ENDOGENOUS LEVERAGE IN A BINOMIAL ECONOMY: THE IRRELEVANCE OF ACTUAL DEFAULT

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Ana Fostel and John Geanakoplos

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COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Endogenous Leverage in a Binomial Economy: The Irrelevance of Actual Default

Ana Fostel* John Geanakoplos†

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Abstract

We show that *binomial* economies with *financial* assets are an informative and tractable model to study endogenous leverage and collateral equilibrium: endogenous leverage can be highly volatile, but it is always easy to compute. The possibility of default can have a dramatic effect on equilibrium, if collateral is scarce, yet we prove the No-Default Theorem asserting that, without loss of generality, there is no default in equilibrium. Thus *potential* default has a dramatic effect on equilibrium, but *actual* default does not. This result is valid with arbitrary preferences, contingent promises, many assets and consumption goods, production, and multiple periods. On the other hand, we show that the theorem fails in trinomial models. For example, in a CAPM model, we find that default is robust. In a model with heterogeneous beliefs, we find that different agents might borrow on the same asset with different $LTVs$.

Keywords: Endogenous Leverage, Collateral Equilibrium, Default, Financial Asset, Binomial Economy, VaR.

JEL Codes: D52, D53, E44, G01, G11, G12

*George Washington University,Washington, DC. New York University. NY. Email: afostel@gwu.edu.

†Yale University, New Haven, CT and Santa Fe Institute, Santa Fe, NM. Email: john.geanakoplos@yale.edu.

1 Introduction

The recent financial crises has brought the impact of leverage on financial system stability to the forefront. The crisis might well be understood as the bottom of a leverage cycle in which leverage and asset prices crashed together. It was preceded by years in which asset prices and the amount of leverage in the financial system increased dramatically. What determines leverage or margin levels in equilibrium? Do these levels involve default? What is the effect of leverage and default on asset prices and the real side of the economy? Needless to say, providing answers to these questions is of key importance.

In this paper we prove a No-Default Theorem for any *binomial* economy with *financial* assets (i.e., assets which give no direct utility to investors and pay the same dividends no matter who owns them). We show that every equilibrium with endogenous leverage can be replaced by another equilibrium with the same asset prices and the same consumption by each agent, in which there is no default.[1] Thus *potential* default has a dramatic effect on equilibrium, but *actual* default does not. The No-Default Theorem is valid in a very general context with arbitrary preferences, contingent promises, many assets, many consumption goods, multiple periods, and production.

The No-Default Theorem shows that for every homogeneous family of promises that (i) use the same asset as collateral, (ii) differ only by a scalar multiplicative factor and (iii) includes the max min promise, we can assume only the max min promise is actively traded in equilibrium.[2] The max min representative for the family is the largest promise in the family which is sure to pay off in full, and is equal to the asset value in at least one state. Thus even with many assets, and many homogeneous families of contracts, there will be no default.

The No-Default Theorem not only shows that actual default is irrelevant, but also provides very simple predictions about how leverage is determined. For example, for the homogenous family of simple (non-contingent) debt contracts, the equilibrium LTV of any actively traded contract can be taken to be the ratio of the worst case return of the collateral divided by the riskless rate of interest. The upshot is that

[1]The choice of leverage might thus be described by a rule which sets the value at risk (VAR) equal to zero.

[2]Though the promises in a homogeneous family are scalar multiples of each other, the deliveries, which are the minimum of the collateral value and the promise, can be highly nonlinearly related.

equilibrium leverage in binomial economies with financial assets is extremely easy to compute.

There are two key assumptions in the theorem. First, we only consider financial assets, that is, assets that do not give direct utility to their holders, and which yield dividends that are independent of who holds them. Second, we assume that the economy is binomial, and that all loans are for one period.[3] A date-event tree in which every state is succeeded by exactly two nodes suggests a world with very short maturity loans and no big jumps in asset values, since Brownian motion can be approximated by binary trees with short intervals. Binomial models might thus be taken as good models of Repo markets, in which the assets do seem to be purely financial, and the loans are extremely short term, usually one day. Repo market loans almost never default, including during the leverage crisis of 2007-9.[4]

The No-Default Theorem implies that if we want to study consumption or production or asset price effects of actual default, we *must* do so in models that either include non-financial assets (like houses or asymmetrically productive land) or that depart from the standard binomial models used in finance. This shows that there is a tremendous difference between physical collateral that generates contemporaneous utility and backs long term promises, and financial collateral that gives utility only through dividends or other cash flows, and backs very short term promises. The No-Default Theorem might explain why there are some markets (like mortgages) in which defaults are to be expected while in others (like Repos) margins are set so strictly that default is almost ruled out.

The No-Default Theorem has a sort of Modigliani-Miller feel to it. But the theorem does *not* assert that the debt-equity ratio is irrelevant. It shows that if we start from any equilibrium, we can move to an equivalent equilibrium, typically with less leverage, in which only no-default contracts are traded. The theorem does not say that starting from an equilibrium with no default, one can construct another equivalent equilibrium with even less leverage. Typically one cannot. In the paper we give an example with a unique equilibrium in which every borrower leverages to the maximum amount without default, but no agent would be indifferent to leveraging any less. In that example, Modigliani-Miller completely fails. Modigliani-Miller does

[3]We could also allow for a long term loan with one payment date, provided that all the states at that date could be partitioned into two events, on each of which the loan promise and the asset value is constant.

[4]Repo defaults, including of the Bear Stearns hedge funds, seem to have totaled a few billion dollars out of the trillions of dollars of repo loans during the period 2007-2009.

not hold in our model because issuers of debt must hold collateral, and because we do not allow short selling of the assets. As we will explain in detail in the paper, the proof of the No-Default Theorem relies on positive spanning, not spanning.

We begin by proving the No-Default Theorem in a very simple static model, with two periods, one asset, and a homogeneous family of simple debt (non-contingent) contracts. The intuition of the proof is based on two ideas. We first show that any two portfolios that give the same payoffs in the two states must cost the same. We then show that in equilibrium each agent is indifferent to replacing his portfolio with another such that on each unit of collateral that he holds, he either leverages to the maximum amount without risk of default, or does not leverage at all. It is important to realize that the new portfolio may involve each agent holding a different amount of the collateral asset than he did in the original equilibrium. Agents are indifferent to switching to the new portfolio because its payoffs are the same and because of the assumption that the asset is a financial asset. The assumption of two states implies that the payoffs are positively spanned by the Arrow security that pays in the good state (which can be obtained by holding the asset and shorting the max min debt using the asset as collateral) and the max min debt contract. If there were three states, it might be impossible for the seller to reproduce his original portfolio payoffs from a portfolio in which he can only hold the asset and issue the max min debt.

One interesting feature of the proof is that it demonstrates the existence of state prices that price the asset and the entire homogeneous family of contracts that use the asset as collateral, even though short-selling is not allowed. The proof therefore shows that even a hypothetical trader who could sell the asset short and did not need to put up collateral could not find an arbitrage opportunity using the asset and all the contracts in the same family. Moreover, we show that those state prices are unique. In short, in a binomial economy with one kind of collateral and one family of debt promises, we always get unique state (Arrow) prices but not necessarily an Arrow-Debreu equilibrium. In binomial economies with many assets, or many different families of loans on the same asset, the No Default Theorem still holds, but state pricing of all contracts is typically impossible.

Another feature of the proof is that it shows that every asset has a maximum debt capacity for each homogeneous family of contracts, given by the value of the max min contract. This debt capacity might be far below the value of the asset. The great advantage of a binomial model with uncertainty is that the debt capacity can fall sharply due to an increase in volatility (that lowers the worst case and increases the best case scenario in the future), even if the asset value does not fall. The loan to value ratio (of loan amount to asset value) or LTV depends not only on first moments, but also on second moments, becoming genuinely endogenous. By contrast, in models of collateral without uncertainty, the debt capacity of a financial asset is always equal to the value of the asset.[5]

The No-Default Theorem shows that the equilibrium LTV for each family of contracts depends on current and future asset prices, but is otherwise independent of the utilities or the endowments of the agents. If nobody wants to borrow beyond the debt capacity of the asset, then the collateral requirements are irrelevant, and debt is determined by the preferences of the agents in the economy, just as in models without collateral. In this case, we might say debt is determined by demand. On the other hand, if collateral is scarce, and agents are borrowing against all of it, then total borrowing is determined by the debt capacities of the assets, independently from agent preferences for borrowing. In this case, we might say debt is determined by the supply of debt capacity. Nonetheless, as the No-Default Theorem states, the equilibrium LTV in each homogeneous family can be taken to be the same in both cases.

Many papers in the literature assume the Value at Risk equals Zero rule (which precludes default), often in contexts where the No-Default Theorem does not apply. In the last section, we study two examples of economies extensively used in the financial literature: i) heterogeneous beliefs and ii) CAPM investors with differences in risk aversion and endowments. First, in order to illustrate the No-Default Theorem, we study both cases in a binomial economy, showing that in the simplest context the No-Default Theorem and the State Pricing Theorem apply, but that in more general binomial models, the No-Default Theorem still applies while state pricing fails. Second, we extend the examples to three states of nature and show that the No-Default Theorem also fails. There is still a debt capacity for each family of contracts, but now the maximum possible loan almost always involves default. Agents who choose to borrow less will use different leverage. In other words, we find two main departures from the the No-Default Theorem. First, both examples show that with enough heterogeneity among investors, equilibrium default is robust. Second, we find that more than one contract can be actively traded in equilibrium on the same collateral, that is, the asset might be bought at different LTVs by different

[5] In the case of financial assets, the value of the asset is the present value of future payments. When there is only one future state, and the entire asset can be costlesly seized upon default, the debt capacity is 100% of the value of the financial asset.

agents. During the period 1997-2007, prime borrowers typically bought houses with high down-payments and low interest rates while subprime borrowers were putting almost no money down but paying a high interest rate on the same kinds of houses.

The paper is organized as follows. Section 2 presents the literature review. Section 3 presents a static model of endogenous leverage and debt with one asset and proves the Simple No-Default and State Pricing Theorems. Section 4 presents the general model of endogenous leverage and proves the general No-Default Theorem. Section 5 presents examples.

2 Literature

To attack the leverage endogeneity problem we follow the techniques developed by Geanakoplos (1997). Agents have access to a menu of contracts, each of them characterized by a promise in future states and one unit of asset as collateral to back the promise. When an investor sells a contract she is borrowing money and putting up collateral; when she is buying a contract, she is lending money. In equilibrium every contract, as well as the asset used as collateral, will have a price. Each collateral-promise pair defines a contract, and every contract has a leverage or loan to value (the ratio of the price of the promise to the price of the collateral). The key is that even if all contracts are priced in equilibrium, because collateral is scarce, only a few will be actively traded. In this sense, leverage becomes endogenous.

Geanakoplos (2003, 2010), Fostel-Geanakoplos (2008, 2012a and 2012b), and Cao (2010), all work with binomial models of collateral equilibrium with financial assets, showing in their various special cases that, as the Binomial No-Default Theorem implies, only the *VaR= 0* contract is traded in equilibrium. These papers generally show that higher leverage leads to higher asset prices.

Geanakoplos (2003) stated a slightly stronger Binomial No-Default theorem (that equilibrium is also unique) in the special case of a continuum of agents with different priors, in which every agent was risk neutral and did not discount the future, and in which the agents' subjective probability of the up state increased monotonically and continuously in the index of the agent. Fostel-Geanakoplos (2012a) formally proved that theorem. The Binomial No-Default Theorem proved in this paper is more general in that it does not depend on the number of agents, or on continuity of preferences across agents, or on identical discount rates, or on risk neutrality, or on any assumption about endowments (for example it does not assume that each agent's endowments in terminal periods is spanned by the asset). It includes the case where there is a finite number of agent types, as well as the case where there is a continuum of heterogeneous agents.

Other papers have already given examples in which the No-Default Theorem does not hold. Geanakoplos (1997) gave a binomial example with a non-financial asset (a house, from which agents derived utility), in which equilibrium leverage was high enough that there was default. Geanakoplos (2003) gave an example with a continuum of risk neutral investors with different priors and three states of nature in which the only contract traded in equilibrium involved default. Simsek (2010) gave an example with two types of investors and a continuum of states of nature with equilibrium default. Araujo, Kubler, and Schommer (2012) provided a two period example of an asset which is used as collateral in two different actively traded contracts when agents have utility over the asset. Fostel and Geanakoplos (2012a) provide an example with three states and multiple contracts traded in equilibrium.

This paper is related to a large and growing literature on collateral equilibrium and leverage. Some of these papers focus on *investor-based leverage* (the ratio of an agent's total asset value to his total wealth) as in Acharya and Viswanathan (2011), Adrian and Shin (2010), Brunnermeier and Sannikov (2011) and Gromb and Vayanos (2002). Other papers, such as Brunnermeier and Pedersen (2009), Cao (2010), Fostel and Geanakoplos (2008, 2012a and 2012b), Geanakoplos (1997, 2003 and 2010) and Simsek (2010), focus on *asset-based leverage* (as defined in this paper).

Not all these models actually make room for endogenous leverage. Often an ad hoc behavioral rule is postulated. To mention just a few, Brunnermeier and Pedersen (2009) assume a VAR rule. Gromb and Vayanos (2002) and Vayanos and Wang (2012) assume a max min rule that prevents default. Some other papers like Garleanu and Pedersen (2011) and Mendoza (2010) assumed a fixed LTV.

In other papers leverage is endogenous, though the modeling strategy is not as in our paper. In the corporate finance approach of Bernanke and Gertler (1986), Holmstrom and Tirole (1997), Acharya and Viswanathan (2011) and Adrian and Shin (2010) the endogeneity of leverage relies on asymmetric information and moral hazard problems between lenders and borrowers. Asymmetric information is important in loan markets for which the borrower is also a manager who exercises control over the value of the collateral. Lenders may insist that the manager puts up a portion of the investment himself in order to maintain his skin in the game. The recent crisis, however, was centered not in the corporate bond world, where managerial control is central, but in the mortgage securities market, where the buyer/borrower generally has no control or specialized knowledge over the cash flows of the collateral. Finally, in Brunnermeier and Sannikov (2011) leverage is endogenous but is determined not by collateral capacities but by agents risk aversion; it is a "demand-determined" leverage that would be the same without collateral requirements. The time series movements of LTV come there from movements in volatility because the added uncertainty makes borrowers more scared of investing, rather than from reducing the debt capacity of the collateral or making lenders more scared to lend.

3 The Irrelevance of Actual Default in a Simple Model of Debt.

3.1 Model

In order to understand the upcoming No-Default Theorem more easily, we restrict attention in this section to two periods, one asset, and non-contingent debt contracts.

3.1.1 Time and Assets

We begin with a simple two-period general equilibrium model, with time $t = 0, 1$. Uncertainty is represented by different states of nature $s \in S$ including a root $s = 0$. We denote the time of s by $t(s)$, so $t(0) = 0$ and $t(s) = 1, \forall s \in S_T$, the set of terminal nodes of S. Suppose there is a single perishable consumption good c and one asset Y which pays dividends d_s of the consumption good in each final state $s \in S_T$. We call the asset a *financial asset* because it gives no direct utility to investors, and pays the same dividends no matter who owns it. Financial assets are valued exclusively because they pay dividends. Houses are not financial assets because they give utility to their owners. Neither is land if its output depends on who owns it and tills it.

3.1.2 Investors

Each investor $h \in H$ is characterized by a utility, u^h, a discount factor, δ_h, and subjective probabilities, γ_s^h, $s \in S_T$. We assume that the utility function for consumption in each state $s \in S$, $u^h : R_+ \rightarrow R$, is differentiable, concave, and monotonic.[6] The expected utility to agent h is

$$U^h = u^h(c_0) + \delta_h \sum_{s \in S_T} \gamma_s^h u^h(c_s). \tag{1}$$

Investor h's endowment of the consumption good is denoted by $e_s^h \in R_+$ in each state $s \in S$. His endowment of the only asset Y at time 0 is $y_{0^*}^h \in R_+$. We assume that the consumption good is present, $\sum_{h \in H} e_0^h > 0, \sum_{h \in H}(e_s^h + d_s y_{0^*}^h) > 0, \forall s \in S_T$.

3.1.3 Collateral and Debt.

We take the consumption good as numeraire and denote the price of Y at time 0 by p. A *debt contract* j promises $j > 0$ units of consumption good in each final state backed by one unit of asset Y serving as collateral. The terms of the contract are summarized by the ordered pair $(j \cdot \widetilde{1}, 1)$. The first component, $j \cdot \widetilde{1} \in R^{S_T}$(the vector of j's with dimension equal to the number of final states), denotes the (non-contingent) promise. The second component, 1, denotes the one unit of the asset Y used as collateral. Let J be the set of all such available debt contracts.

The price of contract j is π_j. An investor can borrow π_j today by selling the debt contract j in exchange for a promise of j tomorrow. Let φ_j be the number of contracts j traded at time 0. There is no sign constraint on φ_j; a positive (negative) φ_j indicates the agent is selling (buying) $|\varphi_j|$ contracts j or borrowing (lending) $|\varphi_j|\pi_j$.

We assume the loan is non-recourse, so the maximum a borrower can lose is his collateral if he does not honor his promise: the actual delivery of debt contract j in state $s \in S_T$ is $min\{j, d_s\}$. If the promise is small enough that $j \leq d_s, \forall s \in S_T$, then the contract will not default. In this case its price defines a riskless rate of interest $(1 + r_j) = j/\pi_j$.

The *Loan to Value* (LTV) associated to debt contract j is given by

$$LTV_j = \frac{\pi_j}{p}. \tag{2}$$

[6] All that matters for the No-Default Theorem is that the utility $U^h : \mathbb{R}^{1+S} \rightarrow \mathbb{R}$ depends only on consumption (and not for example on portfolio holdings). The expected utility representation is done for familiarity, and to emphasize that components such as probabilities or discount factors can differ across agents.

The margin requirement m_j associated to debt contract j is $1 - LTV_j$, and the leverage associated to debt contract j is the inverse of the margin, $1/m_j$.

We define the *average loan to value*, LTV for asset Y, as the trade-value weighted average of LTV_j across all debt contracts actively traded in equilibrium, and the *diluted average loan to value*, LTV_0^Y (which includes assets with no leverage) by

$$LTV^Y = \frac{\sum_h \sum_j \max(0, \varphi_j^h)\pi_j}{\sum_h \sum_j \max(0, \varphi_j^h)p} \geq \frac{\sum_h \sum_j \max(0, \varphi_j^h)\pi_j}{\sum_h y_{0^*}^h p} = LTV_0^Y.$$

3.1.4 Budget Set

Given the asset and debt contract prices $(p, (\pi_j)_{j \in J})$, each agent $h \in H$ decides consumption, c_0, asset holding, y, debt and contract trades, φ_j, at time 0, and also consumption, c_s, in each state $s \in S_T$, in order to maximize utility (1) subject to the budget set defined by

$$B^h(p, \pi) = \{(c, y, \varphi) \in R_+^{1+S} \times R_+ \times R^J :$$

$$(c_0 - e_0^h) + p(y - y_{0^*}^h) \leq \textstyle\sum_{j \in J} \varphi_j \pi_j$$

$$(c_s - e_s^h) \leq y d_s - \textstyle\sum_{j \in J} \varphi_j min(j, d_s), \forall s \in S_T$$

$$\textstyle\sum_{j \in J} max(0, \varphi_j) \leq y\}.$$

At time 0 expenditures on consumption and the asset, net of endowments, must be financed by money borrowed, using the asset as collateral. In the final period, at each state s, consumption net of endowments, can be at most equal to the dividend payment minus debt repayment. Finally, those agents who borrow must hold the required collateral at time 0.[7]

[7] Notice that we are assuming that short selling of assets is not possible. So even with two or more contracts, equilibrium might still be different from Arrow-Debreu. We do not think the assumption of no-short selling is implausible. It is impossible to short sell many assets in the real world, though the CDS market is beginning to change that. In Fostel-Geanakoplos (2012b) we investigate the significance of CDS for asset pricing.

3.1.5 Collateral Equilibrium

A *Collateral Equilibrium* is a set consisting of an asset price, debt contract prices, individual consumptions, asset holdings, and contract trades $((p, \pi), (c^h, y^h, \varphi^h)_{h \in H}) \in (R_+ \times R_+^J) \times (R_+^{1+S} \times R_+ \times R^J)^H$ such that

1. $\sum_{h \in H}(c_0^h - e_0^h) = 0.$
2. $\sum_{h \in H}(c_s^h - e_s^h) = \sum_{h \in H} y^h d_s, \forall s \in S_T.$
3. $\sum_{h \in H}(y^h - y_{0^*}^h) = 0.$
4. $\sum_{h \in H} \varphi_j^h = 0, \forall j \in J.$
5. $(c^h, y^h, \varphi_j^h) \in B^h(p, \pi), \forall h$
   $(c, y, \varphi) \in B^h(p, \pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h.$

Markets for the consumption good in all states clear, assets and promises clear in equilibrium at time 0, and agents optimize their utility in their budget sets. As shown by Geanakoplos and Zame (1997), equilibrium in this model always exists under the assumption we have made so far.

The set H of agents can be taken as finite (in which case we really have in mind a continuum of agents of each of the types), or we might think of $H = [0, 1]$ as a continuum of distinct agents, in which case we must think of all the agent characteristics as measurable functions of h. In the latter case we must think of the summation $\sum$ over agents as an integral over agents, and all the optimization conditions as holding with Lebesgue measure one.

3.2 The Simple Binomial No-Default Theorem

Consider the situation in which there are only two terminal states, $S = \{0, U, D\}$. Asset Y pays d_U units of the consumption good in state $s = U$ and $0 < d_D < d_U$ in state $s = D$.[8] Figure 1 depicts the asset payoff. Default occurs in equilibrium if and only if some contract j with $j > d_D$ is positively traded. One might imagine that

[8] Without loss of generality, $d_U \geq d_D$. If $d_D = 0$ or $d_D = d_U$, then the contracts are perfect substitutes for the asset, so there is no point in trading them. Sellers of the contracts could simply hold less of the asset and reduce their borrowing to zero while buyers of the contracts could buy the asset instead. So we might as well assume $0 < d_D < d_U$.

some agents value the asset much more than others, say because they attach very high probability γ_U^h to the U state, or because they are more risk tolerant, or because they have very low endowments e_U^h in the U state, or because they put a high value δ^h on the future. These agents might be expected to want to borrow a lot, promising $j > d_D$ so as to get their hands on more money to buy more assets at time 0. Indeed it is true that for $j > j^* = d_D$, any agent can raise more money $\pi_j > \pi_{j*}$ by selling contract j rather than j^*. Nonetheless, as the following result shows, we can assume without loss of generality that the only debt contract traded in equilibrium will be the max min contract j^*, on which there is no default.

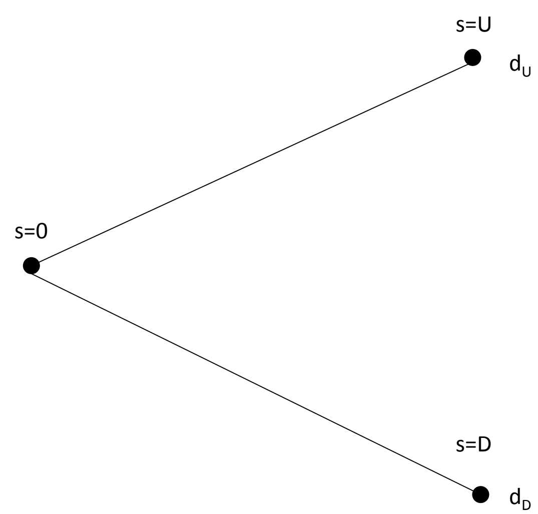

Figure 1: Asset payoff description.

Simple Binomial No-Default Theorem:

Suppose that $S = \{0, U, D\}$, that Y is a financial asset, and that the max min debt contract $j^* = d_D \in J$. Then given any equilibrium $((p, \pi), (c^h, y^h, \varphi^h)_{h \in H})$, we can construct another equilibrium $((p, \pi), (c^h, \bar{y}^h, \bar{\varphi}^h)_{h \in H})$ with the same asset and contract prices and the same consumptions, in which j^* is the only debt contract traded, $\bar{\varphi}_j^h = 0$ if $j \neq j^*$. Hence equilibrium default can be taken to be zero, and equilibrium LTV can be taken equal to $\frac{\pi_j^*}{p} = \frac{d_D/(1+r_{j^*})}{p} = \frac{d_D/p}{1+r_{j^*}}$.

Proof:

The proof is organized in three steps.

1. **Payoff Cone Lemma.**

   The portfolio of assets and contracts that any agent h holds in equilibrium delivers payoff vector (w_U^h, w_D^h) which lies in the cone positively spanned by $(d_U - j^*, 0)$ and (j^*, j^*). The U Arrow security payoff $(d_U - j^*, 0) = (d_U, d_D) - (j^*, j^*)$ can be obtained from buying the asset while simultaneously selling the max min debt contract.

   Any portfolio payoff (w_U, w_D) is the sum of payoffs from individual holdings. The possible holdings include debt contracts $j > j^*, j = j^*, j < j^*$, the asset, and the asset bought on margin by selling some debt contract j. The debt contracts and the asset all deliver at least as much in state U as in state D. So does the leveraged purchase of the asset. In fact, buying the asset on margin using any debt contract with $d_U > j \geq j^*$ is effectively a way of buying the U Arrow payoff $(d_U - j, 0)$. This can be seen in Figure 2.

   In short, we have that the Arrow U security and the max min debt contract positively span all the feasible payoff space, as shown in Figure 3.

2. **State Pricing Lemma.**

   Let $a > 0$ be the price of the synthetic U Arrow security (created by the leveraged purchase of the asset via the contract j^*). Let $b = \pi_{j^*}/j^* - a$. Then if any agent h holds a portfolio delivering (w_U^h, w_D^h), the portfolio costs $aw_U^h + bw_D^h$.

   In step (a) we find state prices for two securities: the asset and the max min debt contract j^*. In step (b) we use the Payoff Cone Lemma to show that the same state prices can be used to price any other debt contract $j \neq j^*$ that is traded in equilibrium. The cost of any portfolio is obtained as the sum of the costs of its constituent parts.

   (a) *Let $a = \frac{p - \pi_{j^*}}{d_U - j^*}$ and $b = \pi_{j^*}/j^* - a$. Consider the max min debt contract and the asset payoff shown in Figure 2. Then $\pi_{j^*} = aj^* + bj^*$ and $p = ad_U + bd_D$.*

   Note first that

$$aj^* + bj^* = aj^* + (\pi_{j^*}/j^* - a)j^* = \pi_{j^*}.$$

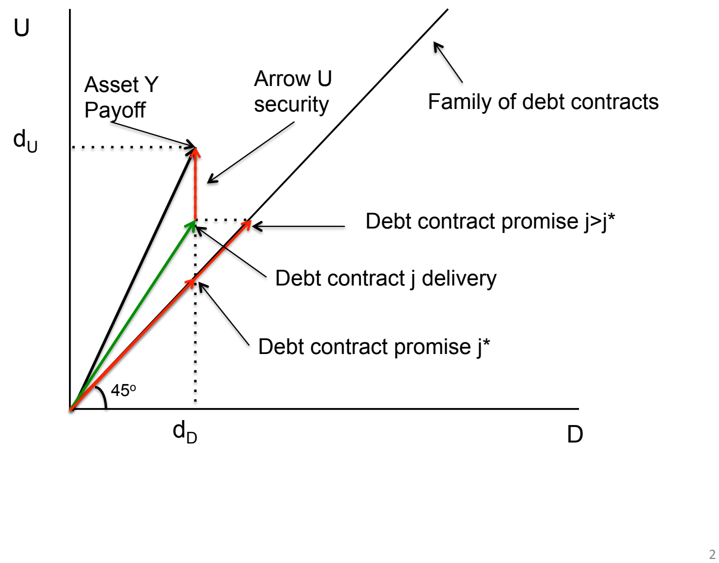

Figure 2: Creating the U Arrow security.

Using the definitions of π_{j^*}, a and j^*

$$ad_U + bd_D - \pi_{j^*} = a(d_U - j^*) + b(d_D - j^*) = (p - \pi_{j^*}) + 0.$$

And hence

$$ad_U + bd_D = p.$$

(b) *Suppose a debt contract j with $j \neq j^* = d_D$ is positively traded in equilibrium. Then $\pi_j = a \cdot min\{d_U, j\} + b \cdot min\{j^*, j\}$.*

The actual delivery of contract j is given by $(min\{d_U, j\}, min\{j^*, j\})$. In case $j \leq j^* = d_D$, the contract fully delivers j in both states, proportionally to contract j^*. If j's price exceeded $\pi_{j^*}(j/j^*) = aj + bj$, its buyers should have bought j/j^* units of j^* instead. Similarly, if its price were less than $\pi_{j^*}(j/j^*) = aj + bj$, its sellers should have sold j/j^* units of j^* instead, which would have been feasible for them as it requires less collateral.

We are left to consider the case in which $j > j^*$, shown in Figure 2. By

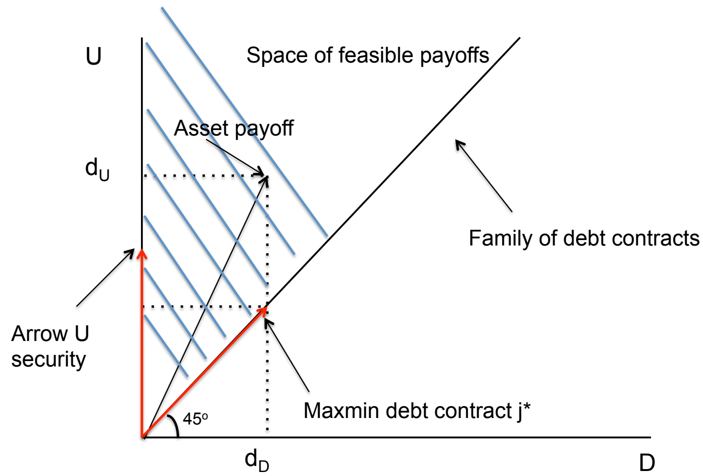

Figure 3: Positive spanning.

the Payoff Cone Lemma, the actual delivery of contract j is in the positive span of $(d_U - j^*, 0)$ and (j^*, j^*):

$$(min\{d_U, j\}, min\{j^*, j\}) = \alpha(d_U - j^*, 0) + \beta(j^*, j^*)$$

with $\alpha, \beta \geq 0$. Hence, $\alpha(d_U - j^*) + \beta j^* = min\{d_U, j\}$ and $\beta j^* = min\{j^*, j\}$. None of the buyers would have purchased j unless

$$\pi_j \leq \alpha a(d_U - j^*) + \beta \pi_{j^*} = a(\alpha(d_U - j^*) + \beta j^*) + b(\beta j^*) =$$
$$= a \cdot min\{d_U, j\} + b \cdot min\{j^*, j\}.$$

On the other hand, any seller of contract j has entered into a double trade, buying (or holding) the asset as collateral at the same time he sold contract j, at a net cost of $p - \pi_j$. Since any contract $j > d_U$ delivers exactly the same way in both states as contract $j = d_U$, we can now without loss of generality restrict attention to contracts j with $d_D < j \leq d_U$. Any agent selling such a contract, while holding the required collateral, receives on net $d_U - j$ in state U, and nothing in state D. He could have obtained exactly the same net payoff by holding $(d_U - j)/(d_U - j^*) < 1$ units of the asset and selling $(d_U - j)/(d_U - j^*) < 1$ units of contract j^*, at a net cost of $(p - \pi_{j^*})(d_U - j)/(d_U - j^*)$. He should have done so unless

$$
\begin{aligned}
p - \pi_j &\le (p - \pi_{j^*})(d_U - j)/(d_U - j^*) \\
p - \pi_j &\le ([ad_U + bj^*] - [(a+b)j^*])(d_U - j)/(d_U - j^*) \\
p - \pi_j &\le a(d_U - j^*)(d_U - j)/(d_U - j^*) = a(d_U - j) \\
ad_U + bj^* - \pi_j &\le a(d_U - j) \\
aj + bj^* &\le \pi_j \\
a \cdot \min\{d_U, j\} + b \cdot \min\{j^*, j\} &\le \pi_j.
\end{aligned}
$$

Hence $\pi_j = a \cdot \min\{d_U, j\} + b \cdot \min\{j^*, j\}$.

3. **Construction of the new default-free equilibrium**

Define

$$
\begin{aligned}
(w_U^h, w_D^h) &= y^h(d_U, d_D) - \sum_j (\min(j, d_U), \min(j, d_D))\varphi_j^h. \\
\bar{y}^h &= \frac{w_U^h - w_D^h}{d_U - d_D}. \\
\bar{\varphi}_{j^*}^h &= [\bar{y}^h d_D - w_D^h]/j^* = \bar{y}^h - w_D^h/j^*.
\end{aligned}
$$

If in the original equilibrium, y^h is replaced by $\bar{y}^h$ and φ_j^h is replaced by 0 for $j \neq j^*$ and by $\bar{\varphi}_{j^*}^h$ for $j = j^*$, and all prices and other individual choices are left the same, then we still have an equilibrium.

(a) *Agents are maximizing in the new equilibrium.*

Note that $\bar{\varphi}_{j^*}^h \le \bar{y}^h$, so this portfolio choice satisfies the collateral constraint.

Using the above definitions, the net payoff in state D is the same as in the original equilibrium,

$$\bar{y}^h d_D - \bar{\varphi}_{j^*}^h j^* = w_D^h$$

and the same is also true for the net payoff in state U,

$$\bar{y}^h d_U - \bar{\varphi}^h_{j^*} j^* = \bar{y}^h(d_U - d_D) + w^h_D = (w^h_U - w^h_D) + w^h_D = w^h_U.$$

Hence the portfolio choice $(\bar{y}^h, \bar{\varphi}^h_{j^*})$ gives the same payoff (w^h_U, w^h_D). From the Lemmas, the newly constructed portfolio must have the same cost as well. Hence every agent is optimizing.

(b) *Markets clear in the new equilibrium.*

Summing over individuals we must get

$$\sum_h \bar{y}^h(d_U, d_D) - \sum_h \bar{\varphi}^h_{j^*}(j^*, j^*) =$$

$$\sum_h (w^h_U, w^h_D) = \sum_h y^h(d_U, d_D) - \sum_h \sum_j \varphi^h_j(\min(j, d_U), \min(j, d_D)) = \sum_h y^h(d_U, d_D).$$

The first equality follows from step (a), the second from the definition of net payoffs in the original equilibrium, and the last equality follows from the fact that $\sum_h \varphi^h_j = 0$ in the original equilibrium for each contract j. Hence we have that

$$\sum_h (\bar{y}^h - y^h)(d_U, d_D) - \sum_h \bar{\varphi}^h_{j^*}(j^*, j^*) = 0.$$

By the linear independence of the vectors (d_U, d_D) and (j^*, j^*) we deduce that

$$\sum_h \bar{y}^h = \sum_h y^h$$
$$\sum_h \bar{\varphi}^h_{j^*} = 0.$$

Hence markets clear. Finally, the formula for equilibrium LTV follows from the fact that the promises are non-contingent, so that the max min contract promises d_D in both states, and from the definition of LTV. The proof is complete.■

3.3 Simple Binomial State Pricing Theorem

We now call attention to an interesting corollary of the proof just given. By modifying the equilibrium prices in the above construction for contracts that are not traded, we can bring them into line with the state prices a, b defined in the proof of the Simple Binomial No-Default Theorem, without affecting equilibrium. More concretely,

Simple Binomial State Pricing Theorem:

Suppose that $S = \{0, U, D\}$, that Y is a financial asset, and that the max min debt contract $j^* = d_D \in J$. Then given any equilibrium $((p, \pi), (c^h, y^h, \varphi^h)_{h \in H})$, we can construct another equilibrium $((p, \bar{\pi}), (c^h, \bar{y}^h, \bar{\varphi}^h)_{h \in H})$ with the same consumptions, the same asset price and the same contract price for j^*, such that every asset and contract is priced by state prices $a > 0$ and $b > 0$ which are uniquely defined if $d_U \neq d_D$. That is, $p = ad_U + bd_D$, and for every $j \in J, \bar{\pi}_j = a \cdot \min\{d_U, j\} + b \cdot \min\{d_D, j\}$. Furthermore, j^* is the only debt contract positively traded, $\bar{\varphi}_j^h = 0$ if $j \neq j^*$.

Proof:

The proof was nearly given in the proof of the Simple Binomial No-Default Theorem. It is straightforward to show that if a previously untraded contract has its price adjusted into line with the state prices, then nothing is affected.■

3.4 Discussion

The Simple Binomial No-Default Theorem shows that in any static binomial model with a single financial asset, we can assume without loss of generality that the *only* debt contract actively traded is the max min debt contract (on which there is no default). Thus in static binomial models, leverage is endogenously determined in equilibrium by the Value at Risk equal zero rule, assumed by many other papers in the literature. Furthermore, by the Simple Binomial State Pricing Theorem, all the contracts (which may be arbitrarily numerous) can be priced by just two state prices. These two theorems make it extremely easy to compute collateral equilibrium for simple binomial collateral economies with debt contracts.

The Binomial No-Default Theorem does not say that equilibrium is unique, only that each equilibrium can be replaced by another with the *same asset price and the same consumption* by each agent, in which there is no default. Thus *potential* default has a dramatic effect on equilibrium, but *actual* default does not.

The Binomial No-Default Theorem has a sort of Modigliani-Miller feel to it. But the theorem does *not* assert that the debt-equity ratio is irrelevant. The theorem shows that if we start from any equilibrium, we can move to an equivalent equilibrium in which only max min debt is traded. If the original equilibrium had default, in the new equilibrium, leverage will be lower. Thus starting from a situation of default, the theorem does say that leverage can be lowered over a range until the point of no default, while leaving all investors indifferent. The theorem does *not* say that starting from a max min equilibrium, one can construct another equilibrium with still lower leverage, or even with higher leverage. Modigliani-Miller does not fully hold in our model because issuers of debt must hold collateral, and because we do not allow short selling of the asset. Our argument relies on positive spanning, not spanning. In Section 5 we give an example with a unique equilibrium in which every borrower leverages to the max min, but no agent would be indifferent to leveraging any less. In that example Modigliani-Miller completely fails.

The Simple Binomial No-Default Theorem not only shows that actual default is irrelevant, but also provides a very simple prediction about equilibrium leverage. According to the theorem, equilibrium LTV for the family of non-contingent debt contracts is the ratio of the worst case return of the asset divided by the riskless rate of interest.

$$LTV = \frac{d_D/(1+r)}{p} = \frac{d_D/p}{(1+r)}.$$

Equilibrium leverage depends on current and future asset prices, but is otherwise independent of the utilities or the endowments of the agents. In the extreme case when the volatility of asset returns is zero, leverage reaches its maximum of 100%.[9] Given a collection of assets with the same price, the asset whose future value has the least bad downside can be leveraged the most.[10] The No-Default Theorem suggests that one reason leverage might have plummeted from 2006-2009 is because the worst case return that lenders imagined got much worse.

Collateralized loans always fall into two categories. In the first category, a borrower is not designating all the assets he holds as collateral for his loans. In this case

[9] This would be the case in any model without uncertainty in which the asset can be costlessly seized in case of default, without moral hazard frictions.

[10] Suppose all the assets can be priced by the state prices a for U and b for D. Define the volatility of a dollar of period one asset as the variance of its second period price plus dividend, computed with respect to the probabilities $(1+r)a$ and $(1+r)b$. Then the asset with the highest volatility can be leveraged the least. Unfortunately even with two states, this volatility rule does not always work, because, as we shall see, different assets may require different state prices to value them.

he would not want to borrow any more at the going interest rates even if he did not need to put up collateral (but was still required, by threat of punishment, to deliver the same payoffs he would had he put up the collateral). His demand for loans is then explained by conventional text book considerations of risk and return. If all borrowers are in this case, then the rate of interest clears the loan market without consideration of default. In the second category, some or all the borrowers might be posting all their assets as collateral. In this case of scarce collateral, the loan market clears at a level determined by the spectre of default. Our contribution is to have proved that in binomial models with financial assets, the equilibrium LTV can be taken to be the same easy to compute number, no matter which category the loan is in.

The No-Default Theorem shows that the equilibrium LTV for each family of contracts is determined by asset values, otherwise independent of the preferences or endowments of the agents. If collateral is scarce, and agents are borrowing against all of it, then total borrowing is also determined independently from agent characteristics. On the other hand, if nobody wants to borrow beyond the debt capacity of the asset, then the collateral requirements are irrelevant, and debt is determined by the preferences of the agents in the economy, just as in models without collateral. Nonetheless, as the No-Default Theorem states, the LTV can be taken to be the same as that determined by the debt capacity. In short, there are two regimes. First, when all the assets fall into the first category, we can say that the debt in the economy is determined by the demand for loans. When all the assets and borrowers fall into the second category, we can say the debt in the economy is determined by the supply of credit, that is, by the maximum debt capacity of the assets.

The distinction between plentiful and scarce capital all supporting loans at the same LTV suggests that is useful to keep track of a second kind of leverage that we call diluted leverage. Consider the following example: if the asset is worth \$100 and its worst case payoff determines a debt capacity of \$80, then in equilibrium we can assume all debt loans written against this asset will have LTV equal to 80%. If an agent who owns the asset only wants to borrow \$30, then she could just as well put up only three eights of the asset as collateral, since that would ensure there would be no default. The LTV would then again be \$30/\$37.50 or 80%. Hence, it is useful to consider what we call *diluted* LTV, namely the ratio of the loan amount to the total value of the asset, even if some of the asset is not used as collateral. The diluted LTV in this example is 30%, because the denominator includes the \$62.50 of asset that was not used as collateral.

In the Simple Binomial State Pricing Theorem, the state prices a, b are like Arrow prices. Their existence implies that there are no arbitrage possibilities in trading the asset and the contracts. Even a trader who had infinite wealth and who was allowed to make promises without putting up any required collateral could not find a trade that made money in some state without ever losing money. However, the equilibrium may not be an Arrow-Debreu equilibrium, even though the state prices are uniquely defined. We shall see an example with unique state prices but Pareto inferior consumptions (coming from the collateral constraints) in Section 5.

We shall see in Section 4 that the Simple Binomial No-Default Theorem continues to hold in more complicated binomial models, but the Simple Binomial State Pricing Theorem does *not* extend to more complex binomial models. In Section 5 we give an example. These subtler binomial models can still be solved fairly easily by the knowledge that actual default is irrelevant, but the pricing of assets and contracts can become more interesting.

There are two key assumptions in the Binary No-Default Theorem. First, we only consider financial assets, that is, assets that do not give direct utility at time 0 to their holders, and which yield dividends at time 1 that are independent from who holds them at time 0. Second, we assume that the tree is binary.

In the first step of the proof, the Payoff Cone Lemma shows that the max min promise plus the U Arrow security (obtained by buying the asset while selling the max min debt contract), positively spans the cone of all feasible portfolio payoffs. The assumption of two states is crucial. If there were three states, it might be impossible for a portfolio holder to reproduce his original net payoffs from a portfolio in which he can only hold the asset and buy or issue the max min debt.

In the second step of the proof, the State Pricing lemma shows that any two portfolios that give the same payoffs in the two states must cost the same. One interesting feature of the proof is that it demonstrates the existence of state prices (that price all the assets) even though short-selling is not allowed. In general, if an instrument (asset or bond) C has payoffs that are a positive combination of the payoffs from instruments A and B, then the price of C cannot be above the positive combination of the prices of A and B. Any buyer could improve on buying C by combining the purchase of A and B. This logic gives an *upper-bound* for prices of all traded instruments. On the other hand, the price of C could be less than the price of the positive combination of A and B because there may be sellers of this instrument, but no agent interested in buying it, and the sellers cannot split C into A and B. Nonetheless, we show that we can also get a *lower-bound* for the price of C. The reason is that in our model, the sellers of the debt contract must own the collateral, and hence on *net* are in fact buyers. What they buy is a positive linear combination of A and B, which gives us an upper bound for the price of what they buy, and hence the missing lower bound on what they sell. In short, the crucial argument in the proof is that sellers are actually buyers of something else that is in the payoff cone. As we will see later, when there are multiple assets, or multiple kinds of loans on the same asset, the sellers of a bond in one family may not be purchasing something in the payoff cone of another family. Each family may require different state prices. That is why the No-Default Theorem holds more generally, but the State Pricing Theorem does not.

In the third step of the proof we use both lemmas to show that in equilibrium each agent is indifferent to replacing his portfolio with another such that on each unit of collateral that he holds, he either leverages to the maximum amount without risk of default, or does not leverage at all. The idea is as follows. If in the original equilibrium the investor leveraged his asset purchases *less* than the max min, he could always leverage some of his holdings to the max min, and the others not at all. If in the original equilibrium the investor was selling *more* debt than the max min, defaulting in the D state, then he could instead reduce his asset holdings and his debt sales to the max min level per unit of asset held, and still end up buying the same amount of the U Arrow security.[11] Let the original buyer of the original risky bond buy instead all of the new max min debt plus all the asset that the original risky bond seller no longer holds. By construction the total holdings of the asset is unchanged, and the total holdings of debt is zero, as before. Furthermore, by construction, the seller of the bond has the same portfolio payoff as before, so he is still optimizing. Since the total payoff is just equal to the dividends from the asset, and that is unchanged, the buyer of the bond must also end up with the same payoffs in the two states, so he is optimizing as well.

It is important to realize that the new portfolio may involve each agent holding a new amount of the collateral asset, while getting the same payoff from his new

[11] If he continued to hold the same assets while reducing his debt to the max min per asset, then he would end up with more of the U Arrow security.

portfolio of assets and contracts. Agents are indifferent to switching to the new portfolio because of the crucial assumption that the asset is a financial asset. If the collateral were housing or productive land for example, the theorem would not hold; it might well be that even with only two states agents would leverage in equilibrium to the point where they would default in one of the states (as shown in an example in Geanakoplos (1997, 2010)).

Finally, it is worth noting that in moving from an old equilibrium in which only contracts $j < j^*$ are traded to the new max min equilibrium, diluted leverage stays the same, but leverage on the margined assets rises. In moving from an old equilibrium with default in which a contract $j > j^*$ is traded to the new max min equilibrium, diluted leverage strictly declines, and leverage on the margined assets also declines.

4 The Irrelevance of Actual Default in a General Binomial Model of Endogenous Leverage.

In this section we show that the irrelevance of actual default is a much more general phenomenon, as long as we maintain our two key assumptions: financial assets and binary payoffs. We shall now present a model in which the conclusion of no default with endogenous leverage still holds even though we allow for the following extensions.

1. Arbitrary contracts.

   Previously we assumed that the only possible contract promise was non-contingent debt. Now we allow for arbitrary promises (j_U, j_D), provided that the max min version of the promise $(\bar{\lambda} j_U, \bar{\lambda} j_D)$ where $\bar{\lambda} = \max\{\lambda \in R_+ : \lambda(j_U, j_D) \leq (d_U, d_D)\}$ is also available.

2. Multiple kinds of contracts.

   Not only can the promises be contingent, there can also be many different (non-colinear) types of promises co-existing. See Figure 4.

3. Multiple assets.

   We can allow for many different kinds of collateral at the same time, each one backing many (possibly) non-collinear promises.

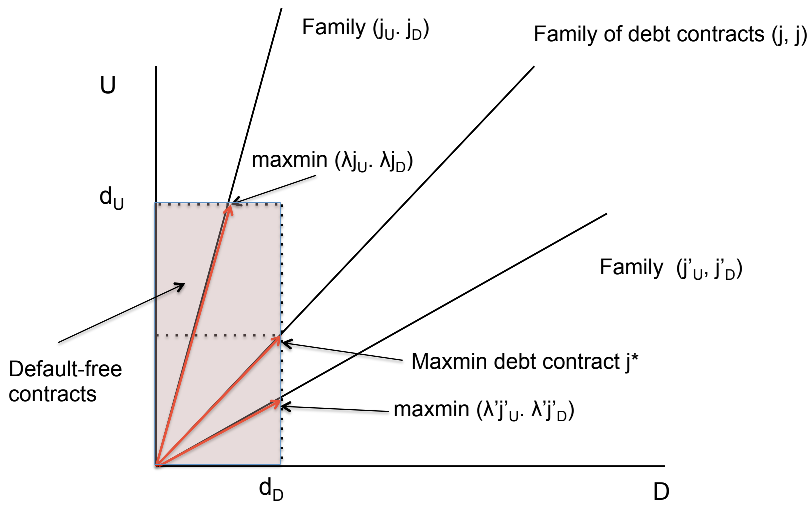

Figure 4: Different types of contingent contracts

4. Production and degrees of durability.

   The model already implicitly includes the storage technology for the asset. Now we allow the consumption goods to be durable, though their durability may be imperfect. We also allow for intra-period production. In fact, we allow for general production sets, provided that the collateral stays sequestered, and prevented from being used as an input.

5. Multiple goods.

   Unlike our previous model, in each state of nature there will be more than one consumption good.

6. Multiple periods.

   We will extend our model to a dynamic model with an arbitrarily (finite) number of periods.

7. Multiple states of nature.

   In each point in time, we will allow for multiple states of nature, as long as each (asset payoff, contract promise) pair takes on at most two values.

4.1 Model

4.1.1 Time and Assets

Uncertainty is represented by the existence of different states of nature in a finite tree $s \in S$ including a root $s = 0$, and terminal nodes $s \in S_T$. We denote the time of s as $t(s)$, so $t(0) = 0$. Each state $s \neq 0$ has a unique immediate predecessor s^*, and each non-terminal node $s \in S \backslash S_T$ has a set $S(s)$ of immediate successors.

Suppose there are $L = \{1, ..., L\}$ consumption goods ℓ and $K = \{1, ..., K\}$ financial assets k which pay dividends $d_s^k \in \mathbb{R}_+^L$ of the consumption goods in each state $s \in S$. The dividends d_s^k are distributed at state s to the investors who owned the asset in state s^*.

Finally, $q_s \in R_+^L$ denotes the vector of consumption goods prices in state s, whereas $p_s \in R_+^K$ denotes the asset prices in state s.

4.1.2 Investors

Each investor $h \in H$ is characterized by a utility, u^h, a discount factor, δ_h, and subjective probabilities γ_s^h denoting the probability of reaching state s from its predecessor s^*, for all $s \in S \backslash \{0\}$. We assume that the utility function for consumption in each state $s \in S$, $u^h : R_+^L \to R$, is differentiable, concave, and weakly monotonic (more of every good is strictly better). The expected utility to agent h is

$$U^h = u^h(c_0) + \sum_{s \in S \backslash 0} \delta_h^{t(s)} \bar{\gamma}_s^h u^h(c_s). \tag{3}$$

where $\bar{\gamma}_s^h$ is the probability of reaching s fom 0 (obtained by taking the product of γ_σ^h over all nodes σ on the path $(0, s]$ from 0 to s).

Investor h's endowment of the consumption good is denoted by $e_s^h \in R_+^L$ in each state $s \in S$. His endowment of the assets at the beginning of time 0 is $y_{0^*}^h \in R_+^K$ (agents have initial endowment of assets only at the beginning). We assume that the consumption goods are all present, $\sum_{h \in H}(e_s^h + d_s y_{0^*}^h) >> 0, \forall s \in S$.

4.1.3 Production

We allow for durable consumption goods (inter-period production) and for intra-period production. For each $s \in S \backslash \{0\}$, let $F_s^h : \mathbb{R}_+^L \to \mathbb{R}_+^L$ be a concave inter-period production function connecting a vector of consumption goods at state s^* that h is consuming with the vector of consumption goods it becomes in state s. In contrast to consumption goods, it is assumed that all financial assets are perfectly durable from one period to the next, independent of who owns them.

For each $s \in S$, let $Z_s^h \subset \mathbb{R}^{L+K}$ denote the set of feasible intra-period production for agent h in state s. Notice, that assets and consumption goods can enter as inputs and outputs of the intra-period production process. Inputs appear as negative components of $z_i < 0$ of $z \in Z^h$, and outputs as positive components $z_i > 0$ of z.

4.1.4 Collateral and Contracts

Contract $j \in J$ is a contract that promises the consumption vector $j_{s'} \in R_+^L$ in each state s'. Each contract j defines its issue state $s(j)$, and the asset $k(j)$ used as collateral. We denote the set of contracts with issue state s backed by one unit of asset k by $J_s^k \subset J$. We suppose that each contract $j \in J_s^k$ delivers only in the immediate successor states of s, i.e. $j_{s'} = 0$ unless $s' \in S(s)$. Contracts are defined extensively by their payment in each successor state. Notice that this definition of contract allows for promises with different baskets of consumption goods in different states. Finally, $J_s = \bigcup_k J_s^k$ and $J = \bigcup_{s \in S \setminus S_T} J_s$.

The price of contract j in state $s(j)$ is $\pi_{s(j)j}$. An investor can borrow $\pi_{s(j)j}$ at $s(j)$ by selling contract j, that is by promising $j_{s'} \in R_+^L$ in each $s' \in S(s(j))$, provided he holds one unit of asset $k(j)$ as collateral.

Since the maximum a borrower can lose is his collateral if he does not honor his promise, the actual delivery of contract j in states $s' \in S(s(j))$ is $min\{q_{s'} \cdot j_{s'}, p_{s'k(j)} + q_{s'} \cdot d_{s'}^k\}$.

The *Loan-to-Value* LTV_j associated to contract j in state $s(j)$ is given by

$$LTV_j = \frac{\pi_{s(j)j}}{p_{s(j)k}}. \tag{4}$$

As before, the *margin* m_j associated to contract j in state $s(j)$ is $1 - LTV_j$. *Leverage* associated to contract j in state $s(j)$ is the inverse of the margin, $1/m_j$ and moves monotonically with LTV_j.

Finally, as in Section 3, we define the *average loan to value*, LTV for asset k, as the trade-value weighted average of LTV_j across all debt contracts actively traded in equilibrium that use asset k as collateral, and the *diluted average loan to value*, LTV_0^k (which includes assets with no leverage) by

$$LTV^k = \frac{\sum_h \sum_j \max(0, \varphi_j^h)\pi_{s(j)j}}{\sum_h \sum_j \max(0, \varphi_j^h)p_{s(j)k}} \geq \frac{\sum_h \sum_j \max(0, \varphi_j^h)\pi_{s(j)j}}{\sum_h y_{0*}^h p_{s(j)k}} = LTV_0^k.$$

4.1.5 Budget Set

Given consumption prices, asset prices, and contract prices (q, p, π), each agent $h \in H$ choses intra-period production plans of goods and assets, $z = (z_c, z_y)$, consumption, c, asset holdings, y, and contract sales/purchases φ in order to maximize utility (3) subject to the budget set defined by

$B^h(q, p, \pi) = \{(z_c, z_y, c, y, \varphi) \in R^{SL} \times R^{SK} \times R_+^{SL} \times R_+^{SK} \times (R^{J_s})_{s \in S \setminus S_T} : \forall s$

$q_s \cdot (c_s - e_s^h - F_s^h(c_{s^*}) - z_{sc}) + p_s \cdot (y_s - y_{s^*} - z_{sy}) \leq$

$q_s \cdot \sum_{k \in K} d_s^k y_{s^*k} + \sum_{j \in J_s} \varphi_j \pi_j - \sum_{k \in K} \sum_{j \in J_{s^*}^k} \varphi_j min\{q_s \cdot j_s, p_{sk} + q_s \cdot d_s^k\};$

$z_s \in Z_s^h;$

$\sum_{j \in J_s^k} max(0, \varphi_j) \leq y_s^k, \forall k\}.$

In each state s, expenditures on consumption minus endowments plus any produced consumption good (either from the previous period or produced in the current period), plus total expenditures on assets minus asset holdings carried over from previous periods and asset output from the intra-period technology, can be at most equal to total asset deliveries plus the money borrowed selling contracts, minus the payments due at s from contracts sold in the past. Intra-period production is feasible. Finally, those agents who borrow must hold the required collateral.

4.1.6 Collateral Equilibrium

A *Collateral Equilibrium* in this economy is a set of consumption good prices, financial asset prices and contract prices, production and consumption decisions, and financial decisions on assets and contract holdings $((q, p, \pi), (z^h, c^h, y^h, \varphi^h)_{h \in H}) \in (R_+^L)_{s \in S} \times (R_+^K \times R_+^{J_s})_{s \in S \setminus S_T} \times (R^{S(L+K)} \times R_+^{SL} \times R_+^{SK} \times (R^{J_s})_{s \in S \setminus S_T})^H$ such that

1. $\sum_{h \in H}(c_s^h - e_s^h - F_s^h(c_{s^*}) - z_{sc}^h) = \sum_{h \in H} \sum_{k \in K} y_{s^*k}^h d_s^k, \forall s.$

2. $\sum_{h \in H}(y_s^h - y_{s^*}^h - z_{sy}^h) = 0, \forall s.$

3. $\sum_{h \in H} \varphi_j^h = 0, \forall j \in J_s, \forall s.$

4. $(z^h, c^h, y^h, \varphi^h) \in B^h(q, p, \pi), \forall h$

   $(z, c, y, \varphi) \in B^h(q, p, \pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h.$

Markets for consumption, assets and promises clear in equilibrium and agents optimize their utility in their budget set.

4.2 A More General No-Default Theorem

It turns out that we can still assume no default in equilibrium without loss of generality in this much more general context as the following theorem shows.

Binomial No-Default Theorem:

Suppose that S is a binomial tree, that is S(s)={sU,sD} for each $s \in S \backslash S_T$. Suppose that all assets are financial assets. Suppose that every contract is a one period contract. Let $((q, p, \pi), (z^h, c^h, y^h, \varphi^h)_{h \in H})$ be an equilibrium. Suppose that for any state $s \in S \backslash S_T$, any asset $k \in K$, and any contract $j \in J_s^k$, the max min promise $(\bar{\lambda} j_{sU}, \bar{\lambda} j_{sD})$ is also available to be traded, where $\bar{\lambda} = \max\{\lambda \in R_+ : \lambda(q_{sU} \cdot j_{sU}, q_{sD} \cdot j_{sD}) \leq (p_{sUk} + q_{sU} \cdot d_{sU},\ p_{sDk} + q_{sD} \cdot d_{sD})\}$. Then we can construct another equilibrium $((q, p, \pi), (z^h, c^h, \bar{y}^h, \bar{\varphi}^h)_{h \in H})$ with the same consumption, asset and contract prices and the same production and consumption choices, in which only max min contracts are traded.

Proof:

The proof of the Simple Binomial No-Default Theorem can be applied in this more general context state by state and ray by ray. Take any $s \in S \backslash S_T$ and any asset $k \in K$. Partition J_s^k into $J_s^k(r_1) \cup ... \cup J_s^k(r_n)$ where the r_i are distinct rays $(\mu_i, \nu_i) \in \mathbb{R}_+^2$ of norm 1 such that $j \in J_s^k(r_i)$ if and only if $(q_{sU} \cdot j_{sU}, q_{sD} \cdot j_{sD}) = \lambda(\mu_i, \nu_i)$ for some $\lambda > 0$. For each agent $h \in H$, consider the portfolio $(y^h(s,k,i), \varphi^h(s,k,i))$ defined by

$$\varphi_j^h(s,k,i) = \varphi_{sj}^h \text{ if } j \in J_s^k(r_i) \text{ and 0 otherwise.}$$
$$y^h(s,k,i) = \sum_{j \in J_s^k(r_i)} \max(0, \varphi_{sj}^h).$$

Denote the portfolio payoffs in each state by

$$w_U^h(s,k,i) = y^h(s,k,i)[p_{sUk} + q_{sU}d_{sU}^k] - \sum_{j \in J_s^k(r_i)} \varphi_j^h(s,k,i) \min(q_{sU} \cdot j_{sU}, p_{sUk} + q_{sU}d_{sU}^k).$$

$$w_D^h(s,k,i) = y^h(s,k,i)[p_{sDk} + q_{sD}d_{sD}^k] - \sum_{j \in J_s^k(r_i)} \varphi_j^h(s,k,i) \min(q_{sD} \cdot j_{sD}, p_{sDk} + q_{sD}d_{sD}^k).$$

If

$$\frac{\mu_i}{\nu_i} < \frac{p_{sUk} + q_{sU}d_{sU}^k}{p_{sDk} + q_{sD}d_{sD}^k}.$$

then the combination of the U Arrow security (which can be obtained by buying the asset k while borrowing on the max min contract of type (s,k,i)) and the max min contract of type (s,k,i) positively spans $(w_U^h(s,k,i), w_D^h(s,k,i))$. Thus we can apply the proof of the Simple Binomial No-Default Theorem to replace all the above trades of contracts in $J_s^k(r_i)$ with a single trade of the max min contract of type (s,k,i). If

$$\frac{\mu_i}{\nu_i} > \frac{q_{sUk} + p_{sU}d_{sU}^k}{q_{sDk} + p_{sD}d_{sD}^k}$$

then exactly the same logic of the Simple Binomial No-Default Theorem applies, but with the D Arrow security instead of the U Arrow security. If there is equality in the above comparison, then the contract and the asset are perfect substitutes, so there is no need to trade the contracts in the family at all. This concludes the proof.■

4.3 Discussion

The main idea of the proof is to apply the simple proof of Section 3 state by state to each asset and each homogeneous family of promises using the asset as collateral. It may now be the case that sometimes the payoff cone is given by the positive span of the max min of the family and the D Arrow security, instead of the U Arrow security. But the logic of the argument stays completely unaltered.

The same proof applies even if there are more than two successor states, provided that for each financial asset the states can be partitioned into two subsets on each of which the collateral value (including dividends of the asset) and the promise value of each contract written on the asset are constant.

The No-Default Theorem can also be extended to contracts with longer maturities. Suppose all the contracts written on some financial asset come due in the same period and that the states in that period can be partitioned into two subsets on each of which the collateral value (including dividends of the asset) and the promise value of each contract written on the asset are constant. Suppose also that the financial asset used as collateral cannot be traded or used for production purposes before maturity. Then the proof of the Binomial No-Default Theorem shows that without loss of generality we can assume no default in equilibrium.

Finally, notice that for each ray, say r_i, we obtain (by the same logic as before), state prices a_i and b_i. However, they need not be the same as the state prices obtained when the argument is applied to a different ray, say r_j. That is why, though the No-Default Theorem still holds, the State Pricing theorem does not. We will give examples of this in the next section.

5 Examples

In this section we study examples of two-period economies extensively used in the financial literature: i) CAPM investors with differences in risk aversion or differences in endowments, and ii) a continuum of risk neutral agents with heterogeneous beliefs. The examples illustrate when the No-Default and State Pricing theorems hold and when they fail.

5.1 Example 1: Binomial CAPM

We present two binomial examples with one financial asset in which the No-Default Theorem and the State Pricing Theorem hold.

We assume one perishable consumption good and one asset which pays dividends $d_U > d_D$ of the consumption good. Consider two types of mean-variance investors, $h = T, A$, characterized by utilities $U^h = u^h(c_0) + \sum_{s \in S_T} \gamma_s u^h(c_s)$, where $u^h(c_s) = c_s - \frac{1}{2}\alpha^h c_s^2$, $s \in \{0, U, D\}$. Agents do not discount the future. Agents have an initial endowment of the asset, $y^h_{0*}, h = T, A$. They also have endowment of the consumption good in each state, $e^h_s, \forall s, h = T, A$. It is assumed that all contract promises are of the form (j, j), $j \in J$, each backed by one unit of the asset as collateral. Agents will never deliver on a promise beyond the value of the collateral since we assume non-recourse loans.

Example 1.1 would satisfy all the assumptions of the classical CAPM provided that we assumed agents always kept their promises, without the need of posting collateral, and so would example 1.2 (extended to untraded endowments). We will present collateral equilibria which illustrate our theorems and the differences from classical CAPM.

5.1.1 Example 1.1: CAPM with Differences in Risk Aversion.

Agents in this case have different levels of risk aversion, so that $\alpha^T < \alpha^A$. Suppose agents each own one unit of the asset, $y^h_{0*} = 1, h = T, A$. Suppose consumption good endowments are given by $e^T = (e^T_0, (e^T_U, e^T_D)) = (1, (1, 1))$ and $e^A = (e^A_0, (e^A_U, e^A_D)) = (1, (1, 1))$. Utility parameters are given by, $\gamma_U = \gamma_D = .5$ and $\alpha^T = .05$ and $\alpha^A = .1$. Finally, asset payoffs are $d_U = 1$ and $d_D = .2$.

Table 1. Collateral Equilibrium with No Default: Prices and Leverage.

Variable	Notation	Value
Asset Price	p	0.5590
State Price	a	0.4585
State Price	b	0.5021
Max min Contract Price	π_{j^*}	0.1921
Leverage	LTV_{j^*}	0.3437

According to the Simple Binomial No-Default Theorem, in searching for equilibrium we never need to look beyond the max min promise $j^* = .2$, for which there will be no default. Tables 1 and 2 present this max min collateral equilibrium. The tolerant agents buy most of the asset in the economy, $y^T = 1.8372$, and use all of their holdings as collateral, leveraging via the max min contract, that is, by promising (.2)(1.8372) in both states U and D. The risk averse investors sell most of their asset endowment and lend to the more tolerant investors, that is, by buying the promises.[12]

[12] To find the equilibrium we guess the regime first and we solve for three variables, p, π_{j*} and ϕ_{j*}, a system of three equations. The first equation is the first order condition for lending corresponding to the risk averse investor:$\pi = \frac{q_U(1-\alpha^A c^A_U)d_D + q_D(1-\alpha^A c^A_D)d_D}{1-\alpha^A c^A_0}$.The second equation is the first order condition of the tolerant investor for purchasing the asset via the max min contract, $p - \pi = \frac{q_U(1-\alpha^T c^T_U)(d_U-d_D)+q_D(1-\alpha^T c^T_D)(d_D-d_D)}{1-\alpha^T c^T_0}$. The third equation is the first order condition for the risk averse investor for holding the asset, $p = \frac{\gamma_U(1-\alpha^A c^A_U)d_U+\gamma_D(1-\alpha^A c^A_D)d_U}{1-\alpha^A c^A_0}$. Finally, we check that the regime is genuine, confirming that the tolerant investor really wants to leverage to the max, for this to be the case, $\pi > \frac{\gamma_U(1-\alpha^T c^T_U)d_D+\gamma_D(1-\alpha^T c^T_D)d_D}{1-\alpha^T c^T_0}$.

Table 2. Collateral Equilibrium with No Default: Allocations.

Asset and Collateral			
	Asset y	Contracts φ_{j^*}	
Tolerant	1.8372	1.8372	
Averse	0.1628	-1.8372	
Consumption			
	$s = 0$	$s = 1$	$s = 2$
Tolerant	0.8850	2.4698	1.000
Averse	1.1150	1.5302	1.400

By the Simple Binomial State Pricing Theorem, all the contracts $j \neq j^*$, as well as $j = j^*$, can be priced by state prices $a = 0.4585$ and $b = 0.5021$. As mentioned before, by the No-Default Theorem, we do not need to investigate trading in any of the contracts $j \neq j^*$. Indeed it is easy to check that this is a genuine equilibrium, and that no agent would wish to trade any of these contracts $j \neq j^*$ at the prices given by a, b. Every agent who leverages chooses to sell the same max min contract, hence asset leverage and contract leverage are the same and described in the table.

This equilibrium is essentially unique, but not strictly unique. In fact, it is easy to check that there is another equilibrium *with* default as shown in Tables 3 and 4, in which the tolerant agents borrow by selling the contract $j = .2651 > j^* = .2$. In the default equilibrium, leverage is higher and the asset holdings of the borrowers are higher (so diluted leverage is much higher). They borrow more money. However, as guaranteed by the Simple Binomial Default Theorem, in both equilibria consumption and asset and contract prices are the same: *actual* default is irrelevant.

Table 3. Collateral Equilibrium with Default: Prices and Leverage.

Variable	Notation	Value
Asset Price	p	0.5590
Promise	j	0.2651
Contract j Price	π_j	0.2219
Leverage	LTV_{j^*}	0.3969

Table 4. Collateral Equilibrium with Default: Allocations.

	Asset and Collateral		
	Asset y	Collateral φ_j	
Tolerant	2	2	
Averse	0	-2	
	Consumption		
	$s = 0$	$s = 1$	$s = 2$
Tolerant	0.8850	2.4698	1.000
Averse	1.1150	1.5302	1.400

Between these two equilibria, the Modigliani-Miller Theorem holds; there is an indeterminacy of debt issuance in equilibrium. However, leverage cannot be reduced below the max min contract level. If the risk tolerant agents were forced to issue still less debt, they would rise in anger. Thus in this example, the No-Default Theorem holds while the Modigliani-Miller Theorem fails beyond a limited range.

Finally, the collateral equilibria do not correspond with the Arrow-Debreu Equilibria or the classical CAPM shown in Table 5.

Table 5. Arrow-Debreu and CAPM equilibrium.

Asset Price	p	0.5629	
State Price	p_U	0.4643	
State Price	p_D	0.4929	
	Consumption		
	$s = 0$	$s = 1$	$s = 2$
Tolerant	0.8951	2.2598	1.1681
Averse	1.1049	1.7402	1.2319
	CAPM Portfolios:	Market	Bond
Tolerant		0.6823	-0.4695
Averse		0.3177	0.4695

State prices in collateral equilibrium are different from the state prices in Arrow-Debreu equilibrium. The asset price in complete markets is slightly higher than in collateral equilibrium. Finally, investors hold shares in the market portfolio $(4, 2.4)$ (aggregate endowment) and in the riskless asset $(1, 1)$.

5.1.2 Example 1.2: CAPM with Differences in Wealth.

Agents in this case have different wealth. Consider the same CAPM model as before but with the following parameter values. Suppose agents each own one unit of the asset, $y^h_{0*} = 1, h = T, A$. Suppose consumption good endowments are given by $e^T = (e^T_0, (e^T_U, e^T_D)) = (1, (1, 5))$ and $e^A = (e^A_0, (e^A_U, e^A_D)) = (3, (5, 5))$. Utility parameters are given by, $\gamma_U = \gamma_D = .5$ and $\alpha^T = .1$ and $\alpha^A = .1$. Finally, asset payoffs are again $d_U = 1$ and $d_D = .2$. In this example agent T has a tremendous desire to buy U Arrow securities and present consumption, and to sell D Arrow securities. But he is limited by the restriction to non-contingent contract promises (j, j).

Tables 6 and 7 present the max min collateral equilibrium. In the collateral equilibrium type-T agents buy all the asset in the economy and use all of their holdings as collateral, leveraging via the max min contract. On the other hand, type-A investors sell all their asset endowment and lend.

Table 6. Collateral Equilibrium with No Default: Prices and Leverage.

Variable	Notation	Value
Asset Price	p	0.4572
State Price	a	0.4027
State Price	b	0.2725
Max min Contract Price	π_{j^*}	0.1350
Leverage	LTV_{j^*}	0.2952

Table 7. Collateral Equilibrium with No Default: Allocations.

	Asset and Collateral		
	Asset y	Contracts φ_{j^*}	
Tolerant	2	2	
Averse	0	-2	
	Consumption		
	$s = 0$	$s = 1$	$s = 2$
Tolerant	0.8122	2.6	5
Averse	3.1872	5.4	5.4

Unlike the previous example the no-default equilibrium in this example is unique. We cannot find another equilibrium involving default with borrowers issuing bigger promises, since there is not enough collateral in the economy. In this case, as before, the collateral equilibrium does not coincide with the complete markets equilibrium shown in Table 8. Unlike before, in this case the complete market asset price is lower than the collateral equilibrium asset price.

Table 8. Arrow-Debreu and CAPM equilibrium.

Asset Price	p	0.4350	
State Price	p_U	0.3750	
State Price	p_D	0.3	
	Consumption		
	$s=0$	$s=1$	$s=2$
Tolerant	0.8024	3.1018	4.4814
Averse	3.1976	4.8982	5.9186
	CAPM Porfolios	Market	Bond
Tolerant		0.5749	1.4970
Averse		0.4251	-1.4970

It is interesting that the asset price in binomial collateral equilibrium can be either higher or lower that the complete market price.

5.2 Example 2: Binomial Economy with Heterogeneous Priors and Two Assets

The following example taken from Fostel-Geanakoplos (2012b) shows how the No-Default theorem holds but the State Pricing Theorem can fail even in a binomial economy, once we add multiple assets.

There are two assets in the economy which produce dividends of the consumption good at time 1. The riskless asset X produces $d_U^X = d_D^X = 1$ unit of the consumption good in each state, and the risky asset Y produces $d_U^Y = 1$ unit in state U and $0 < d_D^Y < 1$ unit of the consumption good in state D. For added simplicity, we suppose that there is no consumption in period 0.

Each investor in the continuum $h \in H = (0,1)$ is risk neutral and characterized by a linear utility for consumption of the single consumption good c at time 1, and subjective probabilities, $(\gamma_U^h, \gamma_D^h = 1 - \gamma_U^h)$. The expected utility to agent $h \in H$ is $U^h(c_U, c_D) = \gamma_U^h c_U + \gamma_D^h c_D$

We shall suppose that γ_U^h is strictly monotonically increasing and continuous in h. Each investor $h \in (0,1)$ has an endowment of one unit of each asset at time 0 and nothing else. Since only the output of Y depends on the state and $1 > d_D^Y$, higher h denotes more optimism. Heterogeneity among the agents stems entirely from the dependence of γ_U^h on h.

In this economy we suppose that the risky asset Y can be used as collateral to back any promise of the form $(0, j)$. From the Binomial No-Default Theorem, we know that any equilibrium is equivalent to one in which the only contract traded is the max min contract $j = d_D^Y$, that is, the contract deliveriyng $(0, d_D^Y)$. The Y-payoff cone positively spanned by the asset Y, the contracts j that use Y as collateral, and the leveraged purchases of the asset Y, is all of $\mathbb{R}_+^2$. In effect, the asset Y can be *tranched* into arbitrary contingent promises. Of course the promise $(0, d_D^Y)$ is like a D Arrow security, and by buying the asset Y and selling off the tranche $(0, d_D^Y)$, any agent can obtain the U Arrow security.

By the first part of the Simple State Pricing Lemma, any contract written on the asset Y can be priced by state prices a and b. But this does not mean that other assets whose payoffs lie in the $Y-$payoff cone must also be priced by these state prices. It might well be that asset X sells for a lower price, precisely because X cannot be used as collateral and thus cannot be tranched into the same pieces as Y.

We calculate the equilibrium for the probabilities $\gamma_U^h = 1 - (1-h)^2$ and $d_D^Y = .4$. Results are shown in Table 9.

Table 9: Collateral Equilibrium.

Asset Y Price	p	1.1413
Price of Arrow U	a	0.8445
Price of Arrow D	b	0.7420
Marginal buyer of Arrow U	h_1	0.6056
Marginal buyer of Arrow D	h_2	0.1386

Without loss of generality, we fix the price of X to be 1 in state 0, and the price of consumption to be 1 in states U and D. We denote the price of the asset at 0 by p. In equilibrium there are two marginal buyers h_1 and h_2. All agents $h > h_1$ will buy all of Y, and sell the down tranche $(0, d_D^Y)$, hence effectively holding only the U Arrow security. Agents $h_2 < h < h_1$ will sell all their endowment of Y and purchase all of the riskless asset X. Finally, agents $h < h_2$ will sell their assets Y and X and buy the down tranche from the most optimistic investors.

In this example the No-Default Theorem holds, and collateral equilibrium does not involve default. But notice that in this example the Binomial State Pricing Theorem fails. The price $a = 0.8445$ of the U Arrow security and the price $b = 0.7420$ of the D Arrow security correctly price the risky asset Y ($1.1314 = .8445 + (.4).7420$), and the contracts promising $(0, j)$. But the state prices a and b *do not* price the riskless asset X, whose price is equal to $1 < p_U + p_D = 1.5865$. The reason for this, as discussed in Section 3, is that though we have an upper bound for the price (1.5865) given by the seller's side, we cannot find a lower bound. The seller of X is not on net buying anything that lies in the payoff cone, since X cannot be tranched. This is why in this more complicated Binomial economy with two assets, we cannot find state prices that price all the securities. This, as explained in Fostel-Geanakoplos (2012b), can generate asset prices bubbles.

The equilibrium is, as usual, not unique because there are other trivially equivalent equilibria. Trade could have happened instead via any promise $(0, j)$ backed by Y with $j > d_D^Y$. Such a contract would have sold for the same price as the promise $(0, d_D^Y)$, and delivered the same amount.

5.3 Example 3: Binomial Economy with One Asset and Two Families of Financial Contracts.

This example shows how the Simple State Pricing Theorem fails when two contract types can be written on the same asset.

Suppose there are four investors $h = A, B, C, D$ with utilities

$$
\begin{aligned}
U^A(c_0, c_U, c_D) &= c_0 + c_U, \\
U^B(c_0, c_U, c_D) &= c_0 + \min(c_U, \frac{c_D}{2}) \\
U^C(c_0, c_U, c_D) &= c_0 + \min(\frac{c_U}{2}, c_D) \\
U^D(c_0, c_U, c_D) &= c_0 + c_D
\end{aligned}
$$

Suppose agents A and D each begin with $1/2$ unit of the consumption good c_0 and nothing else. So $e^A = (1/2, (0, 0)) = e^D$ and $y_{0*}^A = y_{0*}^D = 0$. Agents B and C

each begin with one unit of the asset Y. So $e^B = (0,(0,0)) = e^C$ and $y^B_{0*} = y^C_{0*} = 1$. Suppose Y pays $d_U = d_D = 1$ of the consumption good over the two states. Finally, suppose that there are two contract families, $\mathcal{I}$ and $\mathcal{J}$, where the i-th contract of family $\mathcal{I}$ pays $i(\frac{1}{2}, 1)$ and the j-th contract of family $\mathcal{J}$ pays $j(1, \frac{1}{2})$. It is obvious that the max min contracts are $i^* = 1 = j^*$.

Table 10. Collateral Equilibrium with No Default: Prices and Leverage.

Variable	Notation	Value
Asset Price	p	1
Max min Contract Price	π_{j^*}	0.5
Max min Contract Price	π_{i^*}	0.5
Leverage	LTV_{j^*}	0.5
Leverage	LTV_{i^*}	0.5

Table 11. Collateral Equilibrium with No Default: Allocations.

	Asset and Collateral		
	Asset y	Contracts φ_{j^*}	Contracts φ_{i^*}
A	1	0	1
B	0	0	−1
C	0	−1	0
D	1	1	0
	Consumption		
	$s=0$	$s=1$	$s=2$
A	0	0.5	0
B	0.5	0.5	1
C	0.5	1	0.5
D	0	0	0.5

It is also obvious that there is an equilibrium with $p = 1$ and $\pi_{i^*} = 1/2 = \pi_{j^*}$ described by Tables 10 and 11.[13] Notice that the state prices $(a,b) = (1,0)$ are needed to price the asset and all the j contracts, while different state prices $(a,b) = (0,1)$ are needed to price the asset and all the i contracts.[14] This is the case because agent D, when selling contract j, is effectively buying the D Arrow security which is not

[13]This can most easily be seen by considering the two sub-economies $\{A,B\}$ and $\{C,D\}$ separately, and then realizing that their equilibria can be spliced together to form an equilibrium for the economy with all the agents $\{A,B,C,D\}$.

[14]The Arrow-Debreu equilibrium has state prices $p_U = p_D = 1/4$; agents B and C end up consuming 1/2 each of c_0, agent A consumes 2 units of c_1, and agent D consumes 2 units of c_D.

in the positive span of the max min of the family i and the U Arrow security (the payoff cone for that ray). On the other hand, agent A when selling contract i, is effectively buying the U Arrow security, which is not in the positive span of the max min of the family j and the D Arrow security (the payoff cone for that ray).

5.4 Example 4. CAPM with Default

In this example we extend example 1 to three states of the world, showing that then the No-Default Theorem and the State Pricing theorem fail to hold. In this example actual default matters. We retain the CAPM quadratic utilities, defined in general in example 1. But now $S = \{0, U, M, D\}$. Asset Y pays d_s units of the consumption good in state s, where $d_U \geq d_M \geq d_D$. Only non-contingent promises of the form (j, j, j) are allowed, and each one is collateralized by one unit of asset Y.

We solve the equilibrium for the following parameter values: asset payoffs $(d_U, d_M, d_D) = (3, 2, 1)$, asset endowments, $y^T_{0^*} = 0, y^A_{0^*} = 1$, good endowments, $e^T = (e^T_0, (e^T_U, e^T_M, e^T_D)) = (2, (1, 2, 2))$, and $e^A = (e^A_0, (e^A_U, e^A_M, e^A_D)) = (6, (6, 2, 2))$, risk aversion $\alpha^T = .1, \alpha^A = .1$, finally, probabilities, $\gamma^h_s = 1/3, \forall h, \forall s$. Tables 12 and 13 show the equilibrium.

Table 12. Collateral Equilibrium with Default: Prices and Leverage.

Variable	Notation	Value
Asset Price	p	2.4041
Contract Price	$\pi_{j=d_M}$	2.0836
Leverage	$LTV_{j=d_M}$	0.8666

Table 13. Collateral Equilibrium with Default: Allocations.

	Asset and Collateral			
	Asset y	Contracts $\varphi_{j=d_M}$		
Tolerant	1	1		
Averse	0	-1		
	Consumption			
	$s = 0$	$s = 1$	$s = 2$	$s = 3$
Tolerant	1.6795	2	2	2
Averse	6.3205	8	4	3

In equilibrium type-T agents buy all the asset in the economy and use it all as collateral to issue contracts that promises $d_M = 2$. On the other hand, type-A

agents sell all their asset and lend. There is default in state D in the economy. The No-Default theorem does not apply in this case. Borrowers do not want to borrow any less than what they do in equilibrium. Thus the max min contract $(1,1,1) = (d_D, d_D, d_D)$ is not traded in equilibrium, nor would agents be happy trading it in any equilibrium of this example.

5.5 Example 5: Heterogeneous Beliefs with Default and Multiple LTVs.

This example is analogous to examples in Fostel-Geanakoplos (2012a). Consider a continuum of risk neutral agents $h \in H = (0,1)$ with heterogeneous priors γ_s^h over the states s in period 1. Suppose each agent begins with one unit of asset Y, and one unit of asset X. We suppose there is no consumption at time 0, and that there are three states of nature instead of two. Asset Y pays off $d_U > d_M > d_D$ units of the consumption good in the three states, and asset X pays off 1 unit of the consumption good in each of the three states. Only Y can be used as collateral, and every contract j promises a non-contingent vector (j,j,j) of the consumption good in the three states. Not only do we find actual default, but we also find another departure from the No-Default Theorem. This example shows that two contracts will be traded: a risk-less contract as before that promises d_D, the worst-case or max min scenario in the future, and a risky contract that promises d_M in all states but defaults and delivers only d_D in $s = D$.

For concreteness we display the equilibrium for the following prior probabilities and asset payoffs: $\gamma_U^h = h, \gamma_M^h = h(1-h)$ and $\gamma_D^h = (1-h)^2$, and $d_U = 3, d_M = 2$ and $d_D = 1$. Notice that the higher the h, the more optimistic the agent.

Table 14 shows the results. Without loss of generality, we take the price of asset X to be 1 in state 0, and the price of the consumption good to be 1 in each state $s = U, M, D$. It turns out that the agents can be partitioned into four groups, separated by the three marginal buyers h_M, h_D, h_B. All agents above $h_M = .93$ hold only the risky asset Y, obtained by selling all their X and borrowing all they can on their Y via the risky bond $j = d_M$. Their total holdings of Y are then the $1 - h_M$ they collectively held as endowments, plus the y they collectively bought, where it turns out that $1 - h_M + y = .35$.[15] The next most optimistic agents

[15]Notice that total asset holdings consist of initial endowments, $1 - .93$, plus new purchases, .27.

$.93 = h_M > h > h_D = .66$ buy the remaining .65 units of the risky asset Y, obtained by selling all their X and by selling the riskless contract $j = d_D$. Investors with $.66 = h_D > h > h_B = .48$ sell all their assets X and Y and simply buy (lend) in the risky bond market, i.e. they buy all of the risky contracts $j = d_M$. The most pessimistic investors $h < h_B = .48$ hold all of the X and buy (lend) in the default-free market $j = d_D$. Clearly the riskless contract $j = d_D$ and the asset X are perfect substitutes, and since the most pessimistic agents are buying both, they must sell for the same price per unit promised.

Table 14: Collateral Equilibrium with Default and Multiple Contracts.

Marginal Buyers		
h_M		0.9307
h_D		0.6589
h_B		0.4839
Prices		
Asset price	p	2.4197
Bond price	π_{d_M}	1.7336
Asset purchases in the risky market	y	0.276

When the asset can take on at most two immediate successor values, equilibrium determines a unique actively traded promise (the max min contract) and hence leverage. With three or more successor values, we cannot expect just a single promise to emerge in equilibrium. In this example there is default in equilibrium, and different agents buy the same asset with different leverage. But equilibrium still determines the economy-wide average leverage used to buy the asset. Equilibrium leverage is presented in table 15. There are four securities in total, three risky securities and one risk-less security. Columns 2 and 3 show the holdings and value of such holdings for each of the securities. Most importantly, column 4 shows the LTV of each of the two traded contracts. As was expected, LTV is higher for the risky contracts (they have a higher promise), $LTV_{j=d_M} > LTV_{j=d_D}$. Finally, column 5 shows the asset LTV^Y. As defined in section 2, asset LTV is a weighted average, so it is obtained from the total amount borrowed using all contracts, $.5986 + .6547$ divided by the total value of collateral, 2.4197×1.

Table 15: Equilibrium Leverage.

Security	Holdings	Holdings Value	Contract LTV	Asset LTV
Y lev Medium	0.3453	0.8355	0.7165	0.5180
Y lev Min	0.6547	1.5842	0.4133	
Risky Bond	0.3453	0.5986		
Riskless Bond	0.6547	0.6547		

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