(IR)RATIONAL EXUBERANCE: OPTIMISM, AMBIGUITY AND RISK

By

Anat Bracha and Donald J. Brown

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YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

(Ir)Rational Exuberance: Optimism, Ambiguity and Risk

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Abstract

The equilibrium prices in asset markets, as stated by Keynes (1930): "...will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced." We propose a descriptive theory of finance explicating Keynes' claim that the prices of assets today equilibrate the optimism and pessimism of bulls and bears regarding the payoffs of assets tomorrow.

This equilibration of optimistic and pessimistic beliefs of investors is a consequence of investors maximizing Keynesian utilities subject to budget constraints defined by market prices and investor's income. The set of Keynesian utilities is a new class of non-expected utility functions representing the preferences of investors for optimism or pessimism, defined as the composition of the investor's preferences for risk and her preferences for ambiguity. Bulls and bears are defined respectively as optimistic and pessimistic investors. (Ir)rational exuberance is an intrinsic property of asset markets where bulls and bears are endowed with Keynesian utilities.

JEL Classification: D81, G02, G11

Keywords: Keynes, Bulls and bears, Expectations, Asset markets

1 Introduction

"Irrational exuberance," following Alan Greenspan, describes "unduly escalated asset values." Robert Shiller (2000; second edition 2005), in his now classic book, explains irrational exuberance in the stock- and the real estate-markets by invoking 12 factors taken from sociology, psychology and economics. Despite the contrast between the rich account Shiller proposes for irrational exuberance and the minimalism of our models of financial markets and investor's psychology, we propose a rational model of (ir)rational exuberance in asset markets. That is, the behavior of bulls and bears is rational in the standard economic sense of agents maximizing (non-expected) utility subject to a budget constraint, defined by market prices and the agent's income. This

*Research Department, The Federal Reserve Bank of Boston, 600 Atlantic Avenue, Boston, MA 02210, USA

†Department of Economics, Yale University, Box 208268, New Haven, CT 06520-8268, USA

new class of non-expected utilities, labeled Keynesian utilities, incorporates investor's preferences for optimism. Keynesian utility is an empirically tractable and descriptive characterization of an investor's preferences in financial markets, where she is either a bull or a bear. Simply put, bulls are optimists who believe that market prices will go up, while bears are pessimists who believe that market prices will go down. The equilibrium prices in asset markets, as stated by Keynes (1930): "...will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced." The market prices today therefore equilibrate the odds expected by bulls and bears of the payoffs tomorrow. Keynesian utilities are defined as the composition of the investor's preferences for risk and her preferences for ambiguity, where preferences for risk and preferences for ambiguity are assumed to be independent. If $U(x)$ denotes preferences for risk, and $J(y)$ denotes preferences for ambiguity then

$$U : X \subseteq R^N_{++} \to Y \subseteq R^N_{++}$$

and

$$J : Y \subseteq R^N_{++} \to R$$

where

$$x \to J \circ U(x)$$

is the composition of U and J, denoted $J \circ U(x)$.

In our model, bulls are investors endowed with convex Keynesian utilities and bears are investors endowed with concave Keynesian utilities. It follows from convex analysis that these specifications are equivalent to investors being bulls if and only if they have optimistic beliefs about the future payoffs of state-contingent claims and investors are bears if and only if they have pessimistic beliefs about the future payoffs of state-contingent claims.

Table 1 below summarizes the types of Keynesian utilities, where the cells are investors' preferences for optimism and pessimism. An investor who is both risk averse and ambiguity averse is a bear, i.e., a pessimist. Similarly, an investor who is both risk seeking and ambiguity seeking is a bull, i.e., an optimist. These cases, the diagonal cells of the table, are the symmetric Keynesian utilities and the off-diagonal cells of the table are the asymmetric Keynesian utilities.

Table 1: Keynesian Preferences

Keynesian preferences	Risk-averse	Risk-seeking
Ambiguity-averse	Bears	Asymmetric
Ambiguity-seeking	Asymmetric	Bulls

Economists probably believe bears have Keynesian utilities that are the composition of ambiguity-averse preferences and risk-averse preferences or that bulls have Keynesian utilities that are the composition of ambiguity-seeking preferences and risk-seeking preferences, where we assume that both U and J are monotone. This observation follows from the theorems in convex analysis on the convexity or concavity of the composition of monotone convex or concave functions, see section 3.2 in Boyd and Vandenberghe (2004). It may be surprising to economists that for asymmetric

Keynesian utilities, given α, the proxy for risk, and β, the proxy for ambiguity, that there exists a state-contingent claim $\hat{x}$, "the reference point," where for quadratic utilities of ambiguity and risk, $J \circ U(x)$ is concave or pessimistic on

$$[\hat{x}, +\infty] \equiv \{x \in R_+^N : x \geq \hat{x}\}$$

and $J \circ U(x)$ is convex or optimistic on

$$(0, \hat{x}] \equiv \{x \in R_+^N : x \leq \hat{x}\}.$$

That is, an investor with quadratic utilities of ambiguity and quadratic utilities of risk is a bull for "losses," and a bear for "gains," reminiscent of the shape of risk preferences in prospect theory (see Kahneman 2011).

In the next section we briefly review the theory of affective decision making (ADM) which incorporates optimism bias as proposed by Bracha and Brown (2012). In Sections 3 and 4 we consider separable and quadratic specifications of utilities for risk and ambiguity to illustrate Keynesian utilities where optimists are not always risk and ambiguity seeking and pessimists are not always risk and ambiguity averse. In the final section of the paper, we explicate Keynes' claim that the prices of assets today equilibrate the optimism and pessimism of bulls and bears regarding the payoffs of assets tomorrow. We consider market equilibrium in asset markets where investors are endowed with Keynesian utilities. We prove the existence of a competitive equilibrium, in a exchange economy with two states of the world, where market prices reflect the payoffs of future asset realizations as perceived by bears who diversify and bulls who speculate, i.e., invest only in assets with optimistic payoffs. Nevertheless, with a continuum of bulls their aggregate demand is typically diversified. In a parametric, exchange economy with two states of the world and two types of bulls we derive the comparative statics of the unique competitive equilibrium.

Finally, a few words about the notions of risk, uncertainty, ambiguity, and optimism as they are used in this paper. For Bernoulli (1738) risk means we know the probabilities of tomorrow's state of the world. For Keynes (1937) uncertainty means we do not know the probabilities, in fact the notion of probability of states of the world tomorrow may be meaningless. Ellsberg (1961) introduced the notion of ambiguity as the alternative notion to risk, where we are ignorant of the probability of states of the world tomorrow. For Ellsberg there are two kinds of uncertainty: risk and ambiguity. These are the conventions we follow.

Optimism (pessimism) as proposed in Bracha and Brown refers to the investor's perceived probability, which is skewed towards the more favorable (unfavorable) outcome and is defined with respect to state-utility vectors. The formulation of Bracha and Brown is an elaboration of the notion of optimism bias introduced by Bracha (2005) in her analysis of insurance markets. The contribution of ADM is the presentation of a new class of preferences that provides a formal definition of optimism bias, where the representation of ADM utilities as biconjugates of convex functions is similar to the formal definition of ambiguity aversity in the representation of variational preferences, introduced by Maccheroni, Marinacci, Rustichini (2006), as biconjugates of concave functions. This similarity suggests that in Bracha and Brown optimism bias and ambiguity seeking have the same formal expression.

In this paper, we define optimism (pessimism) with respect to future outcomes of state-contingent claims, not state-utility vectors, where optimism (pessimism) refers to the investor's perceived beliefs today regarding the relative likelihood of large versus small payoffs of a state-contingent claim tomorrow. Optimism (pessimism) is now a function of both the investor's attitudes towards risk and her attitudes towards ambiguity.

2 Optimism Bias and Preferences for Ambiguity

Ambiguity-neutral or subjective expected utility functions, originally proposed by Savage (1954), as the foundation of Bayesian statistics, are a class of preferences where the probability of an event is independent of the prize offered contingent on that event. It therefore "... does not leave room for optimism or pessimism to play any role in the person's judgment" (Savage 1954; p. 68). This claim is inconsistent with the view of Keynes who, as we previously noted, thought of the market price as a balance of the sales of bears, the pessimists, and the purchases of bulls, the optimists. And it is also not the view of Ellsberg (1961), whose explanation of his famous Ellsberg paradox is that subjective probabilities depends on payoffs and is therefore consistent with the notions of optimism or pessimism bias. Influenced by Keynes and Ellsberg, Bracha and Brown (2012) proposed formal definitions of each bias, where

$$J(U(x)) \equiv \max_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)]$$

is the Legendre–Fenchel biconjugate of the optimistic utility function $J(U(x))$ and

$$J(U(x)) \equiv \min_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)]$$

is the Legendre-Fenchel biconjugate of the pessimistic utility function $J(U(x))$. If $F(y)$ is a vector-valued map from R^N into R^N, then F is strictly, monotone increasing if for all x and $y \in R^N$:

$$[x - y] \cdot [F(x) - F(y)] > 0.$$

If $F(y)$ is a vector-valued map from R^N into R^N,then F is strictly, monotone decreasing if for all x and $y \in R^N$:

$$[x - y] \cdot [F(x) - F(y)] < 0.$$

Bracha and Brown observed that $J(z)$ is strictly convex in z where $z = U(x)$ iff $\nabla_z J(z)$ is a strictly, monotone increasing map of z and $J(z)$ is strictly concave in z where $z = U(x)$ iff $\nabla_z J(z)$ is a strictly, monotone decreasing map of z. See section 5.4.3 in Ortega and Rheinboldt (1970) for proof. If $\nabla_z J(z)$ is a strictly monotone increasing map of z, then the investor beliefs are skewed towards the higher utility outcomes of a given action, i.e., she is optimistic and if $\nabla_z J(z)$ is a strictly, monotone decreasing map of z, then the investor beliefs are skewed towards the lower utility outcomes of a given action, i.e., she is pessimistic. It follows from the envelope theorem,

$$\nabla_z J(z) = \arg \max_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)] = \hat{\pi}, \text{ where}$$

$$J(U(x)) = \max_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)] = \sum \widehat{\pi} \cdot U(x) + J^*(\hat{\pi})]$$

and

$$\nabla_z J(z) = \arg \min_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)] = \hat{\pi}, \text{ where}$$

$$J(U(x)) = \min_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)] = \sum \hat{\pi} \cdot U(x) + J^*(\hat{\pi})]$$

If $z = U(x)$, then it follows from ADM that the mathematical equivalence between the convexity of $J(z)$ with respect to z and the increasing monotonicity of the gradient map $\nabla_z J(z)$ implies the behavioral equivalence between ambiguity seeking and optimism bias. It also follows from variational preferences that the mathematical equivalence between the concavity of $J(z)$ with respect to z and the decreasing monotonicity of the gradient map $\nabla_z J(z)$ implies the behavioral equivalence between ambiguity aversity and pessimism bias

In our 2012 paper, optimism (pessimism) is defined with respect to state-utility vectors, $U(x)$, which are unobservable. For empirical applications of optimism bias and pessimism bias in asset markets, we require definitions of optimism or pessimism for investors choosing state-contingent claims x. As we demonstrated in our previous paper, the notions of optimism or pessimism depend on the structure of the investor's utility function. If the Keynesian composite utility function is convex (concave) in x then the investor is optimistic (pessimistic). It follows from the envelope theorem applied to the Legendre–Fenchel biconjugate representation of convex (concave) functions, that the gradient of the investor's utility function with respect to x is a strictly monotone increasing (decreasing) map. The value of the gradient at x is the investor's perceived probability distribution for x. This definition of optimism or pessimism with respect to state-contingent claims is no longer a function of ambiguity attitudes alone; rather, it depends on both the investor's risk and ambiguity attitudes. This is an immediate consequence of the chain rule in computing the gradient of the composite Keynesian utility function. In the next two sections we illustrate this joint dependence for additively separable and quadratic Keynesian utilities. In this section, we define optimistic and pessimistic Keynesian utilities $J \circ U(x)$ over state-contingent claims x.For bulls,

$$J \circ U(x) \equiv \max_{\pi \in R^N_{++}} [\sum \pi \cdot x + J^*(\pi)]$$

the Legendre–Fenchel conjugate of $J \circ U(x)$, where $J^*(\pi)$ is a smooth convex function on R^N_{++} and

$$J^*(\pi) \equiv \max_{x \in R^N_{++}} [\sum \pi \cdot x + J \circ U(x)].$$

For bears,

$$J \circ U(x) \equiv \min_{\pi \in R^N_{++}} [\sum \pi \cdot x + J^*(\pi)]$$

the Legendre–Fenchel conjugate of $J \circ U(x)$, where $J^*(\pi)$ is a smooth concave function on R^N_{++}, and

$$J^*(\pi) \equiv \min_{x \in R^N_{++}} [\sum \pi \cdot x + J \circ U(x)].$$

As in Bracha and Brown, it follows from the envelope theorem,

$$\nabla_x J \circ U(x) = \arg\max_{\pi \in R^N_{++}} [\sum \pi \cdot x + J^*(\pi)] = \hat{\pi}, \text{ where}$$

$$J \circ U(x) = \max_{\pi \in R^N_{++}} [\sum \pi \cdot x + J^*(\pi)] = \sum \hat{\pi} \cdot x + J^*(\hat{\pi})]$$

and

$$\nabla_x J \circ U(x) = \arg\min_{\pi \in R^N_{++}} [\sum \pi \cdot x + J^*(\pi)] = \hat{\pi}, \text{ where}$$

$$J \circ U(x) = \min_{\pi \in R^N_{++}} [\sum \pi \cdot x + J^*(\pi)] = \sum \widehat{\pi} \cdot x + J^*(\hat{\pi})].$$

The odds expected today by bulls or bears of the market prices tomorrow, for the state-contingent claim $\hat{x}$ is derived from the odds defined by $\nabla_x J \circ U(\hat{x})$. That is,

$$\frac{\nabla_x J \circ U(x)}{\|\nabla_x J \circ U(x)\|_1} = \frac{\hat{\pi}}{\|\hat{\pi}\|_1} \in \Delta^0,$$

the interior of the probability simplex . Hence $\nabla_x J \circ U(x)$ and $\frac{\nabla_x J \circ U(x)}{\|\nabla_x J \circ U(x)\|_1}$define the same perceived betting odds that a given payoff of x will be realized. If $J(U(x))$ is ambiguity-seeking and $U(y)$and $U(z)$ differ in only state t of the world, where $u(v_t) > u(z_t)$, the optimistic investor "perceives" that

$$\frac{\Pr(u(v_t)}{1 - \Pr(u(v_t))} > \frac{\Pr(u(z_t))}{1 - \Pr(u(z_t))}$$

and $J(U(v)) > J(U(z))$ consistent with Ellsberg's explanation of ambiguity-seeking behavior. Hence ambiguity-seeking investors are bulls. If $J(U(x))$ is ambiguity-averse and $U(y)$and $U(z)$ differ in only state t of the world, where $u(v_t) > u(z_t)$.The pessimistic investor "perceives" that

$$\frac{\Pr(u(v_t))}{1 - \Pr(u(v_t))} < \frac{\Pr(u(z_t))}{1 - \Pr(u(z_t))}$$

and $J(U(v)) < J(U(z))$ consistent with Ellsberg's explanation of ambiguity-averse behavior. Hence ambiguity-averse investors are bears.

3 Separable Utilities for Risk and Ambiguity

In this section we examine the relationship between optimism/pessimism and risk and ambiguity attitudes in the case of additively-separable Keynesian utility. We start by taking a parametric example. Specifically, we consider utility functions on the space of state-contingent claims and utility functions on the space of state-utility vectors. That is,

$$J \circ U(x) \equiv \sum_{s=1}^{s=N} j \circ u(x_s)$$

$$J(U(x)) \equiv \sum_{s=1}^{s=N} j(u(x_s))$$

where

$$x \equiv (x_1, x_2, .., x_N),\ U(x) \equiv (u(x_1), u(x_2), ..., u(x_N)) \text{ and } j \circ u(x_s) \equiv x_s^{\alpha\beta}.$$

If

$$u(x_s) \equiv x_s^{\beta} \text{ then } j \circ u(x_s) \equiv x_s^{\alpha\beta} \text{ and } j(u(x)) \equiv [(x_s^{\beta})]^{\alpha}.$$

With this example, it is easy to see that optimistic or pessimistic investors, that is investors with convex or concave composite Keynesian utilities, can result from different combinations of risk and ambiguity attitudes. For instance, if $\beta \leqslant 2$, then $u(x_s)$ is concave in x_s. If $\alpha\beta \leqslant 2$, then $\alpha \leqslant 2/\beta$ and $j \circ u(x_s)$ is concave in x_s. Hence, the composite utility function $j \circ u(x_s)$ is pessimistic. If $\alpha \geqslant 2$, then $j(u(x_s))$ is convex in $u(x_s)$. In this case, $J \circ U(x)$ is concave in x and $J(U(x))$is convex in $U(x)$

If $\beta \leqslant 2$, then $u(x_s)$ is concave in x_s. If $\alpha\beta \geqslant 2$, then $\alpha \geqslant 2/\beta$ and $j \circ u(x_s)$ is convex in x_s. Hence, the composite utility function $j \circ u(x_s)$ is optimistic. If $\alpha \geqslant 2$, then $j(u(x_s))$ is convex in $u(x_s)$. In this case, $J \circ U(x)$ is convex in x and $J(U(x))$ is convex in $U(x)$. These examples show that since the value of $\alpha\beta$ determines if the investor is endowed with a pessimistic or optimistic Keynesian utility functions, it is possible that investors can be pessimistic or optimistic over state-contingent claims and be risk averse and ambiguity seeking over state-utility vectors.

We now present a nonparametric example of Keynesian utilities, where we again consider additively-separable utility functions of the form

$$J \circ U(x) \equiv \sum_{s=1}^{s=N} j \circ u(x_s)$$

where $u : R_+ \rightarrow R_+$ and $j : R_+ \rightarrow R_+$. Here is a family of nonparametric examples. If $J \circ U(x)$ is additively separable, then

$$\frac{\partial J(y)}{\partial y_s} = \frac{dj(y_s)}{dy_s}$$

and

$$\frac{\partial J \circ U(x)}{\partial x_s} = \frac{dj(y_s)}{dy_s}\frac{dy_s}{dx_s}$$

where $y_s = u(x_s)$; $x = (x_1, x_2, .., x_N)$ and $y = (y_1, y_2, .., y_N)$.To check if $J \circ U(x)$ is strictly concave in x, we compute the Hessian of $J \circ U(x)$, where we use the chain rule:

$$\frac{\partial^2 J \circ U(x)}{\partial x_s \partial x_r} = \frac{\partial \left(\frac{\partial J \circ U(x)}{\partial x_r} \right)}{\partial x_s} = \frac{\partial \left(\frac{dj(y_r)}{dy_r} \frac{dy_r}{dx_r} \right)}{\partial x_s} = \frac{\partial \left(\frac{dj(y_r)}{dy_r} \frac{dy_r}{dx_r} \right)}{\partial x_s} = 0 \text{ if } s \neq r.$$

The diagonal of the Hessian

$$\frac{\partial^2 J \circ U(x)}{\partial x_s^2} = \left(\frac{d^2 j(y_s)}{dy_s^2} \right) \left(\frac{dy_s}{dx_s} \right)^2 + \left(\frac{dj(y_s)}{dy_s} \right) \left(\frac{d^2 y_s}{dx_s^2} \right) \text{ for } 1 \leq s \leq N$$

where

$$\frac{d^2 j(y_s)}{dy_s^2} > 0 \text{ and } \frac{d^2 y_s}{dx_s^2} < 0 \text{ for } 1 \leq s \leq N.$$

Hence the Hessian matrix

$$\frac{\partial^2 J \circ U(x)}{\partial x_s \partial x_r}$$

is negative definite at x iff

$$\left(\frac{d^2 j(y_s)}{dy_s^2} \right) \left(\frac{dy_s}{dx_s} \right)^2 + \left(\frac{dj(y_s)}{dy_s} \right) \left(\frac{d^2 y_s}{dx_s^2} \right) < 0 \ for \ 1 \leq s \leq N.$$

In fact, every smooth, additively-separable concave (convex) utility function over state-contingent claims can be represented as a Keynesian utility function. That is, we now show: If $F(x_1, x_2, ..., x_N) \equiv \sum_{j=1}^{j=N} f(x_j)$ is a smooth, monotone, concave (convex) utility function on R^N_{++} then $F(x_1, x_2, ..., x_N)$ be represented as the composition of the investor's concave (convex) preferences for risk and her concave (convex) preferences for ambiguity. This representation is unique modulo the investor's preferences for risk.

Theorem 1 *If $f(w)$ is a smooth, monotone concave univariate function of w and $y = g(w)$ is a smooth, monotone concave univariate function of w. Then there exist a smooth, monotone concave univariate function of y, $h(y)$, such that for all w:*

$$f(w) = h \circ g(w).$$

Proof. Applying the chain rule

$$\frac{df}{dw} \equiv \frac{dh}{dy} \frac{dy}{dw}.$$

That is,

$$\frac{dh}{dy} \equiv \left[\frac{df}{dw} \right] \left[\frac{dy}{dw} \right]^{-1}.$$

Hence

$$h(y) - h(y_0) \equiv \int_{y_0}^{y} \frac{dh}{dy} dy \equiv \int_{w_0}^{w} \left[\frac{df}{dw} \right] \left[\frac{dy}{dw} \right]^{-1} dw$$

where

$$y = g(w) \text{ and } y_0 = g(w_0)$$

$h(y)$ is smooth and monotone

$$\text{If } \frac{d^2f}{dw^2} \equiv \frac{d^2h}{dy^2}\left[\frac{dy}{dw}\right]^2 + \frac{dh}{dy}\frac{d^2y}{dw^2} \text{ then} \frac{d^2h}{dy^2} \equiv \left[\frac{d^2f}{dw^2} - \frac{dh}{dy}\frac{d^2y}{dw^2}\right] / \left[\frac{dy}{dw}\right]^2.$$

Hence

$$\frac{d^2h}{dy^2} < 0 \Leftrightarrow \frac{d^2f}{dw^2} < \frac{dh}{dy}\frac{d^2y}{dw^2} = \left[\frac{df}{dw}\right]\left[\frac{dy}{dw}\right]^{-1}\frac{d^2y}{dw^2} \Leftrightarrow \frac{d^2f}{dw^2}/\frac{df}{dw} < \frac{d^2y}{dw^2}/\frac{dy}{dw} \Leftrightarrow$$

$$\frac{d}{dw}\ln\left[\frac{df}{dw}/\frac{dy}{dw}\right] < 0 \Leftrightarrow \ln\left[\frac{df}{dw}/\frac{dy}{dw}\right] < K_0 \Leftrightarrow \frac{df}{dw} < [\exp K_0]\frac{dy}{dw} \text{ where } K_0 = \ln\left[\frac{df}{dw}/\frac{dy}{dw}\right]_{w_0}$$

That is, $h(y)$ is concave on $(y_0, +\infty)$. A similar argument suffices for convex f and g. ■

4 Quadratic Utilities for Risk and Ambiguity

In this section we analyze the case of quadratic utilities and give the conditions under which the investor is optimistic or pessimistic. Here we show that for some class of asymmetric Keynesian utilities, that is when the investor is risk averse yet ambiguity seeking or vice versa, there are regions where the investor is optimistic and others where the investor is pessimistic. The intuition for the analysis in this section derives from the following example of the composition of two quadratic univariate polynomials, where we compute higher order derivatives of the composite function using chain rules for higher derivatives attributed to Faa di Bruno — see Johnson (2000) for the history. Here we follow the exposition in Spindler (2005). If

$$U(x) = \beta_0 + \beta_1 x + \tfrac{\beta}{2}x^2 \Rightarrow \nabla_x U(x) = \beta_1 + \beta x \Rightarrow \nabla_x^2 U(x) = \beta \Rightarrow \nabla_x^3 U(x) = 0$$

$$J(y) = \alpha_0 + \alpha_1 y + \tfrac{\alpha}{2}y^2 \Rightarrow \nabla_y J(y) = \alpha_1 + \alpha y \Rightarrow \nabla_y^2 J(y) = \alpha \Rightarrow \nabla_y^3 J(y) = 0$$

then we denote the composition as $J \circ U(x)$. We use the chain rule to compute the higher order derivatives of $J \circ U(x)$

$$\nabla_x J \circ U(x) = [\nabla_{U(x)} J \circ U(x)][\nabla_x U(x)]$$

$$\nabla_x^2 J \circ U(x) = [\nabla_{U(x)}^2 J \circ U(x)][\nabla_x U(x)]^2 + [\nabla_{U(x)} J \circ U(x)][\nabla_x^2 U(x)]$$

$$\nabla_x^2 J \circ U(x) = \alpha[\nabla_x U(x)]^2 + [\nabla_{U(x)} J \circ U(x)]\beta$$

$$\nabla_x^3 J \circ U(x) = 2\alpha[\nabla_x^2 U(x)][\nabla_x U(x)] + [\nabla_{U(x)}^2 J \circ U(x)][\nabla_x U(x)]\beta = 3\alpha\beta[\nabla_x U(x)]$$

$$\nabla_x^4 J \circ U(x) = 3\alpha\beta^2$$

$$\nabla_x^K J \circ U(x) = 0 \text{ for } K \geq 5.$$

If $[\nabla_{U(x)} J \circ U(x)] > 0$, $\nabla_x U(x) > 0$ and $\text{sign}(\alpha) = \text{sign}(\beta) > 0$, then $\nabla_x J \circ U(x) > 0$ and $\nabla_x^2 J \circ U(x) > 0$.That is, $J \circ U(x)$ is a monotone, convex function of x. If

$\nabla_{U(x)} J \circ U(x)] > 0$, $\nabla_x U(x) > 0$ and $\text{sign}(\alpha) = \text{sign}(\beta) < 0$, then $\nabla_x J \circ U(x) > 0$ and $\nabla_x^2 J \circ U(x) < 0$. That is, $J \circ U(x)$ is a monotone, concave function of x. We reduce the multivariate case to the univariate case by assuming that the symmetric Keynesian utilities of bulls and bears are specified by monotone, convex or concave quadratic multivariate polynomials where the Hessians are positive or negative definite diagonal matrices with equal eigenvalues.

We propose semiparametric specifications of preferences for risk and preferences for ambiguity, defined by scalar proxies for risk and ambiguity: β and α. Piecewise linear-quadratic functions were introduced by Rockafellar (1988). A function $f : R^N \to \bar{R}$ is called piecewise linear-quadratic if the domain of f can be represented as the union of finitely many polyhedral sets, where relative to each set $f(x)$ is of the form $\frac{1}{2}xDx + d \cdot x + \delta$, where $\delta \in R$, $d \in R^N$ and $D \in R^{N \times N}$ is a symmetric matrix. A special case is where the domain of f consists of a single set. Concave quadratic utility functions were introduced by Shannon and Zame (2002) in their analysis of indeterminacy in infinite dimension general equilibrium models. $f(x)$ is a concave quadratic function if for all y and z :

$$f(y) < f(z) + \nabla f(z) \cdot (y - z) - \tfrac{1}{2} K \left\| y - z \right\|^2, \text{ where } K > 0.$$

In addition,we assume that

$$\min_{z \le \Theta} \nabla f(z) > 0, \text{ where } \Theta \text{ is the social endowment.}$$

$J \circ U(x)$ is the composition of a smooth, concave quadratic map $U(x)$,where $U(x)$ is a diagonal $N \times N$ matrix for each $x \in R^N_{++}$ and a smooth, convex quadratic function $J(y)$. If $u : R_+ \to R_+$, then

$$U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$$

is the state-utility vector for the state-contingent claim

$$x = (x_1, x_2, ..., x_N).$$

If $z = [z_1, z_2, ..., z_N]$ and $w = [w_1, w_2, ..., w_N]$, then

$$z \cdot w \equiv [z_1 w_1, z_2 w_2, ..., z_N w_N]$$

is the Hadamard or pointwise product of z and w. If we define the gradient of state-utiliy vector $U(x)$ as the vector

$$\nabla_x U(x) \equiv [\partial u(x_1), \partial u(x_2), ..., \partial u(x_N)]$$

then by the chain rule

$$\nabla_x J \circ U(x) = [\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x))].$$

If

$$G(x) = z(x) \cdot w(x),$$

where $z(x)$ and $w(x) \in R^N_{++}$, then Bentler and Lee (1978) state and Magnus and Neudecker (1985) prove that

$$\nabla_x G(x) = \nabla_x z(x) \text{diag}(w(x)) + \nabla_x w(x) \text{diag}(z(x))$$

The following analysis in this section derives from the following representation of $\nabla^2_x J \circ U(x)$:

$$\begin{aligned} \nabla^2_x J \circ U(x) &= \nabla_x([\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x))]) \\ &= [\nabla^2_{U(x)} J(U(x))](\text{diag}[\nabla_x U(x)])^2 + [\nabla^2_x U(x)]\text{diag}[\nabla_{U(x)} J(U(x))]. \end{aligned}$$

If $U(x)$ is a concave quadratic map and $J(y)$ is a convex quadratic function, then

$$\nabla^2_x U(x) = -\text{diag}(\beta) < 0$$

and

$$\nabla^2_y J(y) = \text{diag}(\alpha) > 0/$$

If A and B are NxN symmetric matrices then $A \precsim B$ iff $A - B$ is negative semidefinite, denoted: $[A - B] \lesssim 0$, where

$$\nabla^2_x J \circ U(x) \lesssim 0 \text{ iff } \text{diag}(\alpha)\text{diag}[\nabla_x U(x)]^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))] \lesssim 0$$

See matrix inequalities in section A.5.2 in Boyd and Vandenberghe for a discussion of the partial ordering $\lesssim$ on the linear vector space of $N \times N$ symmetric matrices. For diagonal $N \times N$ matrices E and F:

$$E \lesssim F \Leftrightarrow E \leq F.$$

If Keynesian utilities are the composition of quadratic utilities for risk and quadratic utilities for ambiguity, then all higher order derivatives, i.e., greater than four, are zero. Hence this class of Keynesian utilities have representations as fourth order multivariate Taylor polynomials. These results follow from repeated application of the chain rule to derivatives of the Keynesian utilities.

Theorem 2 *If $J \circ U(x)$ is the composition of quadratic utilities for risk and quadratic utilities for ambiguity,where*

$$\text{diag}(\beta) = \text{diag}[\nabla^2_x U(x)] \textit{ and } \text{diag}(\alpha) = \text{diag}[\nabla^2_{U(x)} J(U(x)]$$

then $\nabla^K_x J \circ U(x) \equiv 0$ for $K \geq 5$.

Proof. If

$$\nabla_x J \circ U(x) = [\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x))].$$

then

$$\nabla^2_x J \circ U(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x U(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))]$$

$$\nabla_x^3 J \circ U(x) = 3\text{diag}(\beta)\text{diag}(\alpha)\text{diag}[\nabla_x U(x)]$$
$$\nabla_x^4 J \circ U(x) = 3[\text{diag}(\beta)]^2[\text{diag}(\alpha)]$$
$$\nabla_x^K J \circ U(x) = 0 \text{ for } K \geq 5.$$

■

In Theorems 2 and 3, we characterize asymmetric Keynesian utilities, where we prove the existence of a reference point $\hat{x}$ that partitions R_+^N into the standard four quadrants, with the reference point $\hat{x}$ as the origin. $J \circ U(x)$ is concave in quadrant I,where quadrant $I \equiv \{x \in R_+^N : x \geq \hat{x}\}$ and convex in quadrant III, where quadrant $III \equiv \{x \in R_+^N : x \leq \hat{x}\}$ The Hessian of $J \circ U(x)$ is indefinite in quadrants II and IV. That is, $\nabla_x^2 J \circ U(x)$ is indefinite on $R_+^N / \{(\hat{x}, +\infty] \cup (0, \hat{x}]\}$. $J \circ U(x)$ is optimistic for "losses," i.e., $x \leq \hat{x}$ and pessimistic for "gains," i.e., $x \geq \hat{x}$, analogous with the shape of the utility of risk in prospect theory — see figure 10 in Kahneman (2011). In Theorems 4 and 5, we characterize symmetric Keynesian utilities or optimistic and pessimistic investors.

Theorem 3 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$,where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, concave, diagonal quadratic map from R_{++}^N onto R_{++}^N, with the proxy for risk, $-\beta < 0$, (b) $J(y)$ is a monotone, smooth, convex quadratic function from R_+^N into R, with the proxy for ambiguity, $\alpha > 0$, (c)*

$$\nabla_x^2 J \circ \hat{U}(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]: \textit{ Chain Rule}$$

then there exists a reference point $\hat{x}$ such that the financial market data D is rationalized by the composite function $J(U(x))$ with two domains of convexity: $(\hat{x}, +\infty]$ and $(0, \hat{x}]$, where $J \circ U(x)$ is concave on $(\hat{x}, +\infty]$ and $J \circ U(x)$ is convex on $(0, \hat{x}]$.

Proof.

$$\nabla_x^2 U(x) = -\text{diag}(\beta) \text{ where } \beta > 0\text{: Risk-Averse}$$
$$\nabla_{U(x)}^2 J(U(x)) = \text{diag}(\alpha) \text{ where } \alpha > 0\text{: Ambiguity-Seeking}$$
$$\nabla_x^2 J \circ U(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x U(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))]\text{: Chain Rule}$$
$$\lim_{\|x\|_\infty \to \infty} \left\| \text{diag}[\nabla_{U(x)} J(U(x))]^{-1} \text{diag}[\nabla_x U(x)]^2 \right\|_\infty = 0$$
$$\text{diag}[\nabla_{U(x)} J(U(x))]^{-1}\text{diag}[\nabla_x U(x)]^2 \leq \text{diag}[\nabla_{U(x)} J(U(\widehat{x}))]^{-1}\text{diag}[\nabla_x U(\widehat{x})]^2 \leq \text{diag}[\tfrac{\beta}{\alpha}]\text{: Bears}$$
$$\lim_{x \to 0} \left\| \text{diag}[\nabla_{U(x)} J(U(x))]\text{diag}[\nabla_x U(x)]^{-2} \right\|_\infty = 0$$
$$\text{diag}[\nabla_{U(x)} J(U(x))]^{-1}\text{diag}[\nabla_x U(x)]^2 \leq \text{diag}[\nabla_{U(x)} J(U(\widehat{x}))]\text{diag}[\nabla_x U(\widehat{x})]^{-2} \leq \text{diag}[\tfrac{\alpha}{\beta}]\text{: Bulls}$$

■

Theorem 4 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$,where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, convex, diagonal quadratic map from R^N_{++} onto R^N_{++} with the proxy for risk, $\beta > 0$, (b) $J(y)$ is a monotone, smooth, concave quadratic function from R^N_+ into R with the proxy for risk, $-\alpha < 0$, (c)*

$$\nabla_x^2 J \circ \hat{U}(x) = -\text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]\text{: Chain Rule}$$

then there exists a reference point $\hat{x}$ such that the financial market data D is rationalized by the composite function $J(U(x))$ with two domains of convexity: $(\hat{x}, +\infty]$ and $(0, \hat{x}]$, where $J \circ U(x)$ is concave on $(\hat{x}, +\infty]$ and $J \circ U(x)$ is convex on $(0, \hat{x}]$.

Proof.

$$\nabla_x^2 U(x) = \text{diag}(\beta) \text{ where } \beta > 0\text{: Risk-Seeking}$$

$$\nabla^2_{\hat{U}(x)} J(U(x)) = -\text{diag}(\alpha) \text{ where } \alpha > 0\text{: Ambiguity-Averse}$$

$$\nabla_x^2 J \circ \hat{U}(x) = -\text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 + \text{diag}(\beta)\text{diag}\nabla_{U(x)} J(\hat{U}(x))]\text{: Chain Rule}$$

$$\lim_{\|x\|_\infty \to \infty} \left\| \text{diag}[\nabla_{U(x)} J(\hat{U}(x))] \text{diag}[\nabla_x \hat{U}(x)]^{-2} \right\|_\infty = 0$$

$$\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]\text{diag}[\nabla_x \hat{U}(x)]^{-2} \leq \text{diag}[\nabla_{U(x)} J(\hat{U}(\hat{x}))]\text{diag}[\nabla_x \hat{U}(\hat{x})]^{-2} \leq \text{diag}[\tfrac{\alpha}{\beta}]\text{: Bears}$$

$$\lim_{\|x\| \to 0} \left\| \text{diag}[\nabla_{U(x)} J(\hat{U}(x))]^{-1} \text{diag}[\nabla_x \hat{U}(x)]^{2} \right\|_\infty = 0$$

$$\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]\text{diag}[\nabla_x \hat{U}(x)]^{-2} \leq \text{diag}[\nabla_{U(x)} J(\hat{U}(\hat{x}))]\text{diag}[\nabla_x \hat{U}(\hat{x})]^{-2} \leq \text{diag}[\tfrac{\beta}{\alpha}]\text{: Bulls.}$$

■

Theorem 5 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$,where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, concave, diagonal quadratic map from R^N_{++} onto R^N_{++}, with the proxy for risk, $-\beta < 0$, (b) $J(y)$ is a monotone, smooth,concave quadratic function from R^N_{++} into R,with the proxy for ambiguity, $-\alpha < 0$, (c)*

$$\nabla_x^2 J \circ \hat{U}(x) = -\text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]\text{: Chain Rule.}$$

then $J \circ U(x)$ is concave on R^N_{++}.

Proof.

$$\nabla_x^2 U(x) = -\text{diag}(\beta) \text{ where } \beta > 0\text{: Risk-Averse}$$

$$\nabla^2_{\hat{U}(x)} J(U(x)) = -\text{diag}(\alpha) \text{ where } \alpha > 0\text{: Ambiguity-Averse}$$

$$\nabla_x^2 J \circ U(x) = -\text{diag}(\alpha)(\text{diag}[\nabla_x U(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))] < 0\text{: Bears.}$$

■

Theorem 6 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$, where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, convex, diagonal quadratic map from R^N_{++} onto R^N_{++}, with the proxy for risk, $\beta > 0$, (b) $J(y)$ is a monotone, smooth, convex quadratic function from R^N_{++} into R, with the proxy for ambiguity, $\alpha > 0$, (c)*

$$\nabla^2_x J \circ \hat{U}(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]: \textit{ Chain Rule}$$

then $J \circ U(x)$ is convex on R^N_{++}.

Proof.

$$\nabla^2_x U(x) = -\text{diag}(\beta) \text{ where } \beta > 0\text{: Risk-Averse}$$

$$\nabla^2_{\hat{U}(x)} J(U(x)) = -\text{diag}(\alpha) \text{ where } \alpha > 0\text{: Ambiguity-Averse}$$

$$\nabla^2_x J \circ U(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x U(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))] > 0\text{: Bulls.}$$

■

5 Equilibrium Prices in Asset Markets

Recall the Keynesian aphorism: "The equilibrium prices in asset markets will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced." In this final section, we explicate Keynes' claim that the prices of assets today equilibrate the optimism and pessimism of bulls and bears regarding the payoffs of assets tomorrow. We assume, that the consumption sets of investors are convex, open subsets of R^N containing the positive orthant and that investors maximize smooth, monotone concave or smooth, monotone convex Keynesian utility subject to a budget constraint. The budget constraint is defined by market prices and the investor's income.

Bears maximize a smooth, monotone, concave (pessimistic) Keynesian utility; deriving the asset demand of bears is therefore a standard application of the Karush–Kuhn–Tucker (KKT) Theorem, where in our case the Slater constraint qualification is trivially satisfied (see Boyd and Vandenberghe (2004)). For this reason, the first order conditions for a saddle-point of the Lagrangian are necessary and sufficient for optimality. For bears, the utility maximizing optimum may be in the interior of the positive orthant. where the expected odds today, by bears, of tomorrow's market prices are equal to the odds determined by today's market prices.

Bulls maximize a monotone, convex (optimistic) Keynesian utility subject to a budget constraint. This is quite a different problem: the optimum in this case is achieved at an extreme point of the budget set (Rockafellar 1970). If there are only two states of the world, where the market prices are (1,1) and the investor's income is 1, then the extreme points of the budget set are (0,0), (0,1) and (1,0). For a monotone convex utility function, the optimum is achieved at (0,1) or (1,0), the corners of the budget line. More generally, the utility maximizing optimum for bulls is always on the boundary of the positive orthant, i.e., bulls speculate on the most optimistic outcomes.

We consider a two period investment model with two states of the world, where $x = (x_1, x_2)$ is a state-contingent claim and today's state prices are (p_1, p_2).

Example 7 *If the investor's income today is I and she is endowed with convex Keynesian utilities, $U_{\text{Bulls}}(x)$, then her optimal investment problem is (P):*

$$\max\{U_{\text{Bulls}}(x) \mid -x_1 \leq 0, -x_2 \leq 0, p \cdot x - I \leq 0\}$$

where the Fritz John Lagrangian for constrained maximization

$$L(x_1, x_2, \lambda_0, \lambda_1, \lambda_2, \lambda_3) \equiv \lambda_0 U_{\text{Bulls}}(x) - \lambda_1[-x_1] - \lambda_2[-x_2] - \lambda_3[p \cdot x - I]$$

Theorem 8 *[Fritz John]: If $x^*$is a local maximizer of (P) then there exists multipliers $\lambda^* \equiv (\lambda_0^*, \lambda_1^*, \lambda_2^*, \lambda_3^*) \gneqq 0$ such that:*

$$\lambda_0^*(\partial_{x_1} U_{\text{Bulls}}(x^*), \partial_{x_2} U_{\text{Bulls}}(x^*)) = (-\lambda_1^* + \lambda_3^* p_1, -\lambda_2^* + \lambda_3^* p_2),$$

where $\lambda_0^* = 1$, by Theorems 19.12 in Simon and Blume.

Corollary 9 ***Corollary 10*** *(a) If $x^* = (0, x_2^*)$, then*

$$\lambda_0^*(\partial_{x_1} U_{\text{Bulls}}((0, x_2^*)), \partial_{x_2} U_{\text{Bulls}}((0, x_2^*)) = (-\lambda_1^* + \lambda_3^* p_1, \lambda_3^* p_2)$$

It follows that some bulls are more optimistic than the market that tomorrow's state of the world is state 2. That is,

$$\frac{\partial_{x_2} U_{\text{Bulls}}((0, x_2^*))}{\partial_{x1} U_{\text{Bulls}}((0, x_2^*))} = \frac{\lambda_3^* p_2}{-\lambda_1^* + \lambda_3^* p_1} > \frac{p_2}{p_1}$$

(b) If $x^* = (x_1^*, 0)$,then

$$\lambda_0^*(\partial_{x_1} U_{\text{Bulls}}((x_1^*, 0)), \partial_{x_2} U_{\text{Bulls}}((x_1^*, 0)) = (\lambda_3^* p_1, -\lambda_2^* + \lambda_3^* p_2)$$

It follows that the other bulls are more optimistic than the market that tomorrow's state of the world is state 1. That is,

$$\frac{\partial_{x_1} U_{\text{Bulls}}((x_1^*, 0))}{\partial_{x_2} U_{\text{Bulls}}((x_1^*, 0))} = \frac{\lambda_3^* p_1}{-\lambda_2^* + \lambda_3^* p_2} > \frac{p_1}{p_2}$$

Example 11 *If the investor's income today is I and she is endowed with concave Keynesian utilities $U_{\text{Bears}}(x)$, then her optimal investment problem is (P):*

$$\max\{U_{\text{Bears}}(x) \mid x_1 \geq 0, x_2 \geq 0, I - p \cdot x \geq 0\}$$

where the KKT Lagrangian for constrained maximization

$$L(x_1, x_2, \lambda) \equiv U_{\text{Bears}}(x) + \lambda_3[I - p \cdot x] + \lambda_1 x_1 + \lambda_2 x_2.$$

Theorem 12 *[Karush–Kuhn–Tucker] If Slater's constraint qualification is satisfied then* x^* *is a maximizer of* (P)*, where* $x^* \in R_+^N$*, iff there exists a multipliers* $\lambda^* \equiv (\lambda_3^*, \lambda_1^*, \lambda_2^*) \gneqq 0$ *such that:*

$$(\partial_{x_1} U_{\text{Bears}}(x^*), \partial_{x_2} U_{\text{Bears}}(x^*)) = (\lambda_3^* p_1 - \lambda_1^*, \lambda_3^* p_2 - \lambda_2^*).$$

Corollary 13 *(a) If* $x^* = (0, x_2^*)$*,then*

$$(\partial_{x_1} U_{\text{Bears}}((0, x_2^*)), \partial_{x_2} U_{\text{Bears}}((0, x_2^*)) = (\lambda_3^* p_1 - \lambda_1^*, \lambda_3^* p_2).$$

It follows that some bears are more pessimistic than the market that tomorrow's state of the world is state 1*. That is,*

$$\frac{\partial_{x_1} U_{\text{Bears}}((0, x_2^*))}{\partial_{x_2} U_{\text{Bears}}((0, x_2^*))} = \frac{\lambda_3^* p_1 - \lambda_1^*}{\lambda_3^* p_2} < \frac{p_1}{p_2}$$

(b)If $x^* = (x_1^*, 0)$*, then*

$$(\partial_{x_1} U_{\text{Bears}}(x^*), \partial_{x_2} U_{\text{Bears}}(x^*)) = (\lambda_3^* p_1 - \lambda_1^*, \lambda_3^* p_2).$$

It follows that other bears are more pessimistic than the market that tomorrow's state of the world is state 2*. That is,*

$$\frac{\partial_{x_2} U_{\text{Bears}}((x_1^*, 0))}{\partial_{x_1} U_{\text{Bears}}((x_1^*, 0))} = \frac{\lambda_3^* p_2 - \lambda_2^*}{\lambda_3^* p_1} < \frac{p_2}{p_1}$$

Theorem 14 *(a) At the market prices* (p_1, p_2)*, some bulls trade Arrow–Debreu state-contingent claims for state* 2 *with bears for Arrow–Debreu state-contingent claims for state* 1*. That is,*

$$\frac{\partial_{x_2} U_{\text{Bulls}}((0, x_2^*))}{\partial_{x_1} U_{\text{Bulls}}((0, x_2^*))} > \frac{p_2}{p_1} \geq \frac{\partial_{x_2} U_{\text{Bears}}((x_1^*, 0))}{\partial_{x_1} U_{\text{Bears}}((x_1^*, 0))}$$

(b) At the market prices (p_1, p_2)*, other bulls trade Arrow–Debreu state-contingent claims for state* 2 *with other bulls for Arrow–Debreu state-contingent claims for state* 1*. That is,*

$$\frac{\partial_{x_2} U_{\text{Bulls}}((0, x_2^*))}{\partial_{x_1} U_{\text{Bulls}}((0, x_2^*))} > \frac{p_2}{p_1} > \frac{\partial_{x_2} U_{\text{Bulls}}((x_1^*, 0))}{\partial_{x_1} U_{\text{Bulls}}((x_1^*, 0))}.$$

(c) At the market prices (p_1, p_2)*, some bulls trade Arrow–Debreu state-contingent claims for state* 1 *with bears for Arrow–Debreu state-contingent claims for state* 2*. That is,*

$$\frac{\partial_{x_1} U_{\text{Bulls}}((x_1^*, 0))}{\partial_{x_2} U_{\text{Bulls}}((x_1^*, 0))} > \frac{p_1}{p_2} \geq \frac{\partial_{x_1} U_{\text{Bears}}((0, x_2^*))}{\partial_{x_2} U_{\text{Bears}}((0, x_2^*))}.$$

(d) At the market prices (p_1, p_2)*, other bulls trade Arrow-Debreu state-contingent claims for state* 1 *with other bulls for Arrow–Debreu state-contingent claims for state* 2*. That is,*

$$\frac{\partial_{x_1} U_{\text{Bulls}}((x_1^*, 0))}{\partial_{x_2} U_{\text{Bulls}}((x_1^*, 0))} > \frac{p_1}{p_2} > \frac{\partial_{x_1} U_{\text{Bulls}}((0, x_2^*))}{\partial_{x_2} U_{\text{Bulls}}((0, x_2^*))}.$$

Corollary 15 *The important difference between the investments of bulls and bears is that bulls only speculate, but bears may speculate or diversify. That is,*

$$x^*_{\text{Bulls}} = (x^*_1, 0) \text{ or } x^*_{\text{Bulls}} = ((0, x^*_2))$$

and

$$x^*_{\text{Bears}} = (x^*_1, 0) \text{ or } x^*_{\text{Bears}} = ((0, x^*_2)) \text{ or } \frac{\partial_{x_1} U_{\text{Bears}}(x^*)}{\partial_{x_2} U_{\text{Bears}}(x^*)} = \frac{p_1}{p_2}, \text{ where } x^* \in R^N_{++}.$$

The last two examples, suggests a neoclassical model of (ir)rational exuberance in asset markets, where investors are bulls and bears who maximize Keynesian utilities subject to budget constraints, defined by market prices and the investor's income. Consequently, bulls speculate and "bet" on the optimistic realization tomorrow of state 2 or state 1, but bears diversify and choose a portfolio with Arrow–Debreu state-contingent claims for both states. The aggregate demand of bulls is diversified. That is, the aggregate demand of the bulls is the "average" of the demands of the bulls expecting state 2 to be realized tomorrow and the demands of the bulls expecting state 1 to be realized tomorrow. Hence the aggregate demand of bulls is typically in R^N_{++}

Theorem 16 *If there is a finite number of investor classes, where in each class investors have the same income and in each class, there is a continuum of bulls with different utilities and a continuum of bears with the same utilities, then market clearing competitive prices exist in asset markets.*

Proof. For each vector of market prices, the asset demands of optimistic investors are concentrated on the corners of their common budget line. The sum of the means of these distributions over all income classes is the aggregate demand of bulls at the given market prices. In the two states of the world example, we assume a fraction ρ of the subclass of bulls with income I demand $(0, x^*_2(p, I))$ and a fraction $(1 - \rho)$ of the subclass of bulls with income I demand $(x^*_1(p, I), 0)$. Hence the aggregate demands at market prices p for the subclass of optimistic investors with income I is

$$(1 - \rho)(x^*_1(p, I), 0) + \rho(0, x^*_2(p, I)) = [(1 - \rho)(x^*_1(p, I), \rho(0, x^*_2(p, I))] > 0.$$

The aggregate demands of the subclass of pessimistic investors or bears is the sum of the means of the distribution of their demands over all income classes. Neoclassical competitive market prices clear the asset markets if the sum of aggregate demands of bulls and bears at these prices equals the aggregate supply of assets. ■

Example 17 *Here is an example of existence of a competitive equilibrium in an exchange economy with two states of the world.. There is a continuum of bulls indexed on* $[0, 1]$ *and a continuum of bears indexed on* $[0, 1]$*. The sum of the average endowments of the bulls,* Θ_{Bulls}*, and the average endowments of the bears,* Θ_{Bears}*, define the average social endowment* $\Theta \equiv \Theta_{\text{Bulls}} + \Theta_{\text{Bears}}$*. We construct the associated Edgeworth box, where the* X*-axis is the payoff of the average social endowment in state* 1 *and the Y-axis is the payoff of the average social endowment in state 2. Zero is the origin of the positive orthant for bulls, i.e., $x \geq 0$ and Θ is the origin of the positive orthant for bears, i.e., $y \leq \Theta$. If $p = (p_1, p_2)$ is any vector of positive state prices, i.e., the payoffs of the $A - D$ securities tomorrow, where $p \cdot \Theta_{\text{Bulls}} \equiv I$ and $p \cdot \Theta_{\text{Bears}} \equiv J$, then as in the earlier example there exists a fraction $\rho \in (0, 1)$ of bulls who demand the asset with payoffs $(1p_1, 0)$ and a fraction $(1 - \rho) \in (0, 1)$ of bulls who demand the asset with payoffs $(0, 1/p_2)$. Hence aggregate demand of the bulls at state prices p is*

$$z \equiv \left(\rho \frac{I}{p_1}, (1 - \rho)\frac{I}{p_2}\right).$$

In the Edgeworth box, z is a point on the interior of the budget line $p \cdot x = I$, where $x = (x_1, x_2)$ is a state-contingent claim in the positive orthant for bulls. In this example, if every bear maximizes utility subject to the budget constraint $p \cdot y = J$, where $y = \Theta - x$ and z is a state-contingent claim in the positive orthant for the bulls , then $[p; z, \Theta - z]$ is a competitive equilibrium in any exchange economy, where all bears are endowed with the same concave utility function $U(y)$ and

$$\Theta - z = \arg \max_{p \cdot y = J} U(y).$$

Of course, a much less restrictive existence proof of a competitive equilibrium in an exchange economy with a continuum of traders is the well-known existence theorem of Aumann (1969), where agents need not be endowed with convex preferences, i.e., quasi-concave preferences, and consumption sets are R^N_+.

Example 18 *Here is an example of comparative statics in an exchange economy with two states of the world, where there is one type of bears and two types of bulls. Bulls and bears have additively-separable utilities, hence representable as Keynesian utilities by Theorem 1. Both types of bulls have the same income L_{Bull}, hence the same budget set for any pair of market prices. Bears are endowed with Cobb–Douglas utilities, so their demands are never at the corners of the budget line, but bulls are endowed with smooth, monotone quadratic, convex utilities and their demands are only at the corners. The distribution of bulls is defined by $\rho \in [0, 1]$. ρ is the % of type 1 bulls. Type 1 bulls only demand $A - D$ securities for state 1. $(1 - \rho)$ is the % of type 2 bulls. Type 2 bulls only demand $A - D$ securities for state 2. The demand functions for the bears are*

$$X_{\text{Bear}}(p_1, p_2; I_{\text{Bear}}) = \left(\frac{\alpha I_{\text{Bear}}}{p_1}, \frac{\beta I_{\text{Bear}}}{p_2}\right) \text{ where } \alpha + \beta = 1 \text{ and } \alpha, \beta > 0.$$

The demand functions for the bulls are

$$X_{\text{Bull1}}(p_1, p_2; I_{\text{Bull}}) = \rho \left(\left[\frac{I_{\text{Bull}}}{p_1}\right]^2, 0\right) \text{ and } X_{\text{Bull2}}(p_1, p_2; I_{\text{Bull}}) = (1-\rho) \left(0, \left[\frac{I_{\text{Bull}}}{p_2}\right]^2\right).$$

Hence $X_{\text{Bull}}(p_1, p_2; I_{\text{Bull}})$, is the aggregate demand of the bulls ,where

$$X_{\text{Bull}}(p_1, p_2; I_{\text{Bull}}) \equiv X_{\text{Bull1}}(p_1, p_2; I_{\text{Bull}}) + X_{\text{Bull2}}(p_1, p_2; I_{\text{Bull}}) = \left(\rho \left[\frac{I_{\text{Bull}}}{p_1}\right]^2, (1 - \rho) \left[\frac{I_{\text{Bull}}}{p_2}\right]^2\right).$$

If state 2 is the numeraire, i.e.,

$$p_2 = 1 \text{ and } p_1 > 0$$

then it follows from Walras' law that it suffices to find p_1 such that the market for state 1 clears. If

$$\rho[\frac{I_{\text{Bull}}}{p_1}]^2 + \frac{\alpha p_1 I_{\text{Bear}}}{p_1} = \omega_1,$$

where $\omega \equiv (\omega_1, \omega_2)$ is the social endowment,then the equilibrium price

$$p_1 = \frac{\alpha I_{\text{Bear}} \pm ([\alpha I_{\text{Bear}}]^2 + 4\rho[I_{\text{Bull}}]^2\omega_1)^{1/2}}{2\omega_1},$$

where the critical value for ρ is 0. That is,

$$\text{if } \rho \to 0 \text{ then } p_1 \to 0 \text{ or } p_1 \to \frac{\alpha I_{\text{Bear}}}{\omega_1} \text{ where } p_1 \leq 0 \text{ is infeasible.}$$

The unique, feasible equilibrium price

$$p_1(\rho) \equiv \frac{\alpha I_{\text{Bear}} + ([\alpha I_{Bear}]^2 + 4\rho[I_{\text{Bull}}]^2\omega_1)^{1/2}}{2\omega_1},$$

where $p_1(\rho)$ is a strictly concave,monotone function of ρ on $(0, 1]$.

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