DYNAMIC PRICE COMPETITION

By

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Abstract

We consider the model of price competition for a single buyer among many sellers in a dynamic environment. The surplus from each trade is allowed to depend on the path of previous purchases, and as a result, the model captures phenomena such as learning by doing and habit formation in consumption.

We characterize Markovian equilibria for finite and infinite horizon versions of the model and show that the stationary infinite horizon version of the model possesses an equilibrium where all the sellers receive an equilibrium payoff equal to their marginal contribution to the social welfare.

Keywords: Dynamic competition, marginal contribution, Markov perfect equilibrium, common agency.

JEL Classification: D81, D83

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1 Introduction

In this paper, we consider a model of Bertrand price competition for a single buyer among many sellers in a dynamic environment. We analyze the existence and efficiency of equilibria in models where the stage game payoffs to the buyer as well as the sellers may depend on the history of past purchases. Examples of dependence of this type include learning by doing for the sellers and habit formation for the buyer.

Bertrand price competition provides an attractive modeling approach for markets with differentiated commodities as it places the bargaining power in the hands of the players on the long side of the market. This results in a nontrivial sharing of the surplus arising from trades between the buyer and the sellers. If the buyer has unit demand, the equilibrium in the static game is efficient, and the sharing of economic surplus can be studied independently of any economic distortions. We extend the static model to a general dynamic environment. Issues such as surplus sharing and efficiency then require more careful analysis. If current choices have an impact on future surpluses, the intertemporal aspects of surplus sharing gain in importance. Consider for example an industry where an entrant has a technology that will achieve lower costs of production than the incumbent's technology, but whose initial costs are quite high. It may well be that the seller must sell at prices below costs in the initial periods in the expectation of the future profits. The ultimate success of the entrant depends on the degree to which the costs and benefits of the initial periods can be shared between the participants in the market.

In the model of this paper, a finite number of sellers offer differentiated products to a single buyer with unit demand over a discrete time horizon of either finite or infinite length. At the beginning of each period, the sellers choose simultaneously prices for their products and the buyer chooses the seller to supply the product (or possibly she chooses not to buy in that period). Because of assuming unit demands, we have also the alternative interpretation of the model as one where a number of firms compete in spot wage contracts for a given worker over time.[1] All the players discount future with the same discount factor δ. In order to allow for dynamic elements in the model, we allow the surplus from each trade to depend on the sequence of trades made in the previous periods. In the job matching model, such dynamic features arise naturally from learning on the job and participation in training programs. Hence the scope of the model is much larger than in a simple repetition of static price competition across periods.

The finite horizon model is analyzed first. By a simple example, we show that the existence of a pure strategy equilibrium cannot be taken for granted. If the surplus resulting from a purchase from seller i depends on the history of sales by sellers other than i, the model has a direct intertemporal externality. Hence there is really no reason to expect that a model with (spot) prices as the only feasible transfers would be well behaved with respect to the efficiency of the equilibrium allocation. The restrictive element in price competition is that seller i can offer (positive or negative) transfers to the buyer only in conjunction with a purchase of the product of seller i. Yet with externalities, it is conceivable that seller i would sometimes like to induce the buyer to purchase from j and would be willing to support the purchase of product j with a subsidy. To rule out this class of problems, we assume that the surplus generated by the purchase of a given seller's product depends only on the number of past purchases from that seller. This is consistent with the examples of habit formation and learning by doing and it also accommodates job specific learning in the job matching model. Surprisingly, the equilibria in this case may be inefficient as well and there may be a multiplicity of them.

In contrast, the results that we obtain in the stationary infinite horizon version of the model are much more in line with the static model. In particular, the model always has an efficient equilibrium and the payoffs are uniquely determined in a large

[1] Models of this type include Jovanovic (1979) and Miller (1984) in a competitive market and Felli & Harris (1996) in a duopsonistic labor market.

class of games. We show that in the efficient equilibrium, the payoff to each seller coincides with her marginal contribution to the social surplus in the model.

The equilibrium in the infinite horizon model is derived by using the well known "guess and verify" method from dynamic programming. But in stark contrast to the received use of this method, our guess does not rely on any functional or parametric specification of the value function. The novelty in our argument is that the value function of each seller is the guessed to be the difference of two general value functions generated by different socially optimal programs. The equilibrium is then established by demonstrating that the constituting social programs have certain structural properties. This allows us to analyze the value functions of the sellers as if they were the result of single agent optimization problems rather than multi-agent strategic interactions.

In order to keep the arguments as simple as possible, we consider only deterministic models. The techniques we use extend, however, to the stochastic case. Hence e.g. models of learning about match quality can be analyzed with the tools developed in this paper. We defer a fuller discussion of how the argument is extended to the conclusion.

This paper is related to two of our earlier papers. In Bergemann & Välimäki (1996), we analyzed an infinite horizon model of dynamic price competition with *two* sellers and uncertainty. Felli & Harris (1996) considered a similar model, yet set in labour market environment as a job matching model, in which two buyers (the employers) compete for a single seller (the employee) in an infinite horizon continuous time model with Brownian motion. In the current paper, we identify directions in which the results of these two papers extend (the number of sellers), and also dimensions along which the results cannot be pushed any further (finite horizon models).

In Bergemann & Välimäki (2003), we analyze a similar dynamic environment but there the sellers compete in a menu auction rather than in prices. Formally, that model is a dynamic version of the common agency model first analyzed in Bernheim

& Whinston (1986). In a menu auction the sellers are allowed to offer transfers contingent on the purchase decision of the buyer. Thus seller i is allowed to offer a transfer to the buyer in a particular period even in the event that the buyer purchases from seller j in that period. Each seller can therefore cross subsidize the buyer for her purchases with other sellers. This rich set of transfers allows us to establish the efficiency of the dynamic allocation in the presence of intertemporal externalities. As such transfers and subsidies are rarely observed in actual economic situations, we investigate in the current paper when simple spot prices are sufficient for guaranteeing the efficient allocation in the dynamic model.

The paper is organized as follows. Section 2 describes the model, defines the Markov Perfect equilibrium and introduces the notion of marginal contribution. Section 3 analyzes the finite horizon model. Section 4 considers the infinite horizon case. It provides sufficient conditions for the existence of an efficient equilibrium in terms of properties of the marginal contributions. Section 5 considers an example and establishes that the marginal contributions satisfy the desired properties. Section 6 shows that in a wide class of environments, the marginal contribution equilibrium is the unique Markov Perfect equilibrium. Section 7 concludes and suggests further questions for research.

2 The Model

We consider the following stage game model of price competition for a buyer with a unit demand. There are I sellers in the market, and we denote the set of sellers by $\mathcal{I} = \{1, ..., I\}$. Denote the surplus generated in the purchase of seller i's product by x_i. The sellers set simultaneously prices p_i for their products. At the end of the stage game, the buyer chooses either one of the sellers or does not purchase at all. We denote the buyers choice by $a \in \{0, 1, ..., I\}$, where $a = 0$ is interpreted as no purchase and $a = i$ denotes a purchase from seller i. The stage game payoffs for the sellers are given by

$$\pi_i(p_1, ..., p_I, a) = p_i \text{ if } a = i,$$
$$\pi_i(p_1, ..., p_I, a) = 0 \text{ if } a \neq i.$$

The buyer's stage game payoff is given by

$$u_i(p_1, ..., p_I, a) = x_i - p_i \text{ if } a = i.$$

By setting $x_0 = p_0 = 0$, we incorporate the payoffs from the case of no purchases as well.

This (extensive form) stage game is repeated in a discrete time model with $t = 0, 1, ..., T$. We analyze separately the cases where $T < \infty$ and $T = \infty$. All of the players discount future at discount factor $\delta \leq 1$. In the infinite horizon model, we will assume that $\delta < 1$ and that the payoff criterion is the discounted sum of payoffs. At stage t, the actions in all previous periods are observable to all players. A history in the game is a sequence of prices and decisions. More formally, we define histories inductively by letting $h^0 = \emptyset$ and $h^t = h^{t-1} \cup \{p_0^{t-1}, p_1^{t-1}, ...p_I^{t-1}, a^{t-1}\}$, where $\{p_0^{t-1}, p_1^{t-1}, ...p_I^{t-1}, a^{t-1}\}$ are the actions chosen in period $t-1$. Let H^t denote the set of all possible histories in period t and $H = \cup_{t=0}^{\infty} H^t$.

We are interested in a dynamic version of the stage game. Hence we allow the stage game surpluses x_i to depend on h^t. At the same time, we do not want to make the surpluses dependent on past pricing decisions by the sellers. As we also want to avoid calendar time having a direct effect on the surpluses, we are led to consider the vector

$$\theta(h) \triangleq (t_0(h), ..., t_I(h)),$$

where $t_i(h)$ counts the number of times that alternative i was chosen by the buyer along history h, as the relevant state variable summarizing the history. Notice that with this choice for a state variable, we ignore the importance of the order in which the sellers made their sales. Thus we assume that

$$x_i(h) = x_i(\theta(h)) \text{ for all } i \in \{0, ..., I\} \text{ and } h \in H.$$

In fact, in most of this paper we make the stronger assumption that the payoffs from a given seller depend only on the past purchases with that seller.

Definition 1 (Independent Rewards)
The payoffs display independent rewards if for all i and θ: $x_i(\theta(h)) = x_i(t_i(h))$.

From now on, we define a dynamic price competition game Γ to be a collection of functions $\{x_0(\theta), x_1(\theta), ..., x_I(\theta); T\}$ for all θ such that $\Sigma_i t_i(\theta) \leq T$.

A pure behavior strategy for seller i is a sequence of functions $\mathbf{p}_i = \{p_i^t\}_{t=0}^{\infty}$, where

$$p_i^t : H^t \rightarrow \mathbf{R}.$$

The buyer's pure behavior strategy is similarly a sequence $\mathbf{a} = \{a^t\}_{t=0}^{\infty}$, where

$$a^t : \mathbf{R}^I \times H^t \rightarrow \{0, 1, ..., I\}.$$

We are interested in the impact of the payoff relevant history on future play. In other words, we want to consider only Markov perfect equilibria of the game, as defined by Maskin & Tirole (2001). In order to get a precise definition for the payoff relevant state variable, we have to define a set of equivalence classes on the set all possible states Θ. Fix the dynamic price competition game Γ. Each possible θ induces a continuation price competition game $\Gamma(\theta)$ in the standard fashion. We partition Θ into a (possibly infinite) family of subsets $\{\Theta_k\}_{k=1}^{H}$ by the requirement that

$$\Gamma(\theta) = \Gamma(\theta') \Leftrightarrow \theta, \theta' \in \Theta_k \text{ for some } k.$$

In the generic case, a purchase from seller i leads to a payoff relevant state that is different from that following a purchase from seller $j \neq i$. The collection $\{\Theta_k\}_{k=1}^{H}$ forms then the payoff relevant set of states for the dynamic price competition game. We say that a strategy for seller i is Markovian if for all h and h' such that $\theta(h), \theta(h') \in \Theta_k$ for some k, we have $p_i(h) = p_i(h')$. The buyer's strategy is Markovian if for all h and h' such that $\theta(h), \theta(h') \in \Theta_k$ for some k, and for all p, we have $a(h,p) = a(h',p)$.

Definition 2 (Markov Perfect Equilibrium)
A collection $(\mathbf{p}_1, ..., \mathbf{p}_I, \mathbf{a})$ *is a Markov perfect equilibrium if*

1. *for all* i, $\mathbf{p}_i$ *is a best response to* $(\mathbf{p}_{-i}, \mathbf{a})$ *after all histories and* $\mathbf{a}$ *is a best response to* $(\mathbf{p}_1, ..., \mathbf{p}_I)$ *after all histories*;

2. *all players use Markovian strategies.*

In much of what follows, we will concentrate on a refinement of the Markov perfect equilibrium called a cautious equilibrium. For an arbitrary history h we write the continuation payoffs to the buyer and seller i respectively as $V_B(h)$ and $V_i(h)$.

Definition 3 (Cautious Equilibrium)
A Markov perfect equilibrium is a cautious equilibrium if for $a(\theta, p_1, ..., p_I) \in \{0, 1, ..., I\}$, *and all* $i \neq a(\theta, p_1, ..., p_I)$,

$$\delta V_i(\theta, a(\theta, p_1, ..., p_I)) = p_i(\theta) + \delta V_i(\theta, i),$$

where $a(\theta, p_1, ..., p_I)$ *denotes the equilibrium choice rule of the buyer and* (θ, j) *denotes the state vector after state* τ *followed by the choice of alternative* $j \in \{0, 1, ..., I\}$.

The basic idea behind this definition is that no seller should be willing to offer prices that make the seller worse off relative to the equilibrium if accepted. In the static version of this price competition model, cautious equilibrium is equivalent to equilibrium in weakly dominant strategies.[2]

[2] The same equilibrium notion has been used in the equilibrium analysis of Bergemann & Välimäki (1996) and Felli & Harris (1996).

In the equilibrium analysis, we shall make repeated use of arguments based on the social efficiency of paths along the game. We therefore conclude this section by defining socially optimal payoffs for different versions of the game. These concepts will be used repeatedly in the sections that follow. We introduce the indicator function $\mathbf{1}_{\{a^t=i\}}$ to describe the realized payoffs as a function of the choice behavior in period t. More precisely, let $\mathbf{1}_{\{a^t=i\}} = 1$ if $a^t = i$ and $\mathbf{1}_{\{a^t=i\}} = 0$ otherwise. The total surplus along path $\{a^t\}$ is given by

$$\sum_{t=0}^{T} \delta^t \left(\sum_{i=1}^{I} x_i \left(\theta \left(h\right)\right) \mathbf{1}_{\{a^t=i\}} \right).$$

The social surplus is split between the buyer and the sellers. The intertemporal payoff to the buyer from a sequence of choices $\{a^t\}$ is simply

$$\sum_{t=0}^{T} \delta^t \left(\sum_{i=1}^{I} \left(x_i \left(\theta \left(h\right)\right) - p_i \left(\theta \left(h\right)\right)\right) \mathbf{1}_{\{a^t=i\}} \right),$$

and the intertemporal payoff to seller i is given by

$$\sum_{t=0}^{T} \delta^t \left(p_i \left(\theta \left(h\right)\right) \mathbf{1}_{\{a^t=i\}} \right).$$

Because of the quasi-linear payoff specification, Pareto efficiency coincides with total surplus maximization in the game. Therefore we let

$$W\left(\theta_0\right) \triangleq \max_{\{a^t\} \in I^T} \sum_{t=0}^{T} \delta^t \left(\sum_{i=1}^{I} x_i \left(\theta \left(h\right)\right) \mathbf{1}_{\{a^t=i\}} \right),$$

denote the *social value* of the game in period 0 at state θ_0. We can similarly define the continuation values $W(\theta)$ from an arbitrary state vector θ onwards. We will also make use of social values to the game where some sellers have been excluded. We let

$$W^{-S} \triangleq \max_{\{a^t\} \in (\mathcal{I} \setminus S)^T} \sum_{t=0}^{T} \delta^t \left(\sum_{i \in (\mathcal{I} \setminus S)} x_i \left(\theta \left(h\right)\right) \mathbf{1}_{\{a^t=i\}} \right)$$

denote the social value in the game where all sellers in set S have been removed.

The next definition is the key definition for this paper. The *marginal contribution of seller* i to the social welfare is defined as

$$M_i(\theta) \triangleq W(\theta) - W^{-i}(\theta).$$

We may also define the marginal contribution of seller i in the game where seller j has been removed as

$$M_i^{-j}(\theta) \triangleq W^{-j}(\theta) - W^{-i\cup j}(\theta).$$

By $M_S(\theta)$, we denote the marginal contribution of a coalition S of sellers:

$$M_S(\theta) \triangleq W(\theta) - W^{-S}(\theta).$$

Finally, we introduce notation

$$W(\theta\,|k),\ M_i(\theta\,|k),\ \text{etc.}$$

to describe the social values and marginal contributions along paths that start with an arbitrary alternative k in the initial period, but follow the (conditionally) socially optimal path in all subsequent periods.

3 Finite Horizon Equilibrium

In this section, we present three examples that illustrate how changes in the competitive positions of the sellers can have problematic consequences for the efficiency of the equilibrium. The first example addresses direct intertemporal externalities between the sellers. The second and third example consider independent reward payoffs and illustrate the role of fixed finite horizons for the competition of the sellers.

Consider first an example where three sellers, $i \in \{1, 2, 3\}$ are selling a good to a buyer with unit demands per period in a two-period economy. To fix ideas, assume that the first firm sells champagne, the second red wine and the third a dessert wine.

Assume also that the first glass results in a utility of u for the buyer regardless of the specific choice, or:

$$x_i(0) = u > 0 \text{ for all } i.$$

The twist in this example comes from the fact that after consuming the wine of seller i, the consumer only wants to consume the wine of seller $i+1$ whose product yields utility v, and we assume that $\delta v > u$. The second glass of wine thus has a higher discounted utility than the first glass. (All summations and subtractions on the set of sellers are to be interpreted as modulo 3 in this section.) No other choice gives any utility to the buyer in period 1.

Observe that in this model there are no externalities in the stage game between the different sellers, but obviously the sales of one producer affect the values of other sellers in future periods. We solve for equilibria in the model by backwards induction. Denote by $V_B(j)$, $V_i(j)$ the continuation payoffs to the buyer and seller i in the second period, respectively, if j was the seller in period 0. The cautious equilibrium payoffs in period 1 are given by $V_i(j) = v$ if $j = \{i-1\}$ and $V_i(j) = 0$ otherwise. In other words, only the seller who has the desirable product in period 1 makes a positive profit equal to the value of the product. In the cautious equilibrium, we also have that $V_B(j) = u$ if $j = 0$, and $V_B(j) = 0$ otherwise. Notice that if the buyer chose not to make a selection in period 0, then the sellers remain symmetric in period 1 and competition leaves the buyer with all the surplus.

In order to show that this game does not have any cautious equilibria, it is sufficient (by symmetry) to show that there is no cautious equilibrium where seller 1 makes the sale in the first period. To this end, assume to the contrary that seller 1 makes a sale in the first period in a cautious equilibrium. Then by Bertrand pricing, the buyer must be indifferent between buying from seller 1 and another action. There are three possibilities: the indifference can hold vis-a-vis the second seller, the third seller or the option of not buying. The following argument shows that none of these alternatives is consistent with a cautious equilibrium.

First assume that the buyer is indifferent between seller 1 and 2. The equilibrium indifference condition for the buyer is given by

$$u - p_1 + \delta 0 = u - p_2 + \delta 0,$$

and hence requires that:

$$p_1 = p_2.$$

On the other hand, cautious pricing implies for seller 2 that he is indifferent between making a sale today and making a sale tomorrow, or

$$p_2 = \delta v.$$

But by assumption $\delta v > u$, and hence we are lead to

$$u - p_1 = u - p_2 < 0,$$

contradicting optimality for the buyer.

Next assume that the indifference is between sellers 1 and 3. Then again $p_1 = p_3$ by buyer indifference and $p_3 = 0$ by cautious pricing, as seller 3 will not be able to make a sale after seller 1 was chosen in period 0. But then seller 1 can gain by deviating to a higher price, leaving the buyer to seller 3 in period 0 and allowing seller 1 to extract a payoff of $\delta v > u$ in the subsequent period.

The remaining possibility is that the buyer is indifferent between seller 1 and not buying. The buyer's indifference in this case requires that

$$u - p_1 = \delta u, \quad \text{or} \quad p_1 = (1 - \delta)\, u.$$

But then seller 3 can undercut profitably by setting

$$p_3 = (1 - \delta)\, u - \varepsilon.$$

As a result, this two period game does not have cautious equilibria. It should be noted that this game does have a subgame perfect equilibrium in pure strategies.

For example, first period prices $p_1 = p_2 = p_3 = 0$ are consistent with equilibrium if the buyer is to choose seller 1 on equilibrium path and following any deviations by either seller 2 or 3. If seller 1 deviates, then the buyer should buy from seller 2. It is easily verified that this configuration is sequentially rational, but seller 2 is not using a cautious strategy.

The problem arises in this example because sales by seller i in period t have a direct impact on the rewards from the sales by seller j in period $t+1$. Such dependence is a clear manifestation of an intertemporal externality between the sellers, and there is no a priori reason to believe that such externalities can be dealt with in a model where transaction prices are the only transfer instruments. For this reason we ruled out externalities of this type with the assumption of independent rewards. It removes direct intertemporal externalities between the sellers.

The next two examples are meant to show that within finite horizon models, there is a subtle, but important indirect externality even under the assumption of independent rewards. Consider first a model with three firms where the payoffs are given by the following matrix:

	$x_1(\cdot)$	$x_2(\cdot)$	$x_3(\cdot)$
$t_i = 0$	2	2	0
$t_i = 1$	0	0	3

The table reads as follows for alternative 1. The payoff from alternative 1 is 2 when it is has never been used before, or $t_1 = 0$ and 0 when it has been used once before, or $t_1 = 1$. The fixed time horizon is assumed to be given by $T = 1$ and the discount factor δ is close to one. The efficient allocation would prescribe seller 1 and 2 to each realize their high valuation 2, however in the unique cautious equilibrium seller 3 is successful in period 0 and 1. To see why there cannot be an efficient equilibrium note that if alternative 1 or 2 is chosen in the first period, then the outside option value from alternative 3 vanishes altogether. This implies that the buyer cannot guarantee herself any surplus in period 1. By setting price $-2(1-\delta) - \varepsilon$ in $t = 0$, seller 3 can guarantee the buyer a higher total payoff than what is individually rational for either of the two sellers. Two points are worth observing here. First, because of the finite time horizon, the outside option value offered by a given seller changes over time even when that seller is not chosen. Second, the equilibria would be quite different if the sellers were allowed to write binding contracts with the buyer.

The next example shows that even when the game has an efficient equilibrium, it need not be unique. Consider the following two alternatives with a payoff stream as described below:

	$x_1(\cdot)$	$x_2(\cdot)$
$t_i = 0$	$1+\varepsilon$	1
$t_i = 1$	0	0

where $\varepsilon > 0$. The fixed time horizon is again given by $T = 1$ and the discount factor is again δ close to one. In this game, it is possible to support a period 0 choice of 1 as well as a choice of 2 in cautious equilibrium. To see how the inefficient equilibrium arises, we observe first that the buyer can always guarantee herself a payoff of δ by refusing all offers in period 0. By refusing all offers in period 0, the buyer puts the sellers in very competitive position in period 0 and will receive a (discounted) payoff of $\delta \cdot 1$ in the unique cautious equilibrium of the continuation game. We can now establish a cautious equilibrium in period 0 where seller 2 offers a price $p_2 = 1 - \delta$ and seller 1 prices cautiously at $p_1 = \delta(1+\varepsilon)$. It is easy to verify that for $\delta \approx 1$, these prices induce the buyer to select seller 2 in period 0. Obviously a similar equilibrium can be constructed where period 0 sales are made by seller 1. In this example, multiplicity arises since sales by firm i increase future profitability of firm j and as a result, there is little incentive to compete for the buyer in the first period. Notice that in both of the examples above, the buyer has a period 0 choice available that reduces the rents of the efficient sellers. In the first example, this choice (i.e. seller 3) is exercised in equilibrium, in the second example, this choice (seller 0) serves only as an outside option. But in both cases, the competitiveness of the situation was raised by reducing the number of periods in which the sellers could offer their product. In contrast, in an infinite horizon environment, the buyer can only postpone, but never completely eliminate periods of sale.

Finally, it is interesting to note that if the firms were allowed to bid in menu contracts as in Bergemann & Välimäki (2003), then all of the equilibria in both games would satisfy allocative efficiency. Hence we conclude that restricting our attention to equilibria in dynamic (spot) prices may induce efficiency losses in the model, and this motivate the use of more sophisticated contracts in such environments.

4 Infinite Horizon Equilibrium

In demonstrating the existence of an efficient equilibrium, we make use of the guess and verify method of dynamic programming. In this approach, we assume that the equilibrium path is socially efficient after all possible histories. Furthermore, we assume that each seller is paid her marginal contribution in equilibrium. In other words, each seller j gets as her payoff the difference of the social surplus in the model and the social surplus in the model where seller j has been removed. These assumptions pin down the price of the successful seller and the buyer's purchasing decision at all histories. The remaining prices can be recovered from the requirement of cautious pricing. The main task is then to verify that no individual in the model has an incentive to deviate from this guess.

In this section we derive sufficient conditions for the existence of an efficient cautious equilibrium where the sellers' equilibrium payoffs coincide with their marginal contributions. These conditions relate to the properties of the social value function and the marginal contributions. We shall then verify these properties in Section 5 which studies the social programs rather than the equilibrium programs. We assume from now on that the model satisfies independent rewards.

The equilibrium conditions can be written as follows. The (weak) indifference conditions for the buyer with i as the successful seller are:

$$x_i(\theta) - p_i(\theta) + \delta V_B(\theta, i) \geq x_j(\theta) - p_j(\theta) + \delta V_B(\theta, j), \ \forall j \neq i, \tag{1}$$

where the inequality holds for at least one seller, say $k \neq i$, as an equality,

$$x_i(\theta) - p_i(\theta) + \delta V_B(\theta, i) = x_k(\theta) - p_k(\theta) + \delta V_B(\theta, k). \tag{2}$$

The conditions imposed by cautiousness on the pricing policies are

$$p_j(\theta) = \delta V_j(\theta, i) - \delta V_j(\theta, j), \tag{3}$$

and

$$p_i(\theta) \geq \delta V_i(\theta, k) - \delta V_i(\theta, i). \tag{4}$$

In this section, we seek to derive conditions for an efficient equilibrium and i shall always identify the socially efficient seller.

We now guess that the equilibrium value function for each seller j is her marginal contribution, or:

$$V_j(\theta_t) = M_j(\theta_t). \tag{5}$$

We then verify that the guess, expressed by (5), actually satisfies the efficient equilibrium conditions (1)-(4). With the guess we can rewrite the conditions as:

$$x_i(\theta) - p_i(\theta) + \delta\left(W(\theta, i) - \sum_{l=1}^{I} M_l(\theta, i)\right) \tag{6}$$

$$\geq x_j(\theta) - p_j(\theta) + \delta\left(W(\theta, j) - \sum_{l=1}^{I} M_l(\theta, j)\right), \ \forall j \neq i,$$

with at least one inequality holding as equality, and

$$p_j(\theta) = \delta M_j(\theta, i) - \delta M_j(\theta, j), \tag{7}$$

and

$$p_i(\theta) \geq \delta M_i(\theta, k) - \delta M_i(\theta, i).$$

We first derive sufficient conditions for the indifference condition of the buyer. We do this by guessing that the inequality (1) is satisfied as an equality for the seller who would be the socially optimal seller in the absence of i. We denote this seller by j^{-i}. As the marginal contribution property is conjectured to hold in every continuation game, the price of the winning seller i in period t has to be:

$$p_i(\theta) = x_i(\theta) - x_{j^{-i}}(\theta) + \delta W^{-i}(\theta, i) - \delta W^{-i}\left(\theta, j^{-i}\right). \tag{8}$$

When we in insert these equilibrium prices (7) and (8) into the indifference condition (6) of the buyer, we get an expression involving only the social program and the marginal contributions:

$$\begin{aligned} & x_{j^{-i}}(\theta) - \delta W^{-i}(\theta, i) + \delta W^{-i}\left(\theta, j^{-i}\right) + \delta\left(W(\theta, i) - \sum_{l=1}^{I} M_l(\theta, i)\right) \\ & = x_{j^{-i}}(\theta) - \delta M_j(\theta, i) + \delta M_j\left(\theta, j^{-i}\right) + \delta\left(W\left(\theta, j^{-i}\right) - \sum_{l=1}^{I} M_l\left(\theta, j^{-i}\right)\right) \end{aligned}$$

Using the definition of $W^{-i}(\cdot)$ and after cancellations on both sides, we are left with the equality:

$$\sum_{l \neq i, j^{-i}} \delta M_l(\theta, i) = \sum_{l \neq i, j^{-i}} \delta M_l\left(\theta, j^{-i}\right). \tag{9}$$

This equality involves the marginal contributions of all sellers with the exception of the two most efficient sellers, i and j^{-i}. It states that the sum of the marginal contributions of these sellers is the same whether the current selection is i or j^{-i}. As all these sellers are less efficient than either i or j^{-i}, each one of them will be chosen along the efficient path only after i and j^{-i} have been selected initially. From the point of view of these less efficient sellers then, both i and j^{-i} will precede them and hence their contribution will only arise after the selection of i and j^{-i}. Hence from their point of view, it should not matter whether i or j^{-i} is chosen first. We therefore conjecture that the marginal contribution of each seller l is unaffected by the order in which i and j^{-i} are chosen and thus:

$$M_l(\theta, i) = M_l\left(\theta, j^{-i}\right),$$

which would clearly be sufficient to support the equality (9).

Let us next consider the remaining indifference conditions for the buyer, namely his choice between i and all sellers with the exception of j^{-i}. Naturally it is now sufficient to establish that the indifference conditions hold as inequalities rather than equalities. To achieve this, we do not insert the candidate equilibrium price of the winning seller i but a hypothetical price which allows for an easier comparison. Use $p_{i,j}(\theta)$ to denote the following:

$$p_{i,j}(\theta) \triangleq x_i(\theta) - x_j(\theta) + \delta W^{-i}(\theta, i) - \delta W^{-i}(\theta, j). \tag{10}$$

The price $p_{i,j}(\theta)$ represent the differential social value of i compared to j if i were to be removed form the set of alternatives beginning tomorrow. By subtraction, we get the following relation between $p_{i,j}(\theta)$ and $p_i(\theta)$:

$$p_{i,j}(\theta) - p_i(\theta) = x_{j^{-i}}(\theta) + \delta W^{-i}\left(\theta, j^{-i}\right) - x_j(\theta) - \delta W^{-i}(\theta, j) \geq 0.$$

The last inequality holds since alternative j^{-i} is the efficient alternative in the absence of i and hence the social value generated by the choice of j^{-i} is larger than the choice of any other alternative j in the absence of i. As we insert the prices (7) and (10) into the indifference condition (6), we arrive again at an expression which involves only the social values and the marginal contributions, namely:

$$x_{j^{-i}}(\theta) + \delta W^{-i}\left(\theta, j^{-i}\right) - x_j(\theta) - \delta W^{-i}(\theta, j) \geq \delta \sum_{l \neq i,j} \left[M_l(\theta, i) - M_l(\theta, j)\right].$$

By adding $M_i(\theta) - M_i(\theta \,|j\,)$ on both sides, we return to the social value with all sellers, including i and the above inequality becomes:

$$W(\theta) - W(\theta \,|j\,) \geq \sum_{l \neq j} \left[M_l(\theta) - M_l(\theta \,|j\,)\right]. \tag{11}$$

Inequality (11) is our second sufficient condition and it has an intuitive interpretation. It states that the social gains of moving from an inefficient allocation j to the efficient allocation i is larger than the gains arising for the same change in the marginal contributions. Since we want to interpret the marginal contributions as the payoffs to the sellers, the condition simply says that the efficiency loss in choosing an inefficient seller exceeds the reduction in future payoffs to other sellers, or in other words, efficiency losses outweigh rent extraction gains.

The verification of the equilibrium condition for the sellers is straightforward. In a cautious equilibrium, the losing sellers (weakly) prefer sales by i to making sales on their own by construction. It remains to verify that the winning seller i prefers to make a seller rather than to concede the market to another seller j. By the equilibrium hypothesis, seller i receives his marginal contribution in every subgame, and thus a sufficient condition for optimality can be stated as:

$$M_i(\theta) \geq \delta M_i(\theta, l), \ \forall l \neq i.$$

This is the final sufficient condition that we need for our construction of an equilibrium. It simply says that the marginal contribution of agent i is maximized along the efficient path. We thus have established the following result:

Theorem 1 (Existence)
An MPE in cautious strategies which is (i) efficient and (ii) displays marginal contribution payoffs exists provided that the marginal contributions satisfy:

1. $M_i(\theta \,|k) = M_i(\theta)$ *if* k *is chosen prior to* i *on the efficient path;*

2. $M_i(\theta \,|k) \leq M_i(\theta)$ *if* i *is chosen prior to* k *on the efficient path;*

3. $W(\theta) - W(\theta \,|k) \geq \sum_{j \in \mathcal{I} \setminus k} (M_j(\theta) - M_j(\theta \,|k))$.

In the next section we verify that in the model with independent rewards, the marginal contributions satisfy all three properties listed in Theorem 1.

5 Marginal Contributions

The verification of the sufficient conditions derived in Theorem 1 involves the comparisons of four different, yet related, social programs. The determination of the marginal contribution of each alternative occurs through the value of the social program when all alternatives are available and the social program when all alternatives but alternative j are available. The verification of the equilibrium choice of the buyer requires a comparison of the social welfare loss with the private welfare losses of the sellers. The potential social loss is induced by a single suboptimal deviation towards the alternative k, which requires the computation of the conditional optimal value following the choice of k. Finally, as the private welfare losses are measured by the marginal contributions and as those are obtained as the difference of social values, we also have to establish the value of the program without j and an initial suboptimal choice of k. The (conditionally) optimal assignments which arise in these various problems are denoted respectively by $a : \Theta \rightarrow \mathcal{I}$, $a_j : \Theta \rightarrow \mathcal{I} \backslash \{j\}$, $a^k : \Theta \rightarrow \mathcal{I}$ and $a_j^k : \Theta \rightarrow \mathcal{I} \backslash \{j\}$. The relationship between them are represented in the following diagram:

$$\begin{array}{c}\text{marginal}\\ \text{contribution}\\ \text{of } j\end{array} \quad \overbrace{\left\{\begin{array}{ccc} a(\cdot) & \longleftrightarrow & a^k(\cdot) \\ \updownarrow & & \updownarrow \\ a_j(\cdot) & & a_j^k(\cdot)\end{array}\right\}}^{\text{social loss}} \quad \begin{array}{c}\text{marginal}\\ \text{contribution}\\ \text{of } j\\ \text{(with suboptimal } k)\end{array}$$

The main difference between these policies arises with respect to the *calendar time* at which the t_i-th realization of alternative i will be employed. The argument that we present needs careful tracking of the calendar time t as well as the usage time t_i of seller i. We recall that we denote by $t_i(\theta)$ the number of times that alternative i has been used at state θ. We describe by $t(i, t_i)$ the *calendar time* in which the t_i-th realization of alternative i is used in the optimal program. Similarly, we denote by

$t(i, t_i |{-j})$, $t(i, t_i |k)$ and $t(i, t_i |k, -j)$ the calendar time in which the t_i-th realization of alternative i is employed in the program without j, in the suboptimal program starting with k, and in the suboptimal program starting with k and without alternative j, respectively. The comparison between the different programs can then be usefully reduced to a comparison of the calendar times in which a given a realization of alternative i is employed across the different programs.

5.1 Example

Before we set out to verify the validity of the sufficient conditions within the general model, we consider a very simple environment. The simplicity of the example facilitates the tracking of the calendar times at which the alternatives are employed across the different programs. The evident structure of the allocation problem then provides transparency to the proof of the sufficient conditions. Consider the following specification:

$$x_i(t_i) = \begin{cases} x_i, & \text{if} \quad t_i = 0, \\ 0, & \text{if} \quad t_i > 0, \end{cases}$$

and suppose further that

$$x_0 > x_1 > ... > x_I > 0.$$

In other words, each alternative generates a positive value at its first use and thereafter generates zero value. The socially optimal *continuation* policy is therefore to employ in every period the alternative j with the highest remaining valuation. The socially optimal policy starting at $t = 0$ is to select each alternative exactly in the order of their valuations. The path of the optimal policy is thus described by $a(t) = t$. Here, the descending order of the alternatives allows us to identify each alternative i with the time period in which it is employed along the efficient path.

The social value of the efficient program in period 0 can then be written as

$$W(\theta_0) = \sum_{t=0}^{\infty} \delta^t x_t.$$

As there is only a finite number of sellers and hence strictly positive realizations, for all $t \geq I$, we have $x_t = 0$.

The marginal contribution of seller j is given by

$$\begin{aligned} M_j(\theta_0) = W(\theta_0) - W^{-j}(\theta_0) &= \sum_{t=0}^{\infty} \delta^t x_t - \sum_{t=0}^{j-1} \delta^t x_t - \sum_{t=j+1}^{\infty} \delta^{t-1} x_t \\ &= \sum_{t=j}^{\infty} \delta^t (x_t - x_{t+1}). \end{aligned} \tag{12}$$

As one might have expected, the removal of alternative j does not change the value in the programs before the arrival of j in the socially efficient program. However, if we remove alternative j, the immediate consequence is that we have to use the next best alternative, which is alternative $j + 1$. By extension, we will be forced in all future periods to move one of the less efficient alternatives up by exactly one period, and this accounts for the sum of discounted differences starting in period j. In other words, the social benefit of seller j propagates into all future periods as the existence of alternative j permits all subsequent and less valuable alternatives to make their appearance exactly on period later.

The social value of the suboptimal program which starts with alternative k can also be represented as a variation of the efficient social program by simply forwarding the appropriate time indices as:

$$W(\theta_0 | k) = x_k + \sum_{t=0}^{k-1} \delta^{t+1} x_t + \sum_{t=k+1}^{\infty} \delta^t x_t.$$

The suboptimal anticipation of seller k changes the social values in two ways: (i) seller k appears k periods too early and (ii) all sellers ranked before k appear one period too late relative to the social optimum. After $k+1$ periods, the optimal social policy catches up with the suboptimal policy and the values thereafter are identical. The rearrangement of the order in which the alternatives are selected due to the initial suboptimal deviation is depicted in the following figure for $k = 3$:

$t =$	0	1	2	3	4	5	$\cdots$
$a(t) =$	0	1	2	3	4	5	$\cdots$
$a(t\,\|k) =$	3	0	1	2	4	5	$\cdots$

Table 1: Allocation Times along Optimal and Suboptimal Paths

Finally, the marginal contribution of seller j in a suboptimal program can be represented by a combination of the forward and backward operation. Consider first the case in which alternative j would be employed after alternative k in the efficient program, or $j > k$. The marginal contribution of j is given by:

$$\begin{aligned} M_j(\theta_0 | k) &= W(\theta_0 | k) - W^{-j}(\theta_0 | k) \\ &= \sum_{t=j}^{\infty} \delta^t (x_t - x_{t+1}). \end{aligned} \tag{13}$$

As alternative k is used in the optimal as well as in the suboptimal program before alternative j, the marginal contribution of alternative j should remain identical across these two program and this is easily verified by comparing (12) and (13).

If alternative j arrives before alternative k in the efficient program, then the marginal contribution in the suboptimal program is:

$$\begin{aligned} M_j(\theta_0 | k) &= W(\theta_0 | k) - W^{-j}(\theta_0 | k) \\ &= \sum_{t=j}^{k-2} \delta^{t+1} (x_t - x_{t+1}) + \delta^k (x_{k-1} - x_{k+1}) + \sum_{t=k+1}^{\infty} \delta^t (x_t - x_{t+1}). \end{aligned}$$

The effect of a suboptimal allocation on the marginal contribution of seller j is now similar to the one imposed on the social value. The distinction arises due to the fact that the marginal contribution is expressed in differences rather than absolute values: (i) the suboptimal anticipation of k delays the arrival of benefits due to seller j by one period, (ii) as k has been anticipated the marginal benefit in period k is then between x_{k-1} and the next available alternative x_{k+1}, (iii) at $t = k+1$ the socially optimal program catches up with the suboptimal program and the third term is identical in both expressions. The difference in the marginal contributions for $j < k$ can then be expressed as:

$$M_j(\theta_0) - M_j(\theta_0 \,|k) = (1-\delta)\sum_{t=j}^{k-1}\delta^t(x_t - x_{t+1}). \tag{14}$$

An initial and inefficient assignment of seller l then depresses the contribution of seller j through a sequence of one period delays in the accrual of the marginal values of seller k.

Based on these simple computations we can now verify that all three sufficient conditions in Theorem 1 hold. The first sufficient condition, namely that for $j > k$, $M_j(\theta_0) = M_j(\theta_0 \,|k)$, is verified by simply comparing (12) and (13). The second sufficient condition, $M_j(\theta_0) - M_j(\theta_0 \,|k) \geq 0$ for $j < k$, was established by (14) coupled with the observation that $x_t - x_{t+1} > 0$ for all t. The third and final sufficient condition:

$$W(\theta_0) - W(\theta_0 \,|k) \geq \sum_{j\in\mathcal{I}}(M_j(\theta_t) - M_j(\theta_t \,|k)), \tag{15}$$

is readily established as well. The difference in the value between the optimal and the suboptimal program is given by:

$$W(\theta_0) - W(\theta_0 \,|k) = (1-\delta)\sum_{t=0}^{k-1}\delta^t x_t - \left(1-\delta^k\right)x_k. \tag{16}$$

The inequality (15) can be expressed, using (14) and (16), and dividing both sides by $(1-\delta)$ as:

$$\sum_{j=0}^{k-1}\left(\delta^j x_j - \delta^k x_k\right) \geq \sum_{j=0}^{k-1}\sum_{t=j}^{k-1}\delta^t(x_t - x_{t+1}).$$

It is easy to verify that the inequality holds when we consider every element indexed with j separately, or

$$\delta^j x_j - \delta^k x_k \geq \sum_{t=j}^{k-1}\delta^t(x_t - x_{t+1}). \tag{17}$$

We notice that without discounting, i.e. for $\delta = 1$, both sides equalize as the rhs is simply a telescopic expansion of the difference of the lhs. But for $\delta < 1$, the inequality becomes in fact strict. The lhs of (17) expresses for every x_j with $j < k$, the value difference between x_j and x_k *weighted* with the appropriate discount factors of the optimal program:

$$\delta^j x_j - \delta^k x_k > 0.$$

The rhs also presents for every x_j a differential expression between x_j and x_k, but it proceeds in steps $\delta^t (x_t - x_{t+1})$ which are increasingly discounted. This reflects the value difference between x_j and x_k but now in terms of the marginal contribution. Since the marginal contribution only picks up the *inframarginal* differences in every period, it follows directly that the inequality (17) holds.

5.2 Optimal Index Policies

We now proceed to prove the three properties of the marginal contributions within the general payoff environment. The basic argument will follow exactly along the lines suggested by the example. But to pursue this argument, we have to use the structure of the optimal policies to bring the general model closer to the example. We will do this in three steps. The first step, carried out in Lemma 1 and 2, will show that the decreasing sequence of payoffs in the example is in essence without loss of generality. The second step, carried out in Lemma 3 and 4, will show that among all possible continuation paths following an initial and suboptimal use of alternative k, the one which returns immediately to the initial and unconditionally optimal path is the critical path. The third step, carried out in Lemma 5, will show that the marginal contributions *and* the difference in the marginal contribution from optimal and suboptimal path are superadditive. This final argument will allow us to reduce the rent extraction inequality to a particularly simple case. Finally it should be emphasized that all arguments in this section will be based on the properties of optimal policies in single agent allocation problems and no equilibrium arguments will be needed.

We recall that the payoff of each alternative i is a function of its own past use only. We defined the payoff stream $x_i(t_i)$ to depend only on the number of times, t_i, alternative i has been used in the past. For each alternative i and each state t_i we can define an index of its future value through an optimal stopping problem. The optimal stopping time for alternative i in state t_i, denoted by $\tau_i(t_i)$, is defined as the solution to the following problem:

$$\tau_i(t_i) \in \arg\max_{\tau_i \geq t_i} \left\{ \frac{\sum_{s_i=t_i}^{\tau_i} \delta^{s_i} x_i(s_i)}{\sum_{s_i=t_i}^{\tau_i} \delta^{s_i}} \right\}. \quad (18)$$

If the maximization problem (18) allows for multiple solutions for τ_i, then we identify $\tau_i(t_i)$ to be the largest time among the maximizers. We define the index of alternative i, $X_i(t_i)$, as the discounted average under the optimal stopping time, or

$$X_i(t_i) \triangleq \max_{\tau_i \geq t_i} \left\{ \frac{\sum_{s_i=t_i}^{\tau_i} \delta^{s_i} x_i(s_i)}{\sum_{s_i=t_i}^{\tau_i} \delta^{s_i}} \right\}. \quad (19)$$

With these preliminaries in place, it is straightforward to characterize the (conditionally) optimal programs.

Lemma 1 (Optimal Policies)

1. *The (conditionally) optimal assignment in state θ is determined by*

$$\arg\max_i \{X_i(t_i(\theta))\}.$$

2. *For all $i\,j, k, l$ (all distinct) and all t_i, t_j :*

$$\begin{aligned} t(i,t_i) < t(j,t_j) \quad &\Leftrightarrow \quad t(i,t_i\,|k) < t(j,t_j\,|k) \\ &\Leftrightarrow \quad t(i,t_i\,|{-l}) < t(j,t_j\,|{-l}) \\ &\Leftrightarrow \quad t(i,t_i\,|k,-l) < t(j,t_j\,|k,-l). \end{aligned}$$

Proof. See appendix. ■

The first part of Lemma 1 simply restates the celebrated Gittins index theorem for deterministic payoff streams. The index characterization depends only on the properties of the payoff streams of the individual alternatives. As a consequence of the index property, the subsequent parts state that the order in which the alternatives are optimally employed is invariant to the removal of some alternatives or the suboptimal initial use of alternative k.

A further consequence of the optimality of the index policy is stated next. For each alternative i, we define inductively a sequence of stopping times $\{\tau_i^n\}_{n=0}^{\infty}$ as follows. Define $\tau_i^0 \triangleq -1$, and let

$$\tau_i^{n+1} \in \arg\max_{s>\tau_i^n} \left\{ \frac{\sum_{t_i=\tau_i^n+1}^{s} \delta^{t_i} x_i(t_i)}{\sum_{t_i=\tau_i^n+1}^{s} \delta^{t_i}} \right\}. \tag{20}$$

As before, if the maximization problem (20) allows for multiple solutions for τ_i^{n+1}, then we identify τ_i^{n+1} to be the largest time among the maximizers. With this inductively defined sequence of stopping times, we can associate average rewards between stopping times as follows:

$$x_i^{n+1} \triangleq \frac{\sum_{t_i=\tau_i^n+1}^{\tau_i^{n+1}} \delta^{t_i} x_i(t_i)}{\sum_{t_i=\tau_i^n+1}^{\tau_i^{n+1}} \delta^{t_i}}. \tag{21}$$

In contrast to the earlier stopping times and average discounted rewards in (18) and (19), the stopping times defined by (20) are not recalculated at every clock time t_i, but only from stopping time τ_i^n to stopping time τ_i^{n+1}. Similarly, the average discounted rewards defined by (21) are only calculated between stopping times.

Lemma 2 (Decreasing Valuations)

1. *The average returns x_i^n are decreasing in n.*

2. *The tail realizations, with $\tau_i^n < s \leq \tau_i^{n+1}$, satisfy:*

$$\frac{\sum_{t_i=s}^{\tau_i^{n+1}} \delta^{t_i} x_i(t_i)}{\sum_{t_i=s}^{\tau_i^{n+1}} \delta^{t_i}} \geq x_i^{n+1}.$$

Proof. See appendix. ■

The relevance of Lemma 2 is easiest to understand if we start with the second result. It says that between any two stopping times, τ_i^n and τ_i^{n+1}, the discounted average reward until τ_i^{n+1} must be at least as large as it was when evaluated starting from time $t_i = \tau_i^n$. The optimal index policy , stated in Lemma 1, then implies that if it was optimal to use i starting at $t_i = \tau_i^n$, then it will remain optimal to use alternative i uninterrupted at least until $t_i = \tau_i^{n+1}$ is reached. The first result of Lemma 2 then states that when the discounted averages are taken along the stopping times, then the average returns from alternative i are decreasing over time. By the index policy, it follows that the sequence of averages across agents will also decrease over time.

Next we consider a particular alternative k which is used in the suboptimal program $W(\theta_0 | k)$. We know from Lemma 1 that the initial use of the suboptimal k will not change the order in which the remaining alternatives are selected in the optimal continuation program, but it may change repeatedly the time at which they are selected. This is due to the fact that the initial, even if suboptimal use of k, may make the future use of k more desirable and thus further delay the employment of the other alternatives. The source of this complication is the fact that after the initial use of $x_k = x_k(0)$, the average value of the tail realization of $x_k(0)$ may increase and be larger than $x_k(0)$ so that for $s > 0$

$$\frac{\sum_{t_k=s}^{\tau_k^1} \delta^{t_k} x_k(t_k)}{\sum_{t_k=s}^{\tau_k^1} \delta^{t_k}} > x_k(0).$$

In this case, the index policy may recommend the use of alternative k earlier than it would have if $x_k(0)$ had not been removed through its suboptimally early use in period 0. (The payoff $x_k(0)$ now constitutes a sunk cost in order to reach the higher payoffs $x_k(t_k)$.) As the index of alternative k for all tail realizations stays above the original index, we cannot rule out that the initial suboptimal use of k leads to many early (relative to the optimal program) usage times of the alternative

k. The occurrence of such repeated changes makes the comparison between $W(\theta_0)$ and $W(\theta_0 | k)$ notationally cumbersome. It is therefore desirable to find instances for the payoff stream of $x_k(t_k)$ in which the changes between $W(\theta_0)$ and $W(\theta_0 | k)$ are minimal. The only payoff stream for alternative k which can guarantee this and still obtain an average return of x_k^1 for an arbitrary stopping time τ_k^1 is the constant sequence:

$$x_k(0) = x_k(1) = ... = x_k\left(\tau_k^1\right) = x_k^1, \tag{22}$$

where the last equality is the result of the averaging over a constant sequence. With such a constant sequence, the average value and hence index of all tail realization is constant and equal to the initial value. In consequence, the optimal continuation path after the initial suboptimal choice of k is easy to describe. It will simply continue where the optimal path would have started. Moreover, beginning with the time period where the optimal continuation path uses the alternative k again, the path of optimal and suboptimal path will be identical again.

In this respect, if we compare the program $W(\theta_0)$ and $W(\theta_0 | k)$, the only realizations of alternative i which matter are those which are realized optimally before the alternative k is used for the first time. It is then useful to introduce an auxiliary allocation problem, where all realizations of alternative i which would occur after the first use of alternative k are set equal to zero, or:

$$x_i^k(t_i) = \begin{cases} x_i(t_i), & \text{if } t(t_i) < t(t_k = 0) \\ 0 & \text{if } t(t_i) > t(t_k = 0) \end{cases} \tag{23}$$

and all realizations of alternative k are set equal to x_k^1:

$$x_i^k(t_k) = x_k^1 \text{ for all } t_k. \tag{24}$$

We refer to the allocation problem with the payoffs in (23) and (24) as the k- truncated allocation problem. We add the superscript k to the payoff realizations and the social values, $W^k(\theta_0)$, to indicate the modification in the payoffs.

We now establish a relationship between our original allocation model and this specifically modified allocation model.

Lemma 3 (Truncated Allocation Problem)

If $x_k(t_k)$ displays constant payoffs for all t_k with $0 \leq t_k \leq \tau_k^1$, then

1. *for all k,*

$$\frac{1}{1-\delta}\left(W(\theta_0) - W(\theta_0 | k)\right) = W^k(\theta_0) - \frac{1}{1-\delta} x_k^1;$$

2. *for all k and all i,*

$$\frac{1}{1-\delta}\left(M_i(\theta_0) - M_i(\theta_0 | k)\right) = M_i^k(\theta_0).$$

Proof. See appendix. ■

We can now show that the constant sequence of realizations for alternative k is the critical sequence to analyze for the purpose of establishing the sufficient conditions of the marginal contribution equilibrium.

Lemma 4 (Minimal Loss Payoff Stream)

1. *For all k, $W(\theta_0 | k)$ is maximized with $x_k(0) = ... = x_k(\tau_k^1)$.*

2. *For all i and k, $M_i(\theta_0 | k)$ is maximized with $x_k(0) = ... = x_k(\tau_k^1)$.*

Proof. See appendix. ■

A final useful fact about the marginal contributions is stated next.

Lemma 5 (Superadditive Marginal Contributions)

1. *For all i and j,*

$$M_i(\theta_0) + M_j(\theta_0) \leq M_{i\cup j}(\theta_0).$$

2. *For all i, j and k, all distinct;,*

$$M_i(\theta_0) - M_i(\theta_0 | k) + M_j(\theta_0) - M_j(\theta_0 | k) \leq M_{i\cup j}(\theta_0) - M_{i\cup j}(\theta_0 | k).$$

Proof. See appendix. ■

The first results says that the marginal contribution of i and j jointly exceed the sum of the marginal contributions of i and j individually. This is rather intuitive as the removal of alternative i leaves the social program with the possibility to use j, but once i and j are removed jointly, the social program will have to immediately use the possible inferior alternative k. It can be shown that the superadditivity property of the marginal contributions is equivalent to

$$M_i(\theta_0) \leq M_i^{-j}(\theta), \text{ for all } i \text{ and } j.$$

This inequality is perhaps even more intuitive as it says that the contribution of alternative i to the social program becomes more valuable after the removal of alternative j from the choice set. The superadditivity property extends to the difference of the marginal contributions arising from optimal and suboptimal program, and it this inequality which we will use for the next theorem. With these preliminaries in place, we are ready to prove the main result of this section.

Theorem 2 (Marginal Contributions)
The marginal contributions of the infinite horizon game with independent rewards satisfy the sufficient conditions of Theorem 1:

1. $M_i(\theta_0 | k) = M_i(\theta_0)$ *if* $t(k,1) < t(i,1)$;

2. $M_i(\theta_0 | k) \leq M_i(\theta_0)$ *if* $t(i,1) < t(k,1)$;

3. $W(\theta_0) - W(\theta_0 | k) \geq \sum_{j \in \mathcal{I} \setminus k} (M_j(\theta_0) - M_j(\theta_0 | k))$.

Proof. See appendix. ■

This theorem establishes that the infinite horizon model with independent rewards always possesses an equilibrium with strong welfare predictions. The equilibrium path is socially efficient and all the sellers receive their marginal contribution. A further welfare consequence of the marginal contribution property is that (i) the sellers have the socially optimal ex ante incentives for making surplus enhancing investments and (ii) the buyer has the socially correct incentives for investments that increase surplus in all purchases. In the job market context this implies that efficiency results if the firms pay for job specific training and the workers pay for general training.

In the next section, we turn to the issue of uniqueness of these equilibria.

6 Uniqueness

In this section, we address the issue of uniqueness in the infinite horizon model. We assume first that for all i, there exists a $T_i < \infty$ and an $\overline{x}_i$ such that for all $t_i \geq T_i$, $x_i(t_i) = \overline{x}_i$. This assumption implies that along any Markov perfect equilibrium path, the payoffs become static after some finite number of periods. Formally, let

$$T_i = \min\{t_i \,| x_i(s_i) = \overline{x}_i \text{ for some } \overline{x}_i \text{ and all } s_i \geq t_i\}. \tag{25}$$

With this assumption, we analyze the game by backwards induction in the state space.

Theorem 3 (Uniqueness)
The marginal contribution equilibrium is the unique cautious Markov perfect equilibrium of the dynamic price competition game if for all i, there exists $T_i < \infty$ and $\overline{x}_i$ such that for all $t_i \geq T_i$, $x_i(t_i) = \overline{x}_i$.

Proof. See appendix. ■

It should be pointed out that we do not know of any counterexamples to the uniqueness of equilibria in games which do not satisfy the above assumption of eventual constancy. While we have not been able to prove the uniqueness without the use of backward induction arguments based on continuation payoffs, we conjecture that the uniqueness holds even in games where constancy assumption (25) fails, provided, of course, we maintain the independent rewards condition. The basic difficulty in proving uniqueness for the general model stems from the fact that when the continuation paths are inefficient, very little can be said about the exact form of individual continuation payoffs. The best way to see this is to recall the examples in Section 3. For an arbitrary continuation path, there is no reason to believe that the cautious equilibrium choice in period t would be uniquely defined. It might seem that by letting $T_i \to \infty$ for all i, we could make use of arguments based on continuity at infinity.[3] Yet, while this approach would supply us with an alternative way for proving the *existence* of a marginal contribution equilibrium in the general model, it does not provide us with an argument for uniqueness.

The final point to note is that the scope of the uniqueness result is narrower than in the main result of Bergemann & Välimäki (1996). The uniqueness here pertains only to Markov perfect equilibria in cautious strategies whereas the previous result held for *all* MPE of the game with two sellers. The reason for this is that with three or more sellers, continuation payoffs resulting from strategies that do not satisfy cautiousness may violate the marginal contribution property. To see this, consider the following simple example and assume for simplicity that $\delta \approx 1$:

$$\begin{array}{cccc} & x_1(\cdot) & x_2(\cdot) & x_3(\cdot) \\ t_i = 0 & 1+\varepsilon & 1 & \varepsilon \\ t_i = 1 & 0 & 0 & 0 \end{array}.$$

Efficiency requires that 1 is chosen first, 2 next and 3 in the final period. The game has, however, inefficient Markov perfect equilibria in strategies that are not cautious. To see this, observe that seller 3 can transfer all the surplus to the buyer by suitable non-cautious pricing. If he uses such a strategy in the continuation game following the choice of 3 in the first period, and if all other continuation games are played according to the marginal contribution equilibrium, then for ε small, there is an equilibrium where seller 3 is chosen in the first period.

[3] See e.g. Fudenberg & Levine (1983) and Harris (1985) for discussions of such arguments.

7 Conclusion

This paper shows that much of the intuition obtained in the static price competition games extends to the stationary infinite horizon case as long as externalities between sellers are ruled out. In particular, the stationary infinite horizon model possesses an efficient equilibrium where all of the sellers receive their marginal contribution as their equilibrium payoff. The finite horizon case is, however, quite different. Efficient equilibria do not exist in general, and the games may have multiple cautious equilibria.

The arguments in the stationary infinite horizon case are based on a version of the Gittins index theorem for the optimal scheduling of tasks. The most important consequence of this is that the optimal paths in the model satisfy the following invariance property: if along the optimal path, a purchase is made from seller i prior to seller j, then it cannot be the case that j is used prior to i in a game where the stage rewards of players other than i and j have been modified. Since the Gittins index theorem is also valid in the stochastic case (as long as the stage rewards are statistically independent across sellers), the arguments given in the paper remain valid in that case as well. The only modification needed is that in the calculation of the optimal number of times that a seller is used, we must consider random rather than deterministic stopping times.

There are a number of directions for extending our analysis. A possible formulation that would be consistent with our insistence on no static or dynamic externalities between the sellers would be to allow $x_i(\cdot)$ to depend on calendar time in addition to t_i. For example, one could have a time to build before a new sale can be made (this would be relevant for many industries such as ship building where capacity utilization and securing a constant stream of purchases are of primary importance). Such models could also capture many of the dynamic issues arising in the sales of renewable resources. Pursuing these extensions is left for future work.

8 Appendix

The proofs for all results are collected in the appendix.

Proof of Lemma 1. (1.) This is simply a statement of the optimality of the Gittins index policy in the special case of deterministic payoff stream, see Whittle (1982) or Gittins (1989) for a statement in the general stochastic case.

(2.) The relationships follow immediately from the nature of the index policy. As the index of alternative i and j are computed on the basis of their respective payoff streams exclusively, their index is invariant to the addition or removal of additional alternative. As the indices are invariant, the relative order in which alternative i and j are being employed stays invariant too. The same argument applies naturally for the optimal continuation path of i and j, when the initial suboptimal choice was k.■

Proof of Lemma 2. (1.) The argument is by contradiction. Suppose not and hence there exists n such that $x_i^n < x_i^{n+1}$. This directly contradicts the optimality of τ_i^n as a solution to (20) since by hypothesis of $x_i^n < x_i^{n+1}$:

$$\frac{\sum_{t_i=\tau_i^{n-1}+1}^{\tau_i^{n+1}} \delta^{t_i} x_i(t_i)}{\sum_{t_i=\tau_i^{n-1}+1}^{\tau_i^{n+1}} \delta^{t_i}} > x_i^n,$$

but this shows that τ_i^n cannot be a solution to (20) as setting its value to be equal to τ_i^{n+1} is feasible and would obtain a larger average value.

(2.) The argument is again by contradiction. Suppose therefore that there exists an s with $\tau_i^{n-1} < s < \tau_i^n$ such that

$$\frac{\sum_{t_i=s}^{\tau_i^n} \delta^{t_i} x_i(t_i)}{\sum_{t_i=s}^{\tau_i^n} \delta^{t_i}} < x_i^n.$$

It follows from averaging that the average during the complement time has to satisfy:

$$\frac{\sum_{t_i=\tau_i^{n-1}+1}^{s-1} \delta^{t_i} x_i(t_i)}{\sum_{t_i=\tau_i^{n-1}+1}^{s-1} \delta^{t_i}} > x_i^n,$$

for the joint interval to achieve the average x_i^n. But it now follows immediately that τ_i^n cannot be the solution to (20) as $s-1$ achieves a higher average, contradicting the fact that τ_i^n is a solution to the maximization problem.■

Proof of Lemma 3. (1.) For a constant payoff stream of $x_k(t_k)$ for all t_k with $0 \leq t_k \leq \tau_k^1$, the difference $W(\theta_0) - W(\theta_0 | k)$ can be written as

$$W(\theta_0) - W(\theta_0 | k) = (1-\delta) W^{t_k}(\theta_0) - x_k^1 \left(1 - \delta^{t_k}\right),$$

and after dividing by $(1-\delta)$ as

$$\frac{1}{1-\delta}\left(W(\theta_0) - W(\theta_0 | k)\right) = W^k(\theta_0) - \frac{x_k^1}{1-\delta}.$$

(2.) We can write the the difference $M_j(\theta_0) - M_j(\theta_0 | k)$ generally as

$$\begin{aligned} & M_j(\theta_0) - M_j(\theta_0 | k) \\ = \; & \left(W(\theta_0) - W^{-j}(\theta_0)\right) - \left(W(\theta_0 | k) - W^{-j}(\theta_0 | k)\right) \qquad (26) \\ = \; & \left(W(\theta_0) - W(\theta_0 | k)\right) - \left(W^{-j}(\theta_0) - W^{-j}(\theta_0 | k)\right). \end{aligned}$$

For the case of a constant payoff stream of $x_k(t_k)$ for all t_k with $0 \leq t_k \leq \tau_k^1$, the last line in (26) can be written by the first part of this lemma as

$$\left(W^k(\theta_0) - \frac{x_k^1}{1-\delta}\right) - \left(W^{-i,k}(\theta_0) - \frac{x_k^1}{1-\delta}\right) = W^k(\theta_0) - W^{-i,k}(\theta_0),$$

which proves the second part of this lemma.■

Proof of Lemma 4. (1.) Suppose not and thus the maximizing solution is obtained with different payoff stream of alternative k. By the optimality of the stopping times $\{\tau_k^n\}_{n=0}^{\infty}$, this different payoff stream must have some realization $x_k(t_k) > x_k(0)$ with $t_k \leq \tau_k^1$. Consider then a modified version of this payoff stream, denoted by $\hat{x}_k(0)$ and $\hat{x}_k(t_k)$, satisfying:

$$\widehat{x}_k(0) \triangleq x_k(0) + \varepsilon \delta^{t_k} \qquad (27)$$

and

$$\widehat{x}_k(t_k) \triangleq x_k(t_k) - \varepsilon \tag{28}$$

with $\varepsilon > 0$. Clearly, the modified payoff stream has the same average return as the original stream, and for sufficiently small ε, it maintains the same stopping times $\{\tau_k^n\}_{n=0}^{\infty}$ as the original stream.

Consider then the original term $W(\theta_0 |k)$ and the same term under the modification defined by (27) and (28), and denoted by extension as $\widehat{W}(\theta_0 |k)$. As the stopping times and the average rewards of all alternatives remain unchanged, the timing in the employment remains unchanged as well. The difference between the two terms is therefore simply

$$\widehat{W}(\theta_0 |k) - W(\theta_0 |k) = \varepsilon\delta^{t_k} \cdot 1 - \varepsilon \cdot \delta^{t(k,t_k|k)}. \tag{29}$$

As $t_k \leq t(k, t_k |k)$, it follows that the constant stream for alternative k can never decrease the payoff of the program $W(\theta_0 |k)$ and increases it if the t_k-th realization of alternative k comes only after some other alternatives, say j, have been realized as the inequality $t_k < t(k, t_k |k)$ then becomes strict.

(2.) The marginal contribution of i under the suboptimal choice of k is given by $M_i(\theta_0 |k) = W(\theta_0 |k) - W^{-i}(\theta_0 |k)$. Again, suppose that the maximizing solution is obtained with different payoff stream of alternative k. We then look at a modified payoff stream of alternative k as described by (27) and (28). Consider then the difference

$$\begin{aligned}\widehat{M}_i(\theta_0 |k) - M_i(\theta_0 |k) &= \left[\widehat{W}(\theta_0 |k) - \widehat{W}^{-i}(\theta_0 |k)\right] - \left[W(\theta_0 |k) - W^{-i}(\theta_0 |k)\right] \\ &= \left[\widehat{W}(\theta_0 |k) - W(\theta_0 |k)\right] - \left[\widehat{W}^{-i}(\theta_0 |k) - W^{-i}(\theta_0 |k)\right].\end{aligned}$$

By the same argument as in (29), this leads to

$$\left[\varepsilon\delta^{t_k} \cdot 1 - \varepsilon \cdot \delta^{t(k,t_k|k)}\right] - \left[\varepsilon\delta^{t_k} \cdot 1 - \varepsilon \cdot \delta^{t(k,t_k|k,-i)}\right] = \varepsilon \cdot \left(\delta^{t(k,t_k|k,-i)} - \delta^{t(k,t_k|k)}\right) \geq 0,$$

where the last inequality follows from $t(k, t_k |k, -i) \leq t(k, t_k |k)$.■

Proof of Lemma 5. (1.) Since

$$\begin{aligned} M_{i\cup j}(\theta_0) &= W(\theta_0) - W^{-i\cup j}(\theta_0) \\ &= W(\theta_0) - W^{-j}(\theta_0) + W^{-j}(\theta_0) - W^{-i\cup j}(\theta_0) \\ &= M_j(\theta_0) + M_i^{-j}(\theta_0), \end{aligned}$$

the claim follows if we can show:

$$M_i^{-j}(\theta_0) \geq M_i(\theta_0). \tag{30}$$

To establish the above inequality, we replace the payoffs of alternative j by the average payoffs $\{x_j^n\}_{n=1}^{\infty}$ computed along the stopping times $\{\tau_j^n\}_{n=0}^{\infty}$, as defined earlier in (20) and (21):

$$x_j(t_j) = x_j^n \text{ for } \tau_j^{n-1} < t \leq \tau_j^n. \tag{31}$$

By the property of the tail realizations, established in Lemma 2, and the optimal index policy, as stated in Lemma 1, we know that the optimal values in the allocation problem and hence the marginal contribution of alternative i are unaffected if we modify the payoffs of alternative j's as done in (31).

Next we split alternative j into $n = 1, ..., \infty$ (sub-)alternatives, denoted by j_n, along the above mentioned stopping times $\{\tau_j^n\}_{n=0}^{\infty}$. The payoffs by alternative j_n are defined as follows:

$$x_{j_n}(t_{j_n}) = \begin{cases} x_j^n \text{ if } 1 \leq t_{j_n} \leq \tau_j^n - \tau_j^{n-1}, \\ \quad 0 \qquad \text{otherwise.} \end{cases}$$

The marginal contribution of alternative i is unaffected by this split of alternative j.

We now prove inequality (30) in the form of $M_i(\theta_0) \leq M_i^{-j_n}(\theta_0)$ for all j_n. Now,

$$\begin{aligned} M_i(\theta_0) - M_i^{-j_n}(\theta_0) &= W(\theta_0) - W^{-i}(\theta_0) - \left(W^{-j_n}(\theta_0) - W^{-i\cup j_n}(\theta_0)\right) \\ &= W(\theta_0) - W^{-j_n}(\theta_0) - \left(W^{-i}(\theta_0) - W^{-i\cup j_n}(\theta_0)\right). \end{aligned} \tag{32}$$

By the Gittins index theorem as stated in Lemma 1, the order of optimal choices for alternatives different from i and j_n is unaffected by the removal of j_n. Denote by

W_{j_n} and $W_{j_n}^{-i}$ the continuation value to the social program, with and without i, after using alternative j_n. Let $\sigma_j^n \triangleq \tau_j^n - \tau_j^{n-1}$ be the number of periods alternative j_n has a positive contribution of x_j^n. By elementary calculations, it follows that

$$W(\theta_0) - W^{-j_n}(\theta_0) = \delta^{t(j_n,1)} \left(\frac{1-\delta^{\sigma_j^n}}{1-\delta} x_j^n - (1-\delta^{\sigma_j^n}) W_{j_n} \right), \tag{33}$$

and

$$W^{-i}(\theta_0) - W^{-i\cup j_n}(\theta_0) = \delta^{t(j_n,1|-i)} \left(\frac{1-\delta^{\sigma_j^k}}{1-\delta} x_j^n - (1-\delta^{\sigma_j^n}) W_{j_n}^{-i} \right). \tag{34}$$

After inserting (33) and (34) into (32) and observing that $t(j_k, 1|-i) \leq t(j_k, 1)$, by Lemma 2, and that $W_{j_n}^{-i} \leq W_{j_n}$, by virtue of being value functions, the claim is established.

(2.) It is sufficient to show that if we combine any two alternatives j and j' into a single alternative $j \cup j'$, then

$$M_j(\theta_0) - M_j(\theta_0|k) + M_{j'}(\theta_0) - M_{j'}(\theta_0|k) \leq M_{j\cup j'}(\theta_0) - M_{j\cup j'}(\theta_0|k). \tag{35}$$

A repeated application of the same inequality would then eventually lead to merge all alternatives but k under a single identity and lead to the desired result. As

$$M_{j\cup j'}(\theta_0) - M_{j'}(\theta_0) = M_j^{-j'}(\theta_0) \tag{36}$$

we can write the inequality (35) equivalently as

$$M_j(\theta_0) - M_j(\theta_0|k) \leq M_j^{-j}(\theta_0) - M_j^{-j}(\theta_0|k). \tag{37}$$

The identity in (36) follows directly from the definition of marginal contribution:

$$\begin{aligned} M_{j\cup j'}(\theta_0) - M_{j'}(\theta_0) &= \left(W(\theta_0) - W^{-j\cup j'}(\theta_0)\right) - \left(W(\theta_0) - W^{-j'}(\theta_0)\right) \\ &= W^{-j'}(\theta_0) - W^{-j\cup j'}\theta_0 \\ &= M_j^{-j'}(\theta_0). \end{aligned}$$

Rearranging the inequality (37),

$$\left(M_j(\theta_0) - M_j(\theta_0|k)\right) - \left(M_j^{-j}(\theta_0) - M_j^{-j}(\theta_0|k)\right) \leq 0, \tag{38}$$

we show next that the left hand side is maximized for every k when $x_k(0) = ... = x_k(\tau_k^1)$. The argument is by contradiction and uses again the construction of (27) and (28). We thus look at the difference between the payoff generated by the modified and the original sequence of payoffs, or

$$\left(\left(\widehat{M}_j(\theta_0) - \widehat{M}_j(\theta_0 | k)\right) - \left(\widehat{M}_j^{-j'}(\theta_0) - \widehat{M}_j^{-j'}(\theta_0 | k)\right)\right)$$
$$- \left((M_j(\theta_0) - M_j(\theta_0 | k)) - \left(M_j^{-j}(\theta_0) - M_j^{-j}(\theta_0 | k)\right)\right).$$

The payoffs from the marginal contribution of the efficient program remain unchanged, and it suffices to evaluate

$$\left(\widehat{M}_j^{-j'}(\theta_0 | k) - M_j^{-j'}(\theta_0 | k)\right) - \left(\widehat{M}_j(\theta_0 | k) - M_j(\theta_0 | k)\right),$$

or identically expressing it in terms of the social values:

$$\begin{aligned} &\left(\left(\widehat{W}^{-j'}(\theta_0 | k) - W^{-j'}(\theta_0 | k)\right) - \left(\widehat{W}^{-j\cup j'}(\theta_0 | k) - W^{-j\cup j'}(\theta_0 | k)\right)\right) \\ &- \left(\left(\widehat{W}(\theta_0 | k) - W(\theta_0 | k)\right) - \left(\widehat{W}^{-j}(\theta_0 | k) - W^{-j}(\theta_0 | k)\right)\right). \end{aligned} \tag{39}$$

With the construction of (27) and (28), the payoffs resulting from each term inside the respective parenthesis are identical with the exception of the first and the t_k-th realization of alternative k. As the timing of the realizations remains unchanged, it follows that the difference (39) can be written as

$$\left(\left(\varepsilon\delta^{t_k} - \varepsilon\delta^{t(k,t_k | k,-j')}\right) - \left(\varepsilon\delta^{t_k} - \varepsilon\delta^{t(k,t_k | k,-j\cup j')}\right)\right)$$
$$- \left(\left(\varepsilon\delta^{t_k} - \varepsilon\delta^{t(k,t_k | k)}\right) - \left(\varepsilon\delta^{t_k} - \varepsilon\delta^{t(k,t_k | k,-j)}\right)\right),$$

and after the obvious cancellations:

$$\varepsilon\left(\left(\delta^{t(k,t_k | k,-j\cup j')} - \delta^{t(k,t_k | k,-j')}\right) - \left(\delta^{t(k,t_k | k,-j)} - \delta^{t(k,t_k | k)}\right)\right).$$

By Lemma 1, we know that

$$t(k, t_k | k, -j\cup j') - t(k, t_k | k, -j') = t(k, t_k | k, -j) - t(k, t_k | k),$$

as well as

$$t\left(k,t_k\left|k,-j'\right.\right)\leq t\left(k,t_k\left|k\right.\right).$$

It then follows from discounting with $\delta\in(0,1)$, that

$$\left(\delta^{t\left(k,t_k\left|k,-j\cup j'\right.\right)}-\delta^{t\left(k,t_k\left|k,-j'\right.\right)}\right)\geq\left(\delta^{t\left(k,t_k\left|k,-j\right.\right)}-\delta^{t\left(k,t_k\left|k\right.\right)}\right),$$

which leads to the desired contradiction. Hence it is sufficient to evaluate the inequality (38) for constant payoff streams of alternative k. We can write the difference $M_j(\theta_0)-M_j\left(\theta_0\left|k\right.\right)$ by Lemma 3 as

$$M_j(\theta_0)-M_j\left(\theta_0\left|k\right.\right)=M_j^k(\theta_0).$$

It follows that we can rewrite the inequality (38) as

$$M_j^k(\theta_0)\leq M_j^{-j',k}(\theta_0),$$

which is holds by the first part of this lemma.■

Proof of Theorem 2. (1.) By hypothesis, $t(k,1)<t(i,1)$. We can therefore write the marginal contributions of i, $M_i(\theta_0)$ and $M_i\left(\theta_0\left|k\right.\right)$, respectively as

$$M_i(\theta_0)=\delta^{t(i,1)}M_i\left(\theta_{t(i,1)}\right)$$

and

$$M_i\left(\theta_0\left|k\right.\right)=\delta^{t\left(i,1\left|k\right.\right)}M_i\left(\theta_{t\left(i,1\left|k\right.\right)}\right)$$

By Lemma 1, it further follows that $t(i,1)=t\left(i,1\left|k\right.\right)$ and that $\theta_{t(i,1)}=\theta_{t\left(i,1\left|k\right.\right)}$ which establishes the claim.

(2.) By Lemma 4.2, it follows that it is sufficient to evaluate the inequality $M_i(\theta_0)\geq M_i\left(\theta_0\left|k\right.\right)$ when the payoff stream of alternative k is constant for all t_k with $0\leq t_k\leq\tau_k^1$. By Lemma 3.2, the difference then satisfies the relation

$$\frac{1}{1-\delta}\left(M_i(\theta_0)-M_i\left(\theta_0\left|k\right.\right)\right)=M_i^k(\theta_0),$$

and as the marginal contribution of any alternative to any program is weakly positive, the weak inequality follows.

(3.) By Lemma 4.**??**, it follows that it is sufficient to evaluate the inequality in Theorem 1.3 for the case that there are only two sellers, namely k, and $\mathcal{I}/\{k\}$. In this case, the inequality can be written after expressing the marginal contributions through the social values as:

$$W(\theta_0) - W(\theta_0 | k) \geq \left(W(\theta_0) - W_{-\mathcal{I}/\{k\}}(\theta_0)\right) - \left(W(\theta_0 | k) - W_{-\mathcal{I}/\{k\}}(\theta_0 | k)\right),$$

and as all terms cancel in this case, it follows that the inequality is always satisfied and in the extreme case of two sellers in fact as an equality.■

Proof of Theorem 3. The existence of the marginal contribution equilibrium was proved in the previous section. If $t_i(\theta) \geq T_i$ for all i, then the continuation game is in fact a repeated static game of price competition. Define next

$$Z(\theta) \triangleq \Sigma_{i=1}^{I} \min(t_i(\theta), T_i).$$

It is clear that for all states θ' such that $Z(\theta') = \Sigma_{i=1}^{I} T_i$, the marginal contribution equilibrium is the unique cautious equilibrium. The induction hypothesis that we use is that the claim is also true for all states θ such that $Z(\theta) \geq Z'$, where $Z' \leq \Sigma_{i=1}^{I} T_i$. The claim is then proved by induction if we can show that the claim is also true for all states such that $Z(\theta) = Z' - 1$.

Consider any state θ such that $Z(\theta) = Z' - 1$. Denote the equilibrium choice of the buyer at that state by $a(\theta)$. If $a(\theta) = i$ for some i such that $t_i(\theta) < T_i$, then the new state θ' is such that $Z(\theta') = Z'$ and by the induction hypothesis, the continuation game $\Gamma(\theta')$ has a unique cautious Markov perfect equilibrium payoff vector that coincides with the vector of marginal contributions of the players. Let $M_i(\theta, j)$ denote the marginal contribution and hence by the induction hypotheses the equilibrium continuation payoff to seller i if $a(\theta) = j$. If $a(\theta) = i$ for some i such that $t_i(\theta) \geq T_i$, then θ and the new state θ' have the same continuation games, and by the Markov restriction, i will be chosen in all future periods.

We claim first that the cautious Markov perfect equilibrium payoff is unique for each fixed first period choice by the buyer. To see this, it is enough to show that the first period prices are uniquely pinned down by the continuation payoffs. If $a(\theta) = i$, then $p_j(\theta) = \delta V_j(\theta, i) - \delta V_j(\theta, j)$ for all $j \neq i$ by cautiousness. If $t_i(\theta) < T_i$, then $V_j(\theta, i) = M_j(\theta, i)$ by the induction hypothesis. If $t_i(\theta) \geq T_i$, then $V_j(\theta, i) = 0$ since the strategies are Markovian. Similarly, if $t_j(\theta) < T_j$, $V_j(\theta, i) = M_j(\theta, i)$ by the induction hypothesis and if $t_j(\theta) \geq T_j$, then $V_j(\theta, j) = \delta V(\theta, i)$ if $t_i(\theta) < T_i$, and $V_j(\theta, j) = 0$ if $t_i(\theta) \geq T_i$. Hence in all cases, $p_j(\theta)$ is uniquely determined. Finally, $p_i(\theta)$ is determined by the $p_j(\theta)$, $x_i(\theta)$ and the $x_j(\theta)$ and the buyer's indifference condition.

Hence the remaining task is to show that for all θ such that $Z(\theta) = Z' - 1$, in all equilibria, $a(\theta) = i(\theta)$, where $i(\theta)$ is the socially efficient choice at state θ. We argue by contradiction. To do this, we suppose that $a(\theta) = k \notin i(\theta)$ in some equilibrium of the game starting at θ. Since the marginal contribution equilibrium also exists, there must be two separate sets of equilibrium prices at θ. Denote the marginal contribution equilibrium prices by $p_i(\theta)$ and the prices in the other equilibrium by $\widehat{p_i}(\theta)$. We know by construction that in the marginal contribution equilibrium, the buyer is indifferent between $i(\theta)$ and $j^{-i}(\theta)$. Hence we have for the buyer the equilibrium conditions:

$$\begin{aligned} V_B(\theta) &= x_{i(\theta)}(\theta) - p_{i(\theta)}(\theta) + \delta[W(\theta, i(\theta)) - \Sigma_{l \in \mathcal{I}} M_l(\theta, i(\theta))] \\ &= x_{j^{-i}(\theta)}(\theta) - p_{j^{-i}(\theta)}(\theta) + \delta[W\left(\theta, j^{-i}(\theta)\right) - \Sigma_{l \in \mathcal{I}} M_l\left(\theta, j^{-i}(\theta)\right)], \\ &\geq x_k(\theta) - p_k(\theta) + \delta[W(\theta, k) - \Sigma_{l \in \mathcal{I}} M_l(\theta, k)], \qquad \text{for all } k \neq i(\theta), j^{-i}(\theta), \end{aligned}$$

and for the efficient seller:

$$V_{i(\theta)}(\theta) = p_{i(\theta)}(\theta) + \delta M_{i(\theta)}(\theta, i(\theta)) \geq \delta M_{i(\theta)}\left(\theta, j^{-i}(\theta)\right),$$

and for all other sellers:

$$p_k(\theta) = \delta M_k(\theta, i) - \delta M_k(\theta, k) \text{ for all } k \neq i(\theta).$$

In the other equilibrium, let $j(k)$ be the seller that he buyer considers as good as k. Then the buyer's indifference condition is:

$$\begin{aligned} \widehat{V}_B(\theta) &= x_k(\theta) - \widehat{p}_k(\theta) + \delta[W(\theta,k) - \Sigma_{l\in\mathcal{I}} M_l(\theta,k)] \\ &= x_{j(k)}(\theta) - \widehat{p}_{j(k)}(\theta) + \delta[W(\theta,j(k)) - \Sigma_{l\in\mathcal{I}} M_l(\theta,j(k))], \end{aligned}$$

and for the winning seller k

$$\widehat{p}_k(\theta) + \delta M_k(\theta,k) \geq \delta M_k(\theta,j(k)),$$

and all other sellers l:

$$\widehat{p}_l(\theta) = \delta M_l(\theta,k) - \delta M_l(\theta,l), \quad \text{for all } l \neq k.$$

We derive the contradiction for the case where

$$t\left(i(\theta), t_{i(\theta)}\right) < t\left(j^{-i}(\theta), t_{j^{-i}(\theta)}\right) < t(k,t_k) < t\left(j(k), t_{j(k)}\right).$$

The remaining cases, i.e. those with

$$t\left(i(\theta), t_{i(\theta)}\right) < t\left(j^{-i}(\theta), t_{j^{-i}(\theta)}\right) < t\left(j(k), t_{j(k)}\right) < t(j,t_j)$$

and the ones where

$$k = j^{-i}(\theta) \quad \text{or} \quad j(k) \in \{i(\theta), j^{-i}(\theta)\}$$

are handled similarly. By Theorem 2.1 and the equilibrium conditions,

$$\widehat{p}_{j(k)}(\theta) = p_{j(k)}(\theta).$$

As a result, we have

$$V_B(\theta) \geq \widehat{V}_B(\theta). \tag{40}$$

By Theorem 2.2,

$$V_{i(\theta)}(\theta) > \widehat{V}_{i(\theta)}(\theta). \tag{41}$$

But (40) and (41) imply jointly that $i(\theta)$ can capture the buyer and increase the profit by offering $p_{i(\theta)}(\theta) - \varepsilon$ instead of $\widehat{p}_{i(\theta)}(\theta)$, contradicting the equilibrium requirements.■

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