# ALFRED MARSHALL'S CARDINAL THEORY OF VALUE: THE STRONG LAW OF DEMAND

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# Alfred Marshall's Cardinal Theory of Value: The Strong Law of Demand*

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**Abstract**

We show that all the fundamental properties of competitive equilibrium in Marshall's cardinal theory of value, as presented in Note XXI of the mathematical appendix to his *Principles of Economics* (1890), derive from the Strong Law of Demand. That is, existence, uniqueness, optimality, and global stability of equilibrium prices with respect to tatonnement price adjustment follow from the cyclical monotonicity of the market demand function in the Marshallian general equilibrium model.

*Keywords:* Cardinal utility, Quasilinear utility, Cyclical monotonicity

*JEL Classification:* B13, D11, D51

## 1 Introduction

Marshall in NOTE XXI of the mathematical appendix to his *Principles of Economics* (1890) presents a fully articulated theory of general equilibrium in market economies. This is not the partial equilibrium model with only two goods usually associated with Cournot (1838), Dupuit (1844) or Marshall (1890), nor is it the partial equilibrium model exposited in the first chapter of Arrow and Hahn (1971), or in chapter 10 of Mas-Colell, Whinston and Green (MWG) (1995). Marshall's general equilibrium model differs in several essential respects from the general equilibrium model of Walras (1900). In Marshall's model there are no explicit budget constraints for consumers, the marginal utilities of incomes are exogenous constants and market prices are not normalized. He "proves" the existence of market clearing prices, as does Walras, by counting the number of equations and unknowns. Marshall's first order conditions for consumer satisfaction require the gradient of the consumer's utility function to equal the vector of market prices.

A recent modern exposition of the fundamental properties of Marhsall's general equilibrium model in NOTE XXI can be found in sections 8.4, 8.5 and 8.6 of Bewley (2007), where he calls it "short-run equilibrium." As in Marshall, there are no explicit

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budget constraints for consumers, the marginal utilities of incomes are exogenous constants and market prices are not normalized. Consumers in Bewley's model satisfy Marshall's first order conditions in a short-run equilibrium. Bewley proves that: (i) a unique short-run equilibrium exists, (ii) welfare in a short-run equilibrium can be computed using the consumer surplus of a representative consumer and (iii) the short-run equilibrium is globally stable under tatonnment price adjustment.

We show that the fundamental properties of competitive equilibrium in Marshall's theory of value as derived in Bewley are immediate consequences of the market demand function satisfying the Strong Law of Demand, introduced by Brown and Calsamiglia (2007). A demand function is said to satisfy the Strong Law of Demand if it is a cyclically monotone function of market prices. Cyclically monotone demand functions not only have downward sloping demand curves, in the sense that they are monotone functions of market prices, but also their line integrals are path-independent and hence provide an exact measure of the change in consumer's welfare in terms of consumer's surplus for a given multidimensional change in market prices. This is an immediate consequence of Roy's identity applied to the indirect utility function for quasilinear utilities, where the marginal utility of income is one. Following Quah (2000), we show that the Strong Law of Demand is preserved under aggregation across consumers. Hence the area under the market demand curve is an exact measure of the change in aggregate consumer welfare for a given multidimensional change in market prices.

Brown and Calsamiglia prove that a consumer's demand function satisfies the Strong Law of Demand, iff the consumer behaves as if she were maximizing a quasilinear utility function subject to a budget constraint. The defining cardinal property of quasilinear utilities, say for two goods, is that the indifference curves are parallel. Consequently, quasilinear utility is measured on an interval scale. It is in this sense that Marshall's general equilibrium model is a cardinal theory of value, where differences in a consumer's quasilinear utility levels are a proxy for the consumer's intensity of preferences. The assumption of maximizing a quasilinear utility function subject to a budget constraint is made by MWG in their discussion of partial equilibrium analysis in the two good case, but there is no explicit mention of the Strong Law of Demand in their analysis. In Bewley's discussion of short-run equilibrium, there is no explicit mention of the Strong Law of Demand or maximizing a quasilinear utility function subject to a budget constraint.

Brown and Matzkin (1996) define an economic model as refutable if there exists a finite data set consistent with the model and a second finite data set that falsifies the model, where the model is the solution to a finite family of multivariate polynomial inequalities. The unknowns in the inequalities are unobservable theoretical constructs such as utility levels and marginal utilities of income. The parameters in the inequalities are observable market quantities such as market prices, aggregate demands and the income distribution. Using Bewley's (1980) characterization of the short-run equilibrium model as a representative agent model — see also sections 8.5 and 8.6 of his monograph — we propose a refutable model of Marshall's cardinal theory of value. That is, there exists a finite family of multivariate polynomial inequalities, the Afriat inequalities for quasilinear utilities derived by Brown and Calsamiglia, where the parameters are the market prices and aggregate demands and the unknowns are the utility levels and marginal utilities of income of the representative consumer. Brown and Calsamiglia show that these inequalities have a solution iff the finite data set consisting of observations on market prices and associated aggregate market demands is cyclically monotone.

The fundamental difference between the Marshallian and Walrasian theories of value is the measurement scale for utility levels of consumers. In the Marshallian model the measurement scale is cardinal, more precisely an interval scale, where the family of indifference curves is a metric space isometric to the positive real line. That is, fix any open interval, $I \equiv (\bar{x}, +\infty) \in \mathbb{R}^2_{++}$ and assume that the quasilinear utility function $U(x, y) = v(x) + y$ on $\mathbb{R}^2_{++}$ is smooth, monotone and strictly concave. If $(\bar{x}, \bar{y}) \in I$ then define

$$\Phi(\bar{y}) \equiv \{(x, y) \in \mathbb{R}^2_{++} : U(x, y) = U(\bar{x}, \bar{y})\},$$

i.e., the unique indifference curve of $U(x, y)$ passing through $(\bar{x}, \bar{y})$. $\Phi$ is a one-to-one map from the metric space $I$ onto $\Gamma[U]$, the family of indifference curves for $U$. As such, $\Phi$ induces a metric on $\Gamma[U]$, where if $(\alpha, \beta) \in \Gamma[U] \times \Gamma[U]$, then

$$dist(\alpha, \beta) \equiv \left|(\Phi^{-1}(\alpha) - \Phi^{-1}(\beta)\right|.$$

That is, $\Phi^{-1}$ is an isometric imbedding of $\Gamma[U]$ into $\mathbb{R}_{++}$. Of course, this metric representation extends to quasilinear utilities on $\mathbb{R}^{N+1}_{++}$ of the form

$$U(x, y) = v(x) + y \text{ where } x \in \mathbb{R}^N_{++} \text{ and } y \in \mathbb{R}_{++}.$$

That is, $\Phi^{-1}$ is an isometric imbedding of $\Gamma[U]$ into $\mathbb{R}_{++}$.

In the Walrasian model, the measurement scale for utility levels is an ordinal scale, where only properties of consumer demand derivable from indifference curves are admissible in the Walrasian model, e.g., the marginal utility of income is not an admissible property. Ordinal scales are sufficient for characterizing exchange efficiency in terms of Pareto optimality or compensating variation or equivalent variation. Unfortunately, a meaningful discussion of distributive equity requires interpersonal comparisons of aggregate consumer welfare. If there is a representative consumer endowed with a quasilinear utility function, then the equity of interpersonal changes in aggregate consumer welfare is reduced to intrapersonal changes in the consumer surplus of the representative consumer. Hence notions of distributional equity are well defined and exact in the Marshallian cardinal theory of value.

We argue that rationalizing consumer demand with quasilinear cardinal utility functions is comparable to rationalizing consumer demand with neoclassical ordinal utility functions. In the latter case Afriat (1967) proved that neoclassical rationalization is refutable and in the former case, we extended his analysis to show that quasilinear rationalization is also refutable. Hence in both cases, the debate about the efficacy of either the cardinal or ordinal model of utility maximization subject to a budget constraint has been reduced to an empirical question that is resolvable in polynomial time using market data.

For ease of exposition we limit most of our discussion to pure exchange models but, as suggested by the analysis short-run equilibrium in Bewley, all of our results extend to Marshall's general equilibrium model with production.

## 2 A Cardinal Theory of Value

For completeness, we recall Afriat's seminal (1967) theorem on rationalizing consumer demand data $(p_r, x_r)$, $r = 1, 2, ..., N$, with an ordinal utility function and the Brown–Calsamiglia (2007) extension of Afriat's theorem to rationalizing consumer demand data with a cardinal utility function, i.e., a quasilinear utility function. Afriat showed that the finite set of observations of market prices and consumer demands at those prices can be rationalized by an ordinal utility function iff there exists a concave, continuous, non-satiated utility function that rationalizes the data. That is, there exists a concave, continuous, non-satiated utility function $U$, such that for $r = 1, 2, ..., N$:

$$U(x_r) = \max_{p_r \cdot x \leq p_r \cdot x_r} U(x).$$

Moreover, this rationalization is equivalent to two other conditions: (1) The "Afriat inequalities":

$$U_j \leq U_k + \lambda_k p_k \cdot (x_j - x_k) \text{ for } j, k = 1, 2, ..., N$$

are solvable for utility levels $U_r$ and marginal utilities of income $\lambda_r$ and (2) the data satisfies cyclical consistency, a combinatorial condition that generalizes the strong law of revealed preference to allow thick indifference curves. See Varian (1982) for proofs. Brown and Calsamiglia showed that the data can be rationalized by a quasilinear utility function iff the Afriat inequalities have a solution where the $\lambda_r = 1$. Moreover, they show that quasilinear rationalization is equivalent to another combinatorial condition on the data, cyclical monotonicity. Rockafellar (1970) introduced the notion of cyclical monotonicity as a means of characterizing the subgradient correspondence of a convex function. For smooth strictly concave functions $f$ the gradient map $\partial f(x)$ is cylically monotone if for all finite sequences $(p_t, x_t)_{t=1}^{T}$, where $p_t = \partial f(x_t)$:

$$x_1 \cdot (p_2 - p_1) + x_2 \cdot (p_3 - p_3) + \cdots + x_T \cdot (p_1 - p_T) \geq 0.$$

Hildenbrand's (1983) extension of the law of demand to multicommodity market demand functions requires the demand function to be monotone. He showed that it is monotone if the income distribution is price independent and has downward sloping density. Subsequently, Quah (2000) extended Hildenbrand's analysis to individual's demand functions. His sufficient condition for monotone individual demand is in terms of the income elasticity of the marginal utility of income. Assuming that the commodity space is $\mathbb{R}^n_{++}$, we denote the demand function at prices $p \in \mathbb{R}^n_{++}$ by $x(p)$. This demand function satisfies the law of demand or is monotone if for any pair $p, p' \in \mathbb{R}^n_{++}$ of prices

$$(p - p') \cdot [x(p) - x(p')] < 0.$$

This means, in particular, that the demand curve of any good is downward sloping with respect to its own price, i.e., satisfies the law of demand if all other prices are held constant. We denote the Marshallian consumer optimization problem by $(M)$:

$$\max_{x_i \in \mathbb{R}^n_{++}} \frac{1}{\lambda_i} g_i(x_i) - p \cdot x_i$$

where $g_i$ is a smooth, strictly increasing and strictly concave utility function on $\mathbb{R}^n_{++}$, $\lambda_i$ is the exogenous marginal utility of income, $p$ is the vector of market prices and $x_i$ is the consumption bundle. In this model there are no budget constraints and prices are not normalized. This specification of the consumer's optimization problem rationalizes the family of equations defining Marshall's general equilibrium model (absent production) in his NOTE XXI.

**Theorem 1** *If there are $I$ consumers, where each consumer $i$'s optimization problem is given by $(M)$, then the market demand function satisfies the Strong Law of Demand.*

**Proof.** Let $h_i(p) = \frac{1}{\lambda_i} g_i(x_i(p)) - p \cdot x_i(p)$ be the optimal value function for $(M)$ for consumer $i$. Applying the envelope theorem we know that $\partial h_i(p) = -x_i(p)$.Let $H(p) = \sum_{i=1}^{I} h_i(p)$, then $\partial H(p) = \sum_{i=1}^{I} \partial h_i(p) = \sum_{i=1}^{I} -x_i(p)$. Therefore the market demand at prices $p$ is $X(p) = \sum_{i=1}^{I} x_i(p) = -\sum_{i=1}^{I} \partial h_i(p) = -\partial H(p)$. Since $-h_i(p)$ is a concave function, $-\partial h_i(p)$ and $-\partial H(p)$ are cyclically monotone — see Theorem 24.8 in Rockafellar (1970). Hence, the market demand function $X(p)$ satisfies the Strong Law of Demand. ■

**Corollary 2** *The Marshallian general equilibrium model has a unique equilibrium price vector that is globally stable under tatonnment price adjustment.*

**Proof.** Every cyclically monotone map is a monotone map. That is, market demand functions satisfying the Strong Law of Demand a fortiori satisfy the Law of Demand. Hildenbrand (1983) shows that economies satisfying the Law of Demand have a unique equilibrium price vectors that are globally stable under tatonnment price adjustment. ■

**Corollary 3** *Aggregate consumer welfare in the Marshallian general equilibrium model can be computed using consumer surplus.*

**Proof.** Brown and Brown (2007) show that this is a property of cyclically monotone demand functions. To prove that the Marshallian general equilibrium model is refutable, we will first show that it can be described as a representative agent model, as originally suggested by Bewley (1980). The representative agent's utility function in Bewley's Marshallian model is given by the following social welfare function:

$$W(e) = \max_{\{x_1,\ldots,x_I\} \in \mathbb{R}^{nI}_{++}} \left[ \sum_{i=1}^{I} \frac{1}{\lambda_i} g_i(x_i) \right]$$

$$\text{s.t. } \sum_{i=1}^{I} x_i = e.$$

Bewley shows that $(\bar{p}, x(\bar{p}))$ is an equilibrium of the exchange economy with consumers $\{(g_i, \lambda_i)\}_{i=1}^{I}$ and social endowment $\bar{e}$ iff

$$\bar{e} = \arg \max_{e \in R_{++}^{N}} \{W(e) - \bar{p}e\}.$$

Equivalently, for a given $\bar{e}$, the price vector $\bar{p}$ such that $\bar{e} = \arg\max_{e \in R_{++}^{N}} \{W(e) - \bar{p}e\}$ will be the unique competitive equilibrium price vector for this exchange economy. Let $H(\bar{p}) = \max_{e \in \mathbb{R}_{++}^{I}} \{W(e) - \bar{p} \cdot e\}$, then it follows that

$$H(\bar{p}) \equiv \sum_{t=1}^{T} h_t(\bar{p})$$

if $\bar{p}$ is a competitive equilibrium vector of prices. Hence

$$-\left(\frac{\partial H}{\partial p}\right)\bigg|_{\bar{p}} = \sum_{t=1}^{T} -\left(\frac{\partial H}{\partial p}\right)\bigg|_{\bar{p}} = \sum_{t=1}^{T} x_t(\bar{p}) = x(\bar{p}) = e.$$

The equilibrium map $p(e)$ is again the inverse of the demand function of the representative consumer. From Rockafellar (1970, p. 219), Corollary 23.5.1 we know that if $g$ is a continuous concave function on $\mathbb{R}_{++}^{I}$ then $p \in \partial g(x)$ iff $x \in -\partial h(p)$. It follows from this duality relationship that $\bar{p}$ is the unique equilibrium price vector for the social endowment $\bar{e}$ if and only if $\bar{p} = (\partial W/\partial e)|_{e=\bar{e}}$ and $-(\partial H/\partial p)|_{\bar{p}} = \bar{e}$. Given a finite set of observations on social endowments and market clearing prices, we can now characterize the refutable implications of Marshall's theory of value. A given data set rationalizes Marshall's general equilibrium model if and only if it is cyclically monotone. ∎

**Theorem 4** *The equilibrium map, $p(e)$, in Marshall's general equilibrium model is cyclically monotone in $e$, the social endowment.*

**Proof.** Because $g_i$ is strictly concave, $W(e)$ is strictly concave as well. By Theorem 24.8 in Rockafellar (1970) we know that the gradient map of a concave function is cyclically monotone, which implies that the gradient map $\vec{e} \to (\partial W/\partial e)|_{e=\bar{e}} = \bar{p}$ is cyclically monotone. ∎

All of our results: existence, uniqueness, optimality, tatonnment stability and refutability extend to the Marshallian general equilibrium model with production. Optimality, tatonnment stability and refutability follow from the well-known duality result in convex analysis that the supply function is the gradient of the profit function or conjugate of the cost function. As such, the supply function is also cyclically monotone .The cyclical monotonicity of aggregate supply and aggregate demand guarantee (i) that producer and consumer surplus are well defined, (ii) that the excess demand function is cyclically monotone and (iii) that the aggregate demand function and the aggregate supply function are refutable. As in Bewley (2007), existence is shown by maximizing the representative agent's utility function over the compact set of feasible production plans. If this set is strictly convex then the optimum is unique and the supporting prices are the equilibrium prices. See Bewley's chapter 8 on short-run equilibria for detailed proofs of existence, uniqueness, optimality and tatonnment stability. Refutability follows from Brown and Calsamiglia.

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