $\rho_{a\theta}^i$ provides the upper bound while $\rho_{a\theta}^c$ provides the lower bound. If $r < 0$, then the function $\rho_{a\theta}^c$ provides the upper bound while $\rho_{a\theta}^i$ provides the lower bound. ∎

To establish Proposition 15, we first provide an auxiliary result, namely Lemma 3, that describes the set of correlations that can be achieved by the class of information structures (39). We define

the following information to noise ratio:

$$b \triangleq \frac{\sigma^2_{\Delta\theta_i}}{\sigma^2_{\Delta\theta_i} + \sigma^2_{\varepsilon^1}} \in [0, 1]. \tag{60}$$

Lemma 3 (Feasible Outcomes with $\mathbf{s}_i$)

A set of correlations $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ can be achieved by a linear equilibrium in which agents receive a information structure of the form $\mathbf{s}_i$, if and only if, there exists $b \in [0, 1]$ such that:

1. *The following equalities are satisfied:*

$$\sigma_a = \frac{\rho_{a\theta}\sigma_a}{1 - \rho_{aa}r}. \tag{61}$$

$$(1 - r)\rho_{a\phi}\frac{\rho_{a\theta}}{1 - r\rho_{aa}} + \frac{1}{b}(\rho_{a\theta} - \rho_{a\phi})\frac{\rho_{a\theta}}{1 - r\rho_{aa}} = \sigma_\theta. \tag{62}$$

2. *The following inequalities are satisfied:*

$$\rho_{aa}\rho_{\theta\theta} \geq \rho^2_{a\phi}, \quad (1 - \rho_{aa})(1 - \rho_{\theta\theta}) - \frac{1}{b}(\rho_{a\theta} - \rho_{a\phi})^2 \geq 0. \tag{63}$$

Proof. **(If)** Consider a linear Bayes Nash equilibrium in which agents get signals $\{s_i^1, s_i^2, s^3\}$. First, we note that (61) and the first inequality in (63) must be satisfied trivially, as this must be satisfied for any Bayes correlated equilibrium. We now prove that the second inequality in (63) must be satisfied.

In any linear Bayes Nash equilibrium the action of players can be written as follows, $a_i = \alpha_1 s_i^1 + \alpha_2 s_i^1 + \alpha_3 s_i^1$. Therefore, for any constants $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, we know that $\Delta a_i = \alpha_1(\Delta\theta_i + \varepsilon_i^1) + \alpha_2 \varepsilon_i^2$, and thus: $\text{var}(\Delta a_i | \Delta\theta_i) \geq \alpha_1^2 \sigma^2_{\varepsilon^1}$. Yet, note that we can write α_1 as follows: $\alpha_1 = \text{cov}(\Delta\theta_i, \Delta a_i)/\sigma^2_{\Delta\theta_i}$, and $\text{var}(\Delta a_i | \Delta\theta_i)$ is given by:

$$\text{var}(\Delta a_i | \Delta\theta_i) = \sigma^2_{\Delta a} - \left(\frac{\text{cov}(\Delta\theta_i, \Delta a_i)}{\sigma^2_{\Delta\theta_i}}\right)^2 \sigma^2_{\Delta\theta_i} = \sigma^2_a\left((1 - \rho_{aa}) - \frac{(\rho_{a\theta} - \rho_{a\phi})^2}{(1 - \rho_{\theta\theta})}\right).$$

Thus, we have that,

$$\sigma^2_a\left((1 - \rho_{aa}) - \frac{(\rho_{a\theta} - \rho_{a\phi})^2}{(1 - \rho_{\theta\theta})}\right) \geq \left(\frac{\text{cov}(\Delta\theta_i, \Delta a_i)}{\sigma^2_{\Delta\theta_i}}\right)^2 \sigma^2_{\varepsilon^1} = \sigma^2_a\frac{(\rho_{a\theta} - \rho_{a\phi})^2}{(1 - \rho_{\theta\theta})}\frac{\sigma^2_{\varepsilon^1}}{\sigma^2_{\Delta\theta_i}}.$$

Finally, we note that by definition: $\sigma^2_{\varepsilon^1} = \sigma^2_{\Delta\theta_i}(1/b - 1)$. Thus, we get,

$$\left((1 - \rho_{aa}) - \frac{1}{b}\frac{(\rho_{a\theta} - \rho_{a\phi})^2}{(1 - \rho_{\theta\theta})}\right) \geq 0,$$

which is the second inequality in (63).

Finally, we prove that condition (62) must be satisfied. For this, note that in any Bayes Nash equilibrium, we must have that $a_i = \mathbb{E}[\theta_i + rA | s_i^1, s_i^2, s^3]$, and multiplying the equation by s_i^1, we get:

$$s_i^1 a_i = \mathbb{E}[\theta_i \cdot s_i^1 + r \cdot A \cdot s_i^1 | s_i^1, s_i^2, s^3].$$

Taking expectations and using the law of iterated expectations, we get: $\text{cov}(a_i, s_i^1) = \text{cov}(\theta_i, s_i^1) + r \cdot \text{cov}(A, s_i^1)$, and thus:

$$\text{cov}(\theta_i, s_i^1) = \text{cov}(\theta_i, \theta_i) = \sigma_\theta^2, \ \text{cov}(A, s_i^1) = \text{cov}(A, \theta_i) = \rho_{a\phi}\sigma_a\sigma_\theta,$$

and:

$$
\begin{aligned}
\text{cov}(a_i, s_i^1) &= \text{cov}(a_i, \varepsilon_i^1) + \text{cov}(\theta_i, a_i) = \rho_{a\theta}\sigma_a\sigma_\theta + \alpha_1\sigma_{\varepsilon^1}^2 = \rho_{a\theta}\sigma_a\sigma_\theta + \frac{(\rho_{a\theta} - \rho_{a\phi})\sigma_a\sigma_\theta}{\sigma_{\Delta\theta_i}^2}\sigma_{\varepsilon^1}^2 \\
&= \rho_{a\theta}\sigma_a\sigma_\theta + (\rho_{a\theta} - \rho_{a\phi})\sigma_a\sigma_\theta(\frac{1}{b} - 1).
\end{aligned}
$$

Thus, we get:

$$\rho_{a\theta}\sigma_a\sigma_\theta + (\rho_{a\theta} - \rho_{a\phi})\sigma_a\sigma_\theta(\frac{1}{b} - 1) = \sigma_\theta^2 + r\rho_{a\phi}\sigma_a\sigma_\theta,$$

and re-arranging terms we get (62).

(Only If) Let $(\theta_i, \bar{\theta}, a_i, A)$ be a Bayes correlated equilibrium with correlations $(\rho_{a\theta}, \rho_{a\phi}, \rho_{aa})$ satisfying (61)-(63). We will show that there exists a information structure $\mathbf{s}_i$, under which the Bayes Nash equilibrium induces the random variables $(\theta_i, \bar{\theta}, a_i, A)$.

Let ε_i^1 be a random variable that is uncorrelated with $\Delta\theta_i$ and $\bar{\theta}$ (and thus it is a noise term) with variance $\sigma_{\varepsilon^1}^2 = (1 - b)\sigma_{\Delta\theta_i}^2/b$ and such that:

$$\text{cov}(\varepsilon_i^1, a_i) = \frac{\text{cov}(\Delta\theta_i, a_i)}{\sigma_{\Delta\theta_i}^2}\sigma_{\varepsilon^1}^2.$$

Define signals s_i^1 and $\tilde{s}_i$ as follows, $s_i^1 \triangleq \theta_i + \varepsilon_i^1$,

$$\tilde{s}_i \triangleq a_i - \frac{\text{cov}(a_i, \Delta\theta_i)}{\sigma_{\Delta\theta_i}^2}s_1.$$

Thus, by definition s_i^1 and $\tilde{s}_i$ are informationally equivalent to s_i^1 and a_i. Note that by definition $\text{cov}(\tilde{s}_i, \Delta\theta_i) = 0$. Thus, we can define:

$$\tilde{\varepsilon}_i \triangleq \tilde{s}_i - \frac{\text{cov}(\tilde{s}_i, \bar{\theta})}{\sigma_\theta^2}\bar{\theta},$$

60

and write signal $\tilde{s}_i$ as follows: $\tilde{s}_i = \bar{\theta} + \tilde{\varepsilon}_i$, where $\tilde{\varepsilon}_i$ is a noise term (thus independent of $\bar{\theta}$ and $\Delta\theta_i$) with a correlation of $\rho_{\tilde{\varepsilon}\tilde{\varepsilon}}$ across agents. Note that $a_i = \mathbb{E}[\theta_i + rA|s_i^1, a_i]$ holds if and only if:

$$\mu_a = \mu_\theta + r\mu_A, \ \ \sigma_a = \rho_{a\theta}\sigma_\theta + r\rho_{aa}\sigma_a, \ \ \mathrm{cov}(a_i, s_i^1) = \mathrm{cov}(\theta_i, s_i^1) + r\,\mathrm{cov}(A, s_i^1). \quad (64)$$

To show this, just note that we can define a random variable z as follows: $z_i \triangleq \mathbb{E}[\theta_i + rA|s_i^1, a_i]$, and impose $a_i = z_i$. By definition:

$$\mathbb{E}[a_i] = \mathbb{E}[z_i]; \ \ \mathrm{var}(z_i) = \mathrm{cov}(a_i, z_i); \ \ \mathrm{cov}(a_i, z_i) = \mathrm{var}(z_i); \ \ \mathrm{var}(a_i, s_i^1) = \mathrm{cov}(z_i, s_i^1). \quad (65)$$

These corresponds to conditions (64) (obviously, since $a_i = z_i$, $\mathrm{cov}(a_i, z_i) = \mathrm{var}(z_i)$ or $\mathrm{cov}(a_i, z_i) = \mathrm{var}(a_i)$ are redundant). Moreover, any random variable z' that satisfies (65) must also satisfy $a = z'$. This just comes from the fact that these are normal random variables, thus the joint distribution of (z_i', a_i, s_i^1) is completely defined by its first and second moments. Thus, any random variable z_i' that has the same second moments as z_i must satisfy that $a_i - z_i = 0$.

Note that, conditions (64) hold since $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes correlated equilibrium, while $\mathrm{cov}(a_i, s_i^1) = \mathrm{cov}(\theta_i, s_i^1) + \mathrm{cov}(A, s_i^1)$ holds by the assumption (62) (where the calculation is the same as before). Thus, we have that, $(\theta_i, \bar{\theta}, a_i, A)$ is induced by the linear Bayes Nash equilibrium when players receive information structure $\{s_i^1, \tilde{s}_i\}$.

On the other hand, note that the signal $\Delta s_i^1 = \Delta\theta_i + \varepsilon_i^1$ is a sufficient statistic for $\Delta\theta_i$ given that agents only receive signals $(s_i^1, \tilde{s}_i)$. Thus, we have that,

$$\mathbb{E}[\Delta\theta_i|s_i^1, \tilde{s}_i] = \mathbb{E}[\mathbb{E}[\Delta\theta_i|\Delta s_i^1]|s_i^1, \tilde{s}_i] = \mathbb{E}[b\Delta s_i^1|s_i^1, \tilde{s}_i].$$

From the Bayes correlated equilibrium condition, we know that:

$$a_i = \mathbb{E}[\theta_i + rA|s_i^1, a_i] = \mathbb{E}[b\Delta s_i + \bar{\theta} + rA|s_i^1, \tilde{s}_i].$$

Subtracting bs_i^1 from both sides, we get:

$$\tilde{a}_i \triangleq (a_i - bs_i^1) = \mathbb{E}[(1 - b)\bar{\theta} + rA|s_i^1, a_i] = \mathbb{E}[(1 - (1 - r)b)\bar{\theta} + r\tilde{A}|s_i^1, \tilde{s}_i]. \quad (66)$$

Thus, $(\theta_i, \bar{\theta}, \tilde{a}_i, \tilde{A})$ is induced by a linear Bayes Nash equilibrium with common values, in which players get signals $(s_i^1, \tilde{s}_i)$.

We now show that there exists signals (s_i^2, s^3), such that: $s_i^2 = \alpha\bar{\theta} + \varepsilon_i^2$, $s_i^3 = (1 - \alpha)\bar{\theta} + \bar{\varepsilon}$, and

$$\tilde{a}_i = \mathbb{E}[(1 - (1 - r)b)\bar{\theta} + r\tilde{A}|s_i^1, s_i^2, s^3],$$

with $\alpha \in [0,1]$. If we show that such signals (s_i^2, s^3) exists, this directly implies that we must also have that $a_i = \mathbb{E}[\theta_i + rA|s_i^1, s_i^2, s^3]$. Thus, $(\theta_i, \bar{\theta}, a_i, A)$ is induced by the linear Bayes Nash equilibrium when players receive information structure $\{s_i^1, s_i^2, s^3\}$. For this we define ε_i^2 and $\bar{\varepsilon}$ in terms of $\tilde{\varepsilon}_i$ as follows:

$$\bar{\varepsilon} = \mathbb{E}_i[\tilde{\varepsilon}_i], \ \varepsilon_i^2 = \Delta\tilde{\varepsilon}_i = (\tilde{\varepsilon}_i - \mathbb{E}_i[\tilde{\varepsilon}_i]).$$

Note that by (66) we know that:

$$\mathrm{cov}(\tilde{a}_i, \tilde{s}_i) = (1 - (1-r)b)\,\mathrm{cov}(\bar{\theta}, \tilde{s}_i) + r\,\mathrm{cov}(\tilde{A}, \tilde{s}_i).$$

Yet, we can re-write the previous equality using the definition of $\tilde{s}_i$:

$$\mathrm{cov}(\tilde{a}_i, \bar{\theta} + \varepsilon_i^2 + \bar{\varepsilon}) = (1 - (1-r)b)\,\mathrm{cov}(\bar{\theta}, \bar{\theta} + \varepsilon_i^2 + \bar{\varepsilon}) + r\,\mathrm{cov}(\tilde{A}, \bar{\theta} + \varepsilon_i^2 + \bar{\varepsilon}).$$

Yet, we can find α such that the following conditions hold:

$$\mathrm{cov}(\tilde{a}_i, \alpha\bar{\theta} + \varepsilon_i^2) = (1 - (1-r)b)\,\mathrm{cov}(\bar{\theta}, \alpha\bar{\theta} + \varepsilon_i^2) + r\,\mathrm{cov}(\tilde{A}, \alpha\bar{\theta} + \varepsilon_i^2)\ ;$$

$$\mathrm{cov}(\tilde{a}_i, (1-\alpha)\bar{\theta} + \bar{\varepsilon}) = (1 - (1-r)b)\,\mathrm{cov}(\bar{\theta}, (1-\alpha)\bar{\theta} + \bar{\varepsilon}) + r\,\mathrm{cov}(\tilde{A}, (1-\alpha)\bar{\theta} + \bar{\varepsilon}).$$

Thus, the following conditions must hold:

$$\mathrm{cov}(\tilde{a}_i, s_i^2) = (1 - (1-r)b)\,\mathrm{cov}(\bar{\theta}, s_i^2) + r\,\mathrm{cov}(\tilde{A}, s_i^2)\ ; \tag{67}$$

$$\mathrm{cov}(\tilde{a}_i, s_i^3) = (1 - (1-r)b)\,\mathrm{cov}(\bar{\theta}, s_i^3) + r\,\mathrm{cov}(\tilde{A}, s_i^3). \tag{68}$$

Since (66) holds, we must have that:

$$\mathrm{cov}(\tilde{a}_i, s_i^1) = (1 - (1-r)b)\,\mathrm{cov}(\bar{\theta}, s_i^1) + r\,\mathrm{cov}(\tilde{A}, s_i^1), \tag{69}$$

thus, using the same argument as before, we know that (67)-(69) imply that:

$$\tilde{a}_i = \mathbb{E}[(1 - (1-r)b)\bar{\theta} + r\tilde{A}|s_i^1, s_i^2, s^3].$$

Thus, we get the result. ∎

Proof of Proposition 15. We first prove that in any Bayes correlated equilibrium that achieves $(\bar{\rho}(\rho_{aa}), \rho_{aa})$ subject to (61)-(63) must satisfy the following two conditions. If $b < 1$ then both inequalities in (63) must be satisfied with equality. If $b = 1$ then at least one of the inequalities in (63) must be satisfied with equality.

From condition (62), we know that:

$$\rho_{a\phi} = \frac{b}{1 - b(1 - r)} (\frac{\rho_{a\theta}}{b} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}).$$

Replacing $\rho_{a\phi}$ in both inequalities in (63), we get the following two inequalities:

$$\rho_{aa}\rho_{\theta\theta} \geq \left(\frac{b}{1 - b(1 - r)}\right)^2 (\frac{\rho_{a\theta}}{b} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}})^2. \tag{70}$$

$$(1 - \rho_{aa})(1 - \rho_{\theta\theta}) \geq \frac{b^2}{(1 - b(1 - r))^2}((1 - r)\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}})^2. \tag{71}$$

Note that we need to maximize $\rho_{a\theta}$ given the previous two inequalities. Thus, it is clear that at least one of these inequalities must be binding. Thus, inequality (70) or (71) must be binding (it is easy to check that there exists values of $\rho_{a\theta}$ such that either inequality is strict). Thus, at least one of the restrictions must always be satisfied with equality. We now show that, if $b \in (0, 1)$, then both inequalities must be satisfied with equality. To show this just note that the derivative of the right hand side of the inequalities with respect to b is different than 0. Thus, if just one of the constraints is binding and the other one is not, then one can change b and relax both constraints, which allow to get a higher $\rho_{a\theta}$. Note that the argument also holds for $b = 0$, as in this case the second inequality will be satisfied with slack, while the right hand side of the first inequality will be decreasing with respect to b. Thus, we must have that at the maximum $\rho_{a\theta}$ both constraints are binding or $b = 1$.

By the previous argument, $\overline{\rho}_{a\theta}$ is achieved by $b = 1$ or both (70) and (71) are satisfied with equality. If $b = 1$ then agents know their own payoffs. By Proposition 3, if both (70) and (71) are satisfied with equality, then the information structure that achieves this correlations must satisfy that $\sigma_{\varepsilon 2}, \sigma_{\varepsilon 3} \in \{0, \infty\}$. Thus, we can calculate $\overline{\rho}_{a\theta}$ using Propositions 12-14. ∎

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