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A Complete Asymptotic Series for the Autocovariance Function of a Long Memory Process

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Abstract

An infinite-order asymptotic expansion is given for the autocovariance function of a general stationary long-memory process with memory parameter $d \in (-1/2, 1/2)$. The class of spectral densities considered includes as a special case the stationary and invertible ARFIMA(p, d, q) model. The leading term of the expansion is of the order $O\left(1/k^{1-2d}\right)$, where k is the autocovariance order, consistent with the well known power law decay for such processes, and is shown to be accurate to an error of $O\left(1/k^{3-2d}\right)$. The derivation uses Erdélyi's (1956) expansion for Fourier-type integrals when there are critical points at the boundaries of the range of integration - here the frequencies $\{0, 2\pi\}$. Numerical evaluations show that the expansion is accurate even for small k in cases where the autocovariance sequence decays monotonically, and in other cases for moderate to large k. The approximations are easy to compute across a variety of parameter values and models.

1 Introduction

Let $\{X_t, t \in \mathbb{Z}\}$ be a real-valued stationary process with spectral density

$$f_X(w) = f_u(w) \left|1 - e^{-iw}\right|^{-2d}, w \in \mathbb{R}, \tag{1}$$

where $f_u(w)$ is a short-memory spectrum and $d \in (-1/2, 1/2)$. A process satisfying (1) is long-memory persistent if $d \in (0, 1/2)$, short-memory if $d = 0$ and anti-persistent if $d \in (-1/2, 0)$. In the special case where $f_u(w)$ is the spectral density of a stationary and invertible ARMA(p, q) process, X_t is an ARFIMA(p, d, q) process. The present set-up allows for more general models in which $f_u(w)$ is required to satisfy some regularity conditions which are stated below in Assumption 1. The strongest of these is a smoothness requirement that enables an infinite order asymptotic expansion of the autocorrelation function.

Since Hurst's (1951) original article, which was written in a hydrological context, processes satisfying (1) have been studied extensively in a number of disciplines. Early overviews were given in Beran (1994), Robinson (1993) and Baillie (1996), the latter two papers discussing the relevance of these processes in economics and finance. Sowell (1992) derived an expression for the autocovariance function of a stationary and invertible ARFIMA(p, d, q) process which involves hypergeometric functions. One of the applications of this formula is in Gaussian maximum likelihood estimation. Since this early work, many new aspects of the model have been considered and the range of applications have widened considerably in recent years.

Of particular interest in finance, is the applied work with financial datasets involving the memory characteristics of interest rates and the nature of the Fisher equation (Phillips, 2005; Sun and Phillips, 2005), studies of volatility and squared returns (Giraitis et al. 2003, 2007; Robinson and Henry, 1999), and long-range dependence in realized volatility measurements with high frequency data (Lieberman and Phillips, 2006)

The present paper derives a complete (infinite-order) asymptotic expansion of the autocovariance function, $\gamma_X(k)$, corresponding to the long memory spectrum (1). The expansion formula is valid for all $d \in (-1/2, 1/2)$. The leading term of the expansion is of order $O\left(k^{-(1-2d)}\right)$, as $k \to \infty$, consistent with the well known power law decay of the autocorrelogram for such processes. The leading term is shown to have an error order of $O\left(k^{-(3-2d)}\right)$, indicating that it should deliver good accuracy for moderate values of k. Subsequent terms in the expansion decay according to additional powers of k^{-1}.

The derivation of the main formula uses an asymptotic expansion of the Fourier inversion formula for $\gamma_X(k)$, which can be written as

$$\gamma_X(k) = \int_{-\pi}^{\pi} e^{iwk} f_u(w) \left|1 - e^{-iw}\right|^{-2d} dw = \int_{0}^{2\pi} e^{iwk} f_u(w) \left(2\sin(w/2)\right)^{-2d} dw, \tag{2}$$

since $\left|1 - e^{iw}\right|^{-2d} = (2\sin(w/2))^{-2d}$. When $d > 0$, the latter integral has critical points (singularities in the integrand) at both boundaries 0 and 2π. For Fourier integrals of this type asymptotic expansions for large k were originally developed by

Erdélyi (1956) and are described in detail by, among others, Bleistein and Handelsman (1986). Numerical evaluations across different models and parameter values reveal that asymptotic expansions developed in this way provide approximations which are straightforward to compute in the present case and have good accuracy for large values of k and in some cases even for small k, particularly when the autocovariogram is positive and monotonically decreasing.

There are a number of possible applications of the general approximation formula, including standard time series diagnostic plots of theoretical autocovariance functions of long memory processes other than ARFIMA(p, d, q), against sample correlograms, as well as in Gaussian maximum likelihood estimation. The expansion also provides an alternative to Sowell's (1992) formula, in the ARFIMA(p, d, q) framework. One immediate implication of the result is that to $O\left(k^{2d-2}\right)$ the autocovariance function of X_t is equivalent to that of the simple ARFIMA$(0, d, 0)$ model, as noted in Lieberman and Phillips (2006).

The plan for the remainder of the paper is as follows. Section 2 sets up assumptions and presents the main result. Analysis follows in Section 3 and numerical accuracy is considered in Section 4. Section 5 concludes and proofs are in the Appendix.

2 Assumptions and Main Results

We impose the following conditions on the function $f_X(w)$ given in (1).

Assumption A1 *The periodic function* $f_u(w)$*, defined on* $(-\infty, \infty)$*, satisfies the following:*

(i) $f_u(w) = f_u(-w)$

(ii) $f_u(w) \geq 0$

(iii) $\int_{-\pi}^{\pi} f_u(w)\, dw < \infty$

(iv) $f_u(w) = f_u(w + 2\pi)$

(v) $0 < f_u(0) < \infty$

(vi) $f_u(w) \in C^{\infty}[0, 2\pi]$.

Assumption A2 $d \in (-1/2, 1/2)$.

Parts (i)-(iii) of Assumption A1 are necessary and sufficient for $f_u(w)$ to be the spectral density of a real-valued stationary process (e.g., see Brockwell and Davis, 1991, p 122). A1(iv) is a standard 2π periodic condition on $f_u(w)$. A1(v) and (vi) ensure that $f_u(w)$ is infinitely smooth, bounded, and bounded above the origin at the zero frequency, thereby eliminating the possibility of any antipersistent or long memory components or spectral poles away from the zero frequency (as might be caused by integrated seasonal effects, for instance). The smoothness requirement is essential for the development of a complete asymptotic series for $\gamma_X(k)$. Under A2, of course, X_t is stationary. The expansion for $\gamma_X(k)$ requires A1(iv)–A1(vi) and A2. Both A1 and A2 hold for stationary and invertible ARFIMA(p, d, q) processes.

Define

$$F(w) = f_u(w) \left(\frac{2\sin(w/2))}{w(2\pi - w)} \right)^{-2d}, \tag{3}$$

and denote by $F^{(n)}(w)$ the n'th order derivative of $F(w)$. Note that under Assumption A1(vi), $F(w)$ is infinitely differentiable at the two critical points $\{0, 2\pi\}$. The main result of the paper follows.

Theorem 1 *Under Assumptions A1 and A2,*

$$\gamma_X(k) \sim \sum_{n=0}^{\infty} \frac{\Gamma(n+1-2d)}{n! k^{n+1-2d}} \left\{ \frac{d^n}{da^n} \left[(2\pi - a)^{-2d} F(a) \right]_{a=0} (e^{\frac{\pi i}{2}(n+1-2d)} + (-1)^n e^{\frac{\pi i}{2}(n-1+2d)}) \right\}. \tag{4}$$

The expansion gives the following explicit two term approximation for the autocovariance function

$$\gamma_X(k) \sim 2f_u(0) \frac{\Gamma(1-2d)}{k^{1-2d}} \sin\{\pi d\} - \frac{\Gamma(3-2d)}{k^{3-2d}} \left[\frac{2d(1-2d)}{(2\pi)^{2d+2}} F(0) + \frac{1}{(2\pi)^{2d}} F''(0) \right] \sin(\pi d). \tag{5}$$

The proof, given in the Appendix, uses a standard integration by parts technique for Fourier-type integrals, allowing for the presence of singularities at the limits of integration - here the frequencies $\{0, 2\pi\}$. The technique is originally due to Erdélyi (1956) and is commonly used in applied mathematics. Bleistein and Handelsman (1987, Ch 3) provide a thorough treatment.

3 Features of the Expansion

This section provides some analysis of the expansion and some discussion of its implications.

1. As apparent from (5), the leading term of (4) has the very simple form

$$\gamma_X(k) \sim \frac{2f_u(0)\,\Gamma(1-2d)\sin(\pi d)}{k^{1-2d}} + O\left(k^{2d-3}\right), \tag{6}$$

which was given earlier in Lieberman and Phillips (2006) with an error of $O\left(k^{2d-2}\right)$. The implications of (6) are: (i) The autocovariance of X_t decays according to the power law $O\left(k^{2d-1}\right)$, which is well known, but the explicit result (6) in the case of a general short memory component $f_u(\omega)$ seems not to have appeared before; (ii) To order $O\left(k^{2d-3}\right)$, the autocovariance of order-k of an ARFIMA$(0,d,0)$ model with an error variance equal to $2\pi f_u(0)$ is equivalent to the order–k autocovariance of a the more general process X_t; (iii) The second term ($n=1$) in the expansion (4) is shown to be zero (see remark 2 below) in the Appendix and so the leading term has accuracy to order $O\left(k^{2d-3}\right)$. The latter properties justify the use of the simpler ARFIMA$(0,d,0)$ framework for approximate analysis in some more general cases, as discussed in Lieberman and Phillips (2006).

2. The two term expansion (5) involves the first and third terms of the series (4). The term of order $O\left(k^{-(2-2d)}\right)$ is

$$-2(2\pi)^{-2d}\frac{\Gamma(2-2d)}{k^{2-2d}}\left[F'(0)+\frac{d}{\pi}F(0)\right]\cos(\pi d)=0,$$

since $F'(0) = -\frac{d}{\pi} F(0)$, as shown in the Appendix. Thus the leading term (6) has accuracy to an error of $O\left(k^{-(3-2d)}\right)$.

3. The behavior of the series expansion (4) for a given value of k clearly depends on the properties of the sequence of derivatives $\frac{d^n}{da^n}\left[(2\pi - a)^{-2d} F(a)\right]_{a=0}$. The series is majorized by

$$2\sum_{n=0}^{\infty} \frac{\Gamma(n+1-2d)}{n! k^{n+1-2d}} \left[\left| \frac{d^n}{da^n} \left[(2\pi - a)^{-2d} F(a)\right]_{a=0} \right|\right],$$

which converges by the ratio test if

$$\limsup_{n\to\infty} \frac{(n+1-2d)}{(n+1)k} \frac{\left| \frac{d^{n+1}}{da^{n+1}} \left[(2\pi - a)^{-2d} F(a)\right]_{a=0} \right|}{\left| \frac{d^n}{da^n} \left[(2\pi - a)^{-2d} F(a)\right]_{a=0} \right|} < 1.$$

4. The expansion is valid for all $d \in (-1/2, 1/2)$. However, for $d = 0$, it collapses to

$$\begin{aligned} \gamma_X(k) &\sim \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{k^{n+1}} \left\{ e^{\frac{\pi i}{2}(n+1)} + (-1)^n e^{\frac{\pi i}{2}(n-1)} \right\} \\ &= \sum_{n=0, n \text{ odd}}^{\infty} \frac{F^{(n)}(0)}{k^{n+1}} \left\{ e^{\frac{\pi i}{2}(n+1)} + (-1)^n e^{\frac{\pi i}{2}(n-1)} \right\} = 0, \end{aligned}$$

because $F^{(n)}(0) = f_u^{(n)}(0) = 0$ for odd n for ARMA(p, q) processes, as $f_u(w)$ is even. This corresponds to the property that $\gamma_X(k)$ decays faster than any power law when $f_u(w)$ is a short memory spectrum.

5. For the stationary and invertible ARFIMA(p, d, q) case, the expansion (4) provides an alternative to the formula given in Sowell (1992, pp 173–174). Sowell's formula involves hypergeometric functions, whereas the above expression is in terms of derivatives of $F(w)$. Both formulae are easy to calculate on modern computers.

For processes other than ARFIMA(p, d, q), we are not aware of any alternatives to (4) other than direct numerical evaluation of the Fourier integral.

6. The expansion (4) can be applied as a diagnostic tool, in plotting theoretical autocovariance functions against sample correlograms, as well as in Gaussian maximum likelihood estimation when the approximations are good. The formula may also be useful for processes involving spectra besides those for rational ARFIMA(p, d, q) models.

4 Numerical Evaluation

In Tables 1 and 2 we report numerical values of truncated versions of (4) - one consisting of the leading term (6) only, designated app0, and the other consisting of the two-term expansion (5), designated app2. Higher order expansions can easily be computed but are unnecessary in the parameter configurations shown in Tables 1 and 2. The approximations were evaluated for the ARFIMA$(1, d, 1)$ and ARFIMA $(1, d, 2)$ models. The order of the autocovariance, as well as the values of the ARMA and d parameters were chosen at random, with d taking both positive and negative values in $(-1/2, 1/2)$. The benchmark for comparison was taken to be Sowell's (1992) formula, which we verified by numerical integration. All computations were carried out with MATHEMATICA.

Tables 1 and 2 reveal that, across the parameter values considered, the leading term of the approximation is extremely accurate when k is large. The two-term expansion is practically exact for k exceeding 45. In particular, for $k > 60$, app2 is accurate to 9 decimal places. Whether d is positive or negative does not appear to affect the accuracy of the expansions. The same holds for the range of ARMA parameters. In terms of computation time, both Sowell's and our formula are extremely quick to evaluate.

Further evidence on the adequacy of the approximations is given in Figures 1-3, where plots of app0, app2 and Sowell's (1992) formula are shown against k, in two ARFIMA $(1, d, 1)$ models. The case on which Figures 1 and 2 are based reveals that the approximations become indistinguishable from Sowell's (1992) formula for $k \geq 20$, but for $k < 15$ the leading term approximation completely fails to reflect the oscillatory behavior of the covariogram and the next correction term only provides a minor adjustment. The failure arises because the leading term is always non-negative and dominates the approximations. In the second case, shown in figure 3, the correlogram is non-negative and monotonic in k and the approximations seem reliable for all $k \geq 3$.

5 Conclusions

This note provides an explicit expression of the autocovariance function of a general long-memory process in terms of a complete asymptotic series. Outside the ARFIMA(p, d, q) framework, there appears to be no such known formula for the autocovariance sequence, and it seems that there is scope for the use of this formula in autocovariance–based time series applications. Calculations performed in a number of special cases indicate that the first two terms of the expansion provide reliable accuracy for large k in all cases and for small to moderate k when the true autocovariogram is positive and monotonically decreasing.

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APPENDIX

We start by giving the following theorem based on Bleistein and Handelsman (1986, p. 90-91), which is due to Erdélyi (1956, pp. 49-50).

Theorem 2 *If $F(w) \in C^{\infty}[a,b]$, and α and β are not integers, then*

$$I(k) = \int_a^b e^{ikw} (w-a)^{\alpha-1} (b-w)^{\beta-1} F(w)\, dw \tag{7}$$

has the following complete asymptotic series representation as $k \to \infty$

$$I(k) = I_a(k) + I_b(k),$$

where

$$I_a(k) \sim \sum_{n=0}^{\infty} \frac{d^n}{da^n} \left\{ (b-a)^{\beta-1} F(a) \right\} \frac{\Gamma(n+\alpha)}{n! k^{n+\alpha}} e^{\frac{\pi i}{2}(n+\alpha)+ika},$$

and

$$I_b(k) \sim \sum_{n=0}^{\infty} \frac{d^n}{db^n} \left\{ (b-a)^{\alpha-1} F(b) \right\} \frac{\Gamma(n+\beta)}{n! k^{n+\beta}} e^{\frac{\pi i}{2}(n-\beta)+ikb}.$$

Proof of Theorem 1: In view of (2)

$$\begin{aligned} \gamma_X(k) &= \int_0^{2\pi} e^{iwk} f_u(w) \left(2\sin(w/2)\right)^{-2d} dw \\ &= \int_0^{2\pi} e^{iwk} w^{-2d} (2\pi - w)^{-2d} F(w)\, dw, \end{aligned}$$

where

$$F(w) = f_u(w)\left(\frac{2\sin(w/2))}{w(2\pi - w)}\right)^{-2d}.$$

Under Assumption A1(vi), $F(w) \in C^{\infty}[0, 2\pi]$. Setting $a = 0$, $b = 2\pi$, and $\alpha = \beta = 1 - 2d$, it now follows from theorem 2 that

$$\gamma_X(k) = I_0(k) + I_{2\pi}(k),$$

where

$$\begin{aligned} I_0(k) &= \sum_{n=0}^{\infty} \frac{d^n}{da^n}\left[(2\pi - a)^{-2d} F(a)\right]_{a=0} \frac{\Gamma(n+1-2d)}{n!k^{n+1-2d}} e^{\frac{\pi i}{2}(n+1-2d)}, \\ I_{2\pi}(k) &= \sum_{n=0}^{\infty} \frac{d^n}{db^n}\left[b^{-2d} F(b)\right]_{b=2\pi} \frac{\Gamma(n+1-2d)}{n!k^{n+1-2d}} e^{\frac{\pi i}{2}(n-1+2d)+ik2\pi}. \end{aligned}$$

We obtain

$$\begin{aligned} \gamma_X(k) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+1-2d)}{n!k^{n+1-2d}} \left\{ \frac{d^n}{da^n}\left[(2\pi - a)^{-2d} F(a)\right]_{a=0} e^{\frac{\pi i}{2}(n+1-2d)} \right. \\ &\quad \left. + \frac{d^n}{db^n}\left[b^{-2d} F(b)\right]_{b=2\pi} e^{\frac{\pi i}{2}(n-1+2d)} \right\}. \end{aligned} \tag{8}$$

Now,

$$\frac{d^j}{db^j}\left[b^{-2d}\right]_{b=2\pi} = (-1)^j \frac{d^j}{da^j}\left[(2\pi - a)^{-2d}\right]_{a=0}, (j = 0, 1, 2, ...)$$

and noting that

$$F(2\pi - w) = f_u(w)\left(\frac{2\sin(w/2))}{w(2\pi - w)}\right)^{-2d} = F(w),$$

we deduce that

$$F^{(m)}(2\pi) = (-1)^m F^{(m)}(0), (m = 0, 1, 2, ...).$$

Thus,

$$\begin{aligned}
\frac{d^n}{db^n}\left[b^{-2d}F(b)\right]_{b=2\pi} &= \left[\left\{b^{-2d}+F(b)\right\}^{(n)}\right]_{b=2\pi} \\
&= \sum_{j=0}^{n}\binom{n}{j}\left(\frac{d^j}{db^j}\left[b^{-2d}\right]_{b=2\pi}\right)\left(F^{(n-j)}(2\pi)\right) \\
&= \sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{d^j}{da^j}\left[(2\pi-a)^{-2d}\right]_{a=0}(-1)^{n-j}F^{(n-j)}(0) \\
&= (-1)^n\frac{d^n}{da^n}\left[(2\pi-a)^{-2d}F(a)\right]_{a=0}.
\end{aligned}$$

Equation (8) then becomes

$$\begin{aligned}
\gamma_X(k) \sim & \sum_{n=0}^{\infty}\frac{\Gamma(n+1-2d)}{n!k^{n+1-2d}}\left\{\frac{d^n}{da^n}\left[(2\pi-a)^{-2d}F(a)\right]_{a=0}(e^{\frac{\pi i}{2}(n+1-2d)}\right. \\
& \left.+(-1)^n e^{\frac{\pi i}{2}(n-1+2d)})\right\},
\end{aligned}$$

giving the stated result.

We next proceed to calculate the explicit form of the expansion to the first three terms of (8). The first term is:

$$\begin{aligned}
& \frac{\Gamma(1-2d)}{k^{1-2d}}\left\{(2\pi)^{-2d}F(0)\left(e^{\frac{\pi i}{2}(1-2d)}+e^{-\frac{\pi i}{2}(1-2d)}\right)\right\} \\
= \ & (2\pi)^{-2d}F(0)\frac{\Gamma(1-2d)}{k^{1-2d}}2\cos\left\{\frac{\pi}{2}(1-2d)\right\} \\
= \ & 2f_u(0)\frac{\Gamma(1-2d)}{k^{1-2d}}\sin\{\pi d\}, \qquad (9)
\end{aligned}$$

using the fact that $\cos\left\{\frac{\pi}{2}(1-2d)\right\}=\sin(\pi d)$ and since

$$(2\pi)^{-2d}F(0)=f_u(0).$$

Some further calculations reveal that the second term ($n = 1$) in the expansion is zero, as shown below:

$$\begin{aligned}
&\frac{\Gamma(2-2d)}{k^{2-2d}}\left\{\frac{d}{da}\left[(2\pi-a)^{-2d}F(a)\right]_{a=0}\left(e^{\frac{\pi i}{2}(2-2d)}-e^{\frac{\pi i}{2}(2d)}\right)\right\} \\
= \; & (2\pi)^{-2d}\frac{\Gamma(2-2d)}{k^{2-2d}}\left[F'(0)+\frac{2d}{2\pi}F(0)\right]\left[e^{\pi i(1-d)}-e^{\pi id}\right] \\
= \; & -(2\pi)^{-2d}\frac{\Gamma(2-2d)}{k^{2-2d}}\left[F'(0)+\frac{d}{\pi}F(0)\right]\left[e^{-\pi id}+e^{\pi id}\right] \\
= \; & -2(2\pi)^{-2d}\frac{\Gamma(2-2d)}{k^{2-2d}}\left[F'(0)+\frac{d}{\pi}F(0)\right]\cos(\pi d)=0,
\end{aligned} \tag{10}$$

the final line following from the fact that

$$F'(0) = -2d(2\pi)^{2d-1}f_u(0) = -\frac{2d}{2\pi}F(0).$$

The third term in the expansion is

$$\begin{aligned}
&\frac{\Gamma(3-2d)}{2!k^{3-2d}}\left\{\frac{d^2}{da^2}\left[(2\pi-a)^{-2d}F(a)\right]_{a=0}\left(e^{\frac{\pi i}{2}(3-2d)}+e^{\frac{\pi i}{2}(1+2d)}\right)\right\} \\
= \; & \frac{\Gamma(3-2d)}{2!k^{3-2d}}\left[\frac{2d(2d+1)}{(2\pi)^{2d+2}}F(0)+\frac{4d}{(2\pi)^{2d+1}}F'(0)+\frac{1}{(2\pi)^{2d}}F''(0)\right]\left\{(-i)e^{-\pi id}+ie^{\pi id}\right\} \\
= \; & -\frac{\Gamma(3-2d)}{k^{3-2d}}\left[\frac{2d(2d+1)}{(2\pi)^{2d+2}}F(0)+\frac{4d}{(2\pi)^{2d+1}}F'(0)+\frac{1}{(2\pi)^{2d}}F''(0)\right]\sin(\pi d) \\
= \; & -\frac{\Gamma(3-2d)}{k^{3-2d}}\left[\frac{2d(1-2d)}{(2\pi)^{2d+2}}F(0)+\frac{1}{(2\pi)^{2d}}F''(0)\right]\sin(\pi d).
\end{aligned} \tag{11}$$

Combining (9) - (11) gives the following two term approximation to order $O\left(k^{-(3-2d)}\right)$

$$\begin{aligned}
\gamma_X(k) \sim \; & 2f_u(0)\frac{\Gamma(1-2d)}{k^{1-2d}}\sin\{\pi d\} \\
& -\frac{\Gamma(3-2d)}{k^{3-2d}}\left[\frac{2d(1-2d)}{(2\pi)^{2d+2}}F(0)+\frac{1}{(2\pi)^{2d}}F''(0)\right]\sin(\pi d),
\end{aligned} \tag{12}$$

giving the stated result.

Table 1: Autocovariances of the model $(1+\phi B)(1-B)^d X_t = (1+\theta B)\varepsilon_t$

k	ϕ	θ	d	Sowell	App0	App2
100	0.48	−0.71	0.146	0.000266641	0.000266921	0.000266641
20	0.48	−0.71	0.146	0.000813834	0.000834168	0.000812295
47	0.364	0.126	0.373	0.268009	0.268014	0.268009
39	0.619	0.296	0.417	0.599673	0.59968	0.599673
52	0.06	0.718	0.184	0.0534111	0.0534076	0.0534111
28	−0.364	0.188	0.339	0.923406	0.922897	0.923401
41	−0.645	0.285	0.219	0.521648	0.520171	0.521604
61	0.453	0.258	−0.289	−0.000255303	−0.000255311	−0.000255303
83	−0.127	0.721	−0.394	−0.000402104	−0.000401975	−0.000402104
34	0.256	−0.815	−0.483	-5.83159×10^{-6}	-6.63174×10^{-6}	-5.83106×10^{-6}

Sowell: Sowell's (1992) formula, verified with a numerical integration of the spectral density

App0: The one–term approximation

App2: The two–terms expansion

Table 2: Autocovariances of the model

$$(1+\phi B)\,(1-B)^d\,X_t = (1+\theta_1 B+\theta_2 B^2)\,\varepsilon_t$$

k	ϕ	θ_1	θ_2	d	Sowell	App0	App2
24	0.424	0.175	0.392	0.171	0.0334144	0.0333818	0.0334143
45	−0.363	0.275	0.202	−0.247	−0.0036091	−0.00359951	−0.00360902
49	−0.193	0.200	0.400	−0.277	−0.00287325	−0.00286532	−0.0028732
42	0.193	0.200	0.400	−0.450	−0.000448941	−0.000448048	−0.000448939
65	0.373	−0.109	0.608	0.372	0.42406	0.424035	0.42406
89	0.736	0.490	0.287	0.413	0.782226	0.78222	0.782226
76	0.520	0.666	−0.543	−0.476	−0.0000361517	−0.0000362119	−0.0000361517
38	0.412	−0.866	−0.431	0.389	0.0241483	0.024264	0.0241484
27	0.100	0.900	0.050	−0.216	−0.00496688	−0.00496127	−0.00496687
55	0.489	0.327	0.626	0.327	0.723552	0.723512	0.723552

Sowell: Sowell's (1992) formula, verified with a numerical integration of the spectral density

App0: The one–term approximation

App2: The two–terms expansion

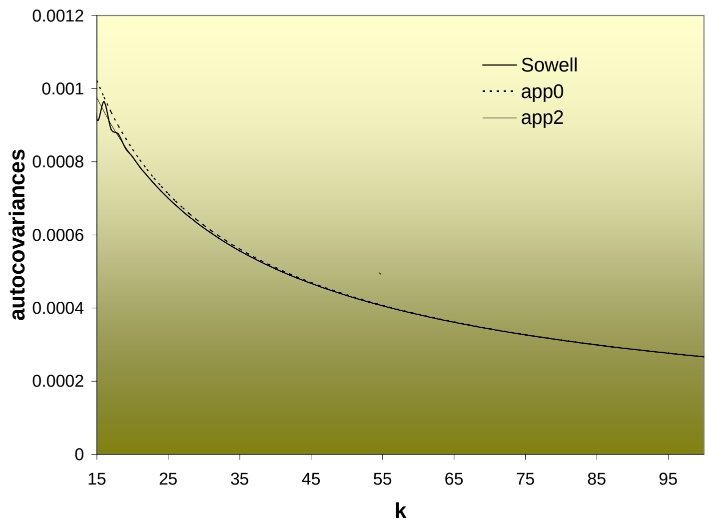

Figure 1: Autocovariogram for $(1 + 0.48B)\,(1 - B)^{0.146}\,X_t = (1 - 0.71B)\,\varepsilon_t$

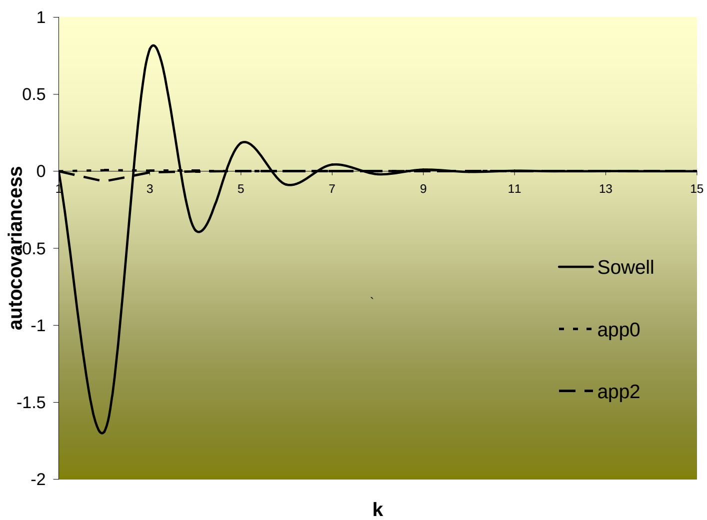

Figure 2: Autocovariogram for $(1 + 0.48B)\,(1 - B)^{0.146}\,X_t = (1 - 0.71B)\,\varepsilon_t$

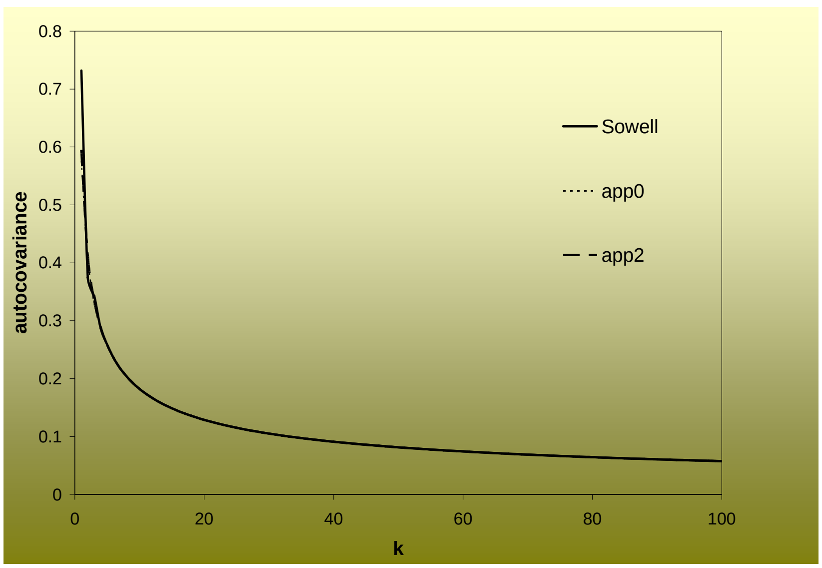

Figure 3: Autocovariogram for $(1 + 0.24B)\,(1 - B)^{0.25}\, X_t = (1 + 0.49B)\,\varepsilon_t$