THE DEMONETIZATION OF GOLD: TRANSACTIONS AND THE CHANGE IN CONTROL

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August 2011

COWLES FOUNDATION DISCUSSION PAPER NO. 1814

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07/11/2011

Contents

Abstract

Three models of a monetary economy are considered, in order to show the effects of a gold demonetization: the first with a gold money, the second with demonetized gold but no central bank, and the third with demonetized gold, but with a central bank. The distinctions between ownership and control are discussed.

JEL Listings: C72, E50, E58

Keywords: Gold demonetization, Gold backed paper, Reserves

1 Money and Transactions

In this essay a series of simple examples is employed to show the relationship between gold and fiat money. Specifically, we study the opening up of an economy to government control via the invention of symbolic money and the demonetization of gold.

We consider three elementary one-period models of money lending. The first represents a simple two-perishable-good economy *before* a gold demonetization. It has gold as the money, no central bank, but has individuals who act as money lenders ("merchant banks" or "individually owned banks"). The second model considers the effects of a demonetization. The transaction use of gold is replaced by paper.[1] All holders of gold are given on a one-to-one basis paper money (but are allowed in addition to keep their gold). In the third model, when gold is demonetized the same amount of paper is issued as in the second model; but it is issued to a central bank instead of to the individuals. As the individuals own the central bank, there has been no change in ownership, just in control that may or may not be justified in terms of expertise and professional role.

All three models are strategic market games, in which there are markets for the two perishable goods; in addition, if it is demonetized, there is a market for gold as well.

Finally, we treat all markets as "buy-sell", rather than "sell all."[2] The one exception is at the end of our discussion of the first, "pre-demonetization" model, where we also compute a sell-all version, in order to illustrate the considerable difference that a change in trading technology can make to the economic distribution.

Our results show a gain in efficiency (in the case of "enough money") when a switch is made from a durable commodity money to a fiat money. This is due to players being able to enjoy both the full service value of gold and transactions value of money — something that cannot be done in the original model with gold money. When we further add in the central bank, there is a somewhat further efficiency gain in the case of "not enough money"; we close the paper with a discussion of the usefulness of central banks.

2 A Playable Game

As the level of abstraction here is so high, it is perhaps useful to consider the one period model as an experimental game that could be played in a gaming laboratory. This helps to clarify the problem of representing an open ended activity that has a history, a future, and "fuzzy" society driven rules, by a finite model that has a well defined beginning, end, and formal rules. In

[1] We use the term"paper" to stand for any form ofsymbolic money with no intrinsic worth as a comodity.

[2] A *sell-all* model of a market is where traders must put all of their initial endowments of goods up for sale, and then buy back all that they wish to consume. It is the simplest price forming mechanism one can construct. In contrast, in the more complex *buy-sell* model, individuals may choose how much to sell at the market (and then buy back any amount). See Shubik (1999) for a full discussion.

particular, both initial and terminal conditions must be defined. The initial conditions cause few difficulties – they are just the "initial endowments" that are given in many economic models. The terminal conditions are another matter. In a T period model we can regard period $T+1$ as a "settlement day" – all accounts are closed, all positions are wound down, and the books are balanced.

Much of economic theory deals with the infinite horizon in either one or both directions. How to balance the books at "infinity" is a well known problem. If one postulates a condition such as rational expectations, at the point of equilibrium the books can be balanced. However no information is supplied about what happens *out* of equilibrium. In an experimental game (and in actual liquidations) such information must be supplied.

In an experimental game the earning of the subjects may depend not merely on their earnings per period, but also upon the evaluation of the assets left over on the day of settlement. Here we have $T = 1$; hence settlement is at the second period. There are no problems with evaluation of perishables at that time, as they will not exist.[3] However, there are problems evaluating any durables left over, such as goods, financial instuments and institutions that are owned. This is addressed in our models below.

3 Model 1: Competitive Lending with Gold as the Money

We first consider a one-period, buy-sell economy:

3.1 Model 1a: Buy-Sell

There are three player types: two types of traders, as well as moneylenders. For the traders, there is a continuum of each type. The traders trade in two perishable goods using a gold money, each with the same utility function $u(x, y, z) = 2\sqrt{xy} + z + \Pi\widehat{z}$. Here x is the amount of the first good consumed, y is the amount of the second good consumed, z is the amount of gold services consumed, and $\widehat{z}$ is the amount of gold owned at the end of the game.[4] Hence

[3] If a production process at time $t = 1$ yields a perishable one period later, then we *would* have to evaluate the possibility of left over perishable at the end of the game. Structuraly this is feasible, but only errors or pathological behavior would call forth residual perishables if the salvage value were zero.

[4] For the purpose at hand (namely the study of the demonetization of gold and the introduction of fiat), utilizing a linear separable term for gold in the utility function presents no restriction. Meanwhile, there is a benefit in that it simplifies considerably the

the first term is the utility derived from the consumption of perishables, the second is from the service value of gold, and the last is from the worth of gold at the end of the game (the parameter Π is the per-unit salvage value for gold[5]).

Traders of Type 1 have a total initial endowment of a of good 1, none of good 2, and m units of the gold money; we write these endowments as $(a, 0, m)$. Type 2 traders have $(0, a, m)$. Hence one would expect that in this model Type 1 traders would want to exchange some of their "Good 1" for Type 2 traders' "Good 2". If there are no transactions costs to trade, then the competitive, general equilibrium level of consumption[6] is for both trader types to end up consuming $a/2$ of each good, plus m units of gold service during the period, plus the terminal value of the ownership of m units of gold. Thus each type would obtain a utility "score" of a from the consumption of the goods, plus m for the gold services consumed, plus Πm from the terminal worth of m units of gold.[7] This compares with a score of $0 + m + \Pi m$ for each type from its initial bundle.

However, the rules of the game require that all trade is intermediated with money. Thus, efficient trade can only take place when the traders are in a position to borrow gold if they do not initially have enough to buy the perishables. We introduce a third agent type, a continuum of perfectly competitive moneylenders who start with an initial supply B of gold and no other commodities. They too have the same utility functions as the traders.

We work with strategic variables. A strategy for a trader α of Type 1 is denoted by $(b^\alpha, q^\alpha, d^\alpha)$, where b_1^α is the amount he bids for Good 2, q^α is the amount of his own Good 1 that he puts up for sale, and d^α is the amount of loan to be paid back to the moneylenders.[8] In what follows, we assume that all of the traders of Type 1 are identical, facing the same utility maximization problems and so acting identically; hence we may also indicate a strategy for the Type 1 traders by (b, q, d), where $b = \int_\alpha b^\alpha =$ the total amount of money bid for good 2, summed across all Type 1 agents,

mathematics involved in the analysis.

[5] It is more consistent with the mathematics to regard Π as a parameter of the system standing for the expected utility of the durable gold at the end of a one period experimental game, In dynamic programming terms it is the salvage value, or it can be regarded as the valuation of any assets left over at the day of final settlement.

[6] Needless to say, this outcome is also Pareto efficient.

[7] We are using m here to stand both for the stock and flow of gold. More properly we could use m and $\dot{m}$, however as they always have a $1:1$ relationship we allow the sloppier notation as the meaning should be clear from context. See Quint-Shubik (200x) for details.

[8] Thus the amount actually borrowed by α is $\frac{d^\alpha}{1+\rho}$, where ρ is the interest rate on loans.

$q = \int_\alpha q^\alpha =$ the total amount of Good 1 offered for sale, also summed across all Type 1 agents, and $d = \int_\alpha d^\alpha =$ the total amount of loan to be paid back by the Type 1 traders to the moneylenders. For now on, we will use this "aggregate" convention for strategies – if one wishes to recover an individual trader's strategy from (b, q, d), all one needs to do is to divide through by the measure of the set of Type 1 traders. We believe this convention for strategies will help make the overall presentation easier to follow.[9]

For further simplicity, we assume that borrowing is essentially instantaneous followed by trade. Thus, when the lenders lend gold, they lend it for the whole period, to be paid back at the "settlement time" at the end of the period.[10]

We assume that the measure of the set of traders of either type, as well as that of the set of lenders, is 1.[11]

The objective function for the traders of Type 1 is:

$$\max_{b,q,d} 2\sqrt{(a-q)\frac{b}{\overline{p}}} + \left(m + \frac{d}{1+\rho} - b\right) + \Pi\left(m + \frac{d}{1+\rho} - b + pq - d\right) \quad (1)$$

Here the parameters p and $\overline{p}$ are the prices for the two goods, ρ is the interest rate on loans, and Π is the per unit salvage value parameter for the gold money.[12] Thus, the first term above is the utility of consumption of the perishables, the second the utility from the service value of gold over the period, and the last term the salvage value of gold. We remark that a more precise model here would also include a default penalty term, but here we assume that default penalties are so great that the traders are essentially forbidden to go bankrupt.

[9] The reader will also note that the quantity m defined before is really a total amount of money initially owned by the continuum of traders, i.e. $m = \int m^\alpha$, where m^α represents the amount of money initially held by trader α. Similarly, a denotes the aggregate endowment of perishable across the continuum of individuals.

[10] See Quint and Shbuik (200x) for a detailed discussion of the distinction between consumption and transactions use of gold.

[11] Realistically the measures sould be of the order of 10,000:1 or more depending upon he society.

[12] If the economy were stationary then we could imagine a discount rate β for which $\Pi = \frac{\beta}{1-\beta}$ and $1 + \rho = \frac{1}{\beta}$.

The constraints on the optimization are:

$$m + \frac{d}{1+\rho} - b \geq 0 \qquad (\lambda) \text{ (cash flow constraint)} \tag{2}$$

$$m + \frac{d}{1+\rho} - b + pq - d \geq 0 \qquad (\mu) \text{ (budget constraint)} \tag{3}$$

$$b, d \geq 0,\ 0 \leq q \leq a \tag{4}$$

The constraint (λ) is the *cash flow constraint*, i.e., the requirement that there must be enough cash on hand to make all bids. The constraint (μ) is the *budget constraint*, i.e., the requirement that all debts must be paid back at the end of the game.

Similarly, the optimization problem for the Type 2 traders is

$$\begin{aligned}
&\max_{\bar{b},\bar{q},\bar{d}}\ 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + \left(m + \frac{\bar{d}}{1+\rho} - \bar{b}\right) + \Pi\left(m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d}\right)\\
&\text{s.t. } m + \frac{\bar{d}}{1+\rho} - \bar{b} \geq 0 \qquad (\bar{\lambda})\\
&\quad m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d} \geq 0 \qquad (\overline{\mu})\\
&\quad \bar{b}, \bar{d} \geq 0,\ \ 0 \leq \bar{q} \leq a.
\end{aligned}$$

where the notation should be apparent.

The lenders in this model are private capitalists. They lend but do not accept deposits. Since they act both as consumers and as moneylenders, their decision variables are b_1^* (the total amount bid by lenders for good 1), b_2^* (the total amount bid by lenders for good 2), and g (the total amount of gold lent to traders). Since there is a continuum of lenders, these variables represent aggregations of identical individual lenders' strategies, much as the traders' variables do.

The optimization for the lenders is:

$$\max_{b_1^*,b_2^*,g} 2\sqrt{\frac{b_1^*}{p}\frac{b_2^*}{\overline{p}}} + (B - b_1^* - b_2^* - g) + \Pi\,(B - b_1^* - b_2^* + \rho g) \tag{5}$$

$$\text{s.t. } B - b_1^* - b_2^* - g \geq 0 \tag{6}$$

$$b_1^*, b_2^*, g \geq 0. \tag{7}$$

Finally, the balance conditions for price are

$$p = \frac{\overline{b} + b_1^*}{q} \tag{8}$$

$$\overline{p} = \frac{b + b_2^*}{\overline{q}} \tag{9}$$

while that for the interest rate is

$$1 + \rho = \frac{d + \overline{d}}{g}. \tag{10}$$

3.2 Results

In Appendix A, we solve the model for two cases. First, in "Case 1," the traders begin with little gold but the lenders have a lot, i.e. m is small and B is large. This gives tight cash flow and budget constraints for the traders, but loose cash flow constraints for the lenders. In "Case 2", both m and B are small – and so all constraints for both traders and lenders are tight. The two cases above allow us to consider set values of m and Π (namely $m = 0$ and $\Pi = 1$), while allowing B to range over an interval of values. The dividing line between Case 1 and Case 2 is where $B = a/2$.

In both cases, we shall see that the money interest rate is always at least one. This reflects the marginal value of consumption of the services of gold, which is 1. When gold is in short supply, the interest rate increases from 1, reflecting the intensity of the shortage.

Finally, in both cases, we note that the consumption levels of the perishables is not Pareto efficient.[13]

Example 1

If $B = 2a$, we are in Case 1 above. Our calculations yield $p = \overline{p} = 1/2$, $\rho = 1$, $d = \overline{d} = a/4$, $b = \overline{b} = a/8$, $q = \overline{q} = a/2$, and the money lenders lend $g = a/4$. Hence the consumption by each trader type is $a/2$ of "their own" perishable, and $a/4$ of the "other" perishable. The bids of the money lenders are $b_1^* = b_2^* = a/8$, and so the consumption of the money lenders is $a/4$ of each of the perishables. The final distribution of resources is $(\frac{a}{2}, \frac{a}{4}, 0)$, $(\frac{a}{4}, \frac{a}{2}, 0)$, and $(\frac{a}{4}, \frac{a}{4}, 2a)$ The utilities to the two trader types are given by $2\sqrt{\frac{a}{2}\frac{a}{4}} + 0 = \frac{\sqrt{2}a}{2}$, while that for the moneylenders is $2\sqrt{\frac{a}{4}\frac{a}{4}} + \frac{3a}{2} + \Pi(2a) = 4a$.

Example 2

[13] In our models, consumption of perishables is Pareto efficient iff each of the three types of consumer (i.e., the two trader types and the lenders) consume equal amounts of Good 1 and Good 2.

Suppose there were far less gold in the system, let us limit it to $B = a$; the solution will differ from Example 1 only in the payoff for the money lenders that is now reduced to $2\sqrt{\frac{a}{4}\frac{a}{4}} + \frac{a}{2} + \Pi(a) = 2a$. We are still in Case 1.

Example 3

Suppose there were still less gold in the system, let it be $B = a/4$; this now puts us into "Case 2" from the analysis in Appendix A. Now we have $p = \overline{p} = 1/4$, $\rho = 3$, $d = \overline{d} = a/8$, $b = \overline{b} = a/32$, $q = \overline{q} = a/2$, and the money lenders lend $g = a/16$. Hence the consumption by each trader type is $a/2$ of "their own" perishable, and $a/8$ of the "other" perishable. The bids of the money lenders are $b_1^* = b_2^* = 3a/32$, and so the consumption of the money lenders is $3a/8$ of each of the perishables. The final distribution of resources is $(\frac{a}{2}, \frac{a}{8}, 0)$, $(\frac{a}{8}, \frac{a}{2}, 0)$, and $(\frac{3a}{8}, \frac{3a}{8}, \frac{a}{4})$ The utilities to the two trader types are given by $2\sqrt{\frac{a}{2}\frac{a}{8}} + 0 = \frac{a}{2}$, while that for the moneylenders is $2\sqrt{\frac{3a}{8}\frac{3a}{8}} + 0 + \Pi(\frac{a}{4}) = a$. The interest rate $\rho = 3$ is above 1 to reflect the shadow price of the shortage of gold.

3.3 Model 1b: Sell-All

Here we consider a sell-all version of Model 1a. Now, when traders come to the perishable goods' market, they must first sell off all of their endowments, and then buy back all they consume. Since they are selling and buying more, their need for money increases.

The notation for the model is similar to that of Model 1a, except for the traders' decision variables. For trader type 1, they are b_1 (the total amount bid for perishable good #1, summed over all type 1 traders), b_2 (the amount bid for perishable good #2, again summed over all type 1 traders), and d (as in Model 1a, the total amount of loan to be paid back to the lenders). Note that we no longer have q as a strategic variable, because the amount of perishables put up for sale is no longer a decision.

The optimization for the traders of type 1 is

$$\begin{aligned} \max_{b_1,b_2,d} 2\sqrt{\frac{b_1}{p}\frac{b_2}{\overline{p}}} + m + \tfrac{d}{1+\rho} - b_1 - b_2 + pa \\ +\Pi\left(m + \tfrac{d}{1+\rho} - b_1 - b_2 + pa - d\right) \end{aligned} \tag{11}$$

$$\text{s.t. } m + \frac{d}{1+\rho} - b_1 - b_2 \geq 0 \quad (\lambda) \text{ (cash flow constraint)} \tag{12}$$

$$m + \frac{d}{1+\rho} - b_1 - b_2 + pa - d \geq 0 \quad (\mu) \text{ (budget constraint)} \tag{13}$$

$$b_1, b_2, d \geq 0 \tag{14}$$

There is a similar optimization for the traders of Type 2, with decision variables $\overline{b}_1$, $\overline{b}_2$, and $\overline{d}$. For the money lenders, the optimization is precisely as before, namely:

$$\max_{b_1^*, b_2^*, g} 2\sqrt{\frac{b_1^*}{p}\frac{b_2^*}{\overline{p}}} + (B - b_1^* - b_2^* - g) + \Pi\left(B - b_1^* - b_2^* + \rho g\right) \quad (15)$$

$$\text{s.t. } B - b_1^* - b_2^* - g \geq 0 \quad (16)$$

$$b_1^*, b_2^*, g \geq 0. \quad (17)$$

Finally, we have the balance conditions $p = \frac{b_1 + \overline{b}_1 + b_1^*}{a}$, $\overline{p} = \frac{b_2 + \overline{b}_2 + b_2^*}{a}$, and $1 + \rho = \frac{d + \overline{d}}{g}$.

3.4 Results

In Appendix B, we solve the model for the special case where m is small. We again have two cases, depending up whether constraint (λ^*) is tight – for high values of B (Case 1) it is not, while for low values of B (Case 2) it is. The operational differences between Models 1a and 1b can be illustrated by comparing the same set of three examples, all with $\Pi = 1$ and $m = 0$. We note that with sell-all, the dividing line between Case 1 and Case 2 falls at a higher value of B, namely $B = a$.

Also, note that in both cases the consumption of perishables is Pareto efficient. This contrasts with the buy-sell model, in which it was *never* efficient.

Example 1

Suppose $B = 2a$. Our calculations (in Case 1) yield $b_1 = b_2 = \overline{b}_1 = \overline{b}_2 = a/8$, $p = \overline{p} = 1/2$, $\rho = 1$, $d = \overline{d} = a/2$, and the money lenders lend $g = a/2$. Hence the consumption by each trader type is $a/4$ of both perishables. The bids of the money lenders are $b_1^* = b_2^* = a/4$ and so the consumption of the money lenders is $a/2$ of each of the commodities. The final distribution of resources is $(\frac{a}{4}, \frac{a}{4}, 0)$, $(\frac{a}{4}, \frac{a}{4}, 0)$, and $(\frac{a}{2}, \frac{a}{2}, 2a)$. The utilities to the two trader types are given by $2\sqrt{\frac{a}{4}\frac{a}{4}} + \frac{a}{2} = a$, while that for the moneylenders is $2\sqrt{\frac{a}{2}\frac{a}{2}} + a + \Pi(2a) = 4a$.

Example 2

Suppose there were far less gold in the system, let us limit it to $B = a$. Now we are at the dividing line between Case 1 and Case 2, so we can use either to compute the variable values. We obtain the same values as in Example 1, except the payoff for the money lenders is now reduced to $2\sqrt{\frac{a}{2}\frac{a}{2}} + 0 + \Pi a = 2a$. In this example when $B = a$ there is precisely enough money to not constrain lending.

Example 3

Suppose there were still less gold in the system, let it be $B = a/4$; now we are in the interior of Case 2. We obtain $b_1 = b_2 = \bar{b}_1 = \bar{b}_2 = a/128$, $p = \overline{p} = 1/8$, $\rho = 7$, $d = \overline{d} = a/8$, $b_1^* = b_2^* = 7a/64$, and the money lenders lend $g = a/32$. The trader types consume $(\frac{a}{16}, \frac{a}{16}, 0)$ and $(\frac{a}{16}, \frac{a}{16}, 0)$, while the lenders consume $(\frac{7a}{8}, \frac{7a}{8}, \frac{a}{4})$.

Finally, we note the high value of ρ – this reflects the high shadow price of the shortage of gold.

3.5 A comparison of Models 1a and 1b

We may glean several key properties of a commodity money from these models, and from other models appearing elsewhere.

1. The key idea associated with "enough money"[14] is that the price of the rental of gold money (ρ) should be exactly its consumption value (which is 1 here).

2. Although "enough money" is a well defined concept, the precise mathematical conditions depend upon the institutional details of trade. The difference between the monetary requirements of the buy-sell and the sell-all models illustrates this.

3. Not only is the specific mechanism relevant, but so too is the speed of operation. If used in trade, the commodity money is unavailable for the whole period for consumption purposes. Quint and Shubik (2005) examine the influence of time-in-trade in detail.

4. The above examples show that a commodity money's elasticity of demand (as a commodity) plays a role. This aspect of utilizing a commodity that does not enter the utility function in a separable form has been studied in detail by Dubey and Shapley (1994).

5. When a commodity money is borrowed, even in a stationary economy, the rate of interest is strictly higher than the money's consumption value. This introduces a wedge between buying and selling prices in the buy-sell model (or any model where the individual controls her offer). This wedge does not appear in the sell-all model.

[14] By "enough money", we mean the case where there is enough money in the economy to finance efficient trade of the perishables. In Models 1a and 1b above, the "enough money" case is Case 1. See Quint-Shubik (xxxx) for details.

3.6 An aside on enough commodity money

The selection of a commodity money depends upon physical properties, such as portability, cognizability, and durability. It also involves problems in the sensitivity of the elasticity of relative prices as its quantity changes. In our models, we assume the commodity money enters the utility function as a linear separable term. If we drop this assumption, new difficulties appear – consider the following example of a sell-all economy where we show that the commodity money can never be in suffiecient supply to support competitive prices and distribution.

Suppose that initial holdings of three goods (by the three agent types) are $(3,0,0)$, $(0,3,0)$, and $(0,0,3)$, and all agents have the same utility function

$$U(x,y,m) = \frac{1}{3}(xy\dot{m})^{\frac{1}{3}} + \Pi m$$

The first two goods are perishables, while the third good is a durable and has been selected as the money. The second term is the expected utility of the left over durable, where Π is an exogenous parameter. For the example assume $\Pi = 1$.

It is easy to see that the general equilibrium solution calls for symmetric prices (p,p,p) and a distribution of $(1,1,1)$ of the commodities to each trader type. Since the transaction structure is sell-all, and the game with money is to achieve the CE outcome, each of the first two trader types would need to borrow $2p+1$ units of the monetary commodity to buy the distribution $(1,1,1)$. But they only have an income of $3p$ each. Thus, if $p < 1$ the types do not have enough money to pay back their loans. On the other hand, if $p \geq 1$, the lenders do not have $2 * (2p+1)$ in cash to lend.

Can this be cured by giving the money lenders more money? We answer "no", by virtually the same argument. Suppose the lenders had some amount $m > 3$. We note that for an arbitrary m the competitive equilibrium prices and quantities become $(\frac{mp}{3}, \frac{mp}{3}, p)$ and $(1,1,\frac{m}{3})$ respectively. The trader types each need $\frac{2mp}{3} + \frac{m}{3}$ in cash in order to buy the distribution $(1,1,\frac{m}{3})$. They each have income mp. Hence if $p < 1$ the types do not have enough money to pay back their loans. And again if $p \geq 1$, the lenders do not have $2 * (\frac{2mp}{3} + \frac{m}{3})$ in cash to lend.

4 Model 2: Trade with Demonetized Gold and Transaction Strips

We now analyze the model of the above (buy-sell) economy after gold has been demonetized. The physical asset gold is stripped of its monetary function. The old money lenders become paper money lenders/gold merchants. We use the term "strips" here to suggest that when a real asset A has more than one function, one might be able to modify the rules of the game to strip A of that function, while creating object B that takes it over. The legal modifications are many and subtle but the principle is relatively simple.

4.1 The three uses of monetary gold

A monetary gold has at least three uses:

1. A store of value (a property shared with all other durables)
2. A provider of consumption or production services (a property shared with all other durables)
3. A provider of transaction services.[15]

At any particular time, a monetary gold can only provide one of the last two services.

4.2 The Demonetization

A society that utilizes gold as money can switch to paper money, maintain full ownership claims of all agents, and provide transaction services to its members. It can accomplish this by stripping gold of its use in transactions and giving all owners of gold a paper (or other) symbolic claim to the gold, on a one-to-one basis. Thus against the B units of gold a piece of paper (a "gold certificate") inscribed with the legend "*This is one unit of transactions gold"* can be used instead of the gold itself to provide payment.

Of course, the gold owners still have the gold, which has value as a durable good (but not as a money). In an ideal world with no exogenous uncertainty and no opportunity for any individuals to print more transaction strips,[16] the gold is now freed up for use in production or consumption

[15] A fourth use is as a numeraire and a fifth use is as scalar measure of value; but these are not particularly germain to this discussion.

[16] Thereby violating the 1 : 1 relationship or the 100% reserves

services, or to lease to others. And while the demonetization has stripped their gold of its use in market transactions, the owners are given two financial instruments that compensate for the loss. The first is the paper described above. The second is a call on gold that can be exercised at the time of settlement. For now we ignore the call feature, but we discuss it later in Section 4.6.

In the new economy, the inital endowments now become $(a, 0, m, m)$ for the Type 1 traders, $(0, a, m, m)$ for the Type 2 traders, and $(0, 0, B, B)$ for the paper moneylenders/gold merchants (hereafter called "lender-merchants"). In each of these endowment vectors, the first two entries are the endowments of the two perishables, the third entry is that of gold (now without its monetary function), and the fourth entry is the amount of "strip" or separated asset issued against the gold to replace it for transactions services. This is in essence a 100% reserves system. The original Bank of Amsterdam appears to have paid in notes backed by its gold in this manner.

The specifics of how the paper is issued are a matter of the rules of the game, which depend upon the laws of the society. There are several ways in which this can happen. Perhaps the most obvious is by means of a warehouse receipt. An individual depositing an amount of gold is presented with a warehouse paper receipt for that amount. But if this paper is non-negotiable, it remains as two-party paper and cannot be utilized in transactions (because it cannot be transferred to a third party). But this restriction is a matter of the rules of the game — a legal system can recognize the legality of third party utilization, at which point the warehouse receipts *can* be utilized as a money if they are universally accepted in payment.

4.3 Demonetized Gold, No Central Bank

We modify the previous buy-sell economy (Model 1a). The initial conditions have been specified above. There is now an extra market for gold, which is also buy-sell. Thus, the traders must specify both the amount of perishable they wish to put up for sale, as well as an amount of gold they wish to put up for sale. All trade must be intermediated with paper money backed by gold.

The objective function for the traders of Type 1 is:

$$\max_{b,q,d,b_3,q_3} 2\sqrt{(a-q)\frac{b}{p}} + \left(m + \frac{b_3}{p_3} - q_3\right) + \Pi_1\left(m + \frac{d}{1+\rho} - b_1 - b_3 + pq + p_3q_3 - d\right)$$
$$+ \Pi_2\left(m + \frac{b_3}{p_3} - q_3\right) \tag{18}$$

Here the decision variables b and b_3 denote the total amounts bid for good 2 and gold respectively, q and q_3 the amounts of good 1 and gold put up for sale respectively, and d is the amount of loan to be repaid to the lenders.[17] The p and $\overline{p}$ are the prices for the two goods, p_3 is the price of gold, ρ is the interest rate on loans, Π_1 is the per unit salvage value parameter for the strips, and Π_2 is the per unit salvage value for gold. Thus, the first term above is the utility of consumption of the perishables, the second the utility from the consumption value of the services of gold for the period, the third term is the salvage (terminal) value for left over gold certificates, and the last term is the salvage value for the asset gold. Since they are modeled as non-depreciating durables, both gold and gold certificates will be left over at the period of final settlement.[18]

The optimizations for the traders and the lender-merchants differ only because their initial endowments. Technically, any individual holding gold could lend the strips she is given and thus could have a lending strategy; however we have specified our initial conditions to be such that the traders would never lend and the lender-merchants would never borrow.

The constraints on the optimization for the traders of Type 1 are:

$$m + \frac{d}{1+\rho} - b - b_3 \geq 0 \qquad (\lambda) \tag{19}$$

$$m + \frac{d}{1+\rho} - b - b_3 + pq + p_3 q_3 - d \geq 0 \qquad (\mu) \tag{20}$$

$$b, b_3, d \geq 0, 0 \leq q_3 \leq m \tag{21}$$

The constraints (λ) and (μ) are the cash flow and budget constraints as in Model 1a.

The optimization problem for the Type 2 traders is similar.

The continuum of lender-merchants act both as consumers and as money-lenders. Their decision variables are b_1^*, b_2^* (the total amount they bid for goods 1 and 2), b_3^* (the total amount bid they bid for gold), q_3^* (the total amount of gold they put up for sale), and g (the total amount of gold strips

[17] We point out that the variables b, q, d, b_3, and q_3 each represent aggregations of identical individual traders' strategies – see Section 3.1 for details.

[18] As before we note that a more precise model here covering all positions in the feasible payoff space would also include a default penalty term; and that here we assume that default penalties are so great that the traders are essentially forbidden to go bankrupt.

lent to traders). Their optimization is:

$$\max_{b_1^*,b_2^*,b_G^*,q_G^*,g} 2\sqrt{\frac{b_1^*}{p}\frac{b_2^*}{\overline{p}}} + \left(B + \frac{b_3^*}{p_3} - q_3^*\right) + \Pi_1\left(B - b_1^* - b_2^* - b_3^* + p_3 q_3^* + \rho g\right) + \Pi_2\left(B + \frac{b_3^*}{p_3} - q_3^*\right) \tag{22}$$

$$\text{s.t. } B - b_1^* - b_2^* - b_3^* - g \geq 0 \tag{23}$$

$$b_1^*, b_2^*, b_{3.}^* g \geq 0,\ 0 \leq q_3^* \leq B \tag{24}$$

Finally, the balance conditions for price are

$$p = \frac{\bar{b} + b_1^*}{q} \tag{25}$$

$$\overline{p} = \frac{b + b_2^*}{\overline{q}} \tag{26}$$

$$p_3 = \frac{b_3 + \overline{b}_3 + b_3^*}{q_3 + \overline{q}_3 + q_3^*} \tag{27}$$

while that for the interest rate is

$$1 + \rho = \frac{d + \overline{d}}{g}. \tag{28}$$

We call the above "Model 2".

4.4 An extra constraint concerning the sale of strips

In our formulation above a conceptual problem emerges concerning the sale of strips. If a strip is backed by gold, then the money lenders cannot be permitted to offer more strips than they have gold on hand – otherwise some of their lending would be unbacked by the 100% reserves of this system. This introduces the extra constraint

$$g \leq B - q_3^*$$

This extra constraint is satisfied in all three examples below.[19]

[19] More generally, if there is a reserve requirement of k (expressed as a proportion), then the extra constraint would be $kg \leq B - q_3^*$. Since the three examples all satisfy this with $k = 1$, necessarily they all satisfy this with any lower values of k.

4.5 Results

In Appendix B we solve the model, again for the case where m is small. With small m, we may assume the traders' cash flow and budget constraints are both tight. In addition, we may assume that the traders do not sell gold ($q_3 = 0$) and the lender-merchants do not buy gold ($b_3^* = 0$).

There are again two cases, depending on the value of B. For higher values of B, we assume that the lender-merchants' cash flow constraint is loose. In the second case, with lower values of B, we assume that this constraint is tight, and that the gold market is inactive.

We now set parameter values so as to most closely match the three examples calculated with Model 1a. To do this, we set $\Pi_1 = \Pi_2 = 1$, $m = 0$, and consider the same three values of B:

Example 1

If $B = 2a$, we are in Case 1 above. Our calculations yield a continuum of possible values for q, namely the interval from $\frac{a}{2}$ to $\frac{2a}{3}$. For each such q, we have $p = \overline{p} = 1$, $\rho = 0$, $d = \overline{d} = q$, $b = \overline{b} = a - q$, and the lender-merchants lend $g = 2q$. Hence the consumption by each trader type is $(a-q, a-q)$ of the perishables. The bids of the lender-merchants are $b_1^* = b_2^* = 2q - a$, and so they consume $2q - a$ of each of the perishables. In addition, we have $p_3 = 2$, $b_3 = \overline{b}_3 = 2q - a$, and $q_3^* = 2q - a$; hence the lender-merchants sell $2q - a$ units of gold ($q - \frac{a}{2}$ to each trader type). The final distribution of resources is $(a-q, a-q, q-\frac{a}{2}, 0)$, $(a-q, a-q, q-\frac{a}{2}, 0)$, and $(2q-a, 2q-a, 3a-2q, 2a)$ for the two trader types and the merchant-lenders respectively. For the traders, their final utility is $2\sqrt{(a-q)(a-q)} + q - \frac{a}{2} + \Pi_2(q - \frac{a}{2}) + 0 = a$, while for the merchant-lenders it is $2\sqrt{(2q-a)(2q-a)} + (3a-2q) + \Pi_2(3a-2q) + 2a = 6a$. These utilities do not depend upon the chosen value of q.

Note that the consumption of perishable is Pareto efficient; recall in the corresponding Example 1 of Model 1a it was not efficient. In fact we can compare the final utilities to the agents in the two examples:

	Trader 1	Trader 2	Lender
Model 1a	$\frac{\sqrt{2}a}{2}$	$\frac{\sqrt{2}a}{2}$	$4a$
Model 2	a	a	$6a$

Table 1

Finally, notice that here that for all values of q we have $g \leq B - q_3^*$, so the reserve requirement is indeed met.

Example 2

If $B = a$, we are on the border of Case 1 and Case 2. There is now only one equilibrium. Our calculations yield $p = \overline{p} = 1$, $\rho = 0$, $d = \overline{d} = a/2$,

$b = \overline{b} = a/2$, $q = \overline{q} = a/2$, and the lender-merchants lend $g = a$. Hence the consumption by each trader type is $(a/2, a/2)$ of the perishables. The bids of the lender-merchants are $b_1^* = b_2^* = 0$, and so they consume none of the perishables. In addition, we have $p_3 = 2$, $b_3 = \overline{b}_3 = 0$, and $q_3^* = 0$; hence the lender-merchants sell no gold to the traders. The final distribution of resources is $(\frac{a}{2}, \frac{a}{2}, 0, 0)$, $(\frac{a}{2}, \frac{a}{2}, 0, 0)$, and $(0, 0, a, a)$ for the two trader types and the merchant-lenders respectively. Notice that here $g \leq B - q_3^*$, so the reserve requirement is indeed met.

Example 3

If $B = a/4$, we are in Case 2. Our calculations yield $p = \overline{p} = \frac{1}{4}$, $\rho = 3$, $d = \overline{d} = a/8$, $b = \overline{b} = a/32$, $q = \overline{q} = a/2$, and the lender-merchants lend $g = a/16$. Hence the consumption by each trader type is $a/2$ of their "own" perishable, and $a/8$ of the "other" perishable. The bids of the lender-merchants are $b_1^* = b_2^* = \frac{3a}{32}$, and so they consume $\frac{3a}{8}$ of each of the perishables. The gold market is inactive, i.e. the lender-merchants sell no gold to the traders. The final distribution of resources is $(\frac{a}{2}, \frac{a}{8}, 0, 0)$, $(\frac{a}{8}, \frac{a}{2}, 0, 0)$, and $(\frac{3a}{8}, \frac{3a}{8}, \frac{a}{4}, \frac{a}{4})$ for the two trader types and the merchant-lenders respectively. This is not efficient.

4.6 The worth of strips at settlement day

At the day of settlement there is a modeling problem concerning end valuation of the gold strips. If viewed as an experimental game, then the only question to a player is how their left over paper money is treated. If conversion to gold is not permitted, then all that matters is its salvage value. Alternatively, if ownership of the paper money includes a call on the gold, there is no reason to convert if the salvage value for the paper is the same as that for gold. Thus these games, without and with convertability, have a solution in common – namely where there is no conversion. This appears to be double counting, but it actually reflects that both the transactions and consumption values are being realized.

A different approach is to consider the infinite horizon version of the game. Here the strip need never be cashed; hence it has the full transaction value over all periods. At equilibrium, this transaction value is equal to the full service value of the gold over all periods. [This is because at a stationary state equilibrium, there can be no advantageous arbitrage opportunites – it cannot pay for an individual with a strip to buy gold or vice-versa.] This justifies our assumption (in the examples) of $\Pi_1 = \Pi_2$.

4.7 Fractional Reserves

Drive for show, but putt for dough
Old golf saying

We have provided a painstakingly precise set of process models within a grotesquely oversimplified economy, in order to show how in the case of enough money, replacing gold by fully backed paper improves efficiency. Our second model has 100% backed gold reserves. The gold strips match the gold. The mathematics of this model does not justify fractional reserves; we conjecture, but do not prove that the model can be modified to work for the infinite horizon with no uncertainty with any fractional reserve ratio. This problem is left for future work.

5 A Disclaimer on Uncertainty

In this paper we do not model uncertainty. We intend to provide a simple example with uncertainty in a further essay, to illustrate that with any exogeneous uncertainty, the meaning of "enough money" (and the ability to supply it) becomes difficult to define, and depends upon the default laws and the society's overall willingness to absorb risk.

6 Model 3: Trade with Central Bank Control of the Strips

A variation of Model 2 is offered where upon the demonetization of the gold the strips are *not* given out to the the gold owners. Instead the society forms a central bank that lends the strips. The bank in turn is owned by holders of the gold, who receive shares in the bank but do not necessarily control it. The original holders of the gold, who were moneylenders in both Model 1 and 2, now are simply dealers in gold, and we call them "gold merchants".

6.1 The negotiability of bank shares?

In this model, the gold and bank shares are packaged together. The individuals holding the gold hold the shares. Another possibility is that the shares are separately negotiable. This is a matter of choice in how a society constructs the rules of the game, either by law or by custom, or by both.

6.2 On natural persons, legal persons, and ownership

In a modern organized society there are two types of "legal persons": a) natural persons, and b) corporate entities, such as for profit firms, universities, hospitals, goverment agencies, etc. All legal persons who are not natural persons are ultimately owned in some form by natural persons.[20] Upon liquidation, their assets must be flowed through to their owners. This even includes governments, although given the ongoing aspects of the nation state this is essentially a fiction – the nation state rarely voluntarily "goes out of business" – it more often ends with war or revolution.

With the luxury of an experimental game with given initial and terminal conditions,[21] we can define a game where at the settlement day the resources of all legal persons who are not natural persons are turned over to the natural persons who are the ultimate owners.

6.3 The central bank

In Model 3, all of the physical and ultimate ownership aspects of the economy for the natural persons are the same as in Model 2. However, there is a change in the number of agents and their strategic power. Instead of giving all agents with gold the strips, a new legal person is created, the central bank. The central bank is funded with all of the gold strips that have been created, but it is owned by those who have supplied the gold backing for the strips.

The initial holdings of the traders, the money lenders and the central bank in this economy are: Traders of Type 1 $(a, 0, m, 0, m)$; Traders of Type 2 $(0, a, m, 0, m)$; gold merchants $(0, 0, B, 0, B)$; the central bank $(0, 0, 0, B + 2m, -(B + 2m))$ where components of these vectors represent

1. the amount held of the first perishable good;
2. the amount held of the second perishable good;
3. the amount of gold held;
4. the amount of gold trading strips held;

[20] In actuality an orderly liquidation of a society raely if ever takes place; thus the liquidation of the central bank suggested is an accounting fiction that stresses ultimate ownership, not control and lays stress on balancing the books.

[21] The terminal conditions at their simplest may be a fixed set of prices for remaining resources, but they also could be books of instructions or algorithms based on the play of the game.

5. the shares held of the central bank.

As gold and central bank shares are traded together, we could easily simplify the notation for the endowments. However, it is perhaps more natural to leave gold and bank shares apart, to emphasize the two roles for gold here – as a commodity and as a backing for paper money.

The balance sheet of the central bank has two items as indicated in Table 2

Assets	Liabilities
$B + 2m$ strips	
	$B + 2m$ shares
$B + 2m$	$B + 2m$

Bank balance sheet

Table 2

Next, we must specify how the central bank makes loans to the various agents. The simplest mechanism is as before, with lenders putting up gold notes and borrowing agents bidding for them; but here there is only one lender (the central bank). In order to fully close the model, we need to specify the motivation of the central bank. Is it profit maximizing (if so it is a monopolist)? Is it a philanthropist concerned with the efficiency of the society? In this model we treat the bank as a strategic dummy, which offers G for loan no matter what.

Conceptually, there is complete freedom for the interest rate in this model, including the taking on of negative values. While logically possible, an outcome with a negative ρ can be ruled out as not occuring in an equilibrium state by using a simple arbitrage argument.

Model 3 As in Models 1a and 2, we consider a buy-sell economy. The optimization problem for the traders of Type 1 is almost as in Model 2:

$$\max_{b,q,d,b_3,q_3} 2\sqrt{(a-q)\frac{b}{\bar{p}}} + \left(m + \frac{b_3}{p_3} - q_3\right) + \Pi_1\left(\frac{d}{1+\rho} - b - b_3 + pq - d + p_3 q_3 + D\right)$$
$$+ \Pi_2\left(m + \frac{b_3}{p_3} - q_3\right) \tag{29}$$
$$\text{s.t. } \frac{d}{1+\rho} - b - b_3 \geq 0 \qquad (\lambda) \text{ (cash flow constraint)} \tag{30}$$
$$\frac{d}{1+\rho} - b - b_3 + pq - d + p_3 q_3 + D \geq 0 \qquad (\mu) \text{ (budget constraint)} \tag{31}$$
$$b, b_3, d \geq 0,\ 0 \leq q \leq a,\ 0 \leq q_3 \leq m \tag{32}$$

The only new notation here is the symbol "D", which stands for the Type 1 traders' share of the liquidation payout from the central bank. Also, note that there is no "m" in either the cash flow or budget constraint. This is because there are now no gold certificates given to traders in recognition of their endowments of m units of gold.

The optimization for the second type of trader is similar:

$$\max_{\bar{b},\bar{q},\bar{d},\bar{b}_3,\bar{q}_3} 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + \left(m + \frac{\bar{b}_3}{p_3} - \bar{q}_3\right) + \Pi_1\left(\frac{\bar{d}}{1+\rho} - \bar{b} - \bar{b}_3 + \bar{p}\bar{q} - \bar{d} + p_3\bar{q}_3 + D\right)$$
$$+ \Pi_2\left(m + \frac{\bar{b}_3}{p_3} - \bar{q}_3\right)$$
$$\text{s.t. } \frac{\bar{d}}{1+\rho} - \bar{b} - \bar{b}_3 \geq 0 \qquad (\bar{\lambda})$$
$$\frac{\bar{d}}{1+\rho} - \bar{b} - \bar{b}_3 + \bar{p}\bar{q} - \bar{d} + p_3\bar{q}_3 + D \geq 0 \qquad (\bar{\mu})$$
$$\bar{b}, \bar{b}_3, \bar{d} \geq 0,\ 0 \leq \bar{q} \leq a,\ 0 \leq \bar{q}_3 \leq m.$$

The former money lenders have become gold merchants. Four of their decision variables are as before b_1^*, b_2^* (the total amount they bid by for goods 1and 2), b_3^* (the total amount they bid for gold), and q_3^* (amount of gold they put up for sale). They no longer have g (the total amount of gold strips lent to traders) as a strategic variable. This has been taken over by the central bank. Instead the merchants will bid d^* for their transaction loans. Their optimization is:

$$\max_{b_1^*,b_2^*,b_3^*,q_3^*,d^*} 2\sqrt{\frac{b_1^*}{p}\frac{b_2^*}{\overline{p}}} + \left(B + \frac{b_3^*}{p_3} - q_3^*\right) + \Pi_1\left(\frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* - d^* + p_3 q_3^* + D^*\right)$$
$$+ \Pi_2\left(B + \frac{b_3^*}{p_3} - q_3^*\right) \tag{33}$$
$$\text{s.t. } \frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* \geq 0 \ (\lambda^*) \tag{34}$$
$$\frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* - d^* + p_3 q_3^* + D^* \geq 0 \ (\mu^*) \tag{35}$$
$$b_1^*, b_2^*, b_3^*, d^* \geq 0,\ 0 \leq q_3^* \leq B \tag{36}$$

Finally, the balance conditions for price are as before

$$p = \frac{\overline{b} + b_1^*}{q} \tag{37}$$
$$\overline{p} = \frac{b + b_2^*}{\overline{q}} \tag{38}$$
$$p_3 = \frac{b_3 + \overline{b}_3 + b_3^*}{q_3 + \overline{q}_3 + q_3^*} \tag{39}$$

while that for the interest rate is

$$1 + \rho = \frac{d + \overline{d} + d^*}{g}. \tag{40}$$

6.4 Results

In Appendix D we solve the model, for small values of m. As in Models 1 and 2, we have two cases, depending on whether G is "large" or "small". Qualitatively, the difference between Case 1 and Case 2 is whether the merchants' cash flow and budget constraints both hold tightly, and whether the gold market shuts down.

We now set parameter values so as to most closely match the examples calculated with Models 1 and 2. So let us now set $\Pi_1 = \Pi_2 = 1$ and $m = 0$.

Example 1

If $G = 2a$ we are in Case 1. We obtain $p = \overline{p} = 1$, $\rho = 0$, $d = \overline{d} = 2a/3$, $b = \overline{b} = a/3$, $q = \overline{q} = 2a/3$. Hence the consumption by each trader type is $(a/3, a/3)$ of the perishables. The bids of the merchants are $b_1^* = b_2^* = a/3$, and so they too consume $a/3$ of each of the perishables. In addition, we have $p_3 = 2$, $b_3 = \overline{b}_3 = a/3$, and $q_3^* = a/3$; hence the merchants sell $a/3$ units of gold ($a/6$ to each trader type). The final distribution of resources is $(\frac{a}{3}, \frac{a}{3}, \frac{a}{6}, 0)$, $(\frac{a}{3}, \frac{a}{3}, \frac{a}{6}, 0)$, $(\frac{a}{3}, \frac{a}{3}, B - \frac{a}{3}, 0)$, and $(0, 0, 0, B)$ for the two trader types, the gold merchants, and the central bank respectively. In addition, the central bank makes no profit; hence $D = D^* = 0$. For each trader type, its final utility is $2\sqrt{\frac{a}{3}\frac{a}{3}} + \frac{a}{6} + \Pi_2 \frac{a}{6} + 0 = a$, while for the merchants it is $2\sqrt{\frac{a}{3}\frac{a}{3}} + B - \frac{a}{3} + \Pi_2(B - \frac{a}{3}) + 0 = 2B$.

The reader will note that these give the same (Pareto efficient) results as in Model 2.

Example 2

If we set $G = a$, we are on the boundary between Case 1 and Case 2. We obtain $p = \overline{p} = 1$, $\rho = 0$, $d = \overline{d} = a/2$, $b = \overline{b} = a/2$, and $q = \overline{q} = a/2$. Hence the consumption by each trader type is $(a/2, a/2)$ of the perishables. The bids of the merchants are $b_1^* = b_2^* = 0$, and so they consume none of the perishables. In addition, we have $p_3 = 2$, $b_3 = \overline{b}_3 = 0$, and $q_3^* = 0$; hence the merchants sell no gold to the traders. The final distribution of resources is $(\frac{a}{2}, \frac{a}{2}, 0, 0)$, $(\frac{a}{2}, \frac{a}{2}, 0, 0)$, $(0, 0, B, 0)$, and $(0, 0, 0, B)$ for the two trader types, the gold merchants, and the central bank respectively. Again, the central bank makes no profit; hence $D = D^* = 0$.

Example 3

If we set $G = \frac{a}{4}$, we are in Case 2. There are a continuum of equilibria, parametrized by ρ, which can take on any value between 0 and 3. For each such ρ, we have $p = \overline{p} = \frac{1}{4}$, $d = \overline{d} = \frac{a}{8}$, $b = \overline{b} = \frac{a}{8(1+\rho)}$, and $q = \overline{q} = \frac{a}{2}$. The consumption of perishables by the trader types is $(\frac{a}{2}, \frac{a}{2(1+\rho)})$ and $(\frac{a}{2(1+\rho)}, a)$. Again, no gold is traded. The merchants bid $\frac{a\rho}{8(1+\rho)}$ for each type of perishable, and end up consuming $\frac{a\rho}{2(1+\rho)}$ of each type of perishable. Consumption of perishable is not efficient here. The "profit" for the central bank (ρG) is $\frac{a\rho}{4}$, all of which goes to the merchants.

The last two models fall under the rubric of Dubey, Mascolell and Shubik [1] they have noncooperative equilibria that give the competitive outcome if there is enough money. If there is not enough money, even in equilibrium the inequalities become binding. The definition of enough money although mathematically well defined depends on institutional detail.

7 The Value of Paper Money in an Economy with Enough Gold

We say that an economy has "enough money" if there is enough money in the economy to finance efficient trade, even if it takes zero-interest loans to those with cash flow constraints. Suppose gold is the money, and gold also has a linear utility as a commodity. If there is enough gold, the marginal *transactions* value of gold must equal its marginal *consumption* value. These both have to be equal to the money rate of interest. When paper money is used, its marginal value in consumption is zero, thus when it is in sufficient supply, its price is zero.

In the examples from Model 2, we selected parameters $\Pi_1 = \Pi_2 = 1$, so the marginal consumption value of gold is 1.[22] Indeed, the value of the strips equals the marginal value of the services of gold, which equals 1.

When $B = a$ the discounted value of the strip equals that of the gold. When $B = 2a$ the discounted value to the lender is zero, but the sum of the discounted worth to the borrowers is still $2a$. It adds $\frac{a}{2}$ to each every period.

8 The Need for a Central Bank?

8.1 Results from our Analysis

Let us summarize our results from the "enough money" (Example 1) cases above. First, in Model 1 (with gold as the money), we had inefficient consumption of the perishables. However, the demonetization of gold via the introduction of the strips permits one to "have one's cake and have the borrowers eat it". Both the consumption and transaction services of gold can be utilized simultaneously, and the consumption of perishable is efficient. In Model 3 we introduce a central bank (under the legal fiction that it is owned by the holders of the gold) which enables a government to control the money supply while limiting the amount of paper in circulation to at most a 1 : 1 ratio with gold. This third model is not only somewhat improbable, but appears to make a libertarian case that the central bank is unneeded – there is no change in efficiency as we move from Model 2 to Model 3.

Thus it seems that the central bank adds no value. But do not forget our simple models assume a stationary economy, with perfect information

[22] Interpreted in terms of an infinite horizon model, this fits with a time discount of $\beta = 1/2$ and the spot price of gold at $p_3 = 2$. Thus, if Π were reinterpreted as $\beta p_3 = \frac{1}{2}2 = 1$ we can connect to the steady state.

flows, costless accounting, safe keeping and many other transaction services. They have no exogenous uncertainty, but with honest error-free individuals in a society without public goods, with law and government provided free of charge. At the very least, in the real world a central bank is useful in policing some of these functions.

In addition, our analysis shows the central bank *does* help increase efficiency in the case where there is not "enough money". For instance, in Example 3 both Model 1 and Model 2 produce inefficient consumption of the perishable. But Model 3 gives a continuum of results, ranging from the Model 2 answer to an efficient result where $\rho = 0$.

8.2 Varying the money supply: Who gets the power?

Of course, our modern economy is not like our simple models here - we *do* have a public sector with a bureaucracy, politics, law, and uncertainty as facts of life. Hence a valid question is whether we should have a central bank and paper money, or should we trust "the market" and the gold miners to take care of everything. It poses a Scylla and Charybdis choice. The choice is between an oligopolistic industry dependent on an arbitrary gold manufacturing technology with relatively little flexibility in increasing or decreasing the supply, and a monopolistist central bank that may be subject to considerable political pressure. The answer is essentially *ad hoc*; but sometimes the economy requires things like an ability to vary the supply of money, a lender of last resort, a bank for the government, and a manager of the national debt. In this case, the central bank, though possibly not necessary, appears to be a sufficient institution that offers many, if not all of these functions.

Possibly the most important question in the allocation of power to private or public institutions is who is in position to vary the money supply in the economy (see Smith and Shubik [6]). In considering a dynamic economy it is easy to construct models in which the causality runs in both directions. The availability of new products or processes may call for new money. Alternatively the availability of funding may call forth innovation. Our formal models above dealt with gold or paper as a means of payment where the paper was competely backed by gold. As soon as the rules are changed in a way that enables all legal persons to issue their own currency (see Sahi and Yao [4] and Sorin [7]) it is possible to design an abstract economy and a formal experimental game (see Huber, Shubik and Sunder) which achieves efficient trade using individual IOU notes as currency. The modeling requirements are so stringent that although logically feasible the information, privacy, accounting and enforcement conditions rule it out at this time.

The central bank as the creator of money appears to be the least bad of all current alternatives. But with this assignment of power goes public need for transparency and safeguards.

References

[1] Dubey, P., A. MasColell, and M. Shubik, 1980. Efficiency properties of strategic market games: an axiomatic approach. *Journal of Economic Theory* 22: 339–362

[2] P. Dubey and L. S. Shapley 1994 "Noncooperative general exchange with a continuum of traders: Two models," *Journal of Mathematical Economics* 23, ...

[3] Quint, T. and M. Shubik 2005 "Multistage Models of Monetary Exchange: An Elementary Discussion of Commodity Money, Fiat Money and Credit, Part 2" *ICFAI Journal of Monetary Economics* 3(2): 6–50.

[4] Sahi, S. and Yao, S., 1989. The noncooperative equilibria of a trading economy with complete markets and consistent prices. *Journal of Mathematical Economics* 18, pp. 325-346.

[5] Shubik, M. 1999a. *The theory of money and financial institutions, Vol. 1.* Cambridge, MA: MIT Press.

[6] Smith, E. and M. Shubik. 2011. Smith, E and M.Shubik 2011. Endogenizing the provision of money:Costs of commodity and fiat. *Journal of Mathematical Economics (forthcoming).*

[7] Sorin, S. 1996. Strategic Market Games with Exchange Rates *Journal of Economic Theory* 68: 431–446

9 Appendix A: A Buy-Sell Model of Competitive Money Lending with Gold

In this model there are the usual two continua of traders, plus a continuum of moneylenders. The trader types are endowed with $(a, 0, m)$ and $(0, a, m)$ respectively, while the lenders have $(0, 0, B)$. However, this time the money is gold. The gold can only be used for transactions or for jewelry (services) during the period, but not both. In the language of Quint-Shubik (2010, Chapter 6), the parameter values (k_1, k_2, k_3) are set equal to $(0, 0, 1)$.

The optimization for the traders of type 1 is

$$\max_{b,q,d} 2\sqrt{(a-q)\frac{b}{\bar{p}}} + \left(m + \frac{d}{1+\rho} - b\right) + \Pi\left(m + \frac{d}{1+\rho} - b + pq - d\right) \quad (41)$$

$$\text{s.t. } m + \frac{d}{1+\rho} - b \geq 0 \qquad (\lambda) \text{ (cash flow constraint)} \quad (42)$$

$$m + \frac{d}{1+\rho} - b + pq - d \geq 0 \qquad (\mu) \text{ (budget constraint)} \quad (43)$$

$$b, d \geq 0,\ 0 \leq q \leq a \quad (44)$$

The first order conditions here are

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-q}{b}} = 1 + \Pi + \lambda + \mu \quad (45)$$

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{b}{a-q}} = (\mu + \Pi)p \quad (46)$$

$$\lambda = (\mu + \Pi)\rho - 1 \quad (47)$$

$$m + \frac{d}{1+\rho} - b = 0 \text{ or } \lambda = 0 \quad (48)$$

$$m + \frac{d}{1+\rho} - b + pq - d = 0 \text{ or } \mu = 0 \quad (49)$$

Similarly, the optimization for the Type 2 traders is

$$\max_{\bar{b},\bar{q},\bar{d}} 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + \left(m + \frac{\bar{d}}{1+\rho} - \bar{b}\right) + \Pi\left(m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d}\right)$$

$$\text{s.t. } m + \frac{\bar{d}}{1+\rho} - \bar{b} \geq 0 \qquad (\overline{\lambda})$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d} \geq 0 \qquad (\overline{\mu})$$

$$\bar{b}, \bar{d} \geq 0,\ 0 \leq \bar{q} \leq a.$$

with first order conditions

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = 1 + \Pi + \bar{\lambda} + \bar{\mu} \tag{50}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = (\bar{\mu} + \Pi)\bar{p} \tag{51}$$

$$\overline{\lambda} = (\bar{\mu} + \Pi)\rho - 1 \tag{52}$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b} = 0 \text{ or } \bar{\lambda} = 0 \tag{53}$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d} = 0 \text{ or } \bar{\mu} = 0. \tag{54}$$

For the continuum of moneylenders, the optimization is

$$\max_{b_1^*, b_2^*, g} 2\sqrt{\frac{b_1^*}{p}\frac{b_2^*}{\overline{p}}} + B - b_1^* - b_2^* - g + \Pi\left(B - b_1^* - b_2^* + \rho g\right) \tag{55}$$

$$\text{s.t. } B - b_1^* - b_2^* - g \geq 0 \ (\lambda^*) \tag{56}$$

$$b_1^*, b_2^*, g \geq 0 \tag{57}$$

Note here that the money lenders have g as a decision variable, and not ρ. The reason is that individually, each lender can decide how much to lend – but they cannot individually influence the interest rate. The first order conditions here are:

$$\sqrt{\frac{b_2^*}{p\overline{p}b_1^*}} = \Pi + \lambda^* + 1 \tag{58}$$

$$\sqrt{\frac{b_1^*}{p\overline{p}b_2^*}} = \Pi + \lambda^* + 1 \tag{59}$$

$$\Pi\rho - \lambda^* - 1 = 0 \tag{60}$$

$$B - b_1^* - b_2^* - g = 0 \text{ or } \lambda^* = 0 \tag{61}$$

Finally, we have the following balance conditions: $p = \frac{\overline{b}+b_1^*}{q}$, $\overline{p} = \frac{b+b_2^*}{\overline{q}}$, and $1+\rho = \frac{d+\overline{d}}{g}$.

Case 1: We first analyze the case where the traders have little gold and the lenders have a lot. Thus the traders' constraints are all tight and the lenders' are loose, i.e. $\lambda > 0$, $\mu > 0$, $\overline{\lambda} > 0$, $\overline{\mu} > 0$, and $\lambda^* = 0$. We also assume a symmetric solution, i.e. $p = \overline{p}$, $b = \overline{b}$, $d = \overline{d}$, $q = \overline{q}$, and $b_1^* = b_2^*$.

Condition (60) gives $\rho = \frac{1}{\Pi}$. And (58) together with symmetry gives $\frac{1}{p} = \Pi + 1$, which is $p = \frac{1}{\Pi+1} = \overline{p}$.

Next, we find an expression for the multiplier μ. We begin with (46), which is $\sqrt{\frac{b}{a-q}} = (\mu+\Pi)p^{\frac{3}{2}}$, or $\sqrt{\frac{a-q}{b}} = \frac{1}{(\mu+\Pi)p^{\frac{3}{2}}}$. Substituting into (45), we have $\frac{1}{\sqrt{p}}\frac{1}{(\mu+\Pi)p^{\frac{3}{2}}} = 1+\Pi+\lambda+\mu = 1+\Pi+(\mu+\Pi)\rho-1+\mu = (1+\rho)(\mu+\Pi)$. But $p = \frac{1}{\Pi+1}$; so we have $(\Pi+1)^2 = (1+\rho)(\mu+\Pi)^2$, which is $(\Pi+1) = \sqrt{1+\rho}(\mu+\Pi) = \sqrt{1+\frac{1}{\Pi}}(\mu+\Pi)$. Solving for μ yields $\mu = \sqrt{\Pi(1+\Pi)} - \Pi$.

Next, we again start with (46), which is $\sqrt{\frac{b}{a-q}} = (\mu+\Pi)p^{\frac{3}{2}}$. Squaring both sides, we have $\frac{b}{a-q} = (\mu+\Pi)^2 p^3$. Now substitute in $p = \frac{1}{\Pi+1}$ and $\mu = \sqrt{\Pi(1+\Pi)} - \Pi$. We end up with $b = (a-q)\frac{\Pi}{(1+\Pi)^2}$.

Next, since the traders' cash flow and budget constraints are both tight, we know $d = pq = \frac{q}{1+\Pi}$. But the tight cash flow constraint also means $d = (b-m)(1+\rho) = (b-m)(1+\frac{1}{\Pi}) = \left((a-q)\frac{\Pi}{(1+\Pi)^2} - m\right)(1+\frac{1}{\Pi})$. Hence $\left((a-q)\frac{\Pi}{(1+\Pi)^2} - m\right)(1+\frac{1}{\Pi}) =. \frac{q}{1+\Pi}$. Solving for q yields

$$q = \frac{a}{2} - \frac{m(1+\Pi)^2}{2\Pi} = \overline{q}. \tag{62}$$

At this point it becomes easy to solve for the other variables: $d = pq = \frac{q}{1+\Pi} = \frac{a}{2(1+\Pi)} - \frac{m(1+\Pi)}{2\Pi}$, $b = (a-q)\frac{\Pi}{(1+\Pi)^2} = \frac{a\Pi}{2(1+\Pi)^2} + \frac{m}{2}$, and $b_1^* = b_2^* = pq - b = d - b = \frac{a}{2(1+\Pi)^2} - \frac{m}{2\Pi} - m$. In addition, $g = \frac{2d}{1+\rho} = \frac{a\Pi}{(1+\Pi)^2} - m$. For the multipliers, we already found $\mu = \sqrt{\Pi(1+\Pi)} - \Pi = \overline{\mu}$ and we are given $\lambda^* = 0$. Finally, we have $\lambda = \rho(\Pi+\mu) - 1 = \frac{1}{\Pi}(\Pi + \sqrt{\Pi(1+\Pi)} - \Pi) - 1 = \sqrt{\frac{1+\Pi}{\Pi}} - 1$.

The results hold if $B - b_1^* - b_2^* - g \geq 0$, i.e. if $B - \frac{a}{(1+\Pi)^2} + \frac{m}{\Pi} + 2m - \frac{a\Pi}{(1+\Pi)^2} + m \geq 0$. This yields a condition of $B \geq \frac{a}{1+\Pi} - \frac{m(1+\Pi)}{\Pi}$. In addition, we need the variables q, d, and $b_1^* = b_2^*$ to be nonnegative. This occurs if $m \leq \frac{\Pi a}{(1+\Pi)^2}$.

Case 2: We analyze the case where B is smaller. Hence the traders' cash flow and budget constraints are tight as before, but the lenders' cash flow constraints are tight, i.e. $\lambda > 0$, $\mu > 0$, and $\lambda^* > 0$. Again assume a symmetric solution, i.e. $p = \overline{p}$, $b = \overline{b}$, $d = \overline{d}$, $q = \overline{q}$, and $b_1^* = b_2^*$.

First, (60) implies $\lambda^* = \Pi\rho - 1$. But also, from (58) and symmetry, we have $\lambda^* = \frac{1}{p} - \Pi - 1$. Hence $\Pi\rho = \frac{1}{p} - \Pi$, which is

$$p = \frac{1}{\Pi(1+\rho)} = \overline{p}. \tag{63}$$

Next, since the traders' cash flow and budget constraints are tight, we have $d = pq$. But by the balance condition pq is equal to $\overline{b} + b_1^*$, and since the lenders' cash flow constraints are tight $b_1^* = b_2^* = \frac{B-g}{2}$. So we have $d = pq = \overline{b} + b_1^* = \overline{b} + \frac{B-g}{2} = \overline{b} + \frac{B - \frac{2d}{1+\rho}}{2} = \frac{B}{2} + \overline{b} - \frac{d}{1+\rho} =. \frac{B}{2} + m$, where the fourth equality follows from the balance constraint, the fifth is just algebra, and the last follows from symmetry $b = \overline{b}$ and the tight traders' cash flow constraint. So we have

$$d = \frac{B}{2} + m = \overline{d}. \tag{64}$$

At this point, we can obtain expressions for all of the variables in terms of ρ: $g = \frac{2d}{1+\rho} = \frac{B+2m}{1+\rho}$, $b = \frac{d}{1+\rho} + m = \frac{\frac{B}{2}+m}{1+\rho} + m = \overline{b}$, $b_1^* = b_2^* = \frac{1}{2}(B-g) = \frac{1}{2}(\frac{B+\rho B-B-2m}{1+\rho}) = \frac{1}{2}(\frac{\rho B-2m}{1+\rho})$, Recalling $p = \frac{1}{\Pi(1+\rho)}$, we also have $q = \overline{q} = \frac{d}{p} = \frac{\frac{B}{2}+m}{\frac{1}{\Pi(1+\rho)}} = \Pi(1+\rho)(\frac{B}{2}+m)$.

So now all that remains to find an expression for ρ. To this end, we observe that (47) implies that

$$\lambda = \rho(\Pi + \mu) - 1. \tag{65}$$

Then

$$(45) \Rightarrow \frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-q}{b}} = 1 + \Pi + \lambda + \mu \tag{66}$$

$$\Rightarrow \frac{1}{\bar{p}}\left(\frac{a-q}{b}\right) = (1 + \Pi + \lambda + \mu)^2 \tag{67}$$

$$\Rightarrow \frac{1}{\bar{p}}\left(\frac{a-q}{b}\right) = (1+\rho)^2(\Pi + \mu)^2 \text{ (using (65))} \tag{68}$$

$$\Rightarrow \frac{1}{\bar{p}}\left(\frac{a-q}{b}\right) = (1+\rho)^2\frac{1}{\bar{p}^3}\left(\frac{b}{a-q}\right) \text{ (using (46))} \tag{69}$$

$$\Rightarrow \left(\frac{a-q}{b}\right)^2 = \frac{(1+\rho)^2}{\bar{p}^2} = \Pi^2(1+\rho)^4 \text{ (using 63)} \tag{70}$$

$$\Rightarrow \frac{a-q}{b} = \Pi(1+\rho)^2 \tag{71}$$

$$\Rightarrow \frac{a - \Pi(1+\rho)(\frac{B}{2}+m)}{\frac{\frac{B}{2}+m}{1+\rho} + m} = \Pi(1+\rho)^2 \tag{72}$$

$$\Rightarrow a - \Pi(1+\rho)(\frac{B}{2}+m) = \Pi(1+\rho)(\frac{B}{2}+m) + \Pi m(1+\rho)^2 \tag{73}$$

$$\Rightarrow \Pi m(1+\rho)^2 + 2\Pi(1+\rho)(\frac{B}{2}+m) - a = 0 \tag{74}$$

The above quadratic expression for $1+\rho$ solves in two cases: first, if $m = 0$, we have $1 + \rho = \frac{a}{\Pi(B+2m)}$. Otherwise, if $m > 0$, we have

$$1 + \rho = \frac{-\Pi(B+2m) + \sqrt{\Pi^2(B+2m)^2 + 4a\Pi m}}{2\Pi m} \tag{75}$$

Finally, we may solve for the multipliers. First, we have (46) $\Rightarrow \mu = \frac{\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{b}{a-q}} - \Pi p}{p} = \frac{1}{p^{1.5}}\sqrt{\frac{1}{\Pi(1+\rho)^2}} - \Pi = \Pi^{1.5}(1+\rho)^{1.5}\sqrt{\frac{1}{\Pi(1+\rho)^2}} = \Pi\sqrt{1+\rho} - \Pi$, where the second equality follows from (71) and the third from (63). Also, $\lambda = \rho(\Pi + \mu) - 1 = \Pi\rho\sqrt{1+\rho} - 1$. Finally, condition (60) directly gives us $\lambda^* = \Pi\rho - 1$.

Note: In order for the above to be valid, we need ρ and all the multipliers to be nonnegative. This requires $\rho \geq \frac{1}{\Pi}$. In the case where $m = 0$, this means $B + 2m \leq \frac{a}{1+\Pi}$. In the case where $m > 0$, the condition is not so easily stated, but does hold whenever a is large compared to $B + 2m$. We also need for b_1^* and b_1^* to be nonegative, which means $\rho B \geq 2m$. Again, if $m = 0$, this holds if $\rho \geq \frac{1}{\Pi}$, i.e. if $B + 2m \leq \frac{a}{1+\Pi}$. And again, if $m > 0$ the condition is more complicated but holds if $a >> B + 2m$.

10 Appendix B: A Sell-All Model of Competitive Money Lending with Gold

Both trader types are endowed with m units of gold money. The continuum of money lenders is endowed with $B = M - 2m$ units of money (where $0 \leq 2m \leq M$). The lenders are utility maximizers, who (just like the traders) derive benefit from consumption of perishables, the service utility of the gold, and the salvage utility of the gold at the end of the game.

The notation we will use is the following. For the Type 1 traders, b_1 and b_2 denote the amount of gold bid in the market for Good 1 and Good 2 respectively. The amount they borrow from the banks is $\frac{d}{1+\rho}$ at interest rate ρ, so the amount they must pay back is d. For the Type 2 traders, the notation $\bar{b}_1$, $\bar{b}_2$, and $\bar{d}$ is defined similarly. The prices for the perishables are p and $\overline{p}$, while the salvage value for the gold is Π per unit. We assume that the service utility of the gold is one per unit-time,

The Type 1 traders face an optimization described by:

$$\max_{b_1,b_2,d} 2\sqrt{\frac{b_1}{p}\frac{b_2}{\overline{p}}} + m + \frac{d}{1+\rho} - b_1 - b_2 + pa + \Pi\left(m + \frac{d}{1+\rho} - b_1 - b_2 + pa - d\right) \tag{76}$$

$$\text{s.t. } m + \frac{d}{1+\rho} - b_1 - b_2 \geq 0 \qquad (\lambda) \text{ (cash flow constraint)} \tag{77}$$

$$m + \frac{d}{1+\rho} - b_1 - b_2 + pa - d \geq 0 \qquad (\mu) \text{ (budget constraint)} \tag{78}$$

$$b_1, b_2, d \geq 0 \tag{79}$$

The first order conditions wrt b_1, b_2, and d yield

$$\sqrt{\frac{b_2}{p\overline{p}b_1}} = 1 + \Pi + \lambda + \mu \tag{80}$$

$$\sqrt{\frac{b_1}{p\overline{p}b_2}} = 1 + \Pi + \lambda + \mu \tag{81}$$

$$\frac{1}{1+\rho} + \frac{\Pi}{1+\rho} - \Pi + \frac{\lambda}{1+\rho} + \frac{\mu}{1+\rho} - \mu = 0 \tag{82}$$

$$m + \frac{d}{1+\rho} - b_1 - b_2 = 0 \text{ or } \lambda = 0 \tag{83}$$

$$m + \frac{d}{1+\rho} - b_1 - b_2 + pa - d = 0 \text{ or } \mu = 0 \tag{84}$$

We also point out that we require the feasibility conditions (77)-(79) for our solution.[23]

Similarly, the Type 2 traders face the optimization below:

$$\max_{\bar{b}_1,\bar{b}_2,\bar{d}} 2\sqrt{\frac{\bar{b}_1}{p}\frac{\bar{b}_2}{\bar{p}}} + m + \frac{\bar{d}}{1+\rho} - \bar{b}_1 - \bar{b}_2 + \bar{p}a + \Pi\left(m + \frac{\bar{d}}{1+\rho} - \bar{b}_1 - \bar{b}_2 + \bar{p}a - \bar{d}\right) \quad (85)$$

$$\text{s.t. } m + \frac{\bar{d}}{1+\rho} - \bar{b}_1 - \bar{b}_2 \geq 0 \qquad (\bar{\lambda}) \text{ (cash flow constraint)} \quad (86)$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b}_1 - \bar{b}_2 + \bar{p}a - \bar{d} \geq 0 \qquad (\bar{\mu}) \text{ (budget constraint)} \quad (87)$$

$$\bar{b}_1, \bar{b}_2, \bar{d} \geq 0 \quad (88)$$

The first order conditions here are

$$\sqrt{\frac{\bar{b}_2}{p\bar{p}\bar{b}_1}} = 1 + \Pi + \bar{\lambda} + \bar{\mu} \quad (89)$$

$$\sqrt{\frac{\bar{b}_1}{p\bar{p}\bar{b}_2}} = 1 + \Pi + \bar{\lambda} + \bar{\mu} \quad (90)$$

$$\frac{1}{1+\rho} + \frac{\Pi}{1+\rho} - \Pi + \frac{\bar{\lambda}}{1+\rho} + \frac{\bar{\mu}}{1+\rho} - \bar{\mu} = 0 \quad (91)$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b}_1 - \bar{b}_2 = 0 \text{ or } \bar{\lambda} = 0 \quad (92)$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b}_1 - \bar{b}_2 + \bar{p}a - \bar{d} = 0 \text{ or } \bar{\mu} = 0 \quad (93)$$

The money lender decision variables are b_1^* (the amount they collectively bid for Good 1), b_2^* (the amount they collectively bid for Good 2), and g (the amount they offer for loan). Their optimization is expressed as:

$$\max_{b_1^*,b_2^*,g} 2\sqrt{\frac{b_1^*}{p}\frac{b_2^*}{\bar{p}}} + B - g - b_1^* - b_2^* + \Pi(B - g - b_1^* - b_2^* + (1+\rho)g) \quad (94)$$

$$\text{s.t. } B - g - b_1^* - b_2^* \geq 0 \qquad (\lambda^*) \quad (95)$$

$$b_1^*, b_2^*, g \geq 0 \quad (96)$$

The first order equations here are

[23] For the Type 2 traders' optimizations, and in Appendices B and C below, we also have similar feasibility conditions, which we don't explicitly write down.

$$\sqrt{\frac{b_2^*}{p\overline{p}b_1^*}} = \Pi + \lambda^* + 1 \tag{97}$$

$$\sqrt{\frac{b_1^*}{p\overline{p}b_2^*}} = \Pi + \lambda^* + 1 \tag{98}$$

$$\Pi\rho - \lambda^* - 1 = 0 \tag{99}$$

$$B - b_1^* - b_2^* - g = 0 \text{ or } \lambda^* = 0 \tag{100}$$

Finally, the balance conditions are $p = \frac{b_1+\bar{b}_1+b_1^*}{a}$, $\bar{p} = \frac{b_2+\bar{b}_2+b_2^*}{a}$ and $1+\rho = \frac{d+\bar{d}}{g}$. We also remark that here the problems for Types 1 and 2 are isomorphic and so we may assume a symmetric solution where $b_1 = b_2 = \bar{b}_1 = \bar{b}_2 \equiv b$, $d = \bar{d}$, $b_1^* = b_2^* \equiv b^*$, and $p = \bar{p}$.

Case 1: m low and B high.

There are several cases, but as before we first consider for the case in which the traders have little money (m small) and the bankers have a lot of money (B large). In terms of our multipliers, we are assuming λ, $\bar{\lambda}$, μ, and $\bar{\mu}$ are positive, while $\lambda^* = 0$. The first observation is that condition (97) or (98), $\lambda^* = 0$, and symmetry together imply that $p = \overline{p} = \frac{1}{1+\Pi}$. Next, conditions ($\lambda$) and ($\mu$) holding tightly together imply that $pa = d$. Hence $d = \frac{a}{1+\Pi} = \bar{d}$. The cash flow constraint holding tightly is $m + \frac{d}{1+\rho} - 2b = 0$, which gives $b = \frac{m+\frac{d}{1+\rho}}{2} = \frac{m(1+\rho)(1+\Pi)+a}{2(1+\rho)(1+\Pi)}$; but then condition (99) and $\lambda^* = 0$ together give $\rho = \frac{1}{\Pi}$, so $b = \frac{m(1+\frac{1}{\Pi})(1+\Pi)+a}{2(1+\frac{1}{\Pi})(1+\Pi)} = \frac{m(1+\Pi)^2+\Pi a}{2(1+\Pi)^2}$. So the traders consume $\frac{b}{p} = \frac{m(1+\Pi)^2+\Pi a}{2(1+\Pi)}$ of each good. Then, we can calculate b^* via the balance condition for price: $b^* = pa - 2b = \frac{a}{1+\Pi} - \frac{m(1+\Pi)^2+\Pi a}{(1+\Pi)^2} = \frac{a-m(1+\Pi)^2}{(1+\Pi)^2}$. So the lenders consume $\frac{b^*}{p} = \frac{a-m(1+\Pi)^2}{(1+\Pi)}$ of each good. They also lend an amount of $g = \frac{2d}{1+\rho} = \frac{2\Pi a}{(1+\Pi)^2}$.

Finally, for the multipliers, condition (80) with $p = \bar{p} = \frac{1}{1+\Pi}$ and $b_1 = b_2$ gives $1 + \Pi = 1 + \Pi + \lambda + \mu$; since λ and μ are nonnegative, this gives $\lambda = \mu = 0$.

We remark that the above results are only valid if: (a) the above expression for b^* is nonnegative, and if (b) the lenders' cash flow constraints (λ^*) hold. These gives the conditions: (a) $m \leq \frac{a}{(1+\Pi)^2}$ and (b) $B \geq \frac{2\Pi a}{(1+\Pi)^2} + 2\frac{a-m(1+\Pi)^2}{(1+\Pi)^2} = \frac{2a}{1+\Pi} - 2m$, which is $B + 2m \geq \frac{2a}{1+\Pi}$.

Case 2: m low and B low. Now we cover the case in which λ, $\bar{\lambda}$, μ, and

$\bar{\mu}$ are again all positive, but $\lambda^* > 0$ also. We first note that condition (97) and symmetry together imply $\lambda^* = \frac{1}{p} - 1 - \Pi$; substituting this expression for λ^* back into (99) gives $\Pi\rho - (\frac{1}{p} - 1 - \Pi) - 1 = 0$, which is

$$p = \frac{1}{\Pi(1+\rho)} = \overline{p}. \tag{101}$$

Next, we note from symmetry, the balance conditions, and the tight (λ^*) constraint that $d = \frac{(1+\rho)g}{2}$, $2b + b^* = pa$, and $b^* = \frac{B-g}{2}$. Hence, starting with the tight cash flow constraint for the traders, we have $m + \frac{d}{1+\rho} - b - b = 0 \Rightarrow m + \frac{g}{2} - 2b = 0 \Rightarrow m + \frac{g}{2} - pa + b^* = 0 \Rightarrow m + \frac{g}{2} - pa + \frac{B-g}{2} = 0 \Rightarrow p = \frac{B+2m}{2a} = \overline{p}$. Not only does this give an expression for p, but this in combination with (101) gives $\frac{1}{\Pi(1+\rho)} = \frac{B+2m}{2a}$, or $\rho = \frac{2a}{\Pi(B+2m)} - 1$.

It is now a simple exercise to calculate expressions for the other variables: $d = pa = \frac{B+2m}{2} = \overline{d}$, $g = \frac{2d}{1+\rho} = \frac{\Pi(B+2m)^2}{2a}$, $b = b_1 = b_2 = \frac{m+\frac{g}{2}}{2} = \frac{4am+\Pi(B+2m)^2}{8a} = \overline{b}_1 = \overline{b}_2$, $b^* = b_1^* = b_2^* = \frac{B-g}{2} = \frac{2aB-\Pi(B+2m)^2}{4a}$, $\lambda^* = \frac{1}{p} - 1 - \Pi = \frac{2a}{B+2m} - \Pi - 1$, $\mu = \frac{1}{p} - 1 - \Pi - \lambda = \frac{1}{p} - 1 - \Pi - (\Pi + \mu)\rho + 1 \Rightarrow \mu = 0$, and $\lambda = \frac{2a}{B+2m} - \Pi - 1$.

In order for these results to hold, we need $b^* \geq 0$, $\lambda^* \geq 0$, and $\lambda \geq 0$. These hold if $\Pi(B+2m)^2 \leq 2aB$ and $(1+\Pi)B \leq 2a$. If $m = 0$, all that is needed is $(1+\Pi)B \leq 2a$.

11 Appendix C: Demonetized Gold, no Central Bank, Lenders also act as Gold Merchants

This model is similar to Appendix A, except now the gold has been demonetized. All holders of x units of gold are now endowed with x units of demonetized gold, plus x units of fiat ("gold strips"). Hence Trader Type 1 is now endowed with a units of good #1, plus m units of gold, plus m units of fiat. Trader Type 2 is endowed with a units of good #1, m units of gold, and m units of fiat. The lenders (now lender-merchants) are endowed with only B units of gold and B units of fiat.

We assume that the financial market (for fiat) operates first, over the time period $[0, k_1]$. This is followed by the goods markets for gold and for perishable, both of which operate over $[k_2, k_3]$ $(0 \leq k_1 \leq k_2 \leq k_3)$. However, while any money borrowed during the financial market is not obtained until the *end* of the market at time k_1, any gold or perishable bought during the second phase is credited at the *beginning* of that phase, at time k_2. This is an important assumption for the trade of gold, as the final owners of gold get to enjoy its service value over the period $[k_2, k_3]$. As in the previous model, we assume $k_1 = 0$, $k_2 = 0$, and $k_3 = 1$.

With the new market for demonetized gold, we have to define some new variables. First, for the traders, the amount they bid for gold is b_3. The amount of gold they put up for sale is q_3. For the lenders, the amount they bid for gold is denoted b_3^*, while the amount they put up for sale is q_3^*. The price of gold is given by p_3 and the parameters Π_1 and Π_2 are the per unit salvage value of strips (fiat) and gold respectively, at the end of the game.

The traders of Type 1 attempt to solve

$$\max_{b,q,d,b_3,q_3} 2\sqrt{(a-q)\frac{b}{\bar{p}}} + \left(m + \frac{b_3}{p_3} - q_3\right) + \Pi_1\left(m + \frac{d}{1+\rho} - b - b_3 + pq - d + p_3 q_3\right) \tag{102}$$

$$+ \Pi_2\left(m + \frac{b_3}{p_3} - q_3\right) \tag{103}$$

$$\text{s.t. } m + \frac{d}{1+\rho} - b - b_3 \geq 0 \qquad (\lambda) \text{ (cash flow constraint)} \tag{104}$$

$$m + \frac{d}{1+\rho} - b - b_3 + pq - d + p_3 q_3 \geq 0 \qquad (\mu) \text{ (budget constraint)} \tag{105}$$

$$b, b_3, d \geq 0,\ 0 \leq q \leq a,\ 0 \leq q_3 \leq m \tag{106}$$

The first order conditions here, with respect to b, q, d, b_3, and q_3 are

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-q}{b}} = \Pi_1 + \lambda + \mu \tag{107}$$

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{b}{a-q}} = (\mu + \Pi_1)p \tag{108}$$

$$\lambda = (\mu + \Pi_1)\rho \tag{109}$$

$$\frac{1+\Pi_2}{p_3} = \Pi_1 + \lambda + \mu \tag{110}$$

$$\Pi_1 p_3 - 1 - \Pi_2 + \mu p_3 = 0 \tag{111}$$

In addition, we have the complementarity constraints

$$m + \frac{d}{1+\rho} - b - b_3 = 0 \text{ or } \lambda = 0 \tag{112}$$

$$m + \frac{d}{1+\rho} - b - b_3 + pq - d + p_3 q_3 = 0 \text{ or } \mu = 0 \tag{113}$$

Similarly, the optimization for the Type 2 traders is

$$
\begin{aligned}
&\max_{\bar b, \bar q, \bar d, \overline{b}_3, \overline{q}_3} \; 2\sqrt{(a-\bar q)\frac{\bar b}{p}} + \left(m + \frac{\overline{b}_3}{p_3} - \overline{q}_3\right) + \Pi_1\left(m + \frac{\bar d}{1+\rho} - \bar b - \overline{b}_3 + \bar p \bar q - \bar d + p_3 \overline{q}_3\right) \\
&+ \Pi_2\left(m + \frac{\overline{b}_3}{p_3} - \overline{q}_3\right) \\
&\text{s.t. } m + \frac{\bar d}{1+\rho} - \bar b - \overline{b}_3 \geq 0 \qquad\qquad (\overline{\lambda}) \\
&\qquad m + \frac{\bar d}{1+\rho} - \bar b - \overline{b}_3 + \bar p \bar q - \bar d + p_3 \overline{q}_3 \geq 0 \qquad\qquad (\overline{\mu}) \\
&\qquad \bar b, \overline{b}_3, \bar d \geq 0, \; 0 \leq \bar q \leq a, \; 0 \leq \overline{q}_3 \leq m.
\end{aligned}
$$

with first order conditions

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-\bar q}{\bar b}} = \Pi_1 + \bar\lambda + \bar\mu \tag{114}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{\bar b}{a-\bar q}} = (\bar\mu + \Pi_1)\bar p \tag{115}$$

$$\overline{\lambda} = (\bar\mu + \Pi_1)\rho \tag{116}$$

$$\frac{1+\Pi_2}{p_3} = \Pi_1 + \overline{\lambda} + \overline{\mu} \tag{117}$$

$$\Pi_1 p_3 - 1 - \Pi_2 + \overline{\mu} p_3 = 0 \tag{118}$$

$$m + \frac{\bar d}{1+\rho} - \bar b = 0 \text{ or } \bar\lambda = 0 \tag{119}$$

$$m + \frac{\bar d}{1+\rho} - \bar b - \overline{b}_3 + \bar p \bar q - \bar d + p_3 \overline{q}_3 = 0 \text{ or } \bar\mu = 0. \tag{120}$$

For the continuum of lender-merchants, the new decision variables are b_3^* (the amount the lenders bid for gold) and q_3^* (the amount of gold they put up for sale). Their optimization is

$$\max_{b_1^*, b_2^*, b_3^*, q_3^*, g} \; 2\sqrt{\frac{b_1^*}{p}\frac{b_2^*}{\overline{p}}} + \left(B + \frac{b_3^*}{p_3} - q_3^*\right) + \Pi_1\left(B - b_1^* - b_2^* + \rho g + p_3 q_3^* - b_3^*\right) \tag{121}$$

$$+ \Pi_2\left(B + \frac{b_3^*}{p_3} - q_3^*\right) \tag{122}$$

$$\text{s.t. } B - b_1^* - b_2^* - b_3^* - g \geq 0 \; (\lambda^*) \tag{123}$$

$$b_1^*, b_2^*, b_3^*, g \geq 0, \; 0 \leq q_3^* \leq B \tag{124}$$

The first order conditions here are:.

$$\sqrt{\frac{b_2^*}{p\overline{p}b_1^*}} = \Pi_1 + \lambda^* \tag{125}$$

$$\sqrt{\frac{b_1^*}{p\overline{p}b_2^*}} = \Pi_1 + \lambda^* \tag{126}$$

$$\Pi_1\rho - \lambda^* = 0 \tag{127}$$

$$-\Pi_1 + \frac{1+\Pi_2}{p_3} - \lambda^* = 0 \tag{128}$$

$$\Pi_1 p_3 - (1+\Pi_2) = 0 \tag{129}$$

$$B - b_1^* - b_2^* - b_3^* - g = 0 \text{ or } \lambda^* = 0 \tag{130}$$

Finally, we have the balance conditions $p = \frac{\overline{b}+b_1^*}{q}$, $\overline{p} = \frac{b+b_2^*}{\overline{q}}$, $p_3 = \frac{b_3+\overline{b}_3+b_3^*}{q_3+\overline{q}_3+q_3^*}$, and $1+\rho = \frac{d+\overline{d}}{g}$.

Case 1: We first consider the case where m is small and B is large, so we assume the traders' cash flow and budget constraints are both tight, but the lender-merchants' cash flow constraint is loose ($\lambda^* = 0$). In addition, we assume m is much smaller than B, and so the lenders will be selling gold to the traders. Mathematically, this means $q_3 = \overline{q}_3 = 0$ and $b_3^* = 0$. However, by doing this we may no longer assume conditions (111), (118), and (128).

We also assume a symmetric solution, i.e. $p = \overline{p}$, $q = \overline{q}$, $b = \overline{b}$, $d = \overline{d}$, and $b_1^* = b_2^*$.

Our analysis is as follows. First, condition (129) implies that $p_3 = \frac{1+\Pi_2}{\Pi_1}$. But then (110) is $\lambda + \mu = \frac{1+\Pi_2}{p_3} - \Pi_1 = \Pi_1 - \Pi_1 = 0$. Since λ and μ are both nonnegative, this implies $\lambda = \mu = 0 = \overline{\lambda} = \overline{\mu}$. But then (109) implies $\rho = 0$. And then (127) gives $\lambda^* = 0$. And then (125) gives $p = \frac{1}{\Pi_1} = \overline{p}$.

We can now get expressions for all of the other variables in terms of q. To begin, note that (107) along with $\lambda = \mu = 0$ gives $\sqrt{\frac{a-q}{b}} = \Pi_1\sqrt{\overline{p}}$; plugging in our previous expression for $\overline{p}$ gives $\sqrt{\frac{a-q}{b}} = \sqrt{\Pi_1}$, or $\frac{a-q}{b} = \Pi_1$. We can rewrite this as

$$b = \frac{a-q}{\Pi_1}. \tag{131}$$

Next, Trader Type 1's cash flow and budget constraints being tight together imply $pq - d + p_3q_3 = 0$; the assumption that $q_3 = 0$ further gives $d = pq = \frac{q}{\Pi_1}$. In addition, the balance condition for p gives $pq = d = \overline{b} + b_1^*$. Hence $b_1^* = d - \overline{b} = d - b = \frac{q}{\Pi_1} - \frac{a-q}{\Pi_1} = \frac{2q-a}{\Pi_1} = b_2^*$. Also, the balance condition for interest $1+\rho = \frac{d+\overline{d}}{g}$ yields $g = 2d$ (because of $\rho = 0$ and symmetry), so $g = \frac{2q}{\Pi_1}$. Furthermore, $b_3 = m + \frac{d}{1+\rho} - b = m + d - b = m + \frac{q}{\Pi_1} - \frac{a-q}{\Pi_1} = m + \frac{2q-a}{\Pi_1} = \overline{b}_3$. This in turn implies $q_3^* = \frac{b_3+\overline{b}_3}{p_3} = \frac{2\left(m+\frac{2q-a}{\Pi_1}\right)\Pi_1}{1+\Pi_2} = \frac{2(\Pi_1 m+2q-a)}{1+\Pi_2}$.

There is no way to pin down an exact value for q, so we have a continuum of solutions parametrized by q. However, we can find upper and lower bounds for q, so that we end up with a "line segment" of solutions. First, since b_1^* and b_2^* are nonnegative, we must have $q \geq \frac{a}{2}$. But also the cash flow constraint for the lenders must be satisfied, i.e. $B - b_1^* - b_2^* - b_3^* - g \geq 0$. This reduces to $B - 2\left(\frac{2q-a}{\Pi_1}\right) - \frac{2q}{\Pi_1} \geq 0$, or $q \leq \frac{2a+\Pi_1 B}{6}$. In addition, we must have $q_3^* \leq B$, which is $\frac{2(\Pi_1 m+2q-a)}{1+\Pi_2} \leq B$ or $q \leq \frac{(1+\Pi_2)B-2\Pi_1 m+2a}{4}$. So the range for q is $\frac{a}{2} \leq q \leq \min(\frac{2a+\Pi_1 B}{6}, \frac{(1+\Pi_2)B-2\Pi_1 m+2a}{4}, a)$. In order for this range to be nonempty we must have $\Pi_1 B \geq a$ and $(1+\Pi_2)B \geq 2\Pi_1 m$.

If one also requires a 100% reserve requirement for lending, i.e. $B - q_3^* \geq g$, this gives the further condition $B \geq \frac{2(\Pi_1 m+2q-a)}{1+\Pi_2} + \frac{2q}{\Pi_1}$. There will be at least one q to satisfy this (namely $q = \frac{a}{2}$) so long as $B \geq \frac{2\Pi_1 m}{1+\Pi_2} + \frac{a}{\Pi_1}$.

Case 2: Now we consider the case where B is smaller (but still significantly larger than m). In this case, we assume that the gold market goes inactive – so in addition to $q_3 = \overline{q}_3 = 0$ and $b_3^* = 0$, we also assume $b_3 = \overline{b}_3 = q_3^* = 0$. This means that we cannot use condition (110), (111), (117), (118), (128), or (129) in our analysis. In addition, we assume all trader and lender-merchant constraints are tight, including now the lender-merchants' cash flow constraints.

We begin with the tight traders' cash flow constraints: $m + \frac{d}{1+\rho} - b - b_3 = 0$. Assuming $b_3 = 0$, the balance condition $1 + \rho = \frac{2d}{g}$, and simplifiying, this yields $m + \frac{g}{2} - b = 0$. But the tight lender-merchants' cash flow constraints (plus symmetry $b_1^* = b_2^*$) imply $g = B - 2b_2^*$; hence we have $m + \frac{B}{2} - b_2^* - b = 0$. Next, the balance constraint for price implies $b + b_2^* = \overline{pq} = pq$, and furthermore the tight cash flow and budget constraitnts (plus assuming $q_3 = 0$) imply $pq = d$. So $m + \frac{B}{2} - b_2^* - b = 0$ implies $m + \frac{B}{2} - d = 0$, or

$$d = m + \frac{B}{2} = \overline{d}. \tag{132}$$

Next, (125) together with symmetry implies that $\frac{1}{p} = \Pi_1 + \lambda^* = \Pi_1 + \Pi_1\rho = \Pi_1(1+\rho)$, where the second equality follows from (127). This is

$$1 + \rho = \frac{1}{\Pi_1 p}. \tag{133}$$

Next, substituting (132), (133), and $b_3 = 0$ into the tight traders' cash flow constraint gives $m + \frac{m+\frac{B}{2}}{\frac{1}{\Pi_1 p}} - b = 0$, which is

$$b = m + \Pi_1 p(m + \frac{B}{2}) = \overline{b}. \tag{134}$$

Next, condition (107) is $\frac{1}{\sqrt{p}}\sqrt{\frac{a-q}{b}} = \Pi_1 + \lambda + \mu$. Substituting in for λ using (109) gives $\frac{1}{\sqrt{p}}\sqrt{\frac{a-q}{b}} = \Pi_1 + \mu + \rho(\Pi_1 + \mu) = (1+\rho)(\Pi_1 + \mu)$. Now (108).implies $(\Pi_1 + \mu) = \frac{1}{p^{\frac{3}{2}}}\sqrt{\frac{b}{a-q}}$, so we have $\frac{1}{\sqrt{p}}\sqrt{\frac{a-q}{b}} = (1+\rho)\frac{1}{p^{\frac{3}{2}}}\sqrt{\frac{b}{a-q}}$, which is $p\left(\frac{a-q}{b}\right) = 1+\rho$. Substituting in for $1+\rho$ using (133) gives $p\left(\frac{a-q}{b}\right) = \frac{1}{\Pi_1 p}$, which is $b = \Pi_1 p^2(a-q)$. Now the traders' cash flow and budget constraints being tight, together with the assumption $q_3 = 0$ imply $pq = d$; hence we can substitute $q = \frac{d}{p} = \frac{m+\frac{B}{2}}{p} = \frac{B+2m}{2p}$ into the last equation for b, obtaining

$$b = \Pi_1 p^2 \left(a - \frac{B+2m}{2p}\right) = \overline{b}. \tag{135}$$

Equations (134) and (135) give two expressions for b; setting them equal to each other gives the following equation, which can be solved for p using computional methods:

$$m + \Pi_1 p\left(m + \frac{B}{2}\right) = \Pi_1 p^2\left(a - \frac{B+2m}{2p}\right). \tag{136}$$

Once we know $p(=\overline{p})$, it is then a simple matter to compute q (and $\overline{q}$) via $q = \overline{q} = \frac{m+\frac{B}{2}}{p}$, $b = \overline{b}$ via (134) or (135), ρ via (133), g from $1+\rho = \frac{2d}{g} = \frac{B+2m}{g}$, and $b_1^* = b_2^*$ from $g = B - 2b_2^*$. Consumption levels and final utilities can then be computed accordingly. Finally, for the multipliers, we may calculate λ^* from $\frac{1}{p} = \Pi_1 + \lambda^*$, μ from (108), and then λ from (109).

In the special case of $m = 0$, equation (136) solves easily, with $p = \overline{p} = \frac{B}{a}$. We also get $q = \overline{q} = \frac{a}{2}$, $b = \overline{b} = \frac{\Pi_1 B^2}{2a}$, $\rho = \frac{a}{\Pi_1 B} - 1$, $g = \frac{\Pi_1 B^2}{a}$, and $b_1^* = b_2^* = \frac{B}{2}(1 - \frac{\Pi_1 B}{a})$. For the multipliers, $\lambda^* = \frac{a}{B} - \Pi_1$, $\mu = \sqrt{\Pi_1}(\sqrt{\frac{a}{B}} - \sqrt{\Pi_1}) = \overline{\mu}$, and $\lambda = \sqrt{\frac{a}{\Pi_1 B}}(\frac{a}{B} - \Pi_1)$. We note that in order for these quantities to be nonnegative, we need for $\Pi_1 B \leq a$.

12 Appendix D: Demonetized Gold with a Central Bank

Now we assume the demonetization of gold in the presence of a central bank. When the demonetization occurs, the holders of x units of gold are now endowed just with x units of demonetized gold. The accompanying x units of "gold strips" (fiat) go to the central bank. For the moneylenders of the previous models, this means that they are endowed only with (B units of) demonetized gold, making them "gold merchants".

The central bank is a strategic dummy, lending an amount G of fiat no matter what. It is essentially owned by the individuals of the society (the traders and the merchants). Thus, at the end of the game, its profits are divided up between the two types of trader and the merchants, in the ratio of m to m to B, reflecting the original ownership of the gold which backs up the bank in the first place.

The traders of Type 1 are endowed with m units of gold, 0 units of fiat, and a units of perishable good #1. Their optimization problem looks like this:

$$\max_{b,q,d,b_3,q_3} 2\sqrt{(a-q)\frac{b}{\bar{p}}} + \left(m + \frac{b_3}{p_3} - q_3\right) + \Pi_1\left(\frac{d}{1+\rho} - b - b_3 + pq - d + p_3q_3 + D\right) \quad (137)$$

$$+ \Pi_2\left(m + \frac{b_3}{p_3} - q_3\right) \quad (138)$$

$$\text{s.t. } \frac{d}{1+\rho} - b - b_3 \geq 0 \qquad (\lambda) \text{ (cash flow constraint)} \quad (139)$$

$$\frac{d}{1+\rho} - b - b_3 + pq - d + p_3q_3 + D \geq 0 \qquad (\mu) \text{ (budget constraint)} \quad (140)$$

$$b, b_3, d \geq 0,\ 0 \leq q \leq a,\ 0 \leq q_3 \leq m \quad (141)$$

Here the new quantity "D" is the amount of fiat that comes back to the Type 1 traders, as a result of their ownership of $\frac{m}{B+2m}$ of the profits of the central bank. Also, note that here the cash flow and budget constraints lack the quantity "m" on the left hand side, reflecting that the traders now are assumed to begin with gold but no fiat as compensation for their gold's demonetization.

The first order conditions in the above, with respect to b, q, d, b_3, and

q_3 are

$$\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{a-q}{b}} = \Pi_1 + \lambda + \mu \tag{142}$$

$$\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{b}{a-q}} = (\mu + \Pi_1)p \tag{143}$$

$$\lambda = (\mu + \Pi_1)\rho \tag{144}$$

$$\frac{1+\Pi_2}{p_3} = \Pi_1 + \lambda + \mu \tag{145}$$

$$\Pi_1 p_3 - 1 - \Pi_2 + \mu p_3 = 0 \tag{146}$$

In addition, we have the complementarity constraints

$$\frac{d}{1+\rho} - b - b_3 = 0 \text{ or } \lambda = 0 \tag{147}$$

$$\frac{d}{1+\rho} - b - b_3 + pq - d + p_3 q_3 + D = 0 \text{ or } \mu = 0 \tag{148}$$

Similarly, the optimization for the Type 2 traders is

$$\begin{aligned}
&\max_{\bar{b},\bar{q},\bar{d},\overline{b}_3,\overline{q}_3} \ 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + \left(m + \frac{\overline{b}_3}{p_3} - \overline{q}_3\right) + \Pi_1\left(\frac{\bar{d}}{1+\rho} - \bar{b} - \overline{b}_3 + \bar{p}\bar{q} - \bar{d} + p_3\overline{q}_3 + D\right) \\
&+ \Pi_2\left(m + \frac{\overline{b}_3}{p_3} - \overline{q}_3\right) \\
&\text{s.t. } \frac{\bar{d}}{1+\rho} - \bar{b} - \overline{b}_3 \geq 0 \qquad\qquad (\overline{\lambda}) \\
&\quad\ \ \frac{\bar{d}}{1+\rho} - \bar{b} - \overline{b}_3 + \bar{p}\bar{q} - \bar{d} + p_3\overline{q}_3 + D \geq 0 \qquad\qquad (\overline{\mu}) \\
&\quad\ \ \bar{b}, \overline{b}_3, \bar{d} \geq 0,\ \ 0 \leq \bar{q} \leq a,\ 0 \leq \overline{q}_3 \leq m.
\end{aligned}$$

with first order conditions

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = \Pi_1 + \bar{\lambda} + \bar{\mu} \tag{149}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = (\bar{\mu} + \Pi_1)\bar{p} \tag{150}$$

$$\overline{\lambda} = (\bar{\mu} + \Pi_1)\rho \tag{151}$$

$$\frac{1+\Pi_2}{p_3} = \Pi_1 + \overline{\lambda} + \overline{\mu} \tag{152}$$

$$\Pi_1 p_3 - 1 - \Pi_2 + \overline{\mu} p_3 = 0 \tag{153}$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b} - \overline{b}_3 = 0 \text{ or } \bar{\lambda} = 0 \tag{154}$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b} - \overline{b}_3 + \bar{p}\bar{q} - \bar{d} + p_3\overline{q}_3 = 0 \text{ or } \bar{\mu} = 0. \tag{155}$$

The gold merchants begin with B units of gold; but like the traders, they begin with no fiat. They have been stripped of their former money lending function. Instead, again like the traders, they have become money borrowers – the new variable d^* stands for the amount of loan (from the central bank) that they must repay. And as a result, they also now have a budget constraint, again just like the traders. Finally, they too have an ownership stake in the central bank, namely $\frac{B}{B+2m}$ of its profits, which we denote by D^*. The merchants' optimization is as below

$$\max_{b_1^*,b_2^*,b_3^*,q_3^*,d^*} 2\sqrt{\frac{b_1^*}{p}\frac{b_2^*}{\overline{p}}} + \left(B + \frac{b_3^*}{p_3} - q_3^*\right) + \Pi_1\left(\frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* - d^* + p_3 q_3^* + D^*\right) \tag{156}$$

$$+ \Pi_2\left(B + \frac{b_3^*}{p_3} - q_3^*\right) \tag{157}$$

$$\text{s.t. } \frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* \geq 0 \ (\lambda^*) \tag{158}$$

$$\frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* - d^* + p_3 q_3^* + D^* \geq 0 \ (\mu^*) \tag{159}$$

$$b_1^*, b_2^*, b_3^*, d^* \geq 0,\ 0 \leq q_3^* \leq B \tag{160}$$

The first order conditions here are:

$$\sqrt{\frac{b_2^*}{p\overline{p}b_1^*}} = \Pi_1 + \lambda^* + \mu^* \tag{161}$$

$$\sqrt{\frac{b_1^*}{p\overline{p}b_2^*}} = \Pi_1 + \lambda^* + \mu^* \tag{162}$$

$$(\Pi_1 + \mu^*)\rho - \lambda^* = 0 \tag{163}$$

$$-\Pi_1 + \frac{1+\Pi_2}{p_3} - \lambda^* - \mu^* = 0 \tag{164}$$

$$\Pi_1 p_3 + \mu^* p_3 - (1+\Pi_2) = 0 \tag{165}$$

$$\frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* = 0 \text{ or } \lambda^* = 0 \tag{166}$$

$$\frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* - d^* + p_3 q_3^* + D^* = 0 \text{ or } \mu^* = 0 \tag{167}$$

The central bank is endowed with only $B + 2m$ units of fiat. It is a strategic dummy – it always just lends a total of G units of fiat to the traders and merchants, where G is a given quantity with $0 \leq G \leq B + 2m$. Its profits are ρG, and so each trader type's share is $D = \frac{m}{B+2m}\rho G$, while the merchants' share is $D^* = \frac{B}{B+2m}\rho G$.

Next, we have the balance conditions a) $p = \frac{\overline{b}+b_1^*}{q}$, b) $\overline{p} = \frac{b+b_2^*}{\overline{q}}$, c) $p_3 = \frac{b_3+\overline{b}_3+b_3^*}{q_3+\overline{q}_3+q_3^*}$, and d)$1+\rho = \frac{d+\overline{d}+d^*}{G}$.

Finally, we make the same symmetry assumptions as usual: $p = \overline{p}$, $q = \overline{q}$, $b = \overline{b}$, $d = \overline{d}$, and $b_1^* = b_2^*$.

We will consider two cases. In both cases, m is relatively small compared to B and G. Since the traders are starting with much less gold than the merchants, we assume that the traders will not be selling gold to the merchants, i.e. $q_3 = \overline{q}_3 = 0$ and $b_3^* = 0$. Also, since the traders and merchants both start with no fiat, we assume the cash flow constraints are tight for both. In Case 1, the central bank lends out a relatively large amount of money (G large). Since B is large compared to m, we assume that when all of the money is recycled back to the merchants they will have loose budget constraints, so $\mu^* = 0$. In Case 2 (G small) there is a lot less money in the economy, so all constraints, for both traders and merchants, are tight.

Case 1: m small, G large

We first analyze the case in which the cash flow constraints (for both types of traders and for merchants) are tight, and the budget constraints for the merchants are loose (so μ^* is equal to zero), $q_3 = \overline{q}_3 = 0$, and $b_3^* = 0$.

The analysis of this case goes as follows. First, $\mu^* = 0$ and (165) together imply that $p_3 = \frac{1+\Pi_2}{\Pi_1}$. Then (145) is $\frac{1+\Pi_2}{p_3} = \Pi_1 + \lambda + \mu$. But $p_3 = \frac{1+\Pi_2}{\Pi_1}$, so this reduces to $\lambda + \mu = 0$. Since the multipliers λ and μ are both constrained to be nonnegative, we have $\lambda = \mu = 0$. Similarly, $\overline{\lambda} = \overline{\mu} = 0$.

Next, $\lambda = 0$ and (144) together imply $\rho = 0$. But this in turn implies $D = D^* = 0$. Also, $\rho = 0$ together with (163) implies $\lambda^* = 0$. And then (161) plus symmetry imply $\frac{1}{p} = \Pi_1 + \lambda^* + \mu^* = \Pi_1$, so we have $p = \frac{1}{\Pi_1} = \overline{p}$. And then (142) (with $\lambda = \mu = 0$) gives $\sqrt{\frac{a-q}{b}} = \Pi_1\sqrt{\overline{p}} = \sqrt{\Pi_1}$, which in turn gives

$$q = a - \Pi_1 b. \tag{168}$$

The next piece is to obtain an expression for b. This will be a somewhat long, tedious process. We begin by working with the budget constraint for Trader Type 1: $\frac{d}{1+\rho} - b - b_3 + pq - d + p_3 q_3 + D \geq 0$. Substituting $\rho = 0$, $q_3 = 0$, and $D = 0$ yields $pq - b - b_3 \geq 0$. But by balancing condition a) $pq - b = pq - \overline{b} = b_1^*$, so we have

$$b_1^* - b_3 \geq 0. \tag{169}$$

But also, the budget constraint for the merchants is $\frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* - d^* + p_3 q_3^* + D^* \geq 0$. Substituting $\rho = 0$, $b_3^* = 0$, and $D^* = 0$ yields $p_3 q_3^* - b_1^* - b_2^* \geq 0$, which (by symmetry) is $p_3 q_3^* - 2b_1^* \geq 0$. But by balancing condition c) (plus the assumptions $q_3 = \overline{q}_3 = b_3^* = 0$ and symmetry) we have $p_3 q_3^* = 2b_3$, and so $2b_3 - 2b_1^* \geq 0$. But this and (169) together imply

$$b_1^* = b_3. \tag{170}$$

Now look at the cash flow constraint for the merchants. We assumed it was tight, i.e. $\frac{d^*}{1+\rho} - b_1^* - b_2^* - b_3^* = 0$. Substituting in $\rho = 0$ and $b_3^* = 0$, and using symmetry gives $d^* - 2b_1^* = 0$, or $b_1^* = \frac{d^*}{2}$. But balancing condition d) (with $\rho = 0$) gives $d^* = G - d - \overline{d} = G - 2d$; hence $b_1^* = \frac{G-2d}{2} = \frac{G}{2} - d = \frac{G}{2} - b - b_3 = \frac{G}{2} - b - b_1^*$. [In the previous, the penultimate equality follows from the tight Trader 1 cash flow constraint (together with $\rho = 0$), while the last equality follows from (170).] Combining the "b_1^*" terms, we have $2b_1^* = \frac{G}{2} - b$, or

$$b_1^* = \frac{G}{4} - \frac{b}{2}. \tag{171}$$

Now we are finally in a position to obtain our expression for b. We start with balancing condition a), which is $pq = b + b_1^*$. Substituting in our expressions $p = \frac{1}{\Pi_1}$, equation (168) for q, and (171) for b_1^*, we have $\frac{a-\Pi_1 b}{\Pi_1} = b + \frac{G}{4} - \frac{b}{2}$, which solves with $b = \frac{4a-\Pi_1 G}{6\Pi_1} = \overline{b}$.

The rest of the variables can now be easily obtained. First, using (171), we have $b_1^* = \frac{G}{4} - \frac{b}{2} = \frac{G}{4} - \frac{a}{3\Pi_1} + \frac{G}{12} = \frac{\Pi_1 G - a}{3\Pi_1} = b_2^*$. Because of (170), we also immediately have $b_3 = \frac{\Pi_1 G - a}{3\Pi_1} = \overline{b}_3$. Next, $q = a - \Pi_1 b = a - \Pi_1 \left(\frac{4a - \Pi_1 G}{6\Pi_1}\right) = \frac{2a + \Pi_1 G}{6} = \overline{q}$. Then $d^* = 2b_1^* = \frac{2(\Pi_1 G - a)}{3\Pi_1}$. And $d = \frac{G - d^*}{2} = \frac{\Pi_1 G + 2a}{6\Pi_1} = \overline{d}$. And finally $q_3^* = \frac{2b_3}{p_3} = \frac{2(\Pi_1 G - a)}{3(1+\Pi_2)}$. For completeness, we also list all of the other values previously found: $p = \frac{1}{\Pi_1} = \overline{p}$, $p_3 = \frac{1+\Pi_2}{\Pi_1}$, and $\rho = D = D^* = \lambda = \overline{\lambda} = \mu = \overline{\mu} = \lambda^* = \mu^* = 0$.

Last, we should state the values over which our calculations are valid. First, all of the calculated expressions for variables above must be nonnegative; this requires $a \leq \Pi_1 G \leq 4a$. Second, the above value for q_3^* must be less than or equal to B, i.e. $\frac{2(\Pi_1 G - a)}{3(1+\Pi_2)} \leq B$.

Case 2: m small, G small

Now we consider what happens when $\Pi_1 G < a$. In this case we assume cash flow and budget constraints for both traders and merchants are all tight, i.e. the multipliers λ, $\overline{\lambda}$, μ, $\overline{\mu}$, λ^*, and μ^* are all positive. The values of q_3^*, b_3, and $\overline{b}_3$ from Case 1 (from the boundary case $\Pi_1 G = a$) lead us to assume the gold trade market shuts down, i.e. in addition to our previous assumptions of $q_3 = \overline{q}_3 = 0$ and $b_3^* = 0$, we also have $q_3^* = b_3 = \overline{b}_3 = 0$.

Our analysis begins with equation (142), namely $\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{a-q}{b}} = \Pi_1 + \lambda + \mu$. Substituting in for λ using (144) yields $\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{a-q}{b}} = (1+\rho)(\Pi_1 + \mu)$. But then substituting in for $\Pi_1 + \mu$ using (143) gives $\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{a-q}{b}} = (1+\rho)\frac{1}{p\sqrt{\overline{p}}}\sqrt{\frac{b}{a-q}}$. Using symmetry $\overline{p} = p$ results in

$$p\left(\frac{a-q}{b}\right) = 1 + \rho. \tag{172}$$

Next, the balancing constraint for price is $b = pq - b_1^*$. But the tight (λ^*) constraint, together with $b_3^* = 0$ and symmetry $b_1^* = b_2^*$ yields $b_1^* = \frac{d^*}{2(1+\rho)}$; substituting this in the previous expression for b gives $b = pq - \frac{d^*}{2(1+\rho)}$. But the tight (λ) constraint implies $b = \frac{d}{1+\rho}$; setting the last two expressions for b equal gives $pq - \frac{d^*}{2(1+\rho)} = \frac{d}{1+\rho}$. This can be rearranged to read

$$q = \frac{d^* + 2d}{2p(1+\rho)}. \tag{173}$$

In addition, we may rewrite (172) as $\frac{p(a-q)}{1+\rho} = b = \frac{d}{1+\rho}$, so $d = ap - pq =$

$ap - \frac{d^*+2d}{2(1+\rho)}$, which gives

$$ap = d + \frac{d^* + 2d}{2(1+\rho)}. \tag{174}$$

Next, the tight cash flow and budget constraints for the merchants together imply that $-d^* + p_3 q_3^* + D^* = 0$. Now q_3^* is assumed to be zero, and D^* is equal to $\frac{B}{B+2m}\rho G$, so this reduces to $d^* = \frac{B}{B+2m}\rho G$. Next, the balance constraint for the interest rate implies that $d = \frac{(1+\rho)G - d^*}{2}$. Substituting in our previous expression for d^*, this simplifies to $d = \frac{(1+\rho)G - \frac{B}{B+2m}\rho G}{2} = \frac{G}{2} + \frac{m}{B+2m}\rho G$. Hence, $d^* + 2d = (1+\rho)G$, and our previous expressions (174) and (173) for p and q reduce to $p = \frac{2d+G}{2a}$ and $q = \frac{G}{2p} = \frac{aG}{2d+G}$ respectively. In addition, substituting in $d = \frac{G}{2} + \frac{m}{B+2m}\rho G$ yields $p = \frac{1}{a}\left(G + \frac{m}{B+2m}\rho G\right) = \overline{p}$ and $q = \frac{a}{2}\left(\frac{B+2m}{B+2m+m\rho}\right) = \overline{q}$. We also have $b = \overline{b} = \frac{d}{1+\rho} = \frac{1}{1+\rho}\left(\frac{G}{2} + \frac{m}{B+2m}\rho G\right)$ and $b_1^* = b_2^* = \frac{d^*}{2(1+\rho)} = \frac{B}{B+2m}\frac{\rho G}{2(1+\rho)}$.

Note that we have a degree of freedom in our solutions here, as all of our variables are in terms of ρ.

Finally, we obtain expressions for the multipliers. First, from (143) we have $\mu = \frac{1}{p\sqrt{p}}\sqrt{\frac{b}{a-q}} - \Pi_1 = \left(\frac{2a}{2d+G}\right)^{\frac{3}{2}}\sqrt{\frac{d/(1+\rho)}{a - \frac{aG}{2d+G}}} = \frac{a}{2d+G}\frac{2}{\sqrt{1+\rho}} - \Pi_1 = \frac{a}{\sqrt{1+\rho}(G + \frac{m}{B+2m}\rho G)} - \Pi_1 = \overline{\mu}$. We also have $\lambda = (\mu + \Pi_1)\rho = \frac{a\rho}{\sqrt{1+\rho}(G+\frac{m}{B+2m}\rho G)} = \overline{\lambda}$. To find $\mu^*$and λ^*, we need to solve the two equations in two unknowns given by (162) and (163). First, (163) gives us $\lambda^* = (\Pi_1 + \mu^*)\rho$. Substituting this expression for λ^* into (162) and using symmetry, we have $\frac{1}{p} = (1+\rho)(\Pi_1 + \mu^*)$, or $\mu^* = \frac{1}{(1+\rho)p} - \Pi_1 = \frac{a}{(1+\rho)\left(G + \frac{m}{B+2m}\rho G\right)} - \Pi_1$. Then $\lambda^* = (\Pi_1 + \mu^*)\rho = \frac{a\rho}{(1+\rho)\left(G+\frac{m}{B+2m}\rho G\right)}$.

In order for the results to be valid, ρ must be nonnegative, and the above expressions for the multipliers μ and μ^* must also be nonnegative. Looking closely at these, we see that there will be a range for ρ, with zero as the lower bound and the upper bound as the value of ρ for which $(1+\rho)\left(G + \frac{m}{B+2m}\rho G\right)$ is equal to $\frac{a}{\Pi_1}$. This range will be nonempty so long as $\Pi_1 G \leq a$.