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By

Peter C. B. Phillips

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COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Long Memory and Long Run Variation*

Peter C. B. Phillips
Cowles Foundation, Yale University,
University of Auckland & University of York

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Abstract

A commonly used defining property of long memory time series is the power law decay of the autocovariance function. Some alternative methods of deriving this property are considered working from the alternate definition in terms of a fractional pole in the spectrum at the origin. The methods considered involve the use of (i) Fourier transforms of generalized functions, (ii) asymptotic expansions of Fourier integrals with singularities, (iii) direct evaluation using hypergeometric function algebra, and (iv) conversion to a simple gamma integral. The paper is largely pedagogical but some novel methods and results involving complete asymptotic series representations are presented. The formulae are useful in many ways including the calculation of long run variation matrices for multivariate time series with long memory and the econometric estimation of such models.

Key words and Phrases: Asymptotic expansion, autocovariance function, fractional pole, Fourier integral, generalized function, long memory, long range dependence, singularity.

JEL Classification: C22, C32

1. Introduction

Time series, semiparametric methods, and nonparametrics now form a vast discipline of knowledge whose applications stretch across the sciences and social sciences. Within this rich scientific domain, multitudinous subfields of research coexist and interlink, fueled by motorways of fast moving research. Along these busy highways we repeatedly see signposts to avenues of knowledge where Peter Robinson has left his mark. One avenue that particularly stands out is time series with long range dependence, where his work has blazed a preeminent trail for others to follow. His 1990 Econometric Society World Congress paper, published in Robinson (1994), was

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a masterly overview of the field that introduced many new econometricians to the field. A few years later, his two 1995 *Annals of Statistics* papers (Robinson, 1995a, 1995b) pioneered the rigorous development of an asymptotic theory of long memory parameter estimation and inference that resolved longstanding technical problems. These contributions laid a new foundation for empirical applications and theoretical developments, many of which are themselves associated with Peter Robinson and his students.

It is a great honor to participate in celebrating these remarkable accomplishments and Peter Robinson's other achievements. Our first contact dates back to 1972 when we were both working on the estimation of continuous time systems with discrete data. Subsequently, we have had regular professional contact in our research, in conferences, and in various editorial endeavours. Throughout, Peter's standards of mathematical rigor, which are now legendary in the profession, his thoroughness in evaluating research, his fairness in adjudicating credit, and the remarkable quality of his scientific writing have set paradigms within the profession. This symposium provides a welcome opportunity for me to thank him for his personal kindness and for us all to applaud his enormous contributions to the econometrics and statistics professions over many decades.

My contribution to the symposium selects an elementary aspect of the field that he has pioneered.and looks at the defining property of long memory time series, which is conventionally expressed in terms of the power law decay of the autocovariance function. The goal of the paper is largely pedagogical. In teaching long memory time series one quickly discovers that the defining property of long memory is not so elementary to establish, at least with any precision. Indeed, standard texts like Beran (1994) and Taniguchi and Kakizawa (2000) do not provide a general proof of the result, but refer readers to Zygmund's (1958) classic text on trigonometric series, a gifted monograph and wonderful research tool but one that is beyond the reach of many students. A second approach, based on Hosking's (1980) original paper and later used in the time series text of Brockwell and Davis (1989), deals with a pure fractional process in which the short memory component is *iid* and derives the autocovariance function by direct calculation in this special case using an integral sourced from Gradshteyn and Ryzhik (1965). That approach falls short of a general result and additionally requires external source material to resolve the integral. So, while that approach is elementary, it is also not as easy for classroom use as might be hoped.

The present paper offers several approaches to obtaining the power law defining property, some of which are new. The author has found that these approaches are suitable for classroom exposition when explicit derivations are given to accompany them. The paper overviews the methods in a form that is intended to be suited to classroom use. The paper also provides complete asymptotic series representations of the autocovariogram in the multivariate case, building on some recent work that has appeared elsewhere (Phillips and Kim, 2007; Lieberman and Phillips, 2007). The resulting formulae are useful in finding the long run variation matrices for multivariate time series with long memory and in multivariate estimation of long memory parameters and cointegrating vectors.

The plan of the paper is as follows. All results are given in Section 2, which is subdivided into sections that detail alternative mathematical approaches. Some concluding remarks are made in Section 3. Proofs and other technical material are placed in the Appendix.

2. Power Law Decay Results

Let X_t be a real-valued covariance stationary m-vector time series generated by the system

$$(1-L)^{d_a}\left(X_{at}-EX_{at}\right)=u_{at};\quad a=1,...,m \tag{1}$$

where $u_t=(u_{1t},\dots,u_{mt})'$ is a covariance stationary process whose spectral density matrix $f_{uu}\left(\lambda\right)\in\mathcal{C}_\infty$ on $[-\pi,\pi]$ and is bounded above zero in the sense of positive definite matrices at the zero frequency $\lambda=0$. Our primary discussion focuses on the case where $d_a\in\left(0,\frac{1}{2}\right)$ that corresponds to stationary long run dependence and this assumption will be maintained below. But some remarks and results on cases where $d_a=0$, corresponding to short memory series, and the antipersistent case $d_a\in\left(-\frac{1}{2},0\right)$ will be given later in the discussion. The smoothness condition on $f_{uu}\left(\lambda\right)$ assists in developing a complete asymptotic expansion of the autocovariance function (acf) defined by a Fourier integral inversion of $f_{uu}\left(\lambda\right)$. The smoothness condition may be relaxed, but any critical values of $f_{uu}\left(\lambda\right)$ within $[-\pi,\pi]$ may contribute to the form of the expansion of the acf.

The process X_t is a multivariate fractionally integrated time series (or vector $I(d)$ process) and each component X_{at} exhibits long-range dependence whenever $d_a>0$. X_t reduces to a multivariate ARFIMA process when u_t is a vector ARMA process, but the specification (1) does not require u_t to be of this or any other parametric form.

Let $f_{xx}(\lambda)$ denote the spectral density of X_t, so that the autocovariance matrix is given by

$$\Gamma_{xx}\left(k\right)=E(X_t-EX_t)(X_{t+k}-EX_t)'=\int_{-\pi}^{\pi}e^{ik\lambda}f_{xx}(\lambda)d\lambda=\int_0^{2\pi}e^{ik\lambda}f_{xx}(\lambda)d\lambda. \tag{2}$$

We seek to exhibit the power law decay property of the elements of $\Gamma_{xx}\left(k\right)$ as $k\to\infty$. Using the transfer function of the individual filters $(1-L)^{d_a}$, the spectrum $f_{xx}(\lambda)$ has the well-known form (e.g., Hannan, 1970, p.61)

$$f_{xx}(\lambda)=\Phi(\lambda)f_{uu}(\lambda)\Phi^*(\lambda),\quad \Phi\left(\lambda\right)=\operatorname{diag}\left((1-e^{i\lambda})^{-d_a}\right), \tag{3}$$

where the affix * signifies complex conjugate transpose. Since

$$(1-e^{i\lambda})^\theta=\begin{cases}|2\sin(\frac{\lambda}{2})|^\theta e^{i(\lambda-\pi)\theta/2} & 0\le\lambda<\pi\\ |2\sin(\frac{\lambda}{2})|^\theta e^{i(\pi-|\lambda|)\theta/2} & -\pi\le\lambda<0\end{cases}.$$

individual elements of the spectrum take the following form over $[-\pi, 0) \cup (0, \pi]$

$$f_{x_a x_b}(\lambda) = \begin{cases} e^{\frac{i\pi(d_a - d_b)}{2}} |\lambda|^{-d_a - d_b} \left|\frac{2\sin(\lambda/2)}{\lambda}\right|^{-d_a - d_b} e^{-\frac{i\lambda(d_a - d_b)}{2}} f_{u_a u_b}(\lambda) & 0 < \lambda < \pi \\ e^{-\frac{i\pi(d_a - d_b)}{2}} |\lambda|^{-d_a - d_b} \left|\frac{2\sin(\lambda/2)}{\lambda}\right|^{-d_a - d_b} e^{\frac{i|\lambda|(d_a - d_b)}{2}} f_{u_a u_b}(\lambda) & -\pi \leq \lambda < 0 \end{cases}, \tag{4}$$

where the spectrum has a single critical point at $\lambda = 0$ that may be approached from the left or right.

The memory parameters $\{d_a\}$ determine the behavior of the spectral matrix $f_{xx}(\lambda)$ around the origin and thereby govern the long-run dynamics of X_t, as is apparent from (4). Note that the phase shift factor $e^{\frac{i\pi(d_a - d_b)}{2}}$ appears even in the local first order approximation to the cross spectrum $f_{x_a x_b}(\lambda)$ in (4), viz.,

$$f_{x_a x_b}(\lambda) \sim \begin{cases} e^{\frac{i\pi(d_a - d_b)}{2}} |\lambda|^{-d_a - d_b} f_{u_a u_b}(0) & 0 < \lambda \\ e^{-\frac{i\pi(d_a - d_b)}{2}} |\lambda|^{-d_a - d_b} f_{u_a u_b}(0) & \lambda < 0 \end{cases}, \tag{5}$$

so that whenever $d_a \neq d_b$ the phase shift factor in (4) or (5) is useful in efficient semiparametric multivariate estimation of the long memory parameters using local Whittle procedures (Robinson, 2007a, 2007b; Shimotsu, 2007).

It is sometimes convenient to write the equivalent form of the spectrum over the interval $(0, 2\pi)$ as

$$f_{x_a x_b}(\lambda) = e^{\frac{i\pi(d_a - d_b)}{2}} \lambda^{-d_a - d_b} (2\pi - \lambda)^{-d_a - d_b} \left|\frac{2\sin(\lambda/2)}{\lambda(2\pi - \lambda)}\right|^{-d_a - d_b} e^{-\frac{i\lambda(d_a - d_b)}{2}} f_{u_a u_b}(\lambda), \tag{6}$$

in which case the spectrum has critical points whenever $d_a + d_b > 0$ at the endpoints $\lambda = \{0, 2\pi\}$ corresponding to the right and left limits to the origin in (4). Lieberman and Phillips (2007) used (6) in developing a complete asymptotic series expansion for the acf in the scalar case.

Our purpose is to demonstrate several different methods of obtaining the asymptotic power law decay property of $\Gamma_{xx}(k) = [(\gamma_{ab}(k))]$ using the form of the spectral density matrix given by (4) and (6). We start with the explicit determination of $\gamma_{ab}(k)$ in the simplest case where the short memory spectrum is constant, viz., $f_{uu}(\lambda) = \Sigma_u / (2\pi)$, and X_t is a pure fractional process. We then consider several alternative methods in the more general case.

Before proceeding further, some general reductions based on (2) and (4) are useful. Define

$$g_{ab}(\lambda) = \left|\frac{1 - e^{i\lambda}}{\lambda(2\pi - \lambda)}\right|^{-d_a - d_b} e^{-\frac{i\lambda(d_a - d_b)}{2}} f_{u_a u_b}(\lambda) = \left|\frac{2\sin(\lambda/2)}{\lambda(2\pi - \lambda)}\right|^{-d_a - d_b} e^{-\frac{i\lambda(d_a - d_b)}{2}} f_{u_a u_b}(\lambda), \tag{7}$$

so that $g_{ab}(\lambda) \in C_\infty[0, 2\pi]$. Using (6) and (7) we have

$$\gamma_{ab}(k) = \int_0^{2\pi} e^{i\lambda k}(1-e^{i\lambda})^{-d_a} f_{u_a u_b}(\lambda)(1-e^{-i\lambda})^{-d_b} d\lambda \tag{8}$$

$$= e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^{2\pi} e^{i\lambda k} \lambda^{-d_a-d_b}(2\pi-\lambda)^{-d_a-d_b} g_{ab}(\lambda)\, d\lambda. \tag{9}$$

Transforming variates in (9) gives

$$\gamma_{ab}(k) = \frac{e^{\frac{i\pi(d_a-d_b)}{2}}}{k^{1-d_a-d_b}} \int_0^{2\pi k} e^{iw} w^{-d_a-d_b}\left(2\pi - \frac{w}{k}\right)^{-d_a-d_b} g_{ab}\left(\frac{w}{k}\right) dw, \tag{10}$$

which in turn suggests that $\gamma_{ab}(k) = O\left(k^{-(1-d_a-d_b)}\right)$ but this reduction does not reveal the constant or the error magnitude in the asymptotic approximation as $k \to \infty$. The constant is particularly important when $a \neq b$ because, as pointed out in Phillips and Kim (2007) and as seen below in (12, 39, 40, 42), there is asymmetry in the covariance asymptotic expansion in this case and the asymmetry is governed by the value of the constant.

2.1 The exact formula in the pure $I(d)$ case

When $u_t \sim iid(0, \Sigma_u = (\sigma_{ab}))$, the acf $\gamma_{ab}(k)$ may be computed directly from the explicit representation of the components of X_t, viz.,

$$X_{at} = \sum_{j=0}^{\infty} \frac{(d_a)_j}{j!} u_{at-j}, \quad (d)_j = d(d+1)\ldots(d+j-1) = \frac{\Gamma(d+j)}{\Gamma(d)}.$$

Specifically, since $(d_b)_{j+k} = \Gamma(d_b+k)(d_b+k)_j / \Gamma(d_b)$ we have

$$\gamma_{ab}(k) = \sum_{j=0}^{\infty} \frac{(d_a)_j (d_b)_{j+k}}{j!(j+k)!}\sigma_{ab} = \frac{1}{\Gamma(k+1)} \sum_{j=0}^{\infty} \frac{(d_a)_j (d_b)_{j+k}}{j!(k+1)_j}\sigma_{ab}$$

$$= \frac{\Gamma(d_b+k)}{\Gamma(d_b)\Gamma(k+1)}\, {}_2F_1(d_a, d_b+k; k+1; 1)\sigma_{ab},$$

where ${}_2F_1(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j!(c)_j} z^j$ is a hypergeometric series. When $z = 1$ and $\mathrm{Re}(c-a-b) > 0$, the series has the closed form expression

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{11}$$

(e.g., Lebedev, p. 244). A simple demonstration of (11) which is useful in lectures follows from the formula for the beta integral by noting that

$$\frac{(b)_j}{(c)_j} = \frac{\Gamma(c)}{\Gamma(b)}\frac{\Gamma(b+j)}{\Gamma(c+j)} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b+j)\Gamma(c-b)}{\Gamma(c+j)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b+j-1}(1-t)^{c-b-1}\, dt,$$

so that

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j! (c)_j} &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \left(\sum_{j=0}^{\infty} \frac{(a)_j}{j!} t^j \right) t^{b-1} (1-t)^{c-b-1} \, dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1-a} \, dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},
\end{aligned}$$

giving (11) directly without recourse to integral tables such as Gradshteyn and Ryhzik (1965), which were employed in Hosking (1980) and Brockwell and Davis (1989).

It follows that the exact formula for the acf is

$$\begin{aligned}
\gamma_{ab}(k) &= \frac{\Gamma(d_b+k)}{\Gamma(d_b)\Gamma(k+1)} \; {}_2F_1(d_a, d_b+k; k+1; 1)\sigma_{ab} \\
&= \frac{\Gamma(d_b+k)}{\Gamma(d_b)\Gamma(k+1)} \frac{\Gamma(k+1)\Gamma(1-d_a-d_b)}{\Gamma(k+1-d_a)\Gamma(1-d_b)}\sigma_{ab} \\
&= \frac{\Gamma(1-d_a-d_b)\Gamma(d_b+k)}{\Gamma(k+1-d_a)\Gamma(1-d_b)\Gamma(d_b)}\sigma_{ab}.
\end{aligned}$$

Using the reflection formula $\Gamma(1-d)\Gamma(d) = \frac{\pi}{\sin(\pi d)}$ we deduce the alternate expression

$$\gamma_{ab}(k) = \frac{\Gamma(1-d_a-d_b)\Gamma(d_b+k)\sin(\pi d_b)}{\pi\Gamma(k+1-d_a)}\sigma_{ab}, \tag{12}$$

noting the asymmetry alluded to earlier. The large k asymptotic formula for ratios of gamma functions[1] (e.g., Erdélyi, 1953)

$$\frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{1}{k^{b-a}} \left\{ 1 + \frac{(a-b)(a+b-1)}{2k} + O\left(\frac{1}{k^2}\right) \right\} \tag{13}$$

then gives the following series expansion as $k \to \infty$

$$\gamma_{ab}(k) = \frac{\sigma_{ab}}{\pi}\Gamma(1-d_a-d_b)\sin(\pi d_b)\frac{\Gamma(k+d_b)}{\Gamma(k+1-d_a)}$$

$$= \frac{\sigma_{ab}\sin(\pi d_b)\Gamma(1-d_a-d_b)}{\pi k^{1-d_a-d_b}} \left\{ 1 + \frac{(d_a+d_b-1)(d_b-d_a)}{2k} + O\left(\frac{1}{k^2}\right) \right\} \tag{14}$$

From (12) and (14) it is clear that $\gamma_{ab}(k)$ is asymmetric and $\gamma_{ab}(k) \neq \gamma_{ba}(k)$ when $a \neq b$. Further, when $d_b = 0$, $X_{bt} - E(X_{bt}) = u_{bt}$, and so $\gamma_{ab}(k) = 0$ in view of the fact that the input process u_t is *iid* and X_t is a pure fractional process.

[1] A simple way of delivering (13) is to use Stirling's approximation $\Gamma(z) \sim (2\pi)^{1/2} z^{z-1} e^{-z} \left\{1 + O\left(z^{-1}\right)\right\}$, an asymptotic formula that is most easily obtained in lectures by using a Laplace approximation to the gamma function integral.

2.2 The Fourier integral approach

Erdélyi (1956) developed a method of finding asymptotic expansions of Fourier integrals with critical points such as (9) using neutralizer functions that smooth out the effects of a function in certain domains. Erdélyi's formulae were used in Lieberman and Phillips (2007) to extract a complete asymptotic series for the acf $\gamma_{aa}(k)$ in the scalar case. The same approach may be used here for the cross covariance function $\gamma_{ab}(k)$, although the resulting formula is more complex as will become apparent.

Let $F(w) \in C^{\infty}[a,b]$, and suppose α and β are not integers. Erdélyi's result implies that the integral

$$I(k) = \int_a^b e^{ikw}(w-a)^{\alpha-1}(b-w)^{\beta-1}F(w)\,dw \tag{15}$$

has the following complete asymptotic series representation as $k \to \infty$

$$I(k) = I_a(k) + I_b(k),$$

where

$$I_a(k) \sim \sum_{n=0}^{\infty} \frac{d^n}{da^n}\left\{(b-a)^{\beta-1}F(a)\right\}\frac{\Gamma(n+\alpha)}{n!k^{n+\alpha}}e^{\frac{\pi i}{2}(n+\alpha)+ika}, \tag{16}$$

and

$$I_b(k) \sim \sum_{n=0}^{\infty} \frac{d^n}{db^n}\left\{(b-a)^{\alpha-1}F(b)\right\}\frac{\Gamma(n+\beta)}{n!k^{n+\beta}}e^{\frac{\pi i}{2}(n-\beta)+ikb}. \tag{17}$$

To apply this formula in the present case we set $a=0$, $b=2\pi$, $\alpha=\beta=1-d_a-d_b$, and $F(w)=g_{ab}(w)$. Then

$$\gamma_{ab}(k) = e^{\frac{i\pi(d_a-d_b)}{2}}\int_0^{2\pi} e^{i\lambda k}\lambda^{-d_a-d_b}(2\pi-\lambda)^{-d_a-d_b}g_{ab}(\lambda)\,d\lambda = e^{\frac{i\pi(d_a-d_b)}{2}}I(k),$$

with

$$I(k) = I_0(k) + I_{2\pi}(k),$$

and

$$\begin{aligned} I_0(k) &= \sum_{n=0}^{\infty}\frac{d^n}{da^n}\left[(2\pi-a)^{-d_a-d_b}F(a)\right]_{a=0}\frac{\Gamma(n+1-d_a-d_b)}{n!k^{n+1-d_a-d_b}}e^{\frac{\pi i}{2}(n+1-d_a-d_b)}, \\ I_{2\pi}(k) &= \sum_{n=0}^{\infty}\frac{d^n}{db^n}\left[b^{-d_a-d_b}F(b)\right]_{b=2\pi}\frac{\Gamma(n+1-d_a-d_b)}{n!k^{n+1-d_a-d_b}}e^{\frac{\pi i}{2}(n-1+d_a+d_b)+ik2\pi}. \end{aligned}$$

Hence

$$\gamma_{ab}(k) = e^{\frac{i\pi(d_a-d_b)}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-d_a-d_b)}{n!k^{n+1-d_a-d_b}} \left\{ \frac{d^n}{da^n} \left[(2\pi - a)^{-d_a-d_b} F(a) \right]_{a=0} e^{\frac{\pi i}{2}(n+1-d_a-d_b)} + \frac{d^n}{db^n} \left[b^{-d_a-d_b} F(b) \right]_{b=2\pi} e^{\frac{\pi i}{2}(n-1+d_a+d_b)} \right\}, \quad (18)$$

giving a complete asymptotic series for the acf . As shown in the Appendix, some further simplifications lead to the following result.

Theorem *If* $d_a + d_b > 0$ *and* $g_{ab}(\lambda) \in C_\infty[0, 2\pi]$, *then* $\gamma_{ab}(k)$ *has the following complete asymptotic series expansion*

$$\gamma_{ab}(k) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1-d_a-d_b)}{n!k^{n+1-d_a-d_b}} \left\{ G_{ab}^{(n)}(0)\, e^{\frac{\pi i}{2}(n+1-2d_b)} + (-1)^n e^{\frac{\pi i}{2}(n-1+2d_b)} G_{ba}^{(n)}(0) \right\}, \quad (19)$$

where $G_{ab}(w) = (2\pi - w)^{-d_a-d_b} g_{ab}(w)$. *An alternative form for the expansion is*

$$\gamma_{ab}(k) = 2\sin\{\pi d_b\} \sum_{j=0}^{\infty} \frac{\Gamma(2j+1-d_a-d_b)(-1)^j}{(2j)!k^{2j+1-d_a-d_b}} \times \left\{ G_{ab}^{(2j)}(0) - \frac{(2j+1-d_a-d_b)\, G_{ab}^{(2j+1)}(0)}{(2j+1)\, ki} \right\}. \quad (20)$$

Remarks

(i) As shown in the Appendix, setting $n = 0$ in (19), the first term in the series reduces to the simple form

$$\gamma_{ab}(k) = \frac{2f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\sin(\pi d_b)}{k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right), \quad (21)$$

as given in Phillips and Kim (2007).

(ii) When $d_b = 0$, it follows from (20) that $\gamma_{ab}(k)$ is zero up to terms that are exponentially small as $k \to \infty$. In this case u_{bt} has only short range dependence and so the covariance of u_{bt+k} with u_{at} is exponentially small as $k \to \infty$. Thus, the asymptotic expansion (20) fully reflects the asymmetry involved in the acf between long memory X_{at} and short memory X_{bt}.

(iii) As shown in the Appendix, the first two terms in the expansion (19) have the explicit form

$$\frac{\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \left\{ \omega_{ab} + \frac{(1-d_a-d_b)\left(\frac{(d_a-d_b)}{2}\omega_{ab} + \omega_{ab}^{(1)}\right)}{2k} \right\}, \quad (22)$$

where $\omega_{ab} = 2\pi f_{u_a u_b}(0)$ and $\omega_{ab}^{(1)} = \sum_{j=-\infty}^{\infty} \gamma_{ab}(j)\, j$. In the pure $I(d)$ case, the inputs u_t are $iid(0, \Sigma = (\sigma_{ab}))$, and we have $\omega_{ab}^{(1)} = 0$ because $\gamma_{ab}(j) = 0$ for all $j \neq 0$, and $\omega_{ab} = \sigma_{ab}$. In that event we get the simplified formula

$$\frac{\sigma_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}}\left\{1+\frac{(d_a+d_b-1)(d_b-d_a)}{2k}\right\}, \tag{23}$$

corresponding to the result (14) obtained directly in that case from the exact formula for the acf. Thus, the first correction term in the expansion is of order $O\left(k^{-2+d_a+d_b}\right)$ and depends on the parameters ω_{ab} and $\omega_{ab}^{(1)}$ in the general case (22). In the scalar case we have $a=b$, $d_a = d_b$ and $\omega_{aa}^{(1)} = \sum_{j=-\infty}^{\infty} \gamma_{aa}(j)\, j = 0$, so that the second term vanishes. Hence, the first correction term in the expansion in the scalar time series case is of $O\left(k^{-3+2d_a}\right)$ – c.f. Lieberman and Phillips (2007).

(iv) When $a=b$, the odd derivatives $G_{ab}^{(2j+1)}(0)$ vanish and we have from (20) the special case

$$\gamma_{aa}(k) = 2\sin\{\pi d_a\}\sum_{j=0}^{\infty}\frac{\Gamma(2j+1-2d_a)(-1)^j G_{aa}^{(2j)}(0)}{(2j)!k^{2j+1-2d_a}},$$

considered in Lieberman and Phillips (2007).

(v) The result (19) continues to hold for $d_a+d_b<0$ by way of analytic continuation. To show this, suppose $d_a = -J_a + d_a'$, $d_b = -J_b + d_b'$ for positive integers J_a and J_b and with $0 < d_a' + d_b' < 1$ and $d_a', d_b' \in (0, 1/2)$. Then, $d_a + d_b = -J + d_a' + d_b'$ where $J = J_a + J_b$ is a positive integer. We can now write (9) as

$$\gamma_{ab}(k) = e^{\frac{i\pi(d_a-d_b)}{2}}\int_0^{2\pi} e^{i\lambda k}\lambda^{-d_a-d_b}(2\pi-\lambda)^{-d_a-d_b} g_{ab}(\lambda)\, d\lambda$$

$$= e^{\frac{i\pi(J_b-J_a)}{2}} e^{\frac{i\pi(d_a'-d_b')}{2}}\int_0^{2\pi} e^{i\lambda k}\lambda^{-d_a'-d_b'}(2\pi-\lambda)^{-d_a'-d_b'} h_{ab}(\lambda)\, d\lambda \tag{24}$$

where $h_{ab}(\lambda) = g_{ab}(\lambda)\lambda^J(2\pi-\lambda)^J \in C_\infty[0,2\pi]$. Applying (19) to (24), we have

$$\begin{aligned}\gamma_{ab}(k) &= e^{\frac{i\pi(J_b-J_a)}{2}} e^{\frac{i\pi(d_a'-d_b')}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-d_a'-d_b')}{n!k^{n+1-d_a'-d_b'}}\Big\{H_{ab}^{(n)}(0)\, e^{\frac{\pi i}{2}\left(n+1-d_a'-d_b'\right)} \\ &\quad + (-1)^n e^{-i\pi(d_a'-d_b')} e^{\frac{\pi i}{2}\left(n-1+d_a'+d_b'\right)} H_{ba}^{(n)}(0)\Big\},\end{aligned}$$

where

$$H_{ab}(\lambda) = (2\pi-\lambda)^{-d_a'-d_b'} h_{ab}(\lambda) = g_{ab}(\lambda)\lambda^J(2\pi-\lambda)^{J-d_a'-d_b'} = \lambda^J G_{ab}(\lambda) \tag{25}$$

since $G_{ab}(\lambda) = (2\pi - \lambda)^{-d_a - d_b} g_{ab}(\lambda) = (2\pi - \lambda)^{J - d'_a - d'_b} g_{ab}(\lambda)$. Note that

$$H_{ab}^{(n)}(0) = \begin{cases} 0 & n < J \\ J! G_{ab}^{(n-J)}(0) & n \geq J \end{cases},$$

and hence

$$\begin{aligned}
\gamma_{ab}(k) &= e^{\frac{i\pi(d_a - d_b)}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-d'_a-d'_b)}{n! k^{n+1-d'_a-d'_b}} \Big\{ H_{ab}^{(n)}(0) e^{\frac{\pi i}{2}\left(n+1-d'_a-d'_b\right)} \\
&\quad + (-1)^n e^{-i\pi(d'_a - d'_b)} e^{\frac{\pi i}{2}\left(n-1+d'_a+d'_b\right)} H_{ba}^{(n)}(0) \Big\} \\
&= e^{\frac{i\pi(d_a - d_b)}{2}} \sum_{n=J}^{\infty} \frac{\Gamma(n-J+1-d_a-d_b)}{n! k^{n-J+1-d_a-d_b}} \Big\{ H_{ab}^{(n)}(0) e^{\frac{\pi i}{2}(n-J+1-d_a-d_b)} \\
&\quad + (-1)^{n+(J_a - J_b)} e^{-i\pi(d_a - d_b)} e^{\frac{\pi i}{2}(n-J-1+d_a+d_b)} H_{ba}^{(n)}(0) \Big\} \\
&= e^{\frac{i\pi(d_a - d_b)}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-d_a-d_b)}{(m+J)! k^{m+1-d_a-d_b}} \Big\{ H_{ab}^{(J+m)}(0) e^{\frac{\pi i}{2}(m+1-d_a-d_b)} \\
&\quad + (-1)^{m+2J_a} e^{-i\pi(d_a - d_b)} e^{\frac{\pi i}{2}(m-1+d_a+d_b)} H_{ba}^{(J+m)}(0) \Big\}
\end{aligned}$$

$$\begin{aligned}
&= e^{\frac{i\pi(d_a - d_b)}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-d_a-d_b)}{(m+J)! k^{m+1-d_a-d_b}} \Big\{ H_{ab}^{(J+m)}(0) e^{\frac{\pi i}{2}(m+1-d_a-d_b)} \\
&\quad + (-1)^m e^{-i\pi(d_a - d_b)} e^{\frac{\pi i}{2}(m-1+d_a+d_b)} H_{ba}^{(J+m)}(0) \Big\} \\
&= \sum_{m=0}^{\infty} \frac{\Gamma(m+1-d_a-d_b)}{(m+J)! k^{m+1-d_a-d_b}} \Big\{ H_{ab}^{(J+m)}(0) e^{\frac{\pi i}{2}(m+1-2d_b)} \\
&\quad + (-1)^m e^{\frac{\pi i}{2}(m-1+2d_b)} H_{ba}^{(J+m)}(0) \Big\}.
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{\Gamma(m+1-d_a-d_b)}{k^{m+1-d_a-d_b}} \Big\{ G_{ab}^{(m)}(0) e^{\frac{\pi i}{2}(m+1-2d_b)} \\
&\quad + (-1)^m e^{\frac{\pi i}{2}(m-1+2d_b)} G_{ba}^{(m)}(0) \Big\},
\end{aligned}$$

which is (19). Observe that the leading term in the expansion is

$$\begin{aligned}
\gamma_{ab}(k) &= \frac{\Gamma(1-d_a-d_b)}{J! k^{1-d_a-d_b}} \Big\{ G_{ab}(0) e^{\frac{\pi i}{2}(1-2d_b)} \\
&\quad + e^{\frac{\pi i}{2}(-1+2d_b)} G_{ba}(0) \Big\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\Gamma(1-d_a-d_b) G_{ab}(0) 2\cos\left\{\frac{\pi}{2}(1-2d_b)\right\}}{k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\Gamma(1-d_a-d_b)\sin\{\pi d_b\}\omega_{ab}}{k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right),
\end{aligned}$$

since $G_{ab}(\lambda) = (2\pi - \lambda)^{-d_a-d_b} g_{ab}(\lambda)$, $g_{ab}(0) = (2\pi)^{d_a+d_b} f_{u_a u_b}(0)$, and therefore $G_{ab}(0) = f_{u_a u_b}(0) = (2\pi)^{-1} \omega_{ab}$. It follows that

$$\begin{aligned} \gamma_{ab}(k) &= \frac{\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \omega_{ab} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\ &= \frac{\Gamma(1-d_a-d_b)\sin\{\pi d_b\}\,\omega_{ab}}{k^{J+1-d_a'-d_b'}} + O\left(\frac{1}{k^{J+2-d_a'-d_b'}}\right), \end{aligned}$$

showing that the acf still follows power law decay but at a faster rate that takes into account the antipersistence.

2.3 Fourier transforms of generalized functions

This approach is based on the theory of asymptotic expansions of Fourier transforms for generalized functions. Lighthill (1959) developed the theory in chapter 4 of his classic monograph and used this theory in chapter 5 to obtain a corresponding development for the coefficients of Fourier series. Lighthill's treatment of asymptotic expansions for Fourier integrals and Fourier coefficients is masterly and seems to be rather neglected in statistical asymptotic theory. Some of the theory was used in Phillips (1985) to obtain tail probability expansions but, to the author's knowledge, no other uses of this theory of asymptotic expansions of generalized functions have appeared in econometrics or statistics. As shown below, the theory is sufficiently powerful to deliver the complete asymptotic series expansion for the autocovariances given in Theorem 1 by immediate use of the generalized function expansion formulae.

We start from (4), which we write as

$$f_{x_a x_b}(\lambda) = \begin{cases} e^{\frac{i\pi(d_a-d_b)}{2}} |\lambda|^{-d_a-d_b} \left|\frac{2\sin(\lambda/2)}{\lambda}\right|^{-d_a-d_b} e^{-\frac{i\lambda(d_a-d_b)}{2}} f_{u_a u_b}(\lambda) & 0 < \lambda < \pi \\ e^{-\frac{i\pi(d_a-d_b)}{2}} |\lambda|^{-d_a-d_b} \left|\frac{2\sin(\lambda/2)}{\lambda}\right|^{-d_a-d_b} e^{\frac{i|\lambda|(d_a-d_b)}{2}} f_{u_a u_b}(\lambda) & -\pi \le \lambda < 0 \end{cases},$$

and by Fourier inversion we have

$$\begin{aligned} \gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^\pi e^{i\lambda k} \lambda^{-d_a-d_b} \left(\frac{2\sin(\lambda/2)}{\lambda}\right)^{-d_a-d_b} e^{-\frac{i\lambda(d_a-d_b)}{2}} f_{u_a u_b}(\lambda)\, d\lambda \\ &\quad + e^{-\frac{i\pi(d_a-d_b)}{2}} \int_0^\pi e^{-i\lambda k} |\lambda|^{-d_a-d_b} \left|\frac{2\sin(\lambda/2)}{\lambda}\right|^{-d_a-d_b} e^{\frac{i|\lambda|(d_a-d_b)}{2}} f_{u_a u_b}(-\lambda)\, d\lambda. \end{aligned}$$

Define

$$\begin{aligned} G_{ab}(w) &= \left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} e^{-\frac{iw(d_a-d_b)}{2}} f_{u_a u_b}(w), \\ H_{ab}(w) &= \left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} e^{\frac{iw(d_a-d_b)}{2}} f_{u_a u_b}(-w). \end{aligned}$$

Then even and odd derivatives satisfy $G_{ab}^{(2m)}(0) = H_{ab}^{(2m)}(0)$ and $G_{ab}^{(2m+1)}(0) = -H_{ab}^{(2m+1)}(0)$, so that

$$\begin{aligned}\gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}}\sum_{j=0}^{\infty}\frac{G_{ab}^{(j)}(0)}{j!}\int_0^{\pi}e^{i\lambda k}\lambda^{j-d_a-d_b}d\lambda \\ &\quad +e^{-\frac{i\pi(d_a-d_b)}{2}}\sum_{j=0}^{\infty}\frac{H_{ab}^{(j)}(0)}{j!}\int_0^{\pi}e^{-i\lambda k}\lambda^{j-d_a-d_b}d\lambda \qquad (26)\\ &= e^{\frac{i\pi(d_a-d_b)}{2}}\sum_{j=0}^{\infty}\frac{G_{ab}^{(j)}(0)}{j!}\int_{-\pi}^{\pi}e^{i\lambda k}|\lambda|^{j-d_a-d_b}h(\lambda)d\lambda \\ &\quad +e^{-\frac{i\pi(d_a-d_b)}{2}}\sum_{j=0}^{\infty}\frac{H_{ab}^{(j)}(0)}{j!}\int_{-\pi}^{\pi}e^{-i\lambda k}|\lambda|^{j-d_a-d_b}h(\lambda)d\lambda, \qquad (27)\end{aligned}$$

where $h(\lambda) = 1$, for $\lambda > 0$ and 0 for $\lambda < 0$. Using Lighthill (1959, equation (13) p. 33)[2] for the Fourier transform of $|\lambda|^{j-d_a-d_b} h(\lambda)$ (treated as a generalized function) we have directly that

$$\int_{-\infty}^{\infty}e^{-i\lambda k}|\lambda|^{j-d_a-d_b}h(\lambda)d\lambda = e^{-\frac{\pi}{2}i(j+1-d_a-d_b)}\frac{\Gamma(j+1-d_a-d_b)}{k^{j+1-d_a-d_b}}. \qquad (28)$$

Further, as shown in Lighthill (1959, theorem 30, p. 72), the same formula (28) provides an asymptotic representation of the Fourier coefficients of a periodic function (treated as a generalized function). The error on this asympotic representation is exponential small in the present case in view of the condition that $g_{ab}(\lambda) \in C_\infty[0, 2\pi]$ in Theorem 1. We may therefore deduce from (27) and (28) the complete asymptotic series

$$\begin{aligned}\gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}}\sum_{j=0}^{\infty}\frac{G_{ab}^{(j)}(0)\,e^{\frac{\pi}{2}i(j+1-d_a-d_b)}}{j!}\frac{\Gamma(j+1-d_a-d_b)}{k^{j+1-d_a-d_b}} \\ &\quad +e^{-\frac{i\pi(d_a-d_b)}{2}}\sum_{j=0}^{\infty}\frac{H_{ab}^{(j)}(0)\,e^{-\frac{\pi}{2}i(j+1-d_a-d_b)}}{j!}\frac{\Gamma(j+1-d_a-d_b)}{k^{j+1-d_a-d_b}} \\ &= e^{-i\pi d_b}\sum_{j=0}^{\infty}\frac{G_{ab}^{(j)}(0)(i)^{j+1}}{j!}\frac{\Gamma(j+1-d_a-d_b)}{k^{j+1-d_a-d_b}} \\ &\quad +e^{i\pi d_b}\sum_{j=0}^{\infty}\frac{H_{ab}^{(j)}(0)(-i)^{j+1}}{j!}\frac{\Gamma(j+1-d_a-d_b)}{k^{j+1-d_a-d_b}} \qquad (29)\end{aligned}$$

[2] Note that since we assume $k > 0$ in our (28), Lighthill's $sgn(y) = 1$ in his equation (13). Note also that Lighthill defines a Fourier transform using $e^{-2\pi i\lambda}$ in place of our $e^{-\iota\pi\lambda}$.

Using $G_{ab}^{(2m)}(0) = H_{ab}^{(2m)}(0)$ and $G_{ab}^{(2m+1)}(0) = -H_{ab}^{(2m+1)}(0)$, we get

$$\begin{aligned}
\gamma_{ab}(k) &= \sum_{j=0}^{\infty} \frac{G_{ab}^{(j)}(0)(i)^{j+1}}{j!} \frac{\Gamma(j+1-d_a-d_b)}{k^{j+1-d_a-d_b}} \left\{ e^{-i\pi d_b} + (-1)^{j+1} e^{i\pi d_b} \right\} \\
&= \sum_{m=0,j=2m}^{\infty} \frac{G_{ab}^{(2m)}(0)(-1)^m}{(2m)!} \frac{\Gamma(2m+1-d_a-d_b)}{k^{2m+1-d_a-d_b}} \frac{1}{-i} \left\{ e^{-i\pi d_b} - e^{i\pi d_b} \right\} \\
&\quad + \sum_{m=0,j=2m+1}^{\infty} \frac{G_{ab}^{(2m+1)}(0)(-1)^{m+1}}{(2m+1)!} \frac{\Gamma(2m+2-d_a-d_b)}{k^{2m+2-d_a-d_b}} \left\{ e^{-i\pi d_b} - e^{i\pi d_b} \right\} \\
&= 2\sin(\pi d_b) \sum_{m=0,j=2m}^{\infty} \frac{G_{ab}^{(2m)}(0)(-1)^m}{(2m)!} \frac{\Gamma(2m+1-d_a-d_b)}{k^{2m+1-d_a-d_b}} \\
&\quad +2\sin(\pi d_b) \sum_{m=0,j=2m+1}^{\infty} \frac{iG_{ab}^{(2m+1)}(0)(-1)^{m+1}}{(2m+1)!} \frac{\Gamma(2m+2-d_a-d_b)}{k^{2m+2-d_a-d_b}} \\
&= 2\sin\{\pi d_b\} \sum_{j=0}^{\infty} \frac{\Gamma(2j+1-d_a-d_b)(-1)^j}{(2j)!k^{2j+1-d_a-d_b}} \\
&\quad \times \left\{ G_{ab}^{(2j)}(0) - \frac{(2j+1-d_a-d_b)\, G_{ab}^{(2j+1)}(0)}{(2j+1)\, ki} \right\},
\end{aligned}$$

which corresponds with the earlier expansion (20) obtain by conventional Fourier expansion methods.

Some further remarks on the approach may be helpful. The asymptotic series (29) effectively makes use of the replacement

$$\int_{-\pi}^{\pi} e^{i\lambda k} |\lambda|^{j-d_a-d_b} h(\lambda)\, d\lambda = \int_{-\infty}^{\infty} e^{i\lambda k} |\lambda|^{j-d_a-d_b} h(\lambda)\, d\lambda \tag{30}$$

$$= i^{j+1} e^{-\frac{\pi}{2} i(d_a+d_b)} \frac{\Gamma(j+1-d_a-d_b)}{k^{j+1-d_a-d_b}} \tag{31}$$

which is justified in Lighthill (1959, theorem 30). Observe that the integral on the left side of (30) may be defined as an ordinary function, while the integral on the right side is only defined as a generalized function. Lighthill's proof uses the Fourier transform for nonperiodic (generalized) functions (as given in (31) above) by noting that the generalized function corresponding to $|\lambda|^{\alpha}$ outside of $[-\pi, \pi]$, which we may consider as $|\mu|^{\alpha}$ outside the interval $(-k\pi, k\pi)$, is well behaved at infinity because it is effectively zero outside this interval. Then the function has any number of derivatives after removing $|\lambda|^{\alpha}$ and the error of the Fourier transform is exponentially small.

2.4 A gamma function representation

As a final approach, we introduce a simple method that uses little more than the gamma function to extract the leading term. It is a convenient method for teaching purposes, avoids the use of integral tables, and involves only elementary methods.

From (4), (8) and (26) we have

$$\begin{aligned}
\gamma_{ab}(k) &= \int_{-\pi}^{\pi} e^{i\lambda k}(1-e^{i\lambda})^{-d_a} f_{u_a u_b}(\lambda)(1-e^{-i\lambda})^{-d_b} d\lambda \\
&= e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{i\lambda k} |\lambda|^{-d_a-d_b} \left|\frac{2\sin(\lambda/2)}{\lambda}\right|^{-d_a-d_b} e^{-\frac{i\lambda(d_a-d_b)}{2}} f_{u_a u_b}(\lambda)\, d\lambda \\
&\quad + e^{-\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{-i\lambda k} |\lambda|^{-d_a-d_b} \left|\frac{2\sin(\lambda/2)}{\lambda}\right|^{-d_a-d_b} e^{\frac{i\lambda(d_a-d_b)}{2}} f_{u_a u_b}(-\lambda)\, d\lambda \\
&= e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{i\lambda k} \lambda^{j-d_a-d_b} G_{ab}(\lambda)\, d\lambda + e^{-\frac{i\pi(d_a-d_b)}{2}} \int_0^{\pi} e^{-i\lambda k} \lambda^{j-d_a-d_b} H_{ab}(\lambda)\, d\lambda \\
&= e^{\frac{i\pi(d_a-d_b)}{2}} \sum_{j=0}^{\infty} \frac{G_{ab}^{(j)}(0)}{j!} \int_0^{\pi} e^{i\lambda k} \lambda^{j-d_a-d_b} d\lambda \\
&\quad + e^{-\frac{i\pi(d_a-d_b)}{2}} \sum_{j=0}^{\infty} \frac{H_{ab}^{(j)}(0)}{j!} \int_0^{\pi} e^{-i\lambda k} \lambda^{j-d_a-d_b} d\lambda.
\end{aligned} \tag{32}$$

Using $G_{ab}(0) = H_{ab}(0) = f_{u_a u_b}(0)$ and variate transformation, (32) becomes

$$\begin{aligned}
& e^{\frac{i\pi(d_a-d_b)}{2}} \sum_{j=0}^{\infty} \frac{G_{ab}^{(j)}(0)}{j! k^{j+1-d_a-d_b}} \int_0^{k\pi} e^{iw} w^{j-d_a-d_b} dw \\
& + e^{-\frac{i\pi(d_a-d_b)}{2}} \sum_{j=0}^{\infty} \frac{H_{ab}^{(j)}(0)(-1)^j}{j! k^{j+1-d_a-d_b}} \int_0^{k\pi} e^{-iw} w^{j-d_a-d_b} dw \\
=\; & \frac{f_{u_a u_b}(0)}{k^{1-d_a-d_b}} \left\{ e^{\frac{i\pi(d_a-d_b)}{2}} \int_0^{k\pi} e^{iw} w^{1-d_a-d_b} dw + e^{-\frac{i\pi(d_a-d_b)}{2}} \int_0^{k\pi} e^{-iw} w^{1-d_a-d_b} dw \right\} \\
& + O\left(\frac{1}{k^{2-d_a-d_b}}\right).
\end{aligned} \tag{33}$$

For $s > 0$ the gamma integral gives

$$\int_0^{\infty} e^{-w(s-i)} w^{1-d_a-d_b} dw = \frac{\Gamma(1-d_a-d_b)}{(s-i)^{1-d_a-d_b}},$$

and then

$$\begin{aligned}
\int_0^{\infty} e^{iw} w^{1-d_a-d_b} dw &= \lim_{s\to 0} \int_0^{\infty} e^{-w(s-i)} w^{1-d_a-d_b} d\lambda = \frac{\Gamma(1-d_a-d_b)}{(-i)^{1-d_a-d_b}} \\
&= \Gamma(1-d_a-d_b)\, e^{\frac{\pi i}{2}(1-d_a-d_b)},
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
\int_0^{\infty} e^{-iw} w^{1-d_a-d_b} dw &= \lim_{s\to 0} \int_0^{\infty} e^{-w(s+i)} w^{1-d_a-d_b} d\lambda = \frac{\Gamma(1-d_a-d_b)}{(i)^{1-d_a-d_b}} \\
&= \Gamma(1-d_a-d_b)\, e^{-\frac{\pi i}{2}(1-d_a-d_b)}.
\end{aligned} \tag{35}$$

The error magnitude in the use of these gamma integrals in (33) can be determined by partial integration (or more precisely by using an asymptotic expansion of the incomplete gamma function) as follows:

$$\begin{aligned}
& e^{\frac{i\pi(d_a-d_b)}{2}}\int_{k\pi}^{\infty} e^{iw}w^{1-d_a-d_b}dw + e^{-\frac{i\pi(d_a-d_b)}{2}}\int_{k\pi}^{\infty} e^{-iw}w^{1-d_a-d_b}dw \\
= & \; 2\int_{k\pi}^{\infty}\cos\left\{\frac{\pi(d_a-d_b)}{2}+w\right\}w^{1-d_a-d_b}dw \\
= & \; 2\left[\left\{\sin\left(\frac{\pi(d_a-d_b)}{2}+w\right)\right\}w^{1-d_a-d_b}\right]_{k\pi}^{\infty} \\
& -(1-d_a-d_b)\int_{k\pi}^{\infty}\left\{\sin\left(\frac{\pi(d_a-d_b)}{2}+w\right)\right\}w^{-d_a-d_b}dw \\
= & \; O\left(\frac{1}{k^{1-d_a-d_b}}\right),
\end{aligned}$$

which leads to an error of smaller order than the leading term upon use of (34) and (35) in (33).

By combining (33) - (35) we have to the first order

$$\begin{aligned}
\gamma_{ab}(k) = & \; \frac{f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}}\left\{e^{\frac{i\pi(d_a-d_b)}{2}}e^{\frac{\pi i}{2}(1-d_a-d_b)} + e^{-\frac{i\pi(d_a-d_b)}{2}}e^{-\frac{\pi i}{2}(1-d_a-d_b)}\right\} \\
& + o\left(\frac{1}{k^{1-d_a-d_b}}\right) \\
= & \; \frac{f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}}\left\{ie^{-i\pi d_b} - ie^{i\pi d_b}\right\} + o\left(\frac{1}{k^{1-d_a-d_b}}\right) \\
= & \; \frac{f_{u_a u_b}(0)\,\Gamma(1-d_a-d_b)\,2\sin(\pi d_b)}{k^{1-d_a-d_b}} + o\left(\frac{1}{k^{1-d_a-d_b}}\right), \qquad (36)
\end{aligned}$$

delivering the leading term of (20) direct methods. The argument may be refined to produce the complete asymptotic series (20).

3. Concluding Remarks

The generalized function approach to developing complete asymptotic series is modern and elegant in its use of applied functional analysis. The applications in the present paper show the power and simplicity of that approach in developing useful expansions for the autocovariance function of long memory time series. On the other hand, the proofs do require some working familiarity with generalized function theory and ideas beyond those which might be expected in normal classroom discussion. A similar comment applies to the approach based on expanding Fourier integrals with singularities, where a deeper understanding of asymptotic methods around singularities is required. In this respect, the gamma function approach seems relatively less demanding and is well suited to the classroom, at least in finding the leading term in the expansion.

The expansion formulae given here are useful in many ways including the calculation of long run variation matrices for multivariate time series with long memory and the econometric estimation of such models. Some applications along these lines are given in Lieberman and Phillips (2007), Phillips and Kim (2007) and Shimotsu (2007).

4. Appendix

Proof of the Theorem

We start by proving (19). From (18) we have

$$\begin{aligned}\gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-d_a-d_b)}{n!k^{n+1-d_a-d_b}}\left\{\frac{d^n}{da^n}\left[(2\pi-a)^{-d_a-d_b}F(a)\right]_{a=0}e^{\frac{\pi i}{2}(n+1-d_a-d_b)}\right.\\ &\quad\left.+\frac{d^n}{db^n}\left[b^{-d_a-d_b}F(b)\right]_{b=2\pi}e^{\frac{\pi i}{2}(n-1+d_a+d_b)}\right\}. \qquad (37)\end{aligned}$$

Observe that

$$\frac{d^j}{db^j}\left[b^{-d_a-d_b}\right]_{b=2\pi}=(-1)^j\frac{d^j}{da^j}\left[(2\pi-a)^{-d_a-d_b}\right]_{a=0},\quad (j=0,1,2,...)$$

and

$$\begin{aligned}F(2\pi-w) &= g_{ab}(2\pi-w)=\left(\frac{2\sin(w/2)}{w(2\pi-w)}\right)^{-d_a-d_b}e^{-\frac{i(2\pi-w)(d_a-d_b)}{2}}f_{u_au_b}(2\pi-w)\\ &= e^{-i\pi(d_a-d_b)}\left(\frac{2\sin(w/2)}{w(2\pi-w)}\right)^{-d_a-d_b}e^{\frac{iw(d_a-d_b)}{2}}f_{u_au_b}(2\pi-w)\\ &= e^{-i\pi(d_a-d_b)}\left(\frac{2\sin(w/2)}{w(2\pi-w)}\right)^{-d_a-d_b}e^{\frac{iw(d_a-d_b)}{2}}f_{u_au_b}(-w)\\ &= e^{-i\pi(d_a-d_b)}g_{ba}(w),\end{aligned}$$

in view of the fact that $f_{u_au_b}(2\pi-w)=f_{u_au_b}(-w)=f_{u_bu_a}(w)$ and

$$g_{ba}(\lambda)=\left(\frac{2\sin(\lambda/2)}{\lambda(2\pi-\lambda)}\right)^{-d_a-d_b}e^{\frac{i\lambda(d_a-d_b)}{2}}f_{u_bu_a}(\lambda)=\left(\frac{2\sin(\lambda/2)}{\lambda(2\pi-\lambda)}\right)^{-d_a-d_b}e^{\frac{i\lambda(d_a-d_b)}{2}}f_{u_au_b}(-\lambda).$$

Further, note that

$$g_{ab}^{(m)}(2\pi)=(-1)^m e^{-i\pi(d_a-d_b)}g_{ba}^{(m)}(0),(m=0,1,2,...),$$

and thus

$$\begin{aligned}
\frac{d^n}{db^n}\left[b^{-d_a-d_b}g_{ab}(b)\right]_{b=2\pi} &= \left[\left\{b^{-d_a-d_b}+g_{ab}(b)\right\}^{(n)}\right]_{b=2\pi} \\
&= \sum_{j=0}^{n}\binom{n}{j}\left(\frac{d^j}{db^j}\left[b^{-d_a-d_b}\right]_{b=2\pi}\right)\left(g_{ab}^{(n-j)}(2\pi)\right) \\
&= \sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{d^j}{da^j}\left[(2\pi-a)^{-d_a-d_b}\right]_{a=0}(-1)^{n-j}e^{-i\pi(d_a-d_b)}g_{ba}^{(n-j)}(0) \\
&= (-1)^n e^{-i\pi(d_a-d_b)}\frac{d^n}{da^n}\left[(2\pi-a)^{-d_a-d_b}g_{ba}(a)\right]_{a=0}.
\end{aligned}$$

Letting $G_{ba}(w)=(2\pi-w)^{-d_a-d_b}g_{ba}(w)$, equation (37) then becomes

$$\begin{aligned}
\gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-d_a-d_b)}{n!k^{n+1-d_a-d_b}}\left\{G_{ab}^{(n)}(0)e^{\frac{\pi i}{2}(n+1-d_a-d_b)}\right. \\
&\quad \left.+(-1)^n e^{-i\pi(d_a-d_b)}e^{\frac{\pi i}{2}(n-1+d_a+d_b)}G_{ba}^{(n)}(0)\right\}, \qquad (38)
\end{aligned}$$

giving the stated formula (19).

Next we prove the simplified formula (20). Note that

$$G_{ab}(w)=(2\pi-w)^{-d_a-d_b}g_{ab}(w)=\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b}e^{-\frac{iw(d_a-d_b)}{2}}f_{u_au_b}(w),$$

so that

$$\begin{aligned}
G_{ba}(w) &= \left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b}e^{-\frac{iw(d_b-d_a)}{2}}f_{u_bu_a}(w) \\
&= \left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b}e^{\frac{iw(d_a-d_b)}{2}}f_{u_au_b}(-w)=G_{ab}(-w),
\end{aligned}$$

and then $G_{ba}^{(2j)}(0)=G_{ab}^{(2j)}(0)$ and $G_{ba}^{(2j+1)}(0)=-G_{ab}^{(2j+1)}(0)$. Observe that when $a=b$ terms involving the odd derivatives drop out as $G_{aa}^{(2j+1)}(0)=-G_{aa}^{(2j+1)}(0)=0$, leading to the formula for the scalar case discussed in Remark A (iv).

Continuing in the general case and working from (38) above, we have

$$\begin{aligned}
\gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-d_a-d_b)}{n!k^{n+1-d_a-d_b}}\left\{G_{ab}^{(n)}(0)e^{\frac{\pi i}{2}(n+1-d_a-d_b)}\right. \\
&\quad \left.+(-1)^n e^{-i\pi(d_a-d_b)}e^{\frac{\pi i}{2}(n-1+d_a+d_b)}G_{ba}^{(n)}(0)\right\} \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(n+1-d_a-d_b)\,i^n}{n!k^{n+1-d_a-d_b}}\left\{G_{ab}^{(n)}(0)e^{\frac{\pi i}{2}(1-2d_b)}+(-1)^n e^{-\frac{\pi i}{2}(1-2d_b)}G_{ba}^{(n)}(0)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \frac{\Gamma(2j+1-d_a-d_b)(-1)^j G_{ab}^{(2j)}(0)}{(2j)! k^{2j+1-d_a-d_b}} 2\sin\{\pi d_b\} \\
&\quad + \sum_{j=0}^{\infty} \frac{\Gamma(2j+2-d_a-d_b)(-1)^{j+1} G_{ab}^{(2j+1)}(0)}{(2j+1)! k^{2j+2-d_a-d_b} i} 2\sin\{\pi d_b\} \\
&= 2\sin\{\pi d_b\} \sum_{j=0}^{\infty} \frac{\Gamma(2j+1-d_a-d_b)(-1)^j}{(2j)! k^{2j+1-d_a-d_b}} \\
&\quad \times \left\{ G_{ab}^{(2j)}(0) - \frac{(2j+1-d_a-d_b) G_{ab}^{(2j+1)}(0)}{(2j+1) ki} \right\}, \qquad (39)
\end{aligned}
$$

giving formula (20). The quantity $G_{ab}^{(2j+1)}(0)/i$ in the second term within the brace of (39) is real. To see this, observe that

$$
G_{ab}(w) = (2\pi - w)^{-d_a-d_b} g_{ab}(w) = \left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} e^{-\frac{iw(d_a-d_b)}{2}} f_{u_a u_b}(w),
$$

and odd derivatives of $\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b}$ at the origin are zero because the function is symmetric about zero. So odd derivatives of $G_{ab}(w)$ at $w = 0$ involve only even derivatives of $\left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b}$ and odd derivatives of $e^{-\frac{iw(d_a-d_b)}{2}} f_{u_a u_b}(w)$. Now odd derivatives of $e^{-\frac{iw(d_a-d_b)}{2}}$ at the origin involve odd powers of $\frac{i(d_a-d_b)}{2}$ and therefore involve a factor of i, whereas even derivatives of $f_{u_a u_b}(w)$ at the origin are real. Similarly, odd derivatives of $f_{u_a u_b}(w)$ at the origin involve odd powers of i and therefore a factor of i. Hence odd derivatives of $G_{ab}(w)$ at $w = 0$ involve a factor of i, so they are purely imaginary and $G_{ab}^{(2j+1)}(0)/i$ is real. It follows that the coefficients in (39) are all real.

First and Second Terms in the Expansion

Since $g_{ab}(0) = (2\pi)^{d_a+d_b} f_{u_a u_b}(0)$ and $G_{ba}(0) = (2\pi)^{-d_a-d_b} g_{ab}(0) = f_{u_b u_a}(0)$, the first term in the expansion (19) is

$$\begin{aligned}
\gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}} \frac{\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} \Big\{ f_{u_a ub}(0)\, e^{\frac{\pi i}{2}(1-d_a-d_b)} \\
&\quad + e^{-i\pi(d_a-d_b)} e^{\frac{\pi i}{2}(-1+d_a+d_b)} f_{u_b u_a}(0) \Big\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} \Big\{ f_{u_a ub}(0)\, e^{\frac{\pi i}{2}(1-2d_b)} \\
&\quad + e^{-\frac{i\pi(d_a-d_b)}{2}} e^{\frac{\pi i}{2}(-1+d_a+d_b)} f_{u_b u_a}(0) \Big\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\Gamma(1-d_a-d_b)}{k^{1-d_a-d_b}} \Big\{ f_{u_a ub}(0)\, e^{\frac{\pi i}{2}(1-2d_b)} + e^{\frac{\pi i}{2}(-1+2d_b)} f_{u_b u_a}(0) \Big\} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\omega_{ab}\Gamma(1-d_a-d_b)\, 2\cos\left\{\frac{\pi}{2}(1-2d_b)\right\}}{2\pi k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right) \\
&= \frac{\omega_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} + O\left(\frac{1}{k^{2-d_a-d_b}}\right), \qquad (40)
\end{aligned}$$

where $\omega_{ab} = \omega_{ba} = 2\pi f_{u_b u_a}(0)$. Expression (40) is the leading term in the expansion as given in Remark A(i).

Taking the expansion to the second term, we have

$$\begin{aligned}
\gamma_{ab}(k) &= e^{\frac{i\pi(d_a-d_b)}{2}} \sum_{n=0}^{1} \frac{\Gamma(n+1-d_a-d_b)}{n! k^{n+1-d_a-d_b}} \Big\{ G_{ab}^{(n)}(0)\, e^{\frac{\pi i}{2}(n+1-d_a-d_b)} \\
&\quad + (-1)^n e^{-i\pi(d_a-d_b)} e^{\frac{\pi i}{2}(n-1+d_a+d_b)} G_{ba}^{(n)}(0) \Big\} + O\left(\frac{1}{k^{3-d_a-d_b}}\right) \\
&= \frac{\omega_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \\
&\quad + e^{\frac{i\pi(d_a-d_b)}{2}} \frac{\Gamma(2-d_a-d_b)}{2k^{2-d_a-d_b}} \Big\{ G_{ab}^{(1)}(0)\, e^{\frac{\pi i}{2}(2-d_a-d_b)} \\
&\quad - e^{-i\pi(d_a-d_b)} e^{\frac{\pi i}{2}(d_a+d_b)} G_{ba}^{(1)}(0) \Big\} + O\left(\frac{1}{k^{3-d_a-d_b}}\right) \qquad (41)
\end{aligned}$$

Now

$$\begin{aligned}
g_{ab}(\lambda) &= \left|\frac{2\sin(\lambda/2)}{\lambda(2\pi-\lambda)}\right|^{-d_a-d_b} e^{-\frac{i\lambda(d_a-d_b)}{2}} f_{u_a u_b}(\lambda) \\
g_{ba}(w) &= \left(\frac{2\sin(w/2)}{w(2\pi-w)}\right)^{-d_a-d_b} e^{\frac{iw(d_a-d_b)}{2}} f_{u_b u_a}(w),
\end{aligned}$$

so

$$\begin{aligned}
G_{ba}(w) &= (2\pi-w)^{-d_a-d_b} g_{ba}(w) \\
&= \left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} e^{\frac{iw(d_a-d_b)}{2}} f_{u_b u_a}(w),
\end{aligned}$$

while

$$
\begin{aligned}
G_{ab}(w) &= (2\pi - w)^{-d_a-d_b}\, g_{ab}(w) \\
&= \left(\frac{2\sin(w/2)}{w}\right)^{-d_a-d_b} e^{-\frac{iw(d_a-d_b)}{2}} f_{u_a u_b}(w),
\end{aligned}
$$

so that

$$
\begin{aligned}
G'_{ba}(0) &= \frac{i(d_a-d_b)}{2} f_{u_b u_a}(0) + f'_{u_b u_a}(0) = \frac{i(d_a-d_b)}{2}\frac{\omega_{ba}}{2\pi} + f'_{u_b u_a}(0) \\
&= \frac{i(d_a-d_b)}{2}\frac{\omega_{ba}}{2\pi} + f'_{u_b u_a}(0) \\
&= \frac{i(d_a-d_b)}{2}\frac{\omega_{ba}}{2\pi} + \frac{i}{2\pi}\omega_{ab}^{(1)} \\
G'_{ab}(0) &= \frac{i(d_b-d_a)}{2} f_{u_a u_b}(0) + f'_{u_a u_b}(0) = \frac{i(d_b-d_a)}{2}\frac{\omega_{ab}}{2\pi} + f'_{u_a u_b}(0) \\
&= \frac{i(d_b-d_a)}{2}\frac{\omega_{ab}}{2\pi} - \frac{i}{2\pi}\omega_{ab}^{(1)}
\end{aligned}
$$

since $f'_{u_a u_b}(w) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_{ab}(h)\, e^{-ihw}(-ih)$ and then

$$
f'_{u_a u_b}(0) = -\frac{i}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_{ab}(h)\, h = -\frac{i}{2\pi}\omega_{ab}^{(1)},
$$

where we define $\omega_{ab}^{(1)} = \sum_{j=-\infty}^{\infty}\gamma_{ab}(j)\, j$, which exists by the smoothness condition on the spectrum f_{uu}. Note that

$$
\begin{aligned}
f'_{u_b u_a}(0) &= -\frac{i}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_{ba}(h)\, h = -\frac{i}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_{ab}(-h)\, h = \frac{i}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_{ab}(h)\, h \\
&= -f'_{u_a u_b}(0) = \frac{i}{2\pi}\omega_{ab}^{(1)} \\
f'_{u_a u_b}(0) &= -\frac{i}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_{ab}(-h)\, h = \frac{i}{2\pi}\sum_{j=-\infty}^{\infty}\gamma_{ab}(j)\, j = -f'_{u_b u_a}(0) = -\frac{i}{2\pi}\omega_{ab}^{(1)}.
\end{aligned}
$$

Hence, $G'_{ba}(0) = \frac{i(d_a-d_b)}{2}\frac{\omega_{ba}}{2\pi} + \frac{i}{2\pi}\omega_{ab}^{(1)} = -G'_{ab}(0)$.

Using these expresssions in (41), we find that

$$\begin{aligned}\gamma_{ab}(k) &= \frac{\omega_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \\ &+e^{\frac{i\pi(d_a-d_b)}{2}}\frac{\Gamma(2-d_a-d_b)}{2k^{2-d_a-d_b}}\left\{G_{ab}^{(1)}(0)\,e^{\frac{\pi i}{2}(2-d_a-d_b)}\right. \\ &\left.-e^{-i\pi(d_a-d_b)}e^{\frac{\pi i}{2}(d_a+d_b)}G_{ba}^{(1)}(0)\right\}+O\left(\frac{1}{k^{3-d_a-d_b}}\right) \\ &= \frac{\omega_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \\ &+e^{\frac{i\pi(d_a-d_b)}{2}}\frac{\Gamma(2-d_a-d_b)\left(\frac{i(d_a-d_b)}{2}\frac{\omega_{ab}}{2\pi}+\frac{i}{2\pi}\omega_{ab}^{(1)}\right)}{2k^{2-d_a-d_b}} \\ &\times\left\{e^{-\frac{\pi i}{2}(d_a+d_b)}-e^{-i\pi(d_a-d_b)}e^{\frac{\pi i}{2}(d_a+d_b)}\right\}+O\left(\frac{1}{k^{3-d_a-d_b}}\right)\end{aligned}$$

$$\begin{aligned}&= \frac{\omega_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \\ &+\frac{\Gamma(2-d_a-d_b)\left(\frac{i(d_a-d_b)}{2}\frac{\omega_{ab}}{2\pi}+\frac{i}{2\pi}\omega_{ab}^{(1)}\right)}{2k^{2-d_a-d_b}}\left\{e^{-\pi i d_b}-e^{i\pi d_b}\right\}+O\left(\frac{1}{k^{3-d_a-d_b}}\right) \\ &= \frac{\omega_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \\ &+\frac{\Gamma(2-d_a-d_b)\sin(\pi d_b)\left(\frac{(d_a-d_b)}{2}\omega_{ab}+\omega_{ab}^{(1)}\right)}{2\pi k^{2-d_a-d_b}}+O\left(\frac{1}{k^{3-d_a-d_b}}\right) \\ &= \frac{\omega_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \\ &+\frac{\Gamma(1-d_a-d_b)\sin(\pi d_b)(1-d_a-d_b)\left(\frac{(d_a-d_b)}{2}\omega_{ab}+\omega_{ab}^{(1)}\right)}{2\pi k^{2-d_a-d_b}}+O\left(\frac{1}{k^{3-d_a-d_b}}\right) \\ &= \frac{\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}} \\ &\left\{\omega_{ab}+\frac{1}{2k}(1-d_a-d_b)\left(\frac{(d_a-d_b)}{2}\omega_{ab}+\omega_{ab}^{(1)}\right)\right\}+O\left(\frac{1}{k^{3-d_a-d_b}}\right),\end{aligned} \tag{42}$$

giving (22) in Remark A(iii). In the pure $I(d)$ case we have $\omega_{ab}^{(1)}=0$ because $\omega_{ab}^{(1)}=\sum_{j=-\infty}^{\infty}\gamma_{ab}(j)\,j=0$ as $\gamma_{ab}(j)=0$ for all $j\neq 0$. Hence, in that case we get the simplified formula

$$\frac{\omega_{ab}\Gamma(1-d_a-d_b)\sin\{\pi d_b\}}{\pi k^{1-d_a-d_b}}\left\{1+\frac{(d_a+d_b-1)(d_b-d_a)}{2k}\right\},$$

corresponding the result (14) obtained directly in that case from the exact formula for the acf.

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