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By

Enriqueta Argones, Itzhak Gilboa and Andrew Weiss

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COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Making Statements and Approval Voting*

Enriqueta Aragones†, Itzhak Gilboa‡, and Andrew Weiss§

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Abstract

We assume that people have a need to make statements, and construct a model in which this need is the sole determinant of voting behavior. In this model, an individual selects a ballot that makes as close a statement as possible to her ideal point, where abstaining from voting is a possible (null) statement. We show that in such a model, a political system that adopts approval voting may be expected to enjoy a significantly higher rate of participation in elections than a comparable system with plurality rule.

1 Introduction

It is well accepted that artists feel the need to express themselves. Psychological studies offer ample evidence that in many situations expressing emotions

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†Institut d'Anàlisi Econòmica, C.S.I.C. enriqueta.aragones@uab.es.

‡Tel-Aviv University and Cowles Foundation, Yale University. igilboa@post.tau.ac.il.

§Department of Economics, Boston University. aweiss@bu.edu

and feelings enhances well-being.[1] Casual observation suggests that many people feel better when they are listened to, when they can air their views, and vent their angers. This paper takes the view that people have a need to express themselves also in the political domain. Moreover, we maintain that the need to make a political statement is an important driving force behind voting and other political activities.

Some people have the privilege of making their views publicly known. For instance, a journalist who writes a daily column in a popular newspaper enjoys this status. But most people realize that their views are heard only by a tiny fraction of people in their society. In order to reach a wider audience, individuals can support or join organizations. A political party, for example, is noticed by many. Being a member of such a party makes one feel that one's voice is heard. Even the mere act of voting for a party already qualifies as a statement.[2]

Voting for a given party, however, does not necessarily make the statement that fully reflects the voter's opinion. It may well be the case that the voter agrees with the parties' platform on some issues, but disagrees with it on others. These points of disagreement may suffice for the voter to dislike any particular statement that voting allows her to make, and to prefer to abstain from voting. We would view such a voter as preferring to make a null statement, because this statement is closer to her ideal point than any statement she could make by voting.

In this paper we formalize the idea of voting as making statements, and we analyze the decision problem of a single voter who is facing a multi-party

[1] Specifically, people who suffered traumas were shown to be psychologically and physiologically heathier as a result of telling the trauma. Pennebaker, Barger, and Tiebout (1989) found that holocaust survivors who disclosed more in interviews about their traumatic experiences reported better health during the period that followed the interview. Foa, Feske, Murdock, Kozak, and McCarthy (1991) showed that post-traumatic stress disorder was effecitvely treated by repetitive recounting of the trauma.

[2] We do not attempt to distinguish between the need to make a statement and the need to be heard. In our model voting would be considered a statement, which is assumed to be heard by the political system.

election. We consider two voting systems and we investigate the consequences of voting as making statements over voter turn-out. The first is plurality vote, in which a votes selects one party, and the party with the most votes wins. The second is approval voting, where each voter may approve of any subset of parties, and the party who gets the approval of the largest number of voters wins.[3]

To compare the probability of abstention for the two voting systems, we assume that the statement made by a voter in an approval voting system is the average of the statements made by each of the party she approves of. Hence approval voting allows voters to make a much richer set of statements than does plurality rule. (This point has been made by Brams and Fishburn, 1983). It follows that less voters would find that the null statement best reflects their views in an approval voting system than in a plurality rule system. Our formal results show that when the parties' and the voter's positions are drawn at random, the difference between the two voting systems can be quite significant. In fact, as the number of issues and the number of parties tend to infinity, the probability of abstention in plurality rule tends to 1, whereas in approval voting it tends to 0. We also show that, when the number of issues is large, much fewer parties are needed to make all voters vote in an approval voting system, as compared to a plurality voting system. Taken together, these results indicate that the difference in voter turn-out between approval and plurality voting may be quite significant.

2 Making Statements and Rationality

Why would voters care about the statements they make rather than about the impact they actually have on policies? Would they not be highly irrational in so doing?

[3]A survey on the origins of approval voting can be found in Weber (1995). Its theoretical properties are analyzed in Brams and Fishburn (1978 and 1983) and its practical applications are discussed in Brams and Fishburn (2005).

Recent advances in formal voting models tend to focus on the prospect of changing the election outcome as the only incentive people have to vote. However, there may obviously be other incentives as well. For instance, Downs (1957) already pointed out that elections are expressions of political preference, and may also be devices for releasing personal aggression in legitimate channels. That is, casting one's vote in an election may be a way of expressing oneself, of making a statement. Indeed, the psychological literature establishes the need people have to make statements and to be heard. In this paper, we focus on the need to make statements as if it were the sole motivation for voting.

We believe that there is nothing inherently irrational about the need to make a statement. On the contrary, we find that this psychological need can generally be functional and adaptive. Moreover, one can see its long terms benefits for large groups even when it may not benefit individuals in the short run.

It is easy to see that an individual who is generally motivated by the need to make a statement will make her wants and demands known to her environment, will protest when she is being mistreated, and so forth. In a variety of environments, especially when one considers repeated interaction in small groups, these types of behavior may result in better treatment by other individuals, and may consequently lead to better functioning. It is therefore possible that the need to make statements has evolved, or has been learned, in repeated interactions within small groups, and that it also affects other types of interactions. According to this view, the need to make a statement in a presidential election is not directly functional, but it may be explained as an extension of the need to make a statement within family interactions, where it is functional.

Furthermore, the need to make a statement in the absence of an immediate personal benefit may be viewed as a type of altruistic behavior. Making a statement in a presidential election may be viewed as a contribution to a public good: society benefits from a high degree of involvement of its citizens in its political life, even though any particular individual may be tempted to free ride the involvement of others. Such contributions often make people feel good, even if they are aware that these contributions do not enhance their personal material well-being. It follows that one can explain the evolution and functionality of the need to make statements along lines similar to the explanations of altruistic behavior.

The need to make a statement in a political set-up can be likened to, or even derived from emotions. Making statements may sometimes be a way to express emotions such as anger, frustration, and so forth. Indeed, much of our discussion above parallels the debate on the rationality and functionality of emotions.

Over the past two decades, many authors have argued for the rationality of emotion (Frank (1989), D'Amasio (1994), LeDoux (1996), and others). The main point of this argument is that emotions need not be impulses that result in sub-optimal functioning. Rather, emotions may generate action tendencies that are functional, at least in the long run. Along the tradition of so-called evolutionary psychology, people have been able to explain the adaptive nature of many emotions, and to envisage an evolutionary mechanism that selects them.

Our argument for the functionality of the need to make statements is in line with this literature. Like emotions, the need to make statements may be learned or may have evolved in environments in which it was directly adaptive, but it has an impact on behavior also in other environments. Like emotions, the need to make statements may lead to behavior that is sub-optimal for the individual in the short run, but optimal for a large society or for the same individual in the long run.

3 Model and Results

The political environment includes a set of parties $T = \{1, ..., m\}$. Each party $j \in T$ is characterized by its positions on the various issues of concern. Let the set of issues be $I = \{1, ..., n\}$. Let $v^j \in \mathbb{R}^n$ be the vector denoting party j's position. That is, $v_i^j \in \mathbb{R}$ is the degree to which organization j supports issue i. Let V be the $n \times m$ matrix whose j-th column is party j's position, $v^j \in \mathbb{R}^n$.

We analyze the decision problem of a single voter. The voter considers every possible ballot she may cast as a vector $x = (x_1, ..., x_m) \in \mathbb{R}_+^m$, where x_j is the degree to which the ballot supports party j. The voting system determines a subset $F \subset \mathbb{R}_+^m$ of feasible values for x.

We wish to compare majority and approval voting. To this end, we need to specify which statement does the voter think she makes when she casts a certain ballot. We will assume that (i) abstention corresponds to the null statement $0 \in \mathbb{R}^n$; (ii) a vote for a single party corresponds to the position $v^j \in \mathbb{R}^n$ of that party; and (iii) a vote approving a non-empty set of parties corresponds to the arithmetic average of their positions.

More precisely, we assume that a ballot $x \in \mathbb{R}_+^m$ makes the statement $Vx = \sum_{j \in T} x_j v^j \in \mathbb{R}^n$, and the sets of feasible ballots (F) are defined below.

(i) **plurality rule**, in which a voter selects a single party. In this case the degree of support is $x^j = 1$ for the selected party and $x^j = 0$ for the others. Hence, the feasible set is $F = F^M = \{0\} \cup \{e^j\}_{j \leq m}$ (where e^j is the j-th unit vector in $\mathbb{R}^m$);

(ii) **Approval voting**, in which a voter may choose any subset of parties as her vote. We model an individual who selects a non-empty subset $S \subset T$ as choosing the vector $x = \frac{1}{|S|} \sum_{j \in S} e^j$, that is, as supporting each party to degree $\frac{1}{|S|}$. This reflects the fact that the strength of the statement made in favor of a party by endorsing it depends on the other parties one endorses. For instance, if the voter endorses all parties but one, she may well feel that her support of each one of the endorsed parties is rather weak. In this case, the feasible set is $F = F^A = \{0\} \cup \{ \frac{1}{|S|} \sum_{j \in S} e^j \}_{\varnothing \neq S \subset T}$.

Observe that in our formulation, a ballot $x \in \mathbb{R}^m_+$ satisfies $\sum_{j \in T} x_j \in \{0, 1\}$. That is, the voter can choose to abstain (equivalently, to cast a blank ballot or to vote for the empty set of parties[4]), in which case $x = 0$, or to vote, and then x is a point in the $(m-1)$ dimensional simplex.

A voter with an ideal point $w \in \mathbb{R}^n$ may be modeled as solving the following problem[5]

$$Min_{x \in F} \left\| \sum_{j \in A} x^j v^j - w \right\| \tag{1}$$

If the solution to this problem is $x = 0 \in \mathbb{R}^m$, the voter will abstain, because making no statement whatsoever will be closer to his views, w, than casting any ballot. Note that we do not assume that abstention results from the cost of voting. Rather, abstention may result from the fact that the voter feels dissatisfied with any statement that the political system allows her to make.

It is obvious that approval voting allows a larger set of possible statements. That is, $F^M \subset F^A$. Since abstention is possible in both systems $(0 \in F^M, F^A)$, it is readily observed that expanding the set F can only result in greater participation in the elections. Specifically, for any choice of positions of the parties and for any choice of positions for the voters, the set of voters who do not vote in plurality rule will be a subset of the same set for approval voting.

[4]In our model we do not distinguish between deciding not to participate in an election and casting an abstention vote. Admittedly, the two differ precisely from the point of view of making statements: an abstention vote may be viewed as making the statement that one does not make a statement. It is, indeed, quite different to make this statement from keeping silent. At this point, however, we do not model meta-statements of this type.

[5]This problem may be likened to the choice of optimal portfolio (see Arrow, 1953), where the parties' positions play the role of financial assets existing in the market, and the degree of support a voter lends to a party is analogous to the amount of an asset an agent wishes to hold.

To see that this inclusion may be strict, let us consider the following example. Suppose that there are two issues. Issue 1 is free trade, whereas issue 2 is legalized abortions. Party R takes the position $(1,1)$, meaning that it supports free trade agreements, as well as legalized abortions. Party L takes the position $(-1,1)$, standing for an opposition to free trade agreement, but support for legalized abortions. Consider a voter who feels that abortions should be legal, but does not have a clear opinion about free trade. Assume that this voter feels uneasy about identifying with either of the extreme views on free trade. She can see the arguments for and against the trade agreements, and she does not quite feel comfortable with them. Let us assume that her ideal point is $w = (0, .8)$.

Assume that parties R and L are the only parties in the system. Under plurality rule, the voter has to choose among the statements $\{(0,0), (1,1), (-1,1)\}$, and she finds that $(0,0)$ is the closest point to $w = (0, .8)$. By contrast, under approval voting, endorsing both R and L results in the statement $0.5(1,1) + 0.5(-1,1) = (0,1)$, which is closer to $w = (0, .8)$ than is $(0,0)$. That is, such an individual will participate in an approval voting, but will not participate in a plurality rule.

In this example, the voter has an ideal point that is rather moderate relative to the positions of the parties. However, F^M might be a strict subset of F^A also when the voter takes positions on each and every issue. Imagine that there are three issues and three parties. The voter's position is $(1,1,1)$. Each party agrees with the voter on two issues, and disagrees on the last. Thus, their position vectors are $(-1,1,1)$, $(1,-1,1)$, and $(1,1,-1)$. As can be easily verified, the voter will prefer to abstain rather than to vote to any single party (for instance, $\|(0,0,0) - (1,1,1)\| = \sqrt{3} < \|(-1,1,1) - (1,1,1)\| = 2$). By contrast, approving of all three parties yields the vector $(1/3, 1/3, 1/3)$ which is closer to $(1,1,1)$ than is $(0,0,0)$.

Since F^A contains exponentially more points than does F^M, one may expect that the number of voters who abstain in an approval voting system will be significantly lower than the corresponding number in a plurality rule system. We now turn to make this argument more precise.

We assume that each party is forced to make a choice on each and every issue. Thus, party j has a position $v^j \in \mathbb{R}^n$ with $v_i^j \in \{-1, 1\}$ for every i. This admittedly extreme assumption is intended to capture the intuition that a party that is silent on too many issues will not be considered credible by the voters. This assumption can be considerably weakened. However, the gist of our argument does require some restriction on the parties' positions. Without it, a party may position itself at the origin, in which case no voter would strictly prefer to abstain under either voting system.

We also assume that each individual has an ideal point $w \in \mathbb{R}^n$ that reflects a position on every issue. That is, $w_i \in \{-1, 1\}$ for every i. This assumption is made for mathematical tractability, and it is evidently highly idealized. However, we suspect that results similar to ours may hold also if this assumption is relaxed.

Under these assumptions, we state two results that attempt to capture the fact that a richer set of possible statements will make more individuals vote. The first result deals with the minimal number of parties one should have to make sure that all individuals vote. The second analyzes the probability that a particular individual would vote.

For a given number of issues, n, let $K_M(n)$ be the minimal number of parties that guarantee that each individual casts a ballot in a plurality rule system, that is, that for no individual is the solution to problem (1) at $x = 0$. Let $K_A(n)$ be the corresponding number for approval voting. We can now state

Theorem 1 *$K_M(n)$ is bounded below by an exponential function of n, whereas $K_A(n)$ is bounded above by 4.*

We now turn to the probabilistic result. Assume that the position of each party, as well as the position of each voter, on each issue, is 1 with probability 50%, and -1 with probability 50%. Assume further that all these random variables are independent. Let $P_M(n,m)$ be the probability that a voter would prefer to vote than to abstain, when there are n issues and m parties, and when the voting system is plurality rule. Let $P_A(n,m)$ be the corresponding probability for approval voting. Our next result deals with the limit of these probabilities when n tends to infinity and $m = n$.

Theorem 2 *As $n \to \infty$, $P_M(n,n) \to 0$ and $P_A(n,n) \to 1$.*

It will be clear from the proof that similar results hold whenever the number of parties, m, grows with the number of issues, but remains significantly smaller than the number of possible positions, 2^n. In this range approval voting will have a clear advantage over plurality rule: almost all voters will prefer to vote in an approval voting system, while they will prefer to abstain in a plurality rule system.

4 Discussion

Our model assumes that voting is the only type of political activity. In reality, however, individuals can resort to other political activities in order to "correct" the statement made by their vote. For example, suppose that, in a presidential election, John prefers the Republican candidate because of his position on free trade agreements. But John supports legal abortions, which the Republican candidate opposes. John may choose to vote for the Republican candidate, but then he may join a pro-choice group to offset his inadvertent support for the Republican candidate's views on abortion.

It is natural to extend our model to deal with a variety of political institutions, including parties, special interest groups, societies, NPOs, and so forth. Each such organization is identified with a certain position vector $v \in \mathbb{R}^n$, and the various ways in which they can be supported generates a richer menu of statements that an individual can make. In such a model, it is natural to assume that the individual has several distinct resources at her disposal, such as a ballot, time, or money.[6]

It stands to reason that the stark differences between majority and approval voting found above will be mitigated in the presence of other organizations, offering additional ways in which individuals can make statements. But since the set of statements that approval voting offers is a super-set of the corresponding set for plurality rule, one may expect that the argument for approval voting remain robust.

This paper takes the extreme view that voting is motivated solely by the need to make statements. As such, it offers a rather extreme alternative to the strategic voter paradigm: rather than assume that voters make realistic calculations about the probability that they affect the election result, we assume that the voters care only about the statements they make, and ignore any potential impact they may have on the result. A more satisfactory model of voting might include both types of motivation and the potential trade-off between them.[7]

[6] This point was made by Jean Tirole.

[7] Note that our only point of departure from the strategic rationality paradigm is in the specification of the utility function. In particular, if one accepts out assumptions about the utility functions, one may embed our model in a standard Bayesian-Nash equilibrium.

5 References

Arrow, K. J. (1953) "The Role of Securities in the Optimal Allocation of Risk-Bearing", *Econometrie*; as translated and reprinted in 1964, *Review of Economic Studies*, 31: 91-96.

Brams, S. J. and P. C. Fishburn (1978) "Approval Voting" *American Political Science Review* 72: 831-847.

Brams, S. J. and P. C. Fishburn (1983) *Approval Voting*, Boston: Birkhauser.

Brams, S. J. and P. C. Fishburn (2005) "Going from Theory to Practice: The Mixed Success of Approval Voting" forthcoming in *Social Choice and Welfare.*

D'Amasio, A. (1994), *Descartes' Error.* New York: Putnam.

Downs, A. (1957) *An Economic Theory of Democracy.* Harper Collins Publishers.

Foa, E. B., Feske, U., Murdock, T., Kozak, M. J., & McCarthy, P. R. (1991). Processing of threat-related information in rape victims. *Journal of Abnormal Psychology*, 100: 156-162.

Frank, R. H. (1988) *Passions within Reason: The Strategic Role of Emotions.* Norton.

LeDoux, J.E. (1996), *The Emotional Brain.* New York: Simon and Schuster.

Pennebaker, J. W., S. D. Barger, and J. Tiebout (1989) "Disclosure of Traumas and Health among Holocaust Survivors", *Psychosomatic Medicine*, 51: 577-589

Weber, R. J. (1995) "Approval Voting" *Journal of Economic Perspectives* 9: 39-49.

6 Appendix: Proofs

6.1 Proof of Theorem 1

We first consider approval voting. Consider the four vectors

$$a = \sum_i e_i;\ b = -\sum_i e_i$$
$$c = e_1 - \sum_{i>1} e_i;\ d = -e_1 + \sum_{i>1} e_i$$

and assume that there are 4 parties, A, B, C, D, with positions a, b, c, d, respectively. Consider a voter with an ideal point w. If $w_1 = 1$, this voter will find that voting for $\{A, C\}$ makes the statement e_1. This statement will be closer to w than will 0. Conversely, if $w_1 = -1$, the vote $\{B, D\}$ makes the statement $-e_1$, which is closer to w than is 0. Hence these four parties suffice to make everyone vote. This implies that $K_A(n) \leq 4$ for every n.

We now turn to bound $K_M(n)$ from below. Observe that an individual with an ideal point $w \in \{-1, 1\}^n$ will prefer to vote for party j, with position $v \in \{-1, 1\}^n$ rather than not to vote if and only if

$$\#\{\, i \,|\, w_i = v_i \} \geq \frac{3n}{4} \tag{2}$$

Indeed, if $\#\{\, i \,|\, w_i = v_i \} < \frac{3n}{4}$, the distance between w and v is greater than $\sqrt{\frac{n}{4}((-1) - (+1))^2} = \sqrt{n}$ whereas the distance between w and 0 is $\sqrt{n}$.

Consider a party with position v. Which voters would prefer voting for it rather than abstaining from voting? Denote $a(w, v) = \#\{\, i \,|\, w_i = v_i \}$. Obviously, a voter with $a(w, v) = 0$, namely, with $w = v$, would prefer voting for the party to abstaining. So would all voters whose w satisfies (2), or, equivalently, $a(w, v) < \frac{n}{4}$. How many such voter positions w exist?

There is one position w for which $a(w, v) = 0$, n positions w for which $a(w, v) = 1$, $\binom{n}{2}$ for which $a(w, v) = 2$, and so forth. Overall, the party will be preferred to zero by

$$N = \sum_{i<\frac{n}{4}} \binom{n}{i}$$

voter positions. However, there are

$$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

possible voter positions. A lower bound on $K_M(n)$ is therefore $\frac{2^n}{N}$. To see this, imagine that k parties are selected. Each is preferred to zero by N voter positions. Even if there is no overlap between these sets of N voter positions (one for each party), one cannot cover more than kN voter positions. To make sure that all voters prefer voting to abstention, we need to have $kN \geq 2^n$, or $k \geq \frac{2^n}{N}$. Hence the minimal number of parties, $K_M(n)$ also satisfies this inequality.

It is left to verify that $\frac{2^n}{N}$ is bounded by an exponential in n. To simplify the calculations that follow, we ignore fractions (and write $\frac{n}{4}$ for $\lfloor \frac{n}{4} \rfloor$, etc.). Let

$$P = \sum_{i \geq \frac{n}{4}}^{\frac{n}{2}} \binom{n}{i}$$

We will estimate the ratio $\frac{P}{N}$. For every $i < \frac{n}{4}$ we compare $\binom{n}{i}$ to $\binom{n}{i+\frac{n}{4}}$. Observe that

$$\frac{\binom{n}{i+\frac{n}{4}}}{\binom{n}{i}} = \frac{(n-i-\frac{n}{4}) \times ... \times (n-i)}{(i+1) \times ... \times (i+\frac{n}{4})} \geq$$

$$\left(\frac{n-i}{i+\frac{n}{4}}\right)^{\frac{n}{4}}$$

for $1 \leq i \leq \frac{n}{4}$, this ratio attains its minimum for $i = \frac{n}{4}$, where it equals $\left(\frac{\frac{3n}{4}}{\frac{n}{2}}\right)^{\frac{n}{4}} = \left(\frac{3}{2}\right)^{\frac{n}{4}}$. In other words, for $i \leq \frac{n}{4}$,

$$\binom{n}{i+\frac{n}{4}} \geq \left(\frac{3}{2}\right)^{\frac{n}{4}} \binom{n}{i}$$

This implies that

$$P = \sum_{i \geq \frac{n}{4}}^{\frac{n}{2}} \binom{n}{i} \geq \left(\frac{3}{2}\right)^{\frac{n}{4}} \sum_{i < \frac{n}{4}} \binom{n}{i} = \left(\frac{3}{2}\right)^{\frac{n}{4}} N$$

Finally,

$$K_M(n) \geq \frac{2^n}{N} \geq \frac{P}{N} \geq \left(\frac{3}{2}\right)^{\frac{n}{4}}$$

and we conclude that the minimal number of parties that are needed to make sure that all voters vote is exponential in n. □

6.2 Proof of Theorem 2

We first wish to show that $P_M(n,n) \to 0$ as $n \to \infty$. Recall that an individual with an ideal point $w \in \{-1,1\}^n$ will prefer to vote for party j, with position $v^j \in \{-1,1\}^n$, rather than not to vote if and only if

$$a(w,v^j) = \#\{\, i \,|\, w_i = v_i^j\} \geq \frac{3n}{4} \tag{3}$$

Fix a voter with an ideal point w. Consider the variable $a(w,v^j)$ for any j. These variables are i.i.d. Binomial random variables, with $a(w,v^j) \sim B(n,.5)$. Thus $E(a(w,v^j)) = .5n$ and $Var(a(w,v^j)) = .25n$. For large n, $\frac{a(w,v^j)}{n}$ is distributed approximately Normal with $\mu = .5$ and $\sigma = \frac{1}{2\sqrt{n}}$. It follows that

$$\Pr(a(w,v^j) \geq \frac{3n}{4}) =$$

$$\Pr(\frac{a(w,v^j)}{n} \geq .5 + \frac{\sqrt{n}}{2}\frac{1}{2\sqrt{n}}) \approx$$

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{\sqrt{n}}{2}}^{\infty} e^{-\frac{x^2}{2}}dx$$

For large enough t, $\int_t^{\infty} e^{-\frac{x^2}{2}}dx \leq e^{-\frac{t^2}{2}}$. Hence, for large n, the probability that the individual will prefer to abstain than to vote for party j is approximately

$$\Pr(a(w,v^j) < \frac{3n}{4}) \geq 1 - e^{-\frac{n}{8}}$$

and the probability that the individual will not find any party j preferable to abstention is approximated by

$$\left(1 - e^{-\frac{n}{8}}\right)^n = \left[\left(1 - e^{-\frac{n}{8}}\right)^{e^{\frac{n}{8}}}\right]^{\frac{n}{e^{\frac{n}{8}}}}$$

Since the expression in the square brackets tends to e^{-1}, and it is raised to a power $\frac{n}{e^{\frac{n}{8}}} \to 0$, we conclude that $\left(1 - e^{-\frac{n}{8}}\right)^n \to 1$ as $n \to \infty$, hence $P_M(n, n) \to 0$ as $n \to \infty$.

We now turn to study the limit behavior of $P_A(n, n)$. Assume without loss of generality that $w_i = 1$ for all $i \leq n$. We wish to show that, with probability that tends to 1, there is a subset of parties $S \subset \{1, ..., n\}$ for which $\frac{1}{|S|} \sum_{j \in S} v_i^j > 0$ for every $i \leq n$. Clearly, the point $\frac{1}{|S|} \sum_{j \in S} v^j$ will be closer to w than is zero, resulting in the individual's preference to vote for the subset S than to abstain.

Consider the vector v^j for a given party j. Its average, $\frac{1}{n} \sum_{i \leq n} v_i^j$, is distributed, for a large enough n, approximately Normal, with $\mu = 0$ and $\sigma = \frac{1}{\sqrt{n}}$. Given n and a realization of $\left(v^j\right)_j$, let S be the subset of parties $j \leq n$ for which $\frac{1}{n} \sum_{i \leq n} v_i^j > \frac{1}{\sqrt{n}}$. Observe that $|S|$ grows linearly in n. In fact, since, by the law of large numbers, $\frac{|S|}{n} \to\sim .1587$, for large enough n we may assume that $\Pr(|S| > .1n)$ is arbitrarily close to 1.

Consider the vectors $v^j \in \mathbb{R}^n$ for $j \in S$. They are independent (even conditional on being in S) and identically distributed. It follows that the distribution of their average, $\frac{1}{|S|} \sum_{j \in S} v^j$, is approximately multinormal. It is easy to see that their expectation is strictly positive in each coordinate. For large enough n, it follows that $\frac{1}{|S|} \sum_{j \in S} v^j$ will be strictly positive (in each coordinate) with probability that is arbitrarily close to 1. $\square$