PROBABILISTIC SOPHISTICATION AND STOCHASTIC MONOTONICITY IN THE SAVAGE FRAMEWORK

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Probabilistic Sophistication and Stochastic Monotonicity in the Savage Framework.

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Abstract

Machina & Schmeidler (1992) show that probabilistic sophistication can be obtained in a Savage setting without imposing expected utility by dropping Savage's axiom **P2** (sure-thing principle) and strengthening his axiom **P4** (weak comparative probability). Their stronger axiom, however, embodies a degree of separability analogous to **P2**. In this note, we obtain probabilistic sophistication using Savage's original axiom **P4** and a weaker analog of Savage's **P2**.

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1 Introduction

In subjective expected utility theory, an agent's preferences under uncertainty can be represented by the expectation of a utility function with respect to the subjective probabilities that represent the individual's beliefs. Savage (1954) axiomatized this in a setting of purely subjective uncertainty. Machina & Schmeidler's (1992) contribution is to provide axioms that continued to characterize beliefs by means of subjective probabilities, but neither assumed nor implied that risk preferences conformed to expected utility theory. In particular, they drop Savage's sure-thing principle (axiom **P2**) and strengthen his comparative-probability axiom (**P4**). They motivate the need to dispense with **P2** by noting that "probabilistically sophisticated non-expected utility preferences" – for example, those that can accommodate the Allais paradox – do not satisfy **P2**.

All preferences over acts that satisfy the Machina-Schmeidler axioms, however, induce preferences over lotteries that respect first-order stochastic dominance. We show that any such preferences over acts must satisfy a weaker version of **P2** we call the "two-outcome sure-thing principle" (**P2′**), which indeed can be interpreted as a notion of stochastic monotonicity for acts. We show that the conjunction of **P2′** and Savage's original weak comparative-probability axiom **P4** is equivalent to Machina and Schmeidler's stronger comparative-probability axiom. Hence, relying on their result enables us to show that, imposing the original Savage axioms but weakening **P2** to **P2′** guarantees probabilistic sophistication.

Machina and Schmeidler (1995) characterize probabilistically sophisticated preferences in an Anscombe-Aumann (1963) setting, by replacing the expected utility hypothesis with a new replacement axiom. Grant and Polak (2006) obtain probabilistic sophistication in this setting by replacing the Machina-Schmeidler replacement axiom with an axiom similar to Savage's **P4** in conjunction with a stochastic monotonicity axiom. The current paper demonstrates that results very similar to those of Grant and Polak (2006) can be obtained in a Savage setting.

Section 2 introduces the notation, Machina and Schmeidler's (1992) rendering of Savage's theory, and their representation result. In Section 3 we introduce our alternative axiom and representation result. Section 4 discusses the relationship between the results of the current paper and those of Sarin and Wakker (2000) who also use ideas of stochastic monotonicity to obtain probabilistic sophistication.

2 Preliminaries: Machina-Schmeidler

Set Up.

Let $\mathcal{X}$ be an arbitrary set of outcomes (finite or infinite). Assume that uncertainty is described by a set of states, $\mathcal{S}$. For any event $E \subset S$, let E^c denote its complement in $\mathcal{S}$.

Let $\mathcal{F}$ denote the set of functions from $\mathcal{S}$ to $\mathcal{X}$ with finite range, i.e. $\mathcal{F}$ is the set of Savage acts. Furthermore, for any pair of acts f and g in $\mathcal{F}$ and any event $E \subset \mathcal{S}$, $h = [f \text{ on } E; g \text{ on } E^c]$ denotes the act in $\mathcal{F}$ formed from 'splicing' and 'recombining' the two acts f and g in such a way that $h(s)$ equals $f(s)$ if $s \in E$, and equals $g(s)$ if $s \in E^c$.

Let $\succsim$ denote the individual's weak preference relation on $\mathcal{F}$ (where, as usual, $\succ$ denotes strict preference and $\sim$ denotes indifference). With slight abuse of notation, x denotes the constant act h, where $h(s) = x$ for all $s \in \mathcal{S}$. Hence we will write $x \succsim y$ if and only if $[x \text{ on } \mathcal{S}] \succsim [y \text{ on } \mathcal{S}]$.

Denote by $\mathcal{L}$ the set of probability measures on $\mathcal{X}$ with finite supports. We will refer to the elements of $\mathcal{L}$ as roulette lotteries or just as lotteries. For each x in $\mathcal{X}$, let δ_x denote the (degenerate) lottery that yields the outcome x with probability one. Any lottery $\mathbf{R}$ in $\mathcal{L}$ may be expressed as a probability weighted mixture of degenerate lotteries, corresponding to the outcomes in its support: that is, $\mathbf{R} = \sum_{i=1}^{m} p_i \delta_{x_i}$, where $p_i = \mathbf{R}(x_i)$ for each $i = 1, \ldots, m$.

Let $\geq^1$ (respectively, $>^1$) denote the partial ordering over roulette lotteries of weak (respectively, strict) first-order stochastic dominance derived from $\succsim$. It is defined as follows: for any pair of lotteries $\mathbf{R} = \sum_{i=1}^{m} p_i \delta_{x_i}$ and $\mathbf{R}^* = \sum_{j=1}^{m^*} q_j \delta_{y_j}$, $\mathbf{R} \geq^1 \mathbf{R}^*$, if

$$\sum_{\{i: x_i \succsim z\}} p_i \geq \sum_{\{j: y_j \succsim z\}} q_j \text{ for all } z \in \mathcal{X}.$$

And $\mathbf{R} >^1 \mathbf{R}^*$, if, in addition, strict inequality holds for some $z \in \mathcal{X}$.

We shall refer to a preference relation defined on $\mathcal{L}$ and denoted by $\succ_L$ as risk preferences. We shall say the risk preferences satisfy stochastic monotonicity (with respect to $\succsim$) if they respect the partial ordering of first-order stochastic dominance, that is, $\mathbf{R} \geq^1 \mathbf{R}^*$ implies $\mathbf{R} \succsim_L \mathbf{R}^*$ and $\mathbf{R} >^1 \mathbf{R}^*$ implies $\mathbf{R} \succ_L \mathbf{R}^*$. We say that a function $V : \mathcal{L} \to \mathbb{R}$ is mixture continuous and stochastically monotonic if it is continuous in probability mixtures and $\mathbf{R} >^1 (\geq^1) \mathbf{R}^*$ implies $V(\mathbf{R}) > (\geq) V(\mathbf{R}^*)$.

Furthermore, we say it is affine if for any pair of roulette lotteries $\mathbf{R}$ and $\mathbf{R}^*$ in $\mathcal{L}$, and any α in $(0,1)$, $V(\alpha\mathbf{R}+(1-\alpha)\mathbf{R}^*)=\alpha V(\mathbf{R})+(1-\alpha)V(\mathbf{R}^*)$.

Subjective Expected Utility

A preference relation $\succsim$ on $\mathcal{F}$ is said to admit a subjective expected utility representation, if there exists a unique subjective probability measure $\pi(.)$ over events and an affine utility function $U(.)$ on $\mathcal{L}$ such that, for all pairs of acts $f=[x_1$ on $A_1;\ldots;x_m$ on $A_m]$ and $g=[y_1$ on $B_1;\ldots;y_n$ on $B_n]$,

$$f \succ g \Leftrightarrow U\left(\sum_{i=1}^{m}\pi(A_i)\,\delta_{x_i}\right) > U\left(\sum_{j=1}^{n}\pi(B_j)\,\delta_{y_j}\right).$$

We follow Machina and Schmeidler's (1992) rendering of Savage's theorem. The following set of axioms is necessary and sufficient for a preference relation to admit a Subjective Expected Utility representation.[1]

Axiom P1 (Ordering): *The relation $\succsim$ is complete, reflexive and transitive.*

Axiom P2 (Sure-Thing Principle) *For all events E and acts f, f^*, g and h,*

$$\begin{aligned}&[f^* \text{ on } E;\ g \text{ on } E^c] \succsim [f \text{ on } E;\ g \text{ on } E^c]\\ \Rightarrow\quad &[f^* \text{ on } E;\ h \text{ on } E^c] \succsim [f \text{ on } E;\ h \text{ on } E^c].\end{aligned}$$

An event E is "null" if any pair of acts that differ only on E are indifferent.

Axiom P3 (Eventwise Monotonicity) *For all non-null events E, outcomes x and y, and acts g,*

$$[x \text{ on } E;\ g \text{ on } E^c] \succsim [y \text{ on } E;\ g \text{ on } E^c] \Leftrightarrow x \succsim y.$$

Axiom P4 (Weak Comparative Probability) *For all events A and B,and outcomes $x^* \succ x$ and $y^* \succ y$,*

$$[x^* \text{ on } A;\ x \text{ on } A^c] \succsim [x^* \text{ on } B;\ x \text{ on } B^c]$$

$$\Rightarrow [y^* \text{ on } A;\ y \text{ on } A^c] \succsim [y^* \text{ on } B;\ y \text{ on } B^c].$$

Axiom P5 (Nondegeneracy) *There exist outcomes x and y such that $x \succ y$.*

[1] Savage also has a state-wise dominance axiom (P7) but it is not needed in this setting in which every act has finite range.

Axiom P6 (Small Event Continuity) *For all outcomes x and acts $f \succ g$, there exists a finite set of events $\{A_1, \ldots, A_n\}$ forming a partition of $\mathcal{S}$ such that*

$$f \succ [x \text{ on } A_i;\ g \text{ on } A_i^c] \ \textit{and}\ [x \text{ on } A_i;\ f \text{ on } A_i^c] \succ g, \textit{ for all } i \in \{1, \ldots, n\}.$$

The first and last axioms are analogous to the standard ordering and continuity axioms in choice under certainty. The fifth axiom simply requires the individual not be indifferent between all outcomes. The fourth axiom allows us to deduce from the agent's preferences over simple bets an induced relative likelihood relation over events. We shall see below that the third axiom is necessary for the induced risk preferences to satisfy first-order stochastic dominance.

As Machina and Schmeidler observe, **P2** may be viewed as being "expected utility-based". That is, not only does it play a critical role in ensuring that beliefs can be represented by a unique subjective probability measure but it also entails that the induced risk preferences are represented by a von Neumann-Morgenstern affine utility function.[2] The aim of Machina and Schmeidler (1992) was to investigate whether it was possible to have the former and without necessarily entailing the latter.

Suppose we fix a probability measure π over events, then we can associate with any act $f = [x_1 \text{ on } A_1; \ldots; x_m \text{ on } A_m]$, the lottery $\mathbf{R}_f^\pi = \sum_{i=1}^m \pi(A_i)\, \delta_{x_i}$, that is, the lottery formed by taking the $\pi-$weighted average of the degenerate lotteries corresponding to the outcomes in the range of f.

Probabilistic sophistication means that there exists a unique subjective probability measure π defined over the set of events, such that the individual ranks any pair of acts only on the basis of the lotteries they induce through π. Formally:

Probabilistic Sophistication A preference relation $\succsim$ on $\mathcal{F}$ is said to be *probabilistically sophisticated* if there exists a unique subjective probability measure $\pi(\cdot)$ over events such that for all pairs of acts f and g, if $\mathbf{R}_f^\pi = \mathbf{R}_g^\pi$ then $f \sim g$.

If an agent is probabilistically sophisticated, we can construct her induced risk preferences as follows: for any pair of lotteries $\mathbf{R}$ and $\mathbf{R}^*$, we set $\mathbf{R} \succsim_L \mathbf{R}^*$ if there exists a pair of acts $f \succsim g$ such that $\mathbf{R} = \mathbf{R}_f^\pi$ and $\mathbf{R}^* = \mathbf{R}_g^\pi$. Furthermore, if an agent is probabilistically sophisticated then knowledge of an individual's subjective beliefs

[2] Karni (1985) points out that Substitution only ensures that the risk preferences are state independent. But this does not preclude the *utility* functions being state-dependent. A richer framework is required to achieve an unambiguous separation of beliefs from potentially state-dependent utility.

π and her risk preferences $\succsim_L$ enables us to recover her entire preference relation over acts.

Assume that the agent's induced preferences over lotteries (her risk preferences) can be represented by a (not necessarily affine) function $V(.)$. Then probabilistic sophistication (with respect to π) is equivalent to:[3] for any pair of acts f and g,

$$f \succ g \Leftrightarrow V\left(\mathbf{R}_f^\pi\right) > V\left(\mathbf{R}_g^\pi\right).$$

If we further assume that the agent's preferences satisfy **P2** then her risk preferences would satisfy the independence axiom of expected utility theory and hence $\succsim_L$ would admit an expected utility representation. Hence in their axiomatization of subjective probability, Machina and Schmeidler (1992) drop **P2,** but they need to strengthen **P4**.

Axiom P4* (Strong Comparative Probability) *For all pairs of disjoint events A and B, outcomes $x^* \succ x$ and $y^* \succ y$, and acts g and h,*

$$\begin{aligned} & [x^* \text{ on } A;\ x \text{ on } B;\ g \text{ on } (A\cup B)^c] \succsim [x \text{ on } A;\ x^* \text{ on } B;\ g \text{ on } (A\cup B)^c] \\ \Rightarrow\ & [y^* \text{ on } A;\ y \text{ on } B;\ h \text{ on } (A\cup B)^c] \succsim [y \text{ on } A;\ y^* \text{ on } B;\ h \text{ on } (A\cup B)^c]. \end{aligned}$$

Any probabilistically sophisticated agent whose preferences satisfy stochastic monotonicity must satisfy this axiom. For such an agent, the first preference indicates $\pi(A) \geq \pi(B)$, which in turn implies the second preference.

Like Savage's **P4**, Machina and Schmeidler's **P4*** stipulates that the relative likelihood assessments of the disjoint events A and B are invariant to the prizes directly used in the bet. But Machina and Schmeidler's axiom is stronger. First, it applies directly to *conditional* relative likelihoods; in particular, to the relative likelihood of the events A and B conditional on the event $A\cup B$. Second, it imposes an additional invariance: these conditional relative likelihood assessments are invariant to the outcomes that obtain on the states outside of A and B. In Savage, both conditioning and this invariance are derived from **P2** (the sure-thing principle). This suggests that weakening **P2** in a suitable way should be sufficient. We do this in the next section by weakening **P2** to an appropriate concept of stochastic monotonicity for acts.

[3] In fact, Machina and Schmeidler (1992) state their definition of probabilistic sophistication in terms of the representation $V(\cdot)$. We favor working with a definition stated purely in terms of preference as the concept of probabilistic sophistication is logically independent of whether or not the preferences admit a functional representation.

3 Probabilities via Stochastic Monotonicity.

We first consider two implications of stochastic monotonicity for risk preferences, and then extend these ideas to preferences over acts. Let $\succsim_L$ be a risk preference relation on the lotteries over $\mathcal{X}$. Stochastic monotonicity for $\succsim_L$ may be expressed in terms of basic substitution operations on lotteries. One basic substitution is to take an outcome in the support of a lottery and change it to a worse outcome. Stochastic monotonicity says we should prefer the lottery before this change.

Stochastic Monotonicity (I) For all $x, y \in \mathcal{X}$, all $\alpha \in (0,1]$, and all $\mathbf{R} \in \mathcal{L}$

$$x \succ y \;\Leftrightarrow\; \alpha\delta_x + (1-\alpha)\,\mathbf{R} \succ \alpha\delta_y + (1-\alpha)\,\mathbf{R}. \tag{1}$$

Another basic substitution is to move a probability mass from a better to a worse outcome within the support of a roulette lottery. Again Stochastic monotonicity says we should prefer the lottery before this change. Put another way: if any two roulette lotteries differ only in the weight they assign to just two outcomes, the agent should prefer the lottery that puts the larger weight on the better outcome.

Stochastic Monotonicity (II) For any pair of outcomes $x, y \in \mathcal{X}$, if two roulette lotteries $\mathbf{R}, \mathbf{R}^* \in \mathcal{L}$ have the property that $\mathbf{R}(z) = \mathbf{R}^*(z)$ for all $z \notin \{x, y\}$, then

$$\begin{aligned} & \alpha\mathbf{R} + (1-\alpha)\,\mathbf{R}^{**} \succsim \alpha\mathbf{R}^* + (1-\alpha)\,\mathbf{R}^{**} \\ \Rightarrow \quad & \alpha\mathbf{R} + (1-\alpha)\,\hat{\mathbf{R}}^{**} \succsim \alpha\mathbf{R}^* + (1-\alpha)\,\hat{\mathbf{R}}^{**} \end{aligned} \tag{2}$$

for all $\mathbf{R}^{**}, \hat{\mathbf{R}}^{**} \in \mathcal{L}$ and $\alpha \in (0,1]$.

Notice that, in expression (2), $\mathbf{R}$ and $\mathbf{R}^*$ differ only in the probability they assign to x and y. Similarly $\alpha\mathbf{R} + (1-\alpha)\,\mathbf{R}^{**}$ and $\alpha\mathbf{R}^* + (1-\alpha)\,\mathbf{R}^{**}$ and $\alpha\mathbf{R} + (1-\alpha)\,\hat{\mathbf{R}}^{**}$ and $\alpha\mathbf{R}^* + (1-\alpha)\,\hat{\mathbf{R}}^{**}$ also differ only in the probability they assign to x and y. Furthermore $\mathbf{R}$ assigns more weight than $\mathbf{R}^*$ to the better outcome if and only if $\alpha\mathbf{R} + (1-\alpha)\,\mathbf{R}^{**}$ (respectively, $\alpha\mathbf{R} + (1-\alpha)\,\hat{\mathbf{R}}^{**}$) assigns more weight than $\alpha\mathbf{R}^* + (1-\alpha)\,\mathbf{R}^{**}$ (respectively, $\alpha\mathbf{R}^* + (1-\alpha)\,\hat{\mathbf{R}}^{**}$) to the better outcome.

Both these stochastic monotonicity axioms resemble the independence axiom in that they preserve preference ordering under the substitution of (sub-)lotteries. But these axioms only involve lotteries that differ just on two outcomes. This is a considerable weakening in that all risk preferences that respect first-order stochastic dominance satisfy both these axioms, regardless of whether or not they satisfy independence.

The two stochastic monotonicity axioms are equivalent if the risk preference $\succsim_L$ is continuous. However, their analogs for acts are not equivalent, so we will impose them separately.

Savage's axiom **P3** (eventwise monotonicity) is an analog of stochastic monotonicity (I) for acts. To see the analogy, let $g = [z \text{ on } E,\ z_1 \text{ on } E_1; \ldots ; z_n \text{ on } E_n]$. Suppose the agent's probability assessments are given by π. Then the terms $[x \text{ on } E;\ g \text{ on } E^c]$ and $[y \text{ on } E;\ g \text{ on } E^c]$ in the statement of the axiom reduce to the lotteries $\pi(E)[\delta_x] + \sum_{i=1}^{n} \pi(E_i)\,\delta_{z_i}$ and $\pi(E)[\delta_y] + \sum_{i=1}^{n} \pi(E_i)\,\delta_{z_i}$, similar to the terms on the right side of expression (1). Therefore, if a probabilistically sophisticated agent satisfies stochastic monotonicity (I), she must satisfy **P3** (eventwise monotonicity).

The following is an analog of stochastic monotonicity (II) for acts. It restricts the sure-thing principle to apply only if, other than for two outcomes, if one of the acts in question assigns an outcome to a state then the other act in question assigns that same outcome to that state.

Axiom P2′ (Two-Outcome Sure-Thing Principle) *For all events E, all outcomes x and y, and any acts f, f^*, g and h, if for all states s and for every outcome $z \notin \{x, y\}$, $f(s) = z$ if and only if $f^*(s) = z$, then*

$$[f \text{ on } E;\ g \text{ on } E^c] \succsim [f^* \text{ on } E;\ g \text{ on } E^c] \ \Rightarrow\ [f \text{ on } E;\ h \text{ on } E^c] \succsim [f^* \text{ on } E;\ h \text{ on } E^c]. \tag{3}$$

To see the analogy to stochastic monotonicity (II), suppose the agent's probability assessments are given by π. The implication trivially follows if E is null or its complement is null, so suppose $\pi(E) \in (0,1)$ and suppose $f = [z_1 \text{ on } A_1; \ldots ; z_m \text{ on } A_m]$, $f^* = [z_1^* \text{ on } B_1; \ldots ; z_n^* \text{ on } B_n]$, $g = [\hat{x}_1 \text{ on } C_1; \ldots ; \hat{x}_{\hat{m}} \text{ on } C_{\hat{m}}]$ and $h = [\hat{y}_1 \text{ on } D_1; \ldots ; \hat{y}_{\hat{n}} \text{ on } D_{\hat{n}}]$. Set

$$\mathbf{R} = \tfrac{1}{\pi(E)}\left[\textstyle\sum_{i=1}^{m} \pi(A_i \cap E)\,\delta_{z_i}\right], \qquad \mathbf{R}^{**} = \tfrac{1}{1-\pi(E)} \textstyle\sum_{k=1}^{\hat{m}} \pi(C_k \cap E^c)\,\delta_{\hat{x}_k},$$

$$\mathbf{R}^{*} = \tfrac{1}{\pi(E)}\left[\textstyle\sum_{j=1}^{n} \pi(B_j \cap E)\,\delta_{z_j^*}\right], \qquad \hat{\mathbf{R}}^{**} = \tfrac{1}{1-\pi(E)} \textstyle\sum_{\hat{k}=1}^{\hat{n}} \pi(D_{\hat{k}} \cap E^c)\,\delta_{\hat{y}_{\hat{k}}}.$$

Hence the reduction of $[f \text{ on } E;\ g \text{ on } E^c]$ is $\pi(E)\,\mathbf{R} + [1-\pi(E)]\,\mathbf{R}^{**}$ and the reduction of $[f^* \text{ on } E;\ g \text{ on } E^c]$ is $\pi(E)\,\mathbf{R}^* + [1-\pi(E)]\,\mathbf{R}^{**}$. Similarly, the reduction of $[f \text{ on } E;\ h \text{ on } E^c]$ is $\pi(E)\,\mathbf{R} + [1-\pi(E)]\,\hat{\mathbf{R}}^{**}$ and the reduction of $[f^* \text{ on } E;\ h \text{ on } E^c]$ is $\pi(E)\,\mathbf{R}^* + [1-\pi(E)]\,\hat{\mathbf{R}}^{**}$. Now suppose that, in each state, $f(s)$ and $f^*(s)$ differ only if they assign the outcomes x or y. Then the 'conditional' reductions $\mathbf{R}$ and $\mathbf{R}^*$ differ only in the probabilities they assign to x and y. Therefore, if a probabilistically sophisticated agent satisfies stochastic monotonicity (II), she must satisfy the two-outcome sure-thing principle.

The following lemma provides a useful reformulation of the two-outcome sure-thing principle.

Lemma 1 Axiom $\mathbf{P2'}$ is equivalent to the following statement: For all disjoint events A and B, outcomes x^* and x, and acts g and h,

$$[x^* \text{ on } A;\ x \text{ on } B;\ g \text{ on } (A \cup B)^c] \succsim [x \text{ on } A;\ x^* \text{ on } B;\ g \text{ on } (A \cup B)^c]$$

implies

$$[x^* \text{ on } A;\ x \text{ on } B;\ h \text{ on } (A \cup B)^c] \succsim [x \text{ on } A;\ x^* \text{ on } B;\ h \text{ on } (A \cup B)^c].$$

Proof. Assume that $\mathbf{P2'}$ holds. Consider any pair of disjoint events A and B, any pair of outcomes x^* and x, and any pair of acts g and h. Set $f := [x^* \text{ on } A;\ x \text{ on } B;\ g \text{ on } (A \cup B)^c]$ and $\bar{f} := [x \text{ on } A;\ x^* \text{ on } B;\ g \text{ on } (A \cup B)^c]$. If $f \succsim \bar{f}$, then the premises of $\mathbf{P2'}$ are satisfied for $E = A \cup B$, and the desired preference follows.

Now assume that the statement in the lemma holds. Consider any event E, outcomes x and y, and acts f, f^*, g and h such that, for all states s and for every outcome $z \notin \{x, y\}$, $f(s) = z$ if and only if $f^*(s) = z$. Suppose that $[f \text{ on } E;\ g \text{ on } E^c] \succsim [f^* \text{ on } E;\ g \text{ on } E^c]$. Let $F = \{s \in E : f(s) = f^*(s)\}$. For each $s \in E \backslash F$, both $f(s)$ and $f^*(s)$ are one of the two outcomes x and y, and $f(s) = x$ if and only if $f^*(s) = y$. Hence, there exist two disjoint events A, B, $A \cup B = E \backslash F$, such that

$$[f \text{ on } E;\ g \text{ on } E^c] = [x \text{ on } A;\ y \text{ on } B;\ g^* \text{ on } (A \cup B)^c]$$

and

$$[f^* \text{ on } E;\ g \text{ on } E^c] = [y \text{ on } A;\ x \text{ on } B;\ g^* \text{ on } (A \cup B)^c],$$

where $g^*(s) = f(s) = f^*(s)$ for $s \in E \backslash F$ and $g^*(s) = g(s)$ for $s \in E^c$. Now, the statement in the lemma implies

$$[x \text{ on } A;\ y \text{ on } B;\ h^* \text{ on } (A \cup B)^c] \succsim [y \text{ on } A;\ x \text{ on } B;\ h^* \text{ on } (A \cup B)^c],$$

where $h^*(s) = f(s) = f^*(s)$ for $s \in E \backslash F$ and $h^*(s) = h(s)$ for $s \in E^c$. Note that the expression on the left reduces to $[f \text{ on } E;\ h \text{ on } E^c]$ and the one on the right to $[f^* \text{ on } E;\ h \text{ on } E^c]$, which ends the proof. ■

While the original formulation of **P2′** illustrates its relationship to **P2**, the above restatement exposes its close connection with **P4***, which is the subject of the next proposition.

Proposition 2 A preference relation $\succsim$ satisfies strong comparative probability (**P4***) if and only if it satisfies the two-outcome sure-thing principle (**P2′**) and weak comparative probability (**P4**).

Proof. **P4** is the restriction of **P4*** to $A \cup B = S$ and **P2′** is the restriction of **P4*** to $y^* = x^*$ and $y = x$.

To show the if part, consider two disjoint events A and B, outcomes $x^* \succ x$ and $y^* \succ y$, and an act g such that

$$[x^* \text{ on } A;\ x \text{ on } B;\ g \text{ on } (A \cup B)^c] \succsim [x \text{ on } A;\ x^* \text{ on } B;\ g \text{ on } (A \cup B)^c].$$

From lemma 1, we have that **P2′** implies that $[x^* \text{ on } A;\ x \text{ on } B;\ x \text{ on } (A \cup B)^c] \succsim [x \text{ on } A;\ x^* \text{ on } B;\ x \text{ on } (A \cup B)^c]$, and **P4** implies $[y^* \text{ on } A;\ y \text{ on } B;\ y \text{ on } (A \cup B)^c] \succsim [y \text{ on } A;\ y^* \text{ on } B;\ y \text{ on } (A \cup B)^c]$. Now, for any act h, invoking lemma 1 we have that **P2′** implies the desired preference $[y^* \text{ on } A;\ y \text{ on } B;\ h \text{ on } (A \cup B)^c] \succsim [y \text{ on } A;\ y^* \text{ on } B;\ h \text{ on } (A \cup B)^c]$. ■

We can now give an alternative characterization of probabilistically sophisticated beliefs, by replacing **P4*** with the conjunction of **P2′** and **P4** in Machina and Schmeidler's (1992) representation result.

Corollary 3 The following two statements are equivalent:

(a) The preference relation $\succsim$ on $\mathcal{F}$ satisfies ordering, the two-outcome sure-thing principle, eventwise monotonicity, weak comparative probability, non-degeneracy and small event continuity.

(b) There exists a unique finitely-additive convex-ranged probability measure π defined over subsets of $\mathcal{S}$ and a non-constant, mixture continuous, stochastically monotonic function V on $\mathcal{L}$, such that for all pairs of acts f and g in $\mathcal{F}$

$$f \succsim g \Leftrightarrow V\left(\mathbf{R}_f^\pi\right) \geq V\left(\mathbf{R}_g^\pi\right).$$

4 Sarin and Wakker

Sarin and Wakker (2000) derive a more-likely-than relationship over events in the following way: for any $A, B \subset \mathcal{S}$, $A \succsim_l B$ if and only if there exist two outcomes $x \succ y$ such that $[x \text{ on } A; y \text{ on } A^c] \succsim [x \text{ on } B; y \text{ on } B^c]$. For a probabilistically sophisticated agent whose subjective beliefs are given by π, $A \succsim_l B$ if and only if $\pi(A) \geq \pi(B)$, i.e. π is an agreeing probability measure for $\succsim_l$. Machina and Schmeidler (1992) observe, however, that the existence of such an agreeing probability measure does not guarantee probabilistic sophistication. Sarin and Wakker (2000) show that the following axiom is sufficient to close the gap, and is also necessary if we require the risk preferences to satisfy stochastic monotonicity.

Cumulative Dominance (Sarin and Wakker [2000]) $f \succsim g$ whenever $\{s \in \mathcal{S} : f(s) \succsim x\}$ $\succsim_l \{s \in \mathcal{S} : g(s) \succsim x\}$ for all outcomes x, where the preference between f and g is strict whenever one of the antecedent $\succsim_l$ orderings is strict ($\succ_l$).

Sarin and Wakker use cumulative dominance in conjunction with the following weakening of **P2** (and of **P2′**) to obtain an alternative characterization of probabilistically sophisticated beliefs.

Axiom P2* (Sarin and Wakker [2000]) For all outcomes $x \succ y$ and events A, B, H with $A \cap H = B \cap H = \varnothing$:

$$[x \text{ on } A; y \text{ on } A^c] \succsim [x \text{ on } B; y \text{ on } B^c] \text{ if and only if}$$

$$[x \text{ on } A \cup H; y \text{ on } (A \cup H)^c] \succsim [x \text{ on } (B \cup H); y \text{ on } (B \cup H)^c].$$

Theorem 4 (Sarin and Wakker [2000]) Assume Savage's (1954) **P1**, **P3**, **P5**, and **P6**. Then the following two statements are equivalent.

(a) Probabilistic sophistication holds and the risk preferences satisfy stochastic monotonicity.

(b) **P2*** and cumulative dominance hold.

Thus, to obtain their representation result, Sarin and Wakker use a weaker version of our sure-thing principle together with cumulative dominance, which is closely related to the stochastic monotonicity of risk preferences. Recall that we also appealed to stochastic monotonicity to motivate our axioms. It is natural then to ask what is the relation of our 'new' axioms to those of Sarin and Wakker (where by 'new' we mean different from Machina and Schmeidler).

The next proposition shows that the conjunction of Sarin and Wakker's **P2*** and cumulative dominance axiom implies the conjunction of our **P2′** and **P4**. But, as a counter example shows, the reverse implication does not hold. Thus our new axioms, though closely related to Sarin and Wakker's new axioms, are weaker.

Proposition 5 If the preference relation $\succsim$ satisfies **P2*** and cumulative dominance , then it satisfies **P2′** and **P4**.

Proof. Assume that the preference relation $\succsim$ satisfies **P2*** and cumulative dominance . Sarin and Wakker (2000) note that the restriction of cumulative dominance to two-outcome acts gives us **P4**. We will show that **P2′** holds by using its equivalent formulation given in Lemma 1. Consider two disjoint events A and B, two outcomes x and y, and an act g. Assume, without loss of generality, that $x \succsim y$ and suppose that $[x \text{ on } A;\ y \text{ on } B;\ g \text{ on } (A \cup B)^c] \succsim [y \text{ on } A;\ x \text{ on } B;\ g \text{ on } (A \cup B)^c]$. We shall show below that $A \succsim_l B$ if and only if $[x \text{ on } A;\ y \text{ on } B;\ h \text{ on } (A \cup B)^c] \succsim [y \text{ on } A;\ x \text{ on } B;\ h \text{ on } (A \cup B)^c]$ for any act h, yielding the desired preference.

First, note that **P2*** is a restriction on the derived likelihood relation: for any events E, F, H with $E \cap H = F \cap H = \varnothing$, we have $E \succsim_l F$ if and only if $E \cup H \succsim_l F \cup H$.

Next, for all acts f and outcomes z, define $U(f, z) = \{s \in S : f(s) \succsim z\}$. Fix an act h and let $f = [x \text{ on } A;\ y \text{ on } B;\ h \text{ on } (A \cup B)^c]$ and $\bar{f} = [y \text{ on } A;\ x \text{ on } B;\ h \text{ on } (A \cup B)^c]$. Below, we show that $A \succsim_l B$ if and only if $U(f, z) \succsim_l U(\bar{f}, z)$ for all outcomes z.

(i) If $z \succ x$, then $U(f, z) = U(h, z) = U(\bar{f}, z)$, and hence $U(f, z) \sim_l U(\bar{f}, z)$.

(ii) If $y \succsim z$, then $U(f, z) = A \cup B \cup U(h, z) = U(\bar{f}, z)$, and hence $U(f, z) \sim_l U(\bar{f}, z)$.

(iii) If $x \succsim z \succ y$, then $U(f, z) = A \cup U(h, z)$ and $U(\bar{f}, z) = B \cup U(h, z)$. **P2*** implies that $A \succsim_l B$ if and only if $U(f, z) \succsim_l U(\bar{f}, z)$.

Cumulative dominance yields $f \succsim \bar{f}$ if and only if $A \succsim_l B$, the desired statement.
■

To see that it is possible for a preference relation to satisfy both **P2′** and **P4**, but not cumulative dominance consider the following example.

Example. Let $\mathcal{S} = \{s_1, s_2\}$ and $\mathcal{X} = \{a, b, c\}$. Let $u : \mathcal{X} \times \mathcal{S} \to \mathbb{R}$ be a state dependent utility function defined by the following payoff matrix:

$u(\cdot,\cdot)$	a	b	c
s_1	4	1	0
s_2	2	0	1

Since there are only two states, we can identify each act f with the vector (x_1, x_2) where $x_s := f(s)$. Consider the preferences $\succsim$ over these acts generated by $V(x_1, x_2) := u(x_1, s_1) + u(x_2, s_2)$. Since these preferences are additive across states, they satisfy **P2′** (in fact, they satisfy **P2**). To show that they satisfy **P4**, first notice that the constant acts are ordered $a \succ b \sim c$. Thus **P4** is satisfied provided that $(a, b) \succ (b, a)$ if and only if $(a, c) \succ (c, a)$, which indeed holds. The 'more likely than' relation $\succsim_\ell$ implied by these preferences is: $\mathcal{S} \succ_\ell s_1 \succ_\ell s_2 \succ_\ell \emptyset$. Cumulative dominance requires in particular that if $\{s \in \mathcal{S} : f(s) \succsim x\} = \{s \in \mathcal{S} : g(s) \succsim x\}$ for all $x \in \mathcal{X}$, then $f \sim g$. To show that our preferences violate this, consider the two acts $f^* = (b, c)$ and $g^* = (c, b)$. We have:

$$\begin{aligned}
\{s \in \mathcal{S} : f^*(s) \succsim a\} &= \emptyset = \{s \in \mathcal{S} : g^*(s) \succsim a\} \\
\{s \in \mathcal{S} : f^*(s) \succsim b\} &= \mathcal{S} = \{s \in \mathcal{S} : g^*(s) \succsim b\} \\
\{s \in \mathcal{S} : f^*(s) \succsim c\} &= \mathcal{S} = \{s \in \mathcal{S} : g^*(s) \succsim c\}
\end{aligned}$$

But $V(b, c) = 2 > V(c, b) = 0$; that is, $f^* \succ g^*$ ■

Underlying this example is the fact that cumulative dominance implies the following restricted version of eventwise monotonicity, **P3**, for the case where constant acts are indifferent: for all outcomes x and y in $\mathcal{X}$, non-null events E and acts g,

$$[x \text{ on } E;\ g \text{ on } E^c] \sim [y \text{ on } E;\ g \text{ on } E^c] \Leftrightarrow x \sim y.$$

Eventwise monotonicity is a type of state-independence axiom. The intuitive appeal of cumulative dominance is that it compares acts by comparing the revealed likelihood of sets on which those acts yield outcomes 'at least as good as' each constant act x. When state independence fails (as in the example above), however, this intuitive appeal is undermined. Suppose an act yields an outcome y in a particular state.

Suppose the constant act y (i.e., the act that yields y in every state) is at least as good as getting x in every state. But, without state independence, it does not follow that getting y in this particular state is at least as good as getting x in that state. In the example, the constant act b is indifferent to the constant act c, but getting b in state s_1 is strictly better than getting c in state s_1 (and vice versa for state s_2).

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