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Bounded Rationality and Limited Datasets*

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Abstract

Theories of bounded rationality are typically characterized over an exhaustive data set. How does one tell whether observed choices are consistent with a theory if the data is incomplete? How can out-of-sample predictions be made? What can be identified about preferences? This paper aims to operationalize some leading bounded rationality theories when the available data is limited, as is the case in most practical settings. We also point out that the recent bounded rationality literature has overlooked a methodological pitfall that can lead to 'false positives' and 'empty' out-of-sample predictions when testing choice theories with limited data.

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1. Introduction

The recent literature has proposed insightful and plausible choice procedures to explain the mounting evidence against rational choice. The axiomatic characterizations offered for these new theories typically assume an exhaustive dataset, that is, one where the choice from every possible problem is recorded. In most cases, however, data will be limited. In empirical settings, the modeler cannot control the choice problems faced by individuals. In experimental settings, generating a complete dataset requires an overwhelming number of decisions by subjects: there are 26 choice problems when the space of alternatives contains 5 elements, 1,013 choice problems when it contains 10 elements, and 32,752 choice problems when it contains 15 elements. In such cases, it is important to understand when observed choices can be explained by a theory, as well as what out-of-sample predictions can be made and what information can be identified about the Decision Maker (henceforth DM).

Understanding the theory of rational choice in the presence of limited data is a classic and well-understood question (Samuelson, 1948; Houthaker, 1950; Richter, 1966; Afriat, 1967; Varian, 1982). This paper contributes to the understanding of recent theories when data is limited, bringing ideas and challenges from the classic literature into the discourse on bounded rationality.

Observed choices are consistent with a theory if there exists a choice function, defined for all choice problems, which is explained by the theory and agrees with observed choices. One need not worry about defining choices for out-of-sample problems when testing for rationality in its standard description. If one has found a complete transitive preference ordering for which observed choices are maximal, then there is no difficulty defining choices for out-of-sample problems in a way that is consistent with rationality: one simply maximizes that same preference. Such extensibility need not hold for other theories. For an obvious illustration of where things can go wrong, consider the theory where the DM's choices emerge through the maximization of a complete *asymmetric* relation. Clearly, this is just an unusual rewriting of rationality, as choices are defined everywhere only if the relation is also transitive. Suppose we observe a DM picking a out of $\{a, b\}$, b out of $\{b, d\}$ and d out of $\{a, d\}$ (or, more generally, any observed choices satisfying WARP but not SARP). It may be tempting to claim that the DM's choices are consistent with this theory, as they coincide in all the *observed* choice problems with the maximization of the asymmetric relation P given by $aPbPdPa$. However, upon further reflection, one would logically conclude otherwise: any extension of observed choices to $\{a, b, d\}$ contradicts the theory.

This point may seem quite abstract, given the nonstandard description of rationality in the above example. However, it highlights a potentially dangerous methodological pitfall, as variants of this extensibility issue affect state-of-the-art theories of bounded rationality. To illustrate, consider a theory which imposes restrictions regarding which alternatives the DM actively considers when maximizing his preference. For instance, Masatlioglu, Nakajima and Ozbay (2012) posit that removing unconsidered alternatives does not change his consideration set. The data might seem consistent with this theory if observed choices are explained by the maximization of a preference over consideration sets satisfying their property for *observed* choice problems. However, one's conclusions regarding the content of these consideration sets could have contradictory implications in *unobserved* problems, making it impossible to extend choices in a way that is consistent with the theory. Hence one cannot limit the test of consistency to finding a story that explains the observed data, without thinking about whether that story extends. We show in Section 3 that previous attempts[1] to study bounded rationality theories with limited datasets overlook out-of-sample restrictions, potentially yielding false positives when testing consistency with the theory and precluding proper identification and out-of-sample prediction.

Thus, despite recent progress, there is still incomplete understanding of how to test bounded rationality theories when data is limited. Importantly, full-data characterizations may be missing some testable implications when applied to incomplete datasets, much in the same way that SARP provides a fuller understanding of the empirical content of rationality than IIA. Section

[1] Those attempts include Manzini and Mariotti (2007, Corollary 1), Manzini and Mariotti (2012) and Tyson (2013).

4 shows how the empirical content of recent prominent theories of Categorization (Manzini and Mariotti, 2012), Limited Attention (Masatlioglu, Nakajima and Ozbay, 2012), and Rationalization (Cherepanov, Feddersen and Sandroni, (2012) can be fully captured by the existence of an acyclic relation satisfying intuitive restrictions, highlighting testable implications that are missing in existing full-data characterizations.

For testing rationality, Samuelson observed that the following revealed-preference restrictions emerge: x must dominate any other option y that is available when x is chosen. Samuelson's Strong Axiom of Revealed Preference (SARP) requires checking that there is an acyclic relation satisfying these restrictions; or in other words, that Samuelson's revealed preference is itself acyclic. While this may still seem hard at first, there are simple procedures allowing to solve this classic problem. For instance, start by making a guess, called x_1, as to which option is least preferred in X. Clearly this guess x_1 cannot dominate another element according to Samuelson's revealed preference. This reduces the problem at hand, allowing to restrict attention to $X \setminus \{x_1\}$. Next, make a guess, x_2, as to which element is least preferred in $X \setminus \{x_1\}$, etc. This *enumeration procedure* succeeds if at least one guess, an option which does not Samuelson-dominate anything that remains, is available in each round. Success is thus equivalent to SARP. Success is easy to check, as it is *path independent* in this case: it does not depend on which guess is chosen when there are multiple possibilities.

We show in Section 5 that this approach can generalize to theories of bounded rationality. The restrictions derived in Section 4 simplify testing, but are typically more elaborate than for rationality. Under Limited Attention, for instance, the data may inform us that option a is preferred to a', or that option b is preferred to b', requiring a decision on the 'or' condition prior to checking for acyclicity. Is testing the theory against the data thus significantly more difficult than testing rationality? Not necessarily. We observe that there is a general class of restrictions that can be tested by enumeration, with success being path independent, just as for rationality. Applying this result, we find that for some theories, including those studied in Section 4, such a simple test exists when the dataset satisfies certain conditions. For these theories, it may be useful to keep such results in mind to guide data collection. In the Online Appendix, we provide several examples of bounded rationality theories which are roughly as easy to test as rationality, regardless of the dataset.

We show in Section 6 how our methodology can be used for identification and out-of-sample prediction. Section 7 concludes with remarks on how our approach can provide insight beyond the theories studied here.

2. Theories of Choice, Limited Data and Predictions

Consider a finite set X of possible alternatives. A *choice problem* is a nonempty subset of X representing the set of feasible alternatives for the DM. The set $\mathcal{P}(X)$ of nonempty subsets of X represents all *conceivable* choice problems the DM may face. A *choice function* $c : \mathcal{P}(X) \to X$ associates to each choice problem S an element $c(S) \in S$. Different theories have been defined to describe how a DM makes choices. Rationality, for instance, posits that the DM picks options by maximizing some preference ordering[2] P:

$$c(S) = \arg\max_P S, \text{ for all } S \in \mathcal{P}(X). \tag{1}$$

Choice theory has developed in recent years to better understand the mounting evidence against rational choice. Consider the following theories, which we will use in this paper to illustrate main points. Under *Shortlisting* (Manzini and Mariotti 2007, henceforth MM07), the DM creates a shortlist of alternatives which are undominated according to some asymmetric preference relation R_1, and picks the maximal alternative in the shortlist according to some asymmetric relation R_2:

$$c(S) = \arg\max_{R_2}\{x \in S \mid \nexists y \in S : \; yR_1x\}, \text{ for all } S \in \mathcal{P}(X). \tag{2}$$

Under *Limited Attention* (Masatlioglu, Nakajima and Ozbay 2012, henceforth MNO12), a DM facing a problem S picks the best element according to some

[2]We use the term *relation* to mean a (possibly incomplete or cyclic) binary relation, and the term *ordering* for a complete, asymmetric and transitive relation. For any relation P and any $S \subseteq X$, let $\arg\max_P S = \{x \in S \mid xPy, \; \forall y \in S \setminus \{x\}\}$.

preference ordering P over a consideration set $\Gamma(S) \subseteq S$, with the restriction that removing ignored alternatives does not change the consideration set:[3]

$$c(S) = \arg\max_P \Gamma(S), \text{ for all } S \in \mathcal{P}(X), \tag{3a}$$

$$\Gamma(S) \subseteq T \subseteq S \Rightarrow \Gamma(T) = \Gamma(S), \text{ for all } S, T \in \mathcal{P}(X). \tag{3b}$$

Under *Categorization* (Manzini and Mariotti 2012, henceforth MM12), the DM uses an asymmetric 'shading' relation $\succ$ defined on categories (which are subsets of X) to eliminate alternatives belonging to an inferior category, and then picks an option by maximizing some asymmetric preference relation P:

$$c(S) = \arg\max_P \{x \in S \mid \nexists R, R' \subseteq S : x \in R \text{ and } R' \succ R\}, \ \forall S \in \mathcal{P}(X) \tag{4}$$

Under *Rationalization* (Cherepanov, Feddersen and Sandroni 2013, henceforth CFS13), the DM maximizes an asymmetric preference relation P over the set of 'rationalizable' options, those which are top-ranked for at least one of his rationales (asymmetric relations $R_1, \ldots, R_K$):

$$c(S) = \arg\max_P \{x \in S \mid \exists i : x = \arg\max_{R_i} S\}, \text{ for all } S \in \mathcal{P}(X). \tag{5}$$

This paper considers the problem of testing whether a DM's choice behavior is consistent with theories such as these. Importantly, we want to allow for the possibility that choices are observed over only some of the conceivable choice problems. The *dataset* $\mathcal{D} \subseteq \mathcal{P}(X)$ consists of those choice problems over which the DM's choice is observed. An *observed choice function* $c_{obs} : \mathcal{D} \to X$ associates to each choice problem $S \in \mathcal{D}$ the alternative in S which has been selected by the DM.

For a theory $\mathcal{T}$, let $\mathcal{C}_\mathcal{T}$ be the set of (complete) choice functions that could emerge under $\mathcal{T}$. Our analysis for limited datasets is grounded on the following principle. For observed choices to be consistent with $\mathcal{T}$, it should be possible to extend observed choices c_{obs} into a complete choice function $c \in \mathcal{C}_\mathcal{T}$.

Definition 1 (Consistency)**.** *An observed choice function $c_{obs} : \mathcal{D} \to X$ is consistent with a theory $\mathcal{T}$ if there exists a choice function $c \in \mathcal{C}_\mathcal{T}$ such that*

[3] MNO12 write $\Gamma(S \setminus \{x\}) = \Gamma(S)$ for all $x \in S \setminus \Gamma(S)$ instead of (3b). As their condition could be vacuous on limited data (which they did not study), condition (3b) is an iterated version that captures all the restrictions consistent with their original motivation.

$c_{obs}(S) = c(S)$ *for every* $S \in \mathcal{D}$.

Given a theory $\mathcal{T}$, there may be multiple ways of extending c_{obs} into a choice function in $\mathcal{C}_{\mathcal{T}}$. Each such extension suggests an option the DM might pick in a given out-of-sample choice problem, leading to a set of possible predictions.

Definition 2 (Prediction)**.** *Fix a theory* $\mathcal{T}$*, observed choices* $c_{obs} : \mathcal{D} \to X$*, and a choice problem* $S \notin \mathcal{D}$*. An option* $x \in S$ *is a prediction for* S *under* $\mathcal{T}$ *if there exists a choice function* $c \in \mathcal{C}_{\mathcal{T}}$ *that coincides with* c_{obs} *on* $\mathcal{D}$ *and has* $c(S) = x$.

The following proposition shows that testing an out-of-sample prediction boils down to testing the consistency of an extended observed choice function.

Proposition 1. *Consider a theory* $\mathcal{T}$*, observed choices* $c_{obs} : \mathcal{D} \to X$*, a choice problem* $S \notin \mathcal{D}$*, and an option* $x \in S$*. Define* $\overline{\mathcal{D}} = \mathcal{D} \cup \{S\}$ *and* $\bar{c}_{obs} : \overline{\mathcal{D}} \to X$ *by* $\bar{c}_{obs}(T) = c_{obs}(T)$ *for all* $T \in \mathcal{D}$ *and* $\bar{c}_{obs}(S) = x$*. Then* x *is a prediction for* S *under* $\mathcal{T}$ *if and only if* $\bar{c}_{obs}$ *is consistent with* $\mathcal{T}$*.*

Therefore, addressing the question of out-of-sample predictions reduces to identifying the testable implications of a theory on limited data.

3. A Potential Pitfall

As is well-known, the Independence of Irrelevant Alternatives (IIA) captures the empirical content of rationality only if the data set is rich enough (e.g. including all choice problems of cardinality two and three). The Strong Axiom of Revealed Preference (SARP) was identified to characterize rationality independently of the data one has collected. Similarly, more work is needed to capture the testable implications of bounded rationality theories, as the full-data characterizations that have been found need not apply to limited data.

Before working towards that end in Section 4, we note that the recent literature on bounded rationality has overlooked a potential pitfall one should keep in mind when testing theories in the presence of limited data. To test Categorization with limited data, MM12 (Definition 4) investigate under which conditions on observed choices do there exist asymmetric relations $\succ, P$ such that (4) holds for $S \in \mathcal{D}$, instead of $S \in \mathcal{P}(X)$. Similarly, for Shortlisting, MM07 (Corollary 1) study when there exist asymmetric relations R_1, R_2 such that (2) holds for $S \in \mathcal{D}$. To test Limited Attention, Tyson (2013) seeks conditions guaranteeing the existence of an ordering P and a consideration set mapping defined on $\mathcal{D}$ such that (3a) and (3b) hold for $S, T \in \mathcal{D}$. In other words, the conditions describing how choices emerge under a theory are checked only over observed problems.

Such an approach may seem natural at first, but one has to be mindful that it may yield 'false positives,' as it may be impossible to extend observed choices into a complete choice function under the theory. This extensibility issue affects prevalent theories of bounded rationality, leading to a potentially dangerous methodological pitfall.

Example 1 (Categorization and Shortlisting)**.** *Take* $X = \{a, b, d, e, f, g\}$, $\mathcal{D} = \{ab, bd, ad, abde, abdf, abdg\}$, *and* $c_{obs} : \mathcal{D} \to X$ *given by:*

S	ab	bd	ad	$abde$	$abdf$	$abdg$
$c_{obs}(S)$	a	b	d	a	b	d

It may seem that this data is consistent with Categorization, since (4) holds for $S \in \mathcal{D}$ *using the shading relation* $\dot{\succ}$ *defined by* $\{e\}\dot{\succ}\{d\}$, $\{f\}\dot{\succ}\{a\}$ *and* $\{g\}\dot{\succ}\{b\}$, *and the preference* $\dot{P}$ *defined by* $a\dot{P}b\dot{P}d\dot{P}a$ *and* $x\dot{P}y$ *for* $x \in \{a, b, d\}$ *and* $y \in \{e, f, g\}$*. However, the data is in fact inconsistent with Categorization. To see this, suppose to the contrary that* c_{obs} *can be extended to a complete choice function* c *which can be derived through (4) via some shading relation* $\succ$ *and some preference* P*. Since* a *does not belong to a dominated category in* $\{a, b, d, e\}$*, it cannot belong to a dominated category in either* $\{a, b, d\}$ *or* $\{a, d\}$*. In view of this and* $c_{obs}(\{a, d\}) = d$*, it must be that* dPa*. Applying the symmetric reasoning for each of* b *and* d*, it must be that* $aPbPdPa$ *and all three alternatives survive elimination in* $\{a, b, d\}$*. This leads to a contradiction, since* $\{a, b, d\}$ *has no* P*-maximal element.*[4] *A similar difficulty arises for*

[4] As seen here, there is an extensibility problem as soon as the set of conceivable problems contains $\{a, b, d\}$ in addition to $\mathcal{D}$. Thus extensibility can be an issue even if the set of conceivable choice problems is a strict subset of $\mathcal{P}(X)$.

Shortlisting: (2) holds for $S \in \mathcal{D}$ using $\dot{R}_2 = \dot{P}$ and $\dot{R}_1$ defined by $e\dot{R}_1 d$, $f\dot{R}_1 a$ and $g\dot{R}_1 b$. However, c_{obs} is inconsistent with Shortlisting, since consistency with that theory implies consistency with Categorization.

Example 2 (Limited Attention)**.** *Take $X = \{a, b, d, e, f\}$, $\mathcal{D} = \{ae, ef, abd, ade, bde, bef\}$, and $c_{obs} : \mathcal{D} \to X$ given by:*

S	ae	ef	abd	ade	bde	bef
$c_{obs}(S)$	e	f	d	a	b	e

It may seem that this data is consistent with Limited Attention, since (3a) and (3b) hold for $S, T \in \mathcal{D}$ using the preference ordering $\dot{P}$ defined by $a\dot{P}d\dot{P}e\dot{P}b\dot{P}f$, and the consideration sets $\dot{\Gamma}(S)$ consisting of $c_{obs}(S)$ and its $\dot{P}$-lower contour set. However, c_{obs} is in fact inconsistent with Limited Attention. To see this, suppose to the contrary that c_{obs} can be extended to a complete choice function c which satisfies (3a) for some preference P and consideration set mapping Γ satisfying (3b). We first show that $d \in \Gamma(\{a, d, e\})$ and $b \in \Gamma(\{b, e, f\})$, which would then imply that aPd and ePb. Suppose instead that $d \notin \Gamma(\{a, d, e\})$; the argument for b is analogous. Property (3b) implies $\Gamma(\{a, e\}) = \Gamma(\{a, d, e\})$, requiring the choice from $\{a, e\}$ to be a, a contradiction. Consider the out-of-sample problem $\{b, d\}$. The ranking aPd implies $a \notin \Gamma(\{a, b, d\})$, which in turn implies that $\Gamma(\{b, d\}) = \Gamma(\{a, b, d\})$. Hence the choice from $\{b, d\}$ is d. At the same time, the ranking ePb implies $e \notin \Gamma(\{b, d, e\})$, which in turn implies that $\Gamma(\{b, d\}) = \Gamma(\{b, d, e\})$. Hence the choice from $\{b, d\}$ is b, a contradiction.

Being subject to this pitfall is not a weakness of a theory. Rather, the moral is that one cannot limit the test of consistency to finding a story that explains the observed data, without thinking whether that story extends. This extensibility problem is precisely avoided (for any theory) by employing Definition 1.

4. Testable Implications of Some Leading Theories

Despite recent progress, Section 3 shows that there is still incomplete understanding of how to test bounded rationality theories when data is incomplete. There is a need for a systematic test of consistency with a given theory, going beyond the specific arguments that were tailor-made for each dataset in the previous section. We now address this matter for some prominent theories, highlighting how their existing full-data characterizations are missing some testable implications when applied to incomplete datasets.

4.1 Limited Attention

Under Limited Attention, a DM maximizes a preference, but only over his consideration set. Thus, observing that x is chosen from S does not imply that x is better than $y \in S$, because the DM may not have paid attention to y. One can only infer that x is revealed preferred to alternatives in his consideration set at S, which itself must be inferred from the choice data. If one wishes to test Limited Attention, then what is all the information about preferences that can be recovered from observed choices?

Suppose the DM picks x out of S, but a different option when dropping z from S. Such an IIA violation (where sets differ by one element) is consistent with Limited Attention only if the DM pays attention to z in S and prefers x to z (if z were ignored, then removing it would not affect choice). Studying full datasets, MNO12 show that these choice patterns contain all the information that can be gleaned. This need not be the case with limited data. Suppose we only see a more general IIA violation, where $c_{obs}(S) \neq c_{obs}(S \setminus \{a, b\})$ for two options $a, b \in S$ that differ from $c_{obs}(S)$. Extending the reasoning above, a or b must receive attention in S and is thus revealed worse than $c_{obs}(S)$.

More subtly, information can be gleaned from any WARP violation. Suppose $c_{obs}(S) \neq c_{obs}(S')$, and yet both options are in $S \cap S'$. If the DM does not pay attention to any option in $S \setminus S'$ (*resp.,* $S' \setminus S$) when choosing from S (*resp.,* S'), then he must choose $c_{obs}(S)$ (*resp.,* $c_{obs}(S')$) from $S \cap S'$. To avoid a contradiction, it must be that the DM pays attention to some option of $S \setminus S'$ when choosing from S (with $c_{obs}(S)$ revealed preferred to it), *or* that he pays attention to some option of $S' \setminus S$ when choosing from S' (with $c_{obs}(S')$ revealed preferred to it). This captures the theory's behavioral content, as seen next.

To introduce the result, let $\mathcal{R}_{LA}(c_{obs})$ be the following set of restrictions on a relation P: for every $S, S' \in \mathcal{D}$ where observed choices differ and belong to $S \cap S'$, we have $c_{obs}(S) P y$ for some $y \in S \setminus S'$ or $c_{obs}(S') P y'$ for some $y' \in S' \setminus S$.

Proposition 2. *The observed choice function $c_{obs} : \mathcal{D} \to X$ is consistent with Limited Attention if and only if there is an acyclic relation satisfying $\mathcal{R}_{LA}(c_{obs})$.*

With a full dataset, as studied in MNO12, the content of the theory is fully captured by those restrictions in $\mathcal{R}_{LA}$ corresponding to nested choice problems which differ by only one element; the other restrictions become redundant.

As can easily be checked, the set of restrictions $\mathcal{R}_{LA}$ associated to observed choices in Example 2 are: (*i*) aPd, (*ii*) ePb, (*iii*) dPb or aPe, (*iv*) bPd or ePf, and (*v*) dPa or bPe. Clearly the first two constraints contradict the last one, confirming our earlier observation that this dataset is not consistent with Limited Attention. Because Proposition 2 provides a systematic test, we can also discover something new: that observed choices in Example 1 *are* consistent with Limited Attention. Indeed, it is not difficult to check that the constraints are: (*i*) aPe or bPf, (*ii*) aPe or dPg, (*iii*) bPf or dPg, (*iv*) bPd or bPf, (*v*) dPa or dPg, and (*vi*) aPb or aPe. Note, for instance, that all of these restrictions are satisfied if aPe, bPf, and dPg (which is acyclic).

4.2 Categorization and Rationalization

As established in the literature, Rationalization and Categorization are both observationally equivalent to maximizing a complete, asymmetric relation P over a *psychological filter* Ψ, i.e., a consideration set mapping satisfying the following 'filter' property: if the DM pays attention to an element in a set, then he also pays attention to it in any subset in which it is contained.

Before studying the DM's preference, we first examine what the choice data reveals about his consideration set. In a choice problem S, the filter property implies that he certainly pays attention to all the elements in $\Psi_{CFS}(S) := \{c_{obs}(T) \mid S \subseteq T,\ T \in \mathcal{D},\ c_{obs}(T) \in S\}$. CFS13 introduced Ψ_{CFS} to characterize their theory under complete data. When data is limited, Ψ_{CFS} may not qualify as a filter, as it may be empty-valued for some out-of-sample choice problems. However, if the data is indeed consistent with Rationalization and Categorization, then there must be some extension c of observed choices under the theory. By the filter property, $c(X)$ must be paid attention to in any choice problem in which it is contained. Similarly, $c(X \setminus \{c(X)\})$ is paid attention to in any subset of $X\setminus\{c(X)\}$ in which it is contained. By iteration, we can construct a valid psychological filter by adding only the O-minimal alternative to Ψ_{CFS}, where O ranks $c(X)$ at the bottom, $c(X\setminus\{c(X)\})$ right above $c(X)$, etc. Of course, the DM might consider additional alternatives. Nonetheless, the next result shows that there is no loss of generality from representing his choices using a 'minimal' filter constructed as above.

Lemma 1. *For any complete asymmetric preference P, there exists a psychological filter Ψ such that c_{obs} is explained by (Ψ, P) if and only if there exists an ordering O such that c_{obs} is explained by (Ψ_O, P), where Ψ_O is given by*

$$\Psi_O(S) = \{\arg\min_O S\} \cup \{c_{obs}(T) \mid S \subseteq T,\ T \in \mathcal{D},\ c_{obs}(T) \in S\}, \quad \forall S \in \mathcal{P}(X).$$

We now turn our attention to the DM's preference. As established in MM12 and CFS13, a choice function c is consistent with Categorization/Rationalization if and only if it is representable using some filter and the preference P derived from pairwise choices: xPy if $c(\{x,y\}) = x$. With limited data, not all pairwise choices may be observed. However, even if $\{x,y\}$ is not in the dataset, we could infer under these theories that the DM *would* pick x from $\{x,y\}$ if $y = c_{obs}(S) \in R \subset S$ and $c_{obs}(R) = x$, for some $R, S \in \mathcal{D}$. This is because for complete datasets, these theories are characterized by the weak-WARP axiom, which requires that if y were the choice from both $\{x,y\}$ and a superset thereof, then x cannot be chosen from a choice set that is 'in between.' Given this, let P^* be the (possibly incomplete) relation defined by xP^*y if $c_{obs}(\{x,y\}) = x$ or the conditions $y = c_{obs}(S) \in R \subset S$ and $x = c_{obs}(R)$ hold for some $R, S \in \mathcal{D}$.

To summarize our analysis so far, if c_{obs} is consistent with Categorization/Rationalization, then there exist an ordering O and a completion P of P^* such that c_{obs} is explained by (Ψ_O, P). Notice that O must satisfy the following set of restrictions, denoted $\mathcal{R}_{CR}(c_{obs})$:

(1) for each $R \in \mathcal{D}$ and $x \in R$, if $xP^*c_{obs}(R)$, then xOy for some $y \in R$;

(2) for each $S, T \in \mathcal{D}$ and $x \in X$, if P^* is cyclic on $\{x, c_{obs}(S), c_{obs}(T)\} \subseteq S \cap T$, then $xOc_{obs}(S)$ or $xOc_{obs}(T)$.

Necessity of (1) follows because the O-minimal element in R belongs to $\Psi_O(R)$, and thus cannot be preferred to $c_{obs}(R)$. As for necessity of (2), observe that if

x were O-minimal in such a triple $\{x, c_{obs}(S), c_{obs}(T)\}$, then it would be impossible to define the choice from that set, as all three options would be paid attention to under Ψ_O. Finding an acyclic relation satisfying $\mathcal{R}_{CR}(c_{obs})$ provides a more stringent test than weak-WARP.[5] The next result shows, moreover, that this test captures the empirical content of Categorization/Rationalization if P^* is complete over the set of chosen elements (that is, if $x = c_{obs}(S)$ and $y = c_{obs}(S')$ for some S, S', then xP^*y or yP^*x). This would be the case, for instance, if $\mathcal{D}$ includes all binary choice problems, as assumed in MM12.

Proposition 3. *If c_{obs} is consistent with Categorization/Rationalization, then there is an acyclic relation satisfying $\mathcal{R}_{CR}(c_{obs})$. The converse holds when P^* is complete over the set of chosen elements.*

As illustration, we can apply this result to confirm our earlier observation that the data in Example 1 is not consistent with Categorization/Rationalization. In this case, P^* is given by $aP^*bP^*dP^*a$, which is indeed complete over the set of chosen elements. Looking just at the restrictions coming from part (2) of the definition of $\mathcal{R}_{CR}$, one obtains (i) dOa or dOb, (ii) aOb or aOd, and (iii) bOa or bOd, which already cannot be satisfied by an acyclic relation.[6] Indeed, if such a relation exists, then the restrictions would be satisfied by an ordering a fortiori, and there would be an element which is bottom-ranked. Yet each of a, b and d are required to be ranked above one of the other elements. The enumeration procedure in the next section, which provides a simple, systematic way of testing whether certain collections of restrictions can be satisfied by an acyclic relation, builds on this idea.

What if P^* is not complete over chosen elements? Then one must make an uninformed guess as to the DM's choice from those unobserved binary problems. Proposition 3 can be extended as follows, to apply to any dataset: *c_{obs} is consistent with Categorization/Rationalization if and only if there is an acyclic*

[5] Consider a weak-WARP violation with y picked from both V and $\{y, z\}$, z picked from U, and $\{y, z\} \subset U \subset V$. This implies zP^*yP^*z, and leads to contradictory constraints (yOz and zOy) when applying (2) using $S = T = U$ with $x = y$ and $S = T = V$ with $x = z$.

[6] Note that (i) comes from taking $S = \{a, b, d, e\}$, $T = \{a, b, d, f\}$ and $x = d$; (ii) comes from taking $S = \{a, b, d, f\}$, $T = \{a, b, d, g\}$ and $x = a$ and (iii) comes from $S = \{a, b, d, e\}$, $T = \{a, b, d, g\}$ and $x = b$.

O satisfying $\mathcal{R}_{CR}(c_{obs})$ for some $\bar{P}^*$ that completes P^* over chosen elements.

To illustrate this last point, we will now show that observed choices from Example 2 *are* consistent with Categorization/Limited Attention. In that case, P^* is not complete over observed choices, as we only have fP^*e and eP^*a. Pick any completion $\bar{P}^*$ of P^* which is transitive. In this case, one can trivially find an acyclic O satisfying $\mathcal{R}_{CR}(c_{obs})$ for $\bar{P}^*$, simply by taking $O = \bar{P}^*$. Indeed, part (2) of the definition of $\mathcal{R}_{CR}$ does not kick in unless there are cycles, and part (1) holds by taking $y = c_{obs}(R)$ in each case. A more general result emerges here: any dataset for which P^* is acyclic is consistent with Categorization/Rationalization (and even the more restrictive theory of Order Rationalization, as seen in the Online Appendix).

5. Testing Generalized SARP Conditions By Enumeration

Section 4 presents testable implications for Limited Attention, Categorization and Rationalization in terms of the existence of an acyclic relation satisfying restrictions inferred from observed choices. This is similar in spirit to the testing of rationality. As is well known, observed choices are consistent with rationality if and only if there exists an acyclic relation P that respects Samuelson's revealed preference, that is, satisfying the set of restrictions $\mathcal{R}_R(c_{obs})$ given by: for all $S \in \mathcal{D}$, if $x = c_{obs}(S)$ then xPy for all $y \in S \setminus \{x\}$.

With all restrictions of the form xPy for pairs x, y, finding whether there exists an acyclic relation satisfying $\mathcal{R}_R$ boils down to checking whether Samuelson's revealed preference is itself acyclic, that is, SARP. Restrictions in $\mathcal{R}_{LA}$ and $\mathcal{R}_{CR}$ are more elaborate (e.g., xPy or zPw), and require taking a decision on the 'or' condition prior to checking for acyclicity. Is testing these theories thus significantly more difficult than testing rationality?

For insight into this question, consider an algorithm to test SARP. Start by making a guess, called x_1, as to which option is least preferred in X. Clearly this guess x_1 cannot dominate another element according to Samuelson's revealed preference. This reduces the problem at hand, allowing to restrict attention to $X \setminus \{x_1\}$. Next, make a guess, x_2, as to which element is least preferred in $X \setminus \{x_1\}$, etc. This *enumeration procedure* succeeds if at least one guess, an option which does not Samuelson-dominate anything that remains, is available in each round. Success is thus equivalent to SARP. More generally, we can define the enumeration procedure for any *guess correspondence* $G : \mathcal{P}(X) \to \mathcal{P}(X) \cup \{\emptyset\}$, where $G(S) \subseteq S$ is the set of possible guesses for each $S \subseteq X$.

Definition 3 (Enumeration procedure using G)**.** Iteratively pick an element in $G(X \setminus \{x_1, \ldots, x_{k-1}\})$ and call it x_k, for each $k = 1, \ldots, |X|$. The enumeration procedure *succeeds* if some choice of x_k in each step results in an enumeration of X, i.e. $G(X \setminus \{x_1, \ldots, x_{k-1}\})$ is nonempty for each k.

In the case of rationality, this procedure has two desirable features which make testing simple.[7] First, determining whether there is a valid guess in each step is easy, since each $G(S)$ is simply the set of alternatives that do not Samuelson-dominate any other element of S. Second, we will show that the procedure's success is *path independent*: it does not depend on which guess is chosen when there are multiple possibilities. Formally, path independence means that success occurs for *any* guess of $x_k \in G(X_k)$ in each step, if and only if success occurs for *some* guess of $x_k \in G(X_k)$ in each step. Thus an experimenter need not worry that the procedure will fail if he makes an unlucky guess along the way, but would have succeeded had he made all the 'right' guesses.

At least under some conditions on the dataset, several bounded rationality theories can be tested using an enumeration procedure with the same desirable properties, making these theories roughly as easy to test as rationality. To establish this result, for any collection of sets $\mathcal{T} \subseteq \mathcal{P}(X)$, denote by $(x, \mathcal{T})$ the restriction on a relation that there exists a set $T \in \mathcal{T}$ for which x is ranked above all the alternatives in T.[8] We call such restrictions *elementary*.

[7] The importance of simple tests is first discussed in Varian (1982), who uses the criterion of computational simplicity. With these two features, the enumeration procedure requires a number of steps which is 'polynomial' instead of 'exponential' in the size of one's dataset.

[8] Finding an ordering satisfying general restrictions can be thought of as an extension of the topological sort problem in computer science (which itself is known to be equivalent to checking SARP). Some such extensions have been studied in the context of job-scheduling problems with waiting conditions; see Möhring et al. (2004) who provide a fast algorithm

Proposition 4. *Let $\mathcal{R}$ be a collection of elementary restrictions. There is an acyclic relation satisfying $\mathcal{R}$ if and only if the enumeration procedure succeeds using the guess correspondence*

$$G(S) = \{x \in S \mid \textit{If } (x, \mathcal{T}) \in \mathcal{R}, \textit{ then } S \cap T = \emptyset \textit{ for some } T \in \mathcal{T}\}. \quad (6)$$

Moreover, success is path independent.

The idea is that the enumeration constructs the relation from the bottom up, by selecting in each step from the set of candidate minimal alternatives.

For Categorization and Rationalization, notice that all the restrictions in $\mathcal{R}_{CR}$ are elementary. So long as P^* is complete over the set of chosen elements (e.g., the dataset contains all binary choice problems, as assumed in MM12), these two theories can be tested using a simple enumeration procedure, just like rationality; the test is path independent thanks to Proposition 4, and determining whether a valid guess exists in each step is easy. Absent of conditions on the dataset, however, a simple test will not always exist: Proposition 10 in the Online Appendix shows that testing consistency of observed choices with these theories is in general NP-hard (that is, there exist worst-case scenario datasets for which testing consistency is difficult). Of course, if there are only few pairs $c_{obs}(S), c_{obs}(S')$ for which P^* is incomplete, then it may well be practical to test consistency by checking whether the enumeration procedure succeeds for some extension of observed choices to those binary choice problems.

For Limited Attention, restrictions in $\mathcal{R}_{LA}$ correspond to pairs of sets that cause a WARP violation. The restrictions are elementary when those sets are related by inclusion, but not otherwise. However, all the non-elementary restrictions become redundant whenever the dataset is closed under intersection (or at least contains the intersection of any two choice problems causing a WARP violation).[9] For the same reasons as above, the testing of Limited Attention is thus tractable not only for full datasets (as in MNO12), but also

for scheduling given conditions of the type "job i comes before at least one of the jobs in a set J." These constraints correspond to the case where the sets in $\mathcal{T}$ are all singletons.

[9]Indeed, if S and S' cause a WARP violation, then $S \cap S'$ causes a WARP violation with S or S'. Suppose it occurs with S. Then $c_{obs}(S)$ must be preferred to some element of $S \setminus S'$, automatically satisfying the 'or' condition from S and S'.

for incomplete datasets satisfying the intersection property. There is no hope, however, of finding a tractable test that applies to all datasets: Proposition 9 in the Online Appendix shows that testing consistency of observed choices with Limited Attention is in general NP-hard. Proposition 2 can still be useful to test consistency of observed choices for moderate datasets, trying different ways of resolving 'or' conditions before applying the enumeration procedure.

6. Identification and Out-of-Sample Predictions

Even under a given theory, there may be multiple ways to describe observed choices (in the same way that a rational choice function may admit multiple utility representations). Nonetheless, the data may reveal conditions that all representations must satisfy. Under Limited Attention, for example, when can a policymaker infer that any preference P rationalizing the data has yPx?

Our methodology is also helpful for such identification problems (in addition to allowing tests for out-of-sample predictions, thanks to Proposition 1). Once a theory's content is captured through the existence of an acyclic relation O satisfying a set of restrictions $\mathcal{R}$, we can check whether yOx must hold in *all* representations. One simple, systematic way to do this is to test whether it is impossible to find an acyclic relation satisfying *both* $\mathcal{R}$ *and* xOy. Since the latter is an elementary restriction, identification is not any more difficult than testing consistency. Inferences about O then translate into identification of the related component(s) of the theory. This is straightforward, for instance, in the case of Limited Attention where O corresponds to the DM's preference P. Translating inferences on O into inferences on preferences is less obvious in the case of Categorization/Rationalization since O does not immediately relate to the DM's preference. However, we prove in Proposition 5 of the Appendix that b is revealed preferred to a under these theories whenever $c_{obs}(\{a,b\}) = b$ and bOa in all acyclic O satisfying $\mathcal{R}_{CR}(c_{obs})$.[10]

The next two examples illustrate identification and out-of-sample predictions for the theories studied so far.

[10]This result holds even if P^* is incomplete.

Example 3 (Limited Attention). *Consider the hypothetical situation with the same choices as in Example 2, except that the choice problem $\{b, e, f\}$ is not observed, that is*

S	ae	ef	abd	ade	bde
$c_{obs}(S)$	e	f	d	a	b

As can easily be checked, the set of restrictions $\mathcal{R}_{LA}$ associated to these observed choices are: (i) aPd, (ii) dPa or bPe, and (iii) dPb or aPe. By Proposition 2, this dataset is consistent with Limited Attention, as the relation P defined by aPd, bPe, and dPb is acyclic and satisfies the restrictions.

If we were to simply apply the full-data revealed preference identified by MNO12, then we would only conclude that the DM prefers a over d. This corresponds to restriction (i). By using the appropriate test for limited data, we can infer two more revealed preferences, namely that bPe and aPe. Indeed, since aPd according to (i), restriction (ii) reduces to bPe. From (iii), either d is preferred to b, or a is preferred to e. Even in the former case, transitivity implies the DM prefers a to e, because he prefers a to d and prefers b to e.

As for out-of-sample prediction, observe from Proposition 2 that only b and f qualify as possible predictions for the problem $\{b, e, f\}$ (we have already shown after Proposition 2 that choosing e is inconsistent with the theory). This is easy to see, since choosing either b or f would not cause any additional WARP violations, leaving the set of restrictions $\mathcal{R}_{LA}$ unchanged.

Example 4 (Categorization/Rationalization). *Consider the observed choices*

S	ab	ad	ae	bd	be	de	abd	abe	bde
$c_{obs}(S)$	b	d	a	d	e	e	b	a	d

We apply Proposition 3 to show that this data is consistent with Categorization/Rationalization. Notice first that observed choices yield dP^*b, bP^*a, aP^*e, eP^*d, eP^*b, and dP^*a, which is complete. Restrictions in $\mathcal{R}_{CR}$ are given by (i) dOa or dOb, (ii) bOa or bOe, and (iii) eOb or eOd.[11] *The enumeration procedure offers a simple way to implement Proposition 3. The procedure first*

[11] Note that (i) comes from taking $R = \{a, b, d\}$ and dP^*b; (ii) comes from taking $R = \{a, b, e\}$ and bP^*a; and (iii) comes from taking $R = \{b, d, e\}$ and eP^*d.

looks for an option not appearing at the 'top' of one of the ranking restrictions, which will serve as the O-minimal element. In this case, a is the only possibility (that is, $G(\{a, b, d, e\}) = \{a\}$), implying that restrictions (i) and (ii) above count as being satisfied in the next steps. The procedure then looks among the remaining options for one that does not appear at the top of the surviving restriction (iii). The possibilities are now b and d (that is, $G(\{b, d, e\}) = \{b, d\}$). Having satisfied the final remaining restriction, the procedure succeeds.

To illustrate identification, note that even though $c_{obs}(\{a, d\}) = d$, options a and d are not comparable according to the revealed preference identified for full data in CFS13. An appropriate test for limited data, however, allows us to correctly infer that d is revealed preferred to a. We just concluded in the paragraph above that any acyclic O satisfying $\mathcal{R}_{CR}(c_{obs})$ must rank a at the very bottom. The revealed preference then immediately follows from Proposition 5, which was discussed before Example 3.

To illustrate how we can make predictions, consider the out-of-sample choice problem $\{a, d, e\}$. Augmenting observed choices by assuming the DM picks option a (resp., d) generates one additional elementary restriction: dOa or dOe (resp., eOa or eOd). The enumeration procedure still succeeds, just as before, and so options a and d are each valid predictions. However, we can also infer that the DM would not pick e. Indeed, the enumeration procedure fails in its first step if one adds the elementary restriction aOd or aOe (as all options would appear at the 'top' of at least one constraint). Notice that the weak-WARP property is satisfied when augmenting c_{obs} by assuming that d is picked from $\{a, c, d\}$. There is thus a risk for wrong predictions if one is not mindful of the fact that weak-WARP does not capture the empirical content of Categorization/Rationalization when data is incomplete, as discussed in Section 3.

7. Concluding Remarks

We have studied some leading theories in this paper, first capturing their testable implications through generalized SARP conditions, and then exploring how their testing compares to that of SARP. This approach should be applicable to other theories as well. A bounded rationality theory often includes an ordering in its description (e.g. as a preference or salience ranking) or may have an observationally equivalent formulation that does (as we saw for Categorization and Rationalization). A natural starting point is to try to express the theory's testable implications in terms of restrictions that data imposes on that relation, and see when they are of an elementary form.

We illustrate this approach further in the Online Appendix, where we provide characterization results for the following theories: (i) *Order Rationalization* (CFS13) restricts the preference in Rationalization to be an ordering; (ii) *Consistent Reference Points* posits that the DM views one alternative in each choice problem as the reference point and maximizes a reference-dependent preference ordering, with the consistency requirement that the reference point in a set remains the reference in subsets; and (iii) the class of *Minimal Consideration* theories, where each theory is indexed by a function k that maps a choice problem S into an integer between 1 and $|S|$. The DM maximizes an ordering over his consideration set $\Gamma(S)$, which must contain at least $k(S)$ elements. The function k bounds the extent of the DM's 'mistakes', and can model, for instance, a DM who always chooses from the top quintile, or is overwhelmed by large choice problems. For all the above theories, Proposition 4 applies, as the testable implications are captured by the existence of an acyclic relation satisfying elementary restrictions, and consistency can be checked using a simple enumeration procedure regardless of the dataset.

Appendix

The simple proof of Proposition 1 is left to the reader.

Proof of Proposition 2 *Necessity.* The argument appears in the main text.

Sufficiency. Suppose an acyclic relation satisfying $\mathcal{R}_{LA}(c_{obs})$ exists, and let P be a transitive completion (so P still satisfies $\mathcal{R}_{LA}(c_{obs})$). We define $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$. For $S \in \mathcal{D}$, $\Gamma(S) = \{c_{obs}(S)\} \cup \{x \in S | c_{obs}(S) P x\}$; for $S \notin \mathcal{D}$,

$$\Gamma(S) = \begin{cases} \Gamma(T) & \text{if } S \subseteq T,\ T \in \mathcal{D},\ \text{and } \Gamma(T) \subseteq S \\ S & \text{otherwise.} \end{cases}$$

Clearly $\Gamma(S) \neq \emptyset$ for any $S \in \mathcal{P}(X)$ and the P-maximal element in $\Gamma(S)$ is $c_{obs}(S)$ for any $S \in \mathcal{D}$. We show Γ is well-defined and satisfies (3b).

Suppose by contradiction that Γ is not well-defined for some S. This means that for some $S \notin \mathcal{D}$, there exist $T, T' \in \mathcal{D}$ such that $S \subseteq T \cap T'$ with $\Gamma(T) \cup \Gamma(T') \subseteq S$, but $\Gamma(T) \neq \Gamma(T')$. This implies $c_{obs}(T) \neq c_{obs}(T')$. Consider any $y \in T \setminus T'$. Then, since $S \subseteq T'$, $y \in T \setminus S$. Moreover, since $\Gamma(T) \subseteq S$, we know $y \in T \setminus \Gamma(T)$. By definition of $\Gamma(T)$ for $T \in \mathcal{D}$, this means $y P c_{obs}(T)$. Similarly, if $y \in T' \setminus T$, we conclude $y P c_{obs}(T')$, contradicting that P satisfies $\mathcal{R}_{LA}(c_{obs})$. To show Γ satisfies (3b), consider $S \in \mathcal{P}(X)$ and $x \in S \setminus \Gamma(S)$. We prove $\Gamma(S \setminus \{x\}) = \Gamma(S)$ in each of the four possible cases:

Case 1: $S \setminus \{x\}, S \in \mathcal{D}$. Since $S \in \mathcal{D}$, and $x \notin \Gamma(S)$, we know $x P c_{obs}(S)$. Suppose that $\Gamma(S \setminus \{x\}) \neq \Gamma(S)$. Then $c_{obs}(S) \neq c_{obs}(S \setminus \{x\})$. Applying $\mathcal{R}_{LA}(c_{obs})$ for choice problems S and $S \setminus \{x\}$, we conclude $c_{obs}(S) P x$, a contradiction.

Case 2: $S \setminus \{x\} \in \mathcal{D}, S \notin \mathcal{D}$. Since $S \setminus \{x\} \in \mathcal{D}$, we know $\Gamma(S \setminus \{x\}) = c_{obs}(S \setminus \{x\}) \cup \{y \in S | c_{obs}(S \setminus \{x\}) P y\}$. Since $S \setminus \Gamma(S) \neq \emptyset$, there exists $T \in \mathcal{D}$ with $S \subseteq T$ and $\Gamma(T) \subseteq S$. Because $T \in \mathcal{D}$, $z P c_{obs}(T)$ for all $z \in T \setminus S$. Since $\Gamma(S) = \Gamma(T)$, we know $x \in T \setminus \Gamma(T)$. Hence $x P c_{obs}(T)$. If $\Gamma(S \setminus \{x\}) \neq \Gamma(S) = \Gamma(T)$, then $c_{obs}(S \setminus \{x\}) \neq c_{obs}(T)$ contradicting $\mathcal{R}_{LA}(c_{obs})$ for the pair of sets T and $S \setminus \{x\}$.

Case 3: $S \setminus \{x\} \notin \mathcal{D}, S \in \mathcal{D}$. Since $S \in \mathcal{D}$, $\Gamma(S) = c_{obs}(S) \cup \{y \in S | c_{obs}(S) P y\}$. If $x \in S \setminus \Gamma(S)$ then $\Gamma(S) \subseteq S \setminus \{x\}$, so by construction $\Gamma(S \setminus \{x\}) = \Gamma(S)$.

Case 4: $S \setminus \{x\}, S \notin \mathcal{D}$. Since $S \setminus \Gamma(S) \neq \emptyset$, there exists $T \in \mathcal{D}$ with $S \subseteq T$ and $\Gamma(T) \subseteq S$. Since $x \in S \setminus \Gamma(S)$, then $\Gamma(T) = \Gamma(S) \subseteq S \setminus \{x\}$ and so $\Gamma(S \setminus \{x\}) = \Gamma(T)$ by construction, and equals $\Gamma(S)$ by transitivity. □

Proof of Lemma 1 It is easy to check that Ψ_O is a filter; hence sufficiency follows by taking $\Psi = \Psi_O$. For necessity, let c be the extension of c_{obs} emerging under (Ψ, P). Define the ordering O by $x_n O \cdots O x_1$, where $x_k = c(X \setminus \{x_1, \ldots, x_{k-1}\})$ for each $k = 1, \ldots, n$. By the filter property and the fact that one pays attention to chosen elements, $\Psi_O(S) \subseteq \Psi(S)$ for all S. This implies that for any $S \in \mathcal{D}$, the observed choice $c_{obs}(S)$ remains P-maximal in $\Psi_O(S)$. To conclude, we show (Ψ_O, P) deliver a well-defined choice everywhere. Fix any $T \subset X$. Applying $\Psi_O \subseteq \Psi$ to $S = \Psi_O(T)$ yields $\Psi_O(\Psi_O(T)) \subseteq \Psi(\Psi_O(T)) \subseteq \Psi_O(T)$. Now using the filter property, $\Psi_O(\Psi_O(T)) = \Psi_O(T)$, and by squeezing, $\Psi(\Psi_O(T)) = \Psi_O(T)$. Then $\Psi_O(T)$ has a well-defined P-maximal element since $\Psi(\Psi_O(T))$ does. □

Proof of Proposition 3 *Necessity.* The argument appears in the main text.

Sufficiency. Suppose an acyclic relation satisfying $\mathcal{R}_{CR}(c_{obs})$ exists. Let O be a transitive completion of this relation (hence O still satisfes $\mathcal{R}_{CR}(c_{obs})$).

We now complete P^* as follows (and use that completion in the remainder of the proof). If there exist $x, y \in X$ such that neither xP^*y nor yP^*x hold, then note, by assumption on P^*, that it cannot be that both x, y are in the image of c_{obs}. If one of these, say x, satisfies $x = c_{obs}(S)$ for some $S \in \mathcal{D}$, then we add xP^*y; otherwise, we can arbitrarily add exactly one of xP^*y or yP^*x.

Using Lemma 1, it remains to show: (i) P^* is asymmetric, (ii) P^* has a well-defined choice on $\Psi_O(S)$ for each $S \subseteq X$, and (iii) this choice coincides with the data for $S \in \mathcal{D}$. For (i), suppose yP^*z and zP^*y. By construction of (the completed) P^*, there must exist $R, R' \in \mathcal{D}$ such that $y = c_{obs}(R)$, $z = c_{obs}(R')$, and $\{y, z\} \subseteq R \cap R'$. Taking $x = y$ in restriction (2) of $\mathcal{R}_{CR}(c_{obs})$ implies yOz; but taking $x = z$ implies zOy, a contradiction. For (ii), suppose by contradiction that P^* has a top-cycle on $\Psi_O(S)$. By completeness, there must be a three-cycle $xP^*yP^*zP^*x$. Suppose without loss that x is O-minimal in the set $\{x, y, z\}$. Since $y, z \in \Psi_O(S)$ they must be the respective choices from some supersets T_y, T_z of S. Since y, z are in the image of c_{obs}, xP^*y could not have arisen from completing P^*; hence x is in the image of c_{obs}, implying that zP^*x also did not arise from the completion procedure. Since $\{x, y, z\} \subseteq S \subseteq T_y \cap T_z$, this contradicts O satisfying restriction (2) in $\mathcal{R}_{CR}(c_{obs})$. For (iii), first note that $c_{obs}(S) \in \Psi_O(S)$ whenever $S \in \mathcal{D}$. Suppose by contradiction that $c_{obs}(T)P^*c_{obs}(S)$, for some $T \in \mathcal{D}$ such that $c_{obs}(T) \in S \subseteq T$. But by definition of P^*, we also have $c_{obs}(S)P^*c_{obs}(T)$, a contradiction to (i). Finally, suppose by contradiction that $xP^*c_{obs}(S)$ for $x = \arg\min_O S$. This cannot be from completing P^*, and contradicts restriction (1) in $\mathcal{R}_{CR}(c_{obs})$. □

Lemma 2. *Let the guess correspondence G satisfy $G(T) \cap S \subseteq G(S)$ for any $S \subseteq T$. Success of the enumeration procedure using G is path independent.*

Proof. Suppose there exists a successful enumeration $x_1, \ldots, x_n$. For any $S \subseteq X$, let x_k be the element of minimal index in S. Then $S \subseteq X_k$ and so $x_k \in G(X_k)$ implies $x_k \in G(S)$. Thus G is non-empty valued and the enumeration procedure succeeds independently of the sequence of guesses made. □

Proof of Proposition 4 Suppose that the enumeration procedure using G succeeds and gives $x_1, \ldots, x_n$. We check that the ordering O defined by yOx if x precedes y in the enumeration satisfies the restrictions in $\mathcal{R}$. Consider a restriction $(x_j, \mathcal{T}) \in \mathcal{R}$. Since $x_j \in G(X_j)$, $X_j \cap T = \emptyset$ for some $T \in \mathcal{T}$. Hence $T \subseteq \{x_1, \ldots, x_{j-1}\}$, or x_jOy for all $y \in T$, and O satisfies $(x_j, \mathcal{T})$, as desired. Finally, suppose there an acyclic relation O satisfying the restrictions in $\mathcal{R}$. In this case, we can complete O into an ordering that still satisfies $\mathcal{R}$. Then $G(S)$ is nonempty for each $S \in \mathcal{P}(X)$, since it contains the O-minimal element in S. Hence the procedure succeeds. Path independence follows from Lemma 2. □

Proposition 5. *Suppose c_{obs} is consistent with Categorization/Rationalization, and $c_{obs}(\{a, b\}) = b$. If bOa for all acyclic relations O satisfying $\mathcal{R}_{CR}(c_{obs})$, then b is revealed preferred to a under these theories.*

Proof. We prove the result by contraposition. Let P be a complete asymmetric relation and Ψ a psychological filter (see the first paragraph of Section 4.2) such that aPb and the maximization of P over the filter generates a complete choice function c which coincides with c_{obs} over observed choice problems. Consider then the following enumeration of X: $x_1 = c(X)$, $x_2 = c(X \setminus \{x_1\})$, $x_3 = c(X \setminus \{x_1, x_2\})$, etc. Let also O be the complete transitive relation defined by $x_n O x_{n-1} O \ldots O x_1$. For convenience, let $X_i = \{x_i, x_{i+1}, \ldots, x_n\}$, for all i.

In this paragraph, we show that O satisfies $\mathcal{R}_{CR}(c_{obs})$, using P^* inferred from c_{obs}. We start by checking restrictions from part (1) of $\mathcal{R}_{CR}$. Suppose $R \in \mathcal{D}$ and $x = x_i \in R$ with $x_i P^* c_{obs}(R)$ and $y O x_i$ for all $y \in R \setminus \{x_i\}$. Hence $c(X_i) = x_i$, $R \subseteq X_i$, and it must be that $c_{obs}(R) P x_i$. By $x_i P^* c_{obs}(R)$, either $c_{obs}(\{x_i, c_{obs}(R)\}) = x_i$ or $x_i P c_{obs}(R)$. Either way, the asymmetry of P is violated. We then check restrictions from part (2) of $\mathcal{R}_{CR}$. Suppose that P^* is cyclic over $R = \{x, c_{obs}(S), c_{obs}(T)\} \subseteq S \cap T$, with $c_{obs}(S) O x$ and $c_{obs}(T) O x$. Each of the pairwise relationships in $x P^* c_{obs}(S) P^* c_{obs}(T) P^* x$ comes either from a binary choice problem, or the revealed preference of CFS13. Notating $x = x_i$, we know $R \subseteq X_i$ and $c(X_i) = x_i$, so using either source of P^* we must also have $x_i P c_{obs}(S) P c_{obs}(T) P x_i$. Moreover, it must be that $\Psi(R) = R$, meaning that the choice from $\Psi(R)$ using P is not well-defined.

We conclude the proof by showing, by contradiction, that aOb. Suppose instead that bOa, and let $a = x_j$, and $b = x_k$ with $k > j$. But then $c(X_j) = x_j$, which is a. Hence $\Psi(\{a, b\}) = \{a, b\}$ and it must be that bPa, contradicting the original premise of the proof. $\square$

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Online Appendix

This appendix presents further results and technical details for *Bounded Rationality and Limited Datasets* (by Geoffroy de Clippel and Kareen Rozen).

Contents

A Our Methodology Applied to Additional Theories

We introduce additional theories in Section A.1, and show in Section A.2 how their empirical content can be captured by generalized SARP conditions, that is, the existence of an acyclic relation that satisfies a set of restrictions inferred from choices. Such results thus mirror those presented in Section 3 of the paper for Limited Attention and Categorization/Rationalization. In Section A.3, we show that, for all these additional theories, restrictions are elementary, and that making a valid guess in each step of the enumeration is easy. As a consequence of our results in the paper, all these theories can thus be tested through enumeration in a way that is roughly as easy as rationality.

A.1 Additional Theories

Order Rationalization (CFS13) is simply the variant of Rationalization where the preference is required, in addition, to be an ordering.

Under *Consistent Reference Points*, the DM views one alternative in each choice problem as his reference, and picks the best alternative according to his reference-dependent preference ordering. Reference points are assumed to be consistent in the following sense: if x is the reference point in a choice problem S, then x remains the reference point in subsets of S containing x. This theory is essentially equivalent to Rubinstein and Salant's theory of Triggered Rationality, where the most salient alternative triggers the rationale used to make a choice.[1] It can also be seen as capturing a form of Ariely, Lowenstein and Prelec's 'coherent arbitrariness'.[2]

The class of *Minimal Consideration* theories extends rational choice by bounding from below the number of options considered. Theories in this class are indexed by a function k that associates to each problem S an integer between 1 and $|S|$. The DM uses an ordering P to pick the best element in his consideration set $\Gamma(S)$, which must contain at least $k(S)$ elements. The function k, which fixes a theory, limits the extent of a DM's 'mistakes.' If $k(S) = |S| - 1$ for all S, then the DM always picks from the top two options in a choice problem; if $k(S) = \lceil (1-\alpha)|S| \rceil$ for all S, then the DM always picks from the top α-percentile. Theories in this class can also capture a DM who becomes overwhelmed in large choice problems, with $k(S)$ decreasing in $|S|$.

A.2 Testable Implications via Generalized SARP Conditions

We start by studying the testable implications of Order Rationalization. As in Rationalization, the choice from a set is also considered in subsets. Thus y is revealed preferred to x, denoted $y \succ^*_{OR} x$, if y is chosen in the presence of x, which itself is the choice from a superset.[3] Let $\mathcal{R}_{OR}(c_{obs})$ be the collection of restrictions that y is ranked above x for any x, y with $y \succ^*_{OR} x$. The next result shows that CFS13's full-data characterization of Order Rationalization in terms of the acyclicity of $\succ^*_{OR}$ extends to limited datasets.

[1] Rubinstein, Ariel and Yuval Salant (2006), Two Comments on the Principle of Revealed Preference, *mimeo*. In addition, studying choice from lists, Rubinstein and Salant [*Theoretical Economics*, **1**, 3 (2006)] propose a model (their Example 4) where the DM's preference depends on the first element presented. Consistent Reference Points can be seen as the case where the list is unknown, or subjectively determined.

[2] Ariely, D., G. Loewenstein, and D. Prelec. (2003), 'Coherent Arbitrariness': Stable Demand Curves Without Stable Preferences, *Quarterly Journal of Economics*, 118, 73–106.

[3] This revealed preference is identified by CFS13 when $\mathcal{D} = \mathcal{P}(X)$.

Proposition 6. *The observed choice function c_{obs} is consistent with Order Rationalization if and only if there is an acyclic relation satisfying $\mathcal{R}_{OR}(c_{obs})$.*

We now turn our attention to Consistent Reference Points. Since the DM's reference point satisfies IIA, it can be interpreted as maximal for a 'salience ordering' $\succ_{REF}$. If x is the reference point in a choice problem, then it remains the reference point in smaller problems containing it; and the DM's choices in those problems all arise from maximizing the same preference ordering $\succ_x$. For any choice problem S and any $x \in S$, we say that $a \succ^*_{S,x} b$ if $c_{obs}(R) = a$ for some $R \subseteq S$ that contains b and x. This is the revealed preference under the *supposition* that x is the DM's reference point in S. Observe that when $\succ^*_{S,x}$ is cyclic, then x cannot be the reference point in S, and therefore cannot be the most salient alternative in S. Let $\mathcal{R}_{REF}$ be the following collection of restrictions on $\succ_{REF}$: for each $S \in \mathcal{P}(X)$ and $x \in S$ such that $\succ^*_{S,x}$ is cyclic, there is $y \in S \setminus \{x\}$ with $y \succ_{REF} x$.

Proposition 7. *The observed choice function c_{obs} is consistent with Consistent Reference Points if and only if there is an acyclic relation satisfying $\mathcal{R}_{REF}$.*

Interestingly, the proof reveals that it is without loss of generality to require that if x is preferred to y when the reference point is y, then x is also preferred to y when the reference point is x.[4]

To understand the testable implications of Minimal Consideration theories, we start by fixing a theory in this class, which is described by a given function $k : \mathcal{P}(X) \to \mathbb{N}$. If the DM picks x from S, then there must exist at least $k(S)-1$ alternatives in S that are inferior to x. These restrictions are summarized by $\mathcal{R}_k = \{(c_{obs}(S), \mathcal{T}_S) \mid S \in \mathcal{D}\}$, where $\mathcal{T}_S$ denotes the set of subsets of $S \setminus \{c_{obs}(S)\}$ with exactly $k(S) - 1$ elements.

Proposition 8. *For each $k : \mathcal{P}(X) \to \mathbb{N}$, the observed choice function c_{obs} is consistent with k-Minimal Consideration if and only if there is an acyclic relation satisfying $\mathcal{R}_k$.*

The proofs of Propositions 6-8 appear in Section A.4 below.

[4] Such a reference effect is related to status quo bias; see Tversky and Kahneman [*The Quarterly Journal of Economics*, **106**, 1039 (1991)] and Masatlioglu and Ok [*Journal of Economic Theory*, **121**, 1 (2005)].

A.3 Enumeration

In this section, we apply Proposition 4 from the paper with Propositions 6-8 to show that testing each of the theories discussed in this appendix is roughly as easy as testing rationality.

For Order Rationalization and Minimal Consideration theories, one sees that Proposition 4 applies since, for any c_{obs}, all the restrictions in $\mathcal{R}_{OR}(c_{obs})$ and $\mathcal{R}_k(c_{obs})$ are elementary. Thus, in each case, the existence of an acyclic relation satisfying the restrictions can be tested using a path-independent enumeration procedure. The associated guess correspondences derived from (6) can be written as $G_{OR}(S) = \{x \in S \mid \text{If } x \succ^*_{OR} y \text{ then } y \notin S\}$ and $G_k(S) = \{x \in S \mid \text{For all } R \in \mathcal{D}, \text{ if } x = c_{obs}(R) \text{ then } |R \setminus S| \geq k(R) - 1\}$,[5] and hence it is easy to determine whether a valid guess exists in each step.

Strictly speaking, restrictions associated to Consistent Reference Points are not elementary, but become elementary when reversed. Formally, let $\mathcal{R}^*_{REF}(c_{obs})$ be the following set of restrictions on a relation O: for each $S \in \mathcal{P}(X)$ and $x \in S$ such that $\succ^*_{S,x}$ is cyclic, there is $y \in S \setminus \{x\}$ with xOy. Existence of an acyclic relation satisfying $\mathcal{R}^*_{REF}(c_{obs})$ is equivalent to the existence of an acyclic relation satisfying $\mathcal{R}_{REF}(c_{obs})$ (simply by reversing the relation). Restrictions in $\mathcal{R}^*_{REF}(c_{obs})$ are elementary, and hence Proposition 4 applies once again. Thus consistency can be checked using a path-independent enumeration procedure. The associated guess correspondence derived from (6) can be written as $G_{REF}(S) = \{x \in S \mid \succ^*_{S,x} \text{ is acyclic}\}$,[6] and hence it is easy to determine whether a valid guess exists in each step.

A.4 Proofs

Proof of Proposition 6 Necessity was given earlier. For sufficiency, suppose there is an acyclic relation satisfying $\mathcal{R}_{OR}$, and let P be a transitive com-

[5]Indeed, there exists $T \in \mathcal{T}_R$ such that $S \cap T = \emptyset$ if and only if one can find $k(R) - 1$ elements that are in R but not S.

[6]To see whether $x \in G_{REF}(S)$, first note that it suffices to check only those restrictions $(x, \{y\}_{y \in R \setminus \{x\}})$ corresponding to $R \subseteq S$ (as there trivially exists $y \in R \setminus \{x\}$ such that $S \cap \{y\} = \emptyset$ when $R \nsubseteq S$). Next, if $\succ^*_{R,x}$ is cyclic for some $R \subset S$, then so is $\succ^*_{S,x}$, and of course there is no $y \in S \setminus \{x\}$ such that $S \cap \{y\} = \emptyset$.

pletion (hence P still satisfies $\mathcal{R}_{OR}$). Define the filter Ψ_P as in Lemma 1 (using P for O). CFS13 (Section 4.1) show that a filter is the set of rationalizable elements for some rationales $\{R_k\}_k$. Let c be the choice function arising from $(P, \{R_k\}_k)$ under the theory. For any $S \in \mathcal{D}$, we show $c(S) = c_{obs}(S)$. Suppose otherwise; then $\Psi_P(S)$ contains at least two elements, and $c(S)$ must be the observed choice from some $T \in \mathcal{D}$ with $S \subset T$. This implies $c_{obs}(S) \succ^*_{OR} c(S)$. But then $c_{obs}(S) P c(S)$, contradicting P-maximality of $c(S)$ in $\Psi_P(S)$. □

Proof of Proposition 7 Necessity was given earlier. For sufficiency, suppose an acyclic relation satisfying $\mathcal{R}_{REF}$ exists, and let $\succ_{REF}$ be a transitive completion (hence $\succ_{REF}$ still satisfies $\mathcal{R}_{REF}$). Let x_i denote the i-th maximal element according to $\succ_{REF}$. For each i, let $\succ_{x_i}$ be a transitive completion of $\succ^*_{X_i,x_i}$. Such a completion exists, because x_i being $\succ_{REF}$-maximal in $X_i = \{x_i, x_{i+1}, \ldots, x_n\}$ implies $\succ^*_{X_i,x_i}$ is acyclic. The choice function $c : \mathcal{P}(X) \to X$ generated by these primitives will now be shown to coincide with c_{obs} on $\mathcal{D}$. Take any $S \in \mathcal{D}$. Let k be the smallest index such that $x_k \in S$. Then $S \subseteq X_k$. By definition of $\succ_{x_k}$, $c_{obs}(S) \succ_{x_k} y$ for all $y \in S \setminus \{c_{obs}(S)\}$. □

Proof of Proposition 8 Necessity was given earlier. For sufficiency, suppose an acyclic relation satisfying $\mathcal{R}_k$ exists, and let P be a transitive completion (P still satisfies $\mathcal{R}_k$). Let $\Gamma(S)$ be the (weak) P-lower contour set of $c_{obs}(S)$ for $S \in \mathcal{D}$, and $\Gamma(S) = S$ otherwise. The choice function obtained by maximizing P over Γ clearly extends c_{obs}. Since P satisfies $\mathcal{R}_k$, for any $S \in \mathcal{D}$ there exists $T \in \mathcal{T}_S$ with $k(S) - 1$ elements such that $c_{obs}(S) P x$ for all $x \in T$. The condition $|\Gamma(S)| \geq k(S)$ thus holds for $S \in \mathcal{D}$ (it is trivial for $S \notin \mathcal{D}$). □

B Complexity Results

Proposition 9. *The classic SAT problem is reducible in polynomial time into the problem of determining whether observed choices are consistent with Limited Attention.*

Proof. Fix an instance of SAT with a set $\mathcal{L}$ of literals and a set $\mathcal{C}$ of clauses. Consider the abstract set of options X that contains all literals and their negations, all clauses, plus three options denoted x, y, and z. Let $\mathcal{L}_c$ denote the set of literals in clause c and let the literal $\bar{\ell}$ denote the negation of literal ℓ. Construct the following observed choice function:

S	cy	xz	yz	cxz	xyz	$\ell\bar{\ell}x$	$\ell\bar{\ell}xz$	$\ell\bar{\ell}y$	$cy\mathcal{L}_c$
$c_{obs}(S)$	y	x	z	z	y	ℓ	x	$\bar{\ell}$	c

for all $c \in \mathcal{C}$ and all $\ell \in \mathcal{L}$. Applying Proposition 2, c_{obs} is consistent with Limited Attention if and only if there is an acyclic relation O on X such that:

(i) yOx, from $c_{obs}(\{y,z\}) = z$ and $c_{obs}(\{x,y,z\}) = y$.

(ii) xOz, from $c_{obs}(\{\ell,\bar{\ell},x\}) = \ell$ and $c_{obs}(\{\ell,\bar{\ell},x,z\}) = x$.

(iii) For all $c \in \mathcal{C}$: zOc, from $c_{obs}(\{x,z\}) = x$ and $c_{obs}(\{c,x,z\}) = z$.

(iv) For all $\ell \in \mathcal{L}$: ℓOx or $\bar{\ell}Oy$, from $c_{obs}(\{\ell,\bar{\ell},x\}) = \ell$ and $c_{obs}(\{\ell,\bar{\ell},y\}) = \bar{\ell}$.

(v) For all $c \in \mathcal{C}$, there exists $\ell \in \mathcal{L}_c$ such that $cO\ell$, from $c_{obs}(\{c,y\}) = y$ and $c_{obs}(\{c,y\} \cup \mathcal{L}_c) = c$.

(vi) For all $\ell \in \mathcal{L}$ and $c \in \mathcal{C}$, $xO\ell$ or $xO\ell'$ or zOc, from $c_{obs}(\{c,x,z\}) = x$ and $c_{obs}(\{\ell,\bar{\ell},x,z\}) = x$.

Note that (vi) is redundant in view of (ii). These conditions are exhaustive, since we have used all the pairs $R, R' \in \mathcal{D}$ which cause a WARP violation.

We show SAT has a truthful assignment if and only if there exists an accylic relation O satisfying (i)-(v). Suppose SAT has a truthful assignment. We construct an ordering O by putting the false literals (in any order) at the top of the ordering; then y; then x; then z; then the clauses; and then the true literals (in any order). It is easy to check that O satisfies (i) to (v). Conversely, suppose an acyclic relation satisfying (i)-(v) exists, and let O be a transitive completion. We construct an assignment for SAT: if $xO\ell$ then ℓ is true; if $xO\bar{\ell}$ then ℓ is false; and if both ℓOx and $\bar{\ell}Ox$, then assign ℓ an arbitrary value. This is well-defined since, by (i) and (iv), it cannot be that both $xO\ell$ and $xO\bar{\ell}$. By (v), for all $c \in \mathcal{C}$, there exists $\ell \in \mathcal{L}_c$ such that $cO\ell$. Combined with (ii)-(iii), we conclude ℓ is true. Hence the assignment is truthful for SAT. □

Proposition 10. *The classic SAT problem is reducible in polynomial time into the problem of determining whether observed choices are consistent with psychological filter theory.*

Proof. Fix an instance of SAT with a set $\mathcal{L}$ of literals and a set $\mathcal{C}$ of clauses. Consider the abstract set of options X that contains all literals and their negations, all clauses, plus options denoted w, w', w'', x, y, and one option z_c for each clause c. Let V be the sets of variables defining the literals, let $\mathcal{L}_c$ be the set of literals appearing in clause c, and let $\bar{\ell}$ be the negation of the literal ℓ. Construct the following observed choice function:

S	cx	cy	cz_c	ℓc	vx	vy	$\bar{v}x$	$\bar{v}y$	wx	xz_c	cxy	wxz_c
$c_{obs}(S)$	x	y	c	c	x	v	$\bar{v}$	y	w	z_c	c	x

S	$vwxy$	$vw''xy$	$\bar{v}w''xy$	$\bar{v}w'xy$	$cz_c\mathcal{L}_c$
$c_{obs}(S)$	x	y	x	y	z_c

for all $c \in \mathcal{C}$, all $v \in V$, and all $\ell \in \mathcal{L}_c$.

We show that SAT has a truthful assignment if and only if there exist a filter Ψ and a relation P that generates a choice function c which coincides with c_{obs} on $\mathcal{D}$. Suppose first that SAT has a truthful assignment. First we pick a relation P such that yPx, xPw'', yPw'', and $z_cP\ell$, for each clause c and true literal ℓ in c, and aPb, for all $a, b \in X$ such that $c_{obs}(\{a, b\}) = a$. Next we consider the enumeration of the elements in X that starts with w''; followed by all true literals (in any order); followed by all clauses (in any order); followed by x, y, w, and w' (in that order); followed by z_c for each clause c (in any order); followed by all literals and their negations that did not already appear (in any order). For each choice problem R, let $\Psi(R)$ be the set containing the first element in the enumeration that belongs to R plus any element $a \in R$ such that $a = c_{obs}(S)$ for some $S \in \mathcal{D}$ containing R. It is easy to check that Ψ is a filter. It remains to show that the choice function generated by (Ψ, P) coincides with c_{obs} on $\mathcal{D}$. This follows by definition of P on pairs. By definition of Ψ, we have $\Psi(\{c, x, y\}) = \{c\}$ and $\Psi(\{w, x, z_c\}) = \{x\}$, and hence $c = c_{obs}$ on these two choice problems as well. For each variable v, $\Psi(\{v, w, x, y\}) = \{x\}$ or $\{v, x\}$ depending on whether v comes after or before x. In either case, the choice is x since xPv. For each variable v, $\Psi(\{\bar{v}, w'', x, y\}) = \{w'', x\}$ and $\Psi(\{v, w'', x, y\}) = \{w'', y\}$. The choices are x and y, respectively, since xPw''

and yPw''. For each variable v, $\Psi(\{\bar{v}, w', x, y\}) = \{x, y\}$ or $\{\bar{v}, y\}$ depending on whether $\bar{v}$ comes after or before x. In either case, the choice is y since yPx and $yP\bar{v}$. Finally, for each clause c, $\Psi(\{c, z_c\} \cup \mathcal{L}_c)$ contains z_c and a true literals appearing in c. By definition of P, $c(\{c, z_c\} \cup \mathcal{L}_c) = z_c$, as desired.

Conversely, suppose the filter Ψ and relation P generate a choice function c which coincides with c_{obs} on $\mathcal{D}$. We can assume without loss of generality that the DM pays attention to both options in each pair under Ψ, so that aPb if and only if $c(\{a, b\}) = a$ for all $a, b \in X$. We construct an assignment for SAT as follows. Consider the enumeration of X defined by $x_1 = c(X)$ and $x_k = c(X \setminus \{x_1, \ldots, x_{k-1}\})$, for all $k \geq 2$. Say a literal is true if it appears before both x and y. (It need not be that every literal or its negation is true, but that will not matter.) We show it is impossible to have both a literal and its negation true. Suppose, on the contrary, that there is a variable v such that both v and $\bar{v}$ come before both x and y in the enumeration. Assume that $c(\{x, y\}) = x$ so that xPy (a similar reasoning applies in the other case where yPx). From the corresponding pairwise choices, we infer $\bar{v}Px$ and $yP\bar{v}$. Notice that $x, y \in \Psi(\{\bar{v}, x, y\})$ since x is picked out of $\{\bar{v}, w'', x, y\}$ and y is picked out of $\{\bar{v}, w', x, y\}$. Also, $\bar{v} \in \Psi(\{\bar{v}, x, y\})$ since $\bar{v}$ is picked from the set consisting of elements of X that succeed $\bar{v}$ in the enumeration. We reach a contradiction since P is cyclic over $\{\bar{v}, x, y\}$ and all three receive attention.

Given this well-defined truth assignment, we check that all clauses in SAT are satisfied. Let c be a clause. Since x is picked out of $\{w, x, z_c\}$, but also z_cPx and wPx, x must precede both w and z_c in the enumeration. Otherwise, the first element in $\{w, x, z_c\}$ appearing in the enumeration is the first element in $\{w, z_c\}$ in the enumeration. That element must be paid attention when choosing from $\{w, x, z_c\}$, contradicting that x is picked. Similarly, c precedes both x and y in the enumeration since c is picked out of $\{c, x, y\}$, xPc and yPc. From $c(\{c, z_c\} \cup \mathcal{L}_c) = z_c$ and cPz_c, we conclude that c is not the first element of that choice problem to appear in the enumeration. If z_c comes first, then z_c precedes c. This would contradict the fact that c precedes x and x precedes z_c. Hence, one of the literals in $\mathcal{L}_c$ appears first, and precedes c. That literal is true since c precedes x and y. The assignment is thus truthful for SAT. $\square$