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December 2001

COWLES FOUNDATION DISCUSSION PAPER NO. 1341

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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October 2001

Abstract

A decision maker has to choose one of several random variables, with uncertaintly known distributions. As a Bayesian she behaves as if she new the distributions. In his paper we suggest an axiomatic derivtion of these (subjective) distributions, which is much more economical than the derivations by de Finetti or Savage. They derive the whole joint distribution of all the available random variables.

1 Introduction

This paper provides an axiomatic derivation of subjective probabilities in the context of expected value maximization without a state space. As opposed to the usual approaches (Ramsey (1931), de Finetti (1937), and Savage (1954)), our model does not assume that the decision maker can or need estimate probabilities beyond those that are directly used for expected value maximization.

In many decision problems under uncertainty acts are not given as functions from states of nature to outcomes. For instance, suppose that the decision maker has to select one of several investments options (portfolios)

*Personal acknowledgements to be added. This work was partially supported by ISF Grant No. 790/00.

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of equal present value. For each portfolio she would like to know the distribution of its values at, say, a year later. To present our approach we turn to the stylized example of m urns containing colored balls. A ball will be drawn at random from each urn, and the outcome will be determined by the color of the ball drawn from the urn chosen by the decision maker. There are n possible colors, but the distributions of colors in the urns are unknown. The standard approach calls for the definition of states of nature as functions from acts to outcomes. Thus, there are n^m states in this problem, and $n^m - 1$ parameters for the decision maker to assess. However, expected value (or expected utility) maximization will only make use of $m(n-1)$ numbers, namely, the distribution of colors in each urn separately. If we view the color of the ball being drawn from each urn as a random variable, only the marginal distributions of these random variables are needed. In the standard approach, by contrast, the decision maker is asked to assess the entire joint distribution.

An obvious question arises: can one provide an axiomatic derivation of subjective probabilities (coupled with expected value maximization) that would *not* resort to a probability on an entire algebra of events (defining a large state space), but would make do with the probability values that are of actual use?

The purpose of this paper is to provide such an axiomatization. We assume as given a set of acts (such as the urns), and a set of physical consequences (such as the color of the balls drawn out of the urns). To each physical consequence we can attach a real-valued payoff, to be thought of as a monetary prize, or a utility index. Given such a payoff assignment, we assume that the decision maker can express preferences over the set of acts. We provide conditions under which there exist, for each act, a (subjective) distribution (over physical consequences), such that, for each payoff assignment, the decision maker ranks acts according to their expected payoff with respect to these distributions. In other words: each act is identified with a probability vector over the set of physical consequences. An entry in such a vector is interpreted as the subjective probability that, should this act be chosen, the corresponding consequence would result. We show that for every payoff assignment, the corresponding preference relation over acts agrees with maximization of expected value with respect to the given payoff assignment and the subjective distributions we derive.

Treating each available act as a random variable, taking values in the set of physical consequences, our result identifies a subjective distribution for each such variable. We do not identify their joint distribution, as do de Finetti and Savage. Rather, we only deal with the marginal distributions, which are those that will be used in choice between actually available acts. It also follows that our model does not address the question of (subjective) independence of or correlation between these random variables.

Our result serves several purposes. First, it relates the notion of subjective probability to preferences in a new way. As such, it might make this notion meaningful even when Savage questionnaires are too complex and counter-intuitive to be considered observable data. Second, it outlines the elicitation of subjective marginal distributions of a set of random variables, without resorting to the elicitation of the joint distribution thereof. Lastly, it can be used to judge the expected payoff paradigm, both descriptively and normatively, in situations where the de Finetti-Savage approach requires too detailed preference data. In particular, it shows that the expected payoff paradigm might be cognitively plausible in situations in which its known axiomatic derivations are not. That is, it is possible that decision makers generate subjective marginal distributions and choose a random variable that maximizes the corresponding subjective expected utility, without having Bayesian beliefs over the entire algebra of events generated by the random variables in question. In the language of preferences, decision makers may only have well-defined preferences between acts that are actually available to them, but not between hypothetical acts that can be defined by the former.

The next section presents the model and results. Proofs are relegated to an appendix. We defer further discussion to Section 3.

2 Model and Results

Assume that a decision maker is facing a decision problem with a finite and non-empty set of *acts,* A. Each act will result in one (and only one) physical *consequence* from the set $N = \{1, ..., n\}$ for $n \geq 1$. A *context* is a real-valued function on N. The space of all contexts, $\mathbb{R}^N$, is identified with $\mathbb{R}^n$, endowed with the natural topology and the standard algebraic operations. Given a context $x \in \mathbb{R}^n$, $\succsim_x \subset A \times A$ is a binary relation over acts.

The interpretation is as follows. The physical consequences are abstract, and they do not determine the decision maker's utility. Rather, it is the context x which associates a utility value to each possible consequence. Put differently, the set of contexts is the set of possible utility functions on the abstract set of consequences N. As in the introduction, a consequence might be, for instance, "a ball drawn from urn 1 is red". We do not assume that different acts have disjoint sets of possible consequences. Rather, every physical consequence might, a-priori, result from any act. It is assumed that we can observe the decision maker's preferences over acts given *any* utility function.

We now formulate axioms on $\{\succsim_x\}_{x \in \mathbb{R}^n}$:

A1 Order: For every $x \in \mathbb{R}^n$, $\succsim_x$ is complete and transitive on A.

A2 Additivity: For every $x, y \in \mathbb{R}^n$ and every $a, b \in A$, if $a \succsim_x b$ and $a \succ_y b$, then $a \succ_{x+y} b$.

A3 Continuity: For every $a, b \in A$ the sets $\{x \mid a \succ_x b\}$ and $\{x \mid b \succ_x a\}$ are open.

A4 Diversity: For every list (a, b, c, d) of distinct elements of A there exists $x \in \mathbb{R}^n$ such that $a \succ_x b \succ_x c \succ_x d$. If $|A| < 4$, then for any strict ordering of the elements of A there exists $x \in \mathbb{R}^n$ such that $\succ_x$ is that ordering.

A5 Neutrality: For every constant $c \in \mathbb{R}^n$ (i.e., $c_i = c_j$ for all $i, j \in N$), and every $a, b \in A$, $a \sim_c b$.

Axiom 1 is standard. Axiom 2 is the most crucial axiom, as it guarantees that the set of contexts (utility functions) for which act a is preferred to act b is convex. Axiom 3 states that this set is also open. The diversity axiom (A4) rules out certain preferences. For instance, it does not allow one lottery to be always preferred to another. Finally, A5 is a weak consequentialism axiom. It holds whenever the decision maker cares only about the final utility derived from consequences: if this utility function happens to be constant, no act should be preferred to any other.

The statement of the theorem requires two additional definitions. A function $P : N \to [0,1]$ with $\sum_{i \in N} P(i) = 1$ is called a *lottery*. Algebraic operations on lotteries are performed pointwise. In particular, the α-mixture of lotteries P and Q, $\alpha P + (1-\alpha)Q$ is also a lottery. A collections of lotteries $\{P_a\}_{a \in A}$ is called *4-independent* if every four (or fewer) lotteries in it are linearly independent.[1]

Theorem 1 *: The following two statements are equivalent:*

(i) $\{\succsim_x\}_{x \in \mathbb{R}^n}$ *satisfy A1 - A5;*

(ii) There is a collection of 4-independent lotteries $P = \{P_a\}_{a \in A}$ *such that:*

$$(*) \quad \begin{array}{c} \text{for every } x \in \mathbb{R}^n \text{ and every } a, b \in A, \\ a \succsim_x b \quad \text{iff} \quad \sum_{i \leq n} P_a(i)x(i) \geq \sum_{i \leq n} P_b(i)x(i) \ , \end{array}$$

To what extent are the lotteries $\{P_a\}_{a \in A}$ unique? Clearly, if $\{P_a\}_{a \in A}$ satisfy $(*)$, then, for any lottery R and any $\alpha \in (0,1]$, the collection $\{\alpha P_a +$

[1] That is, if $|A| < 4$, $\{P_a\}_{a \in A}$ is 4-independent if it is linearly independent.

$(1-\alpha)R\}_{a\in A}$ also satisfies $(*)$.[2] Observe that in $\{P_a\}_{a\in A}$ differences between lotteries are more pronounced than in $\{\alpha P_a + (1-\alpha)R\}_{a\in A}$. This gives rise to the following definition.

For two collections of lotteries, $P = \{P_a\}_{a\in A}$ and $Q = \{Q_a\}_{a\in A}$, we say that P is *more extreme than* Q if there exists a lottery R and $\alpha \in (0,1)$ such that $\alpha P_a + (1-\alpha)R = Q_a$ for all $a \in A$. We can now state the uniqueness result.

Proposition 2 *There exists a unique collection of 4-independent lotteries P that satisfies $(*)$ and that is more extreme than any other collection Q that satisfies $(*)$.*

We do not know of a set of axioms that are necessary and sufficient for a representation as in $(*)$ by a collection of lotteries that need not be 4-independent. We do know that dropping A4 will not do.[3] It is clear from the proof of the main result in Gilboa and Schmeidler (1997) that weaker versions of A4 suffice for a representation as in $(*)$. Ashkenazi and Lehrer (2001) also offer a condition that is weaker than A4, and that also suffices for such a representation. The diversity axiom is stated here in its simplest and most elegant form, rather than in its mathematically weakest form.

3 Discussion

Our derivation of expected utility with subjective lotteries assumes that legitimate data are only preferences between acts that are actually available, but that these can be observed for any assignment of utility values to consequences. One obvious drawback of this axiomatization is that, like that of

[2] This is tantamount to saying that the indepence axiom of vNM (1944) is necessary for expected utility maximization.

[3] In Gilboa and Schmeidler (1997) we provide two counter-examples that show that A1-A3, coupled with a weaker neutrality axiom, do not imply a representation as in $(*)$. The first counter-example uses a finite set A and satisfies the stronger version of A5 used here, and therefore applies to our case as well.

de Finetti (1937), it assumes linearity (or convexity) in payoffs. This can be interpreted in several ways. First, one may assume that payoffs are monetary, and that the decision maker is risk neutral. This makes the data easily observable, but the axiom becomes rather implausible since decision makers are typically not assumed to be risk neutral, especially when large sums of money are involved.

Second, numerical payoffs can be viewed as utility values. That is, one may assume that there is an independent measurement of a cardinal utility function, and that contexts x refer to values of this utility function. No loss of generality is involved here, since every expected utility maximizer will satisfy this axiom in this formulation. But the allegedly observable data are then supposed to be choices given such assignments of utility values, and it is not entirely clear where one can get such a utility function to begin with.

A third interpretation involves small amounts of money, and it relies on the fact that an expected utility maximizer (with a differentiable utility function) behaves like a risk neutral decision maker when small amounts of money are involved. To be more concrete, assume that there are preference orders $\{\succsim'_x\}_{x\in\mathbb{R}^n}$ on the set of act A for any monetary payoff function $x \in \mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$ define a new relation, $\succsim_x$ as follows: $a \succsim_x b$ iff there exists $\widehat{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \widehat{\varepsilon})$, $a \succsim'_{\varepsilon x} b$. Obviously, $a \succsim_x b$ iff $a \succsim_{\lambda x} b$ for every $x \in \mathbb{R}^n$ and every $\lambda > 0$. One can now assume that $\{\succsim_x\}_{x\in\mathbb{R}^n}$ satisfy A1-A5. In this interpretation, the completeness axioms becomes less obvious. However, the additivity axiom (A2) is more palatable. Naturally, the preferences $\{\succsim_x\}_{x\in\mathbb{R}^n}$, namely, preferences given arbitrarily small monetary prizes, suffice for the derivation of subjective lotteries. But this approach suggests that elicitation of probabilities be made using prizes about which the decision maker does not really care. It is not clear that one would like to base the derivation of subjective probabilities entirely on preferences between asymptotically small prizes.

Rather than adopting one of these interpretation, one can also attempt to derive the utility function in conjunction with the subjective lotteries. Building on the techniques of Wakker (1989), one may re-state the additivity axiom so that it correspond to addition of utilities, rather than of prizes. Indeed, in Gilboa, Schmeidler, and Wakker (1999) we follow a similar tack in adapting the original result of Gilboa and Schmeidler (1997), which assumed a given utility function, to a result that derives the utility function as well. In fact, one may start with the first axiomatization in Gilboa, Schmeidler, and Wakker (1999), strengthen axiom A5 as above, and continue to obtain a derivation of expected utility with subjective lotteries and a general utility function.

Our approach assumes that there are physical consequences, to which various payoffs can be assigned. For instance, the physical consequences could be colors of balls drawn from urns, whereas the monetary payoffs attached to them are arbitrary. There are applications in which this arbitrariness is unwarranted. For instance, if the physical consequence of death cannot be ascribed arbitrary utility value.

Yet, there are many economic applications in which one may define a physical consequence that does not uniquely define the decision maker's well-being. For instance, the physical consequences could be the success or failure of certain new technologies, while the payoffs attached to them are defined by market conditions. Alternatively, physical consequences might correspond to prices of stocks, whereas the decision maker's payoff is defined by a various derivatives she holds. In situations such as this, it might be easier for a decision maker to imagine various payoffs attached to physical consequences, than to imagine conceivable acts defined on the entire state space in which actual acts are embedded.

Observe that, if one starts with acts and consequences and uses them to define states of nature, all preference questions in our model have corresponding questions in de Finetti's model. Whereas we assume that physical consequences might yield arbitrary utility values, de Finetti would use states that define the actual utility (through the consequence) and then attach to them hypothetical utility. By contrast, the de Finetti questionnaire would include many hypothetical questions that are not needed in our model.

It might be insightful to embed our model in a de Finetti-Savage model. Let there be given m acts, each of which may result in one of n physical consequences. Given an act a and a payoffs vector $x \in \mathbb{R}^n$, the payoff x_i will result if a is chosen by the decision maker and consequence i is chosen by nature. The corresponding de Finetti model has n^m states. An act a and a payoff vector x define a de Finetti act $y \in \mathbb{R}^{n^m}$ as follows: for a state of nature j, where the choice of act a results in consequence i, $y_j = x_i$. Act b with the same context $x \in \mathbb{R}^n$ is represented by $z \in \mathbb{R}^{n^m}$ where, for a state of nature k in which act b results in consequence l, $z_k = x_l$. The de Finetti acts y and z in $\mathbb{R}^{n^m}$ assume the same coordinate values, which are those of $x \in \mathbb{R}^n$, but not at the same coordinates. Conversely, taking the same act a with another payoff vector $x' \in \mathbb{R}^n$ would yield a de Finetti act $y' \in \mathbb{R}^{n^m}$ that is obtained from y by replacing each x_i by x'_i. If we fix an act a and consider all the vectors $y \in \mathbb{R}^{n^m}$ that correspond to a and to some $x \in \mathbb{R}^n$, we obtain a subspace of $\mathbb{R}^{n^m}$ whose dimension is n. Ranging over all acts a, the entire sets of de Finetti acts that result in the union of m such subspaces. This is clearly a much smaller set that $\mathbb{R}^{n^m}$. Moreover, in our axiomatization an act a in the context $x \in \mathbb{R}^n$ is compared only with the other $m-1$ acts in the same context x. By contrast, in a de Finetti-Savage axiomatization, each vector $y \in \mathbb{R}^{n^m}$ is compared with all other vectors in $\mathbb{R}^{n^m}$.

Appendix: Proof

Proof of Theorem 1:

We first quote the main result in Gilboa and Schmeidler (1997). In this theorem, A1-A4 are identical to ours. A5 is weaker:

A5* Weak Neutrality: For every $a, b \in A$, $a \sim_0 b$.
where 0 denotes the origin in $\mathbb{R}^n$. The theorem states

Theorem 3 *(Gilboa-Schmeidler, 1997): The following two statements are equivalent if* $|A| \geq 4$*:*

(i) $\{\succsim_x\}_{x \in \mathbb{R}^n}$ *satisfy A1 - A4, A5*;*

(ii) There is a collection $\{v^a\}_{a \in A} \subset \mathbb{R}^n$ *such that:*

$$(**) \quad \begin{array}{c} \text{for every } x \in \mathbb{R}^n \text{ and every } a, b \in A, \\ a \succsim_x b \quad \text{iff} \quad \sum_{i \leq n} v_i^a x_i \geq \sum_{i \leq n} v_i^b x_i \ , \end{array}$$

and, for every distinct $a, b, c, d \in A$*, the vectors* $\{v^a - v^b, v^b - v^c, v^c - v^d\}$ *are linearly independent.*

Furthermore, $\{v^a\}_{a \in A}$ *are unique in the following sense:* $\{u^a\}_{a \in A}$ *also satisfy* $(**)$ *iff there exists* $\alpha > 0$ *and* $\beta \in \mathbb{R}^n$ *such that* $u^a = \alpha v^a + \beta$ *for all* $a \in A$*.*

Clearly, the representation in $(**)$ is basically identical to that in $(*)$, with $P_a(i) = v_i^a$. As in Theorem 1, Theorem 3 restricts the type of collections of vectors $\{v^a\}_{a \in A}$ that may arise. We now show that, if v^a is a lottery on $\{1, ..., n\}$, this restriction is equivalent to that of Theorem 1.

Proposition 4 *Assume first* $|A| \geq 4$*. A collection of lotteries* $\{v^a\}_{a \in A}$ *is 4-independent iff for every list* (a, b, c, d) *of distinct elements of* A*, the vectors* $\{v^a - v^b, v^b - v^c, v^c - v^d\}$ *are linearly independent. Similar equivalence holds for the case* $|A| < 4$*.*

Proof. Assume that $|A| \geq 4$. Let a collection of lotteries $\{v^a\}_{a\in A}$ be given, and assume that it is 4-independent. Consider distinct a, b, c, d. If the vectors $\{v^a - v^b, v^b - v^c, v^c - v^d\}$ are dependent, so are $\{v^a, v^b, v^c, v^d\}$, contrary to our assumption.

For the converse direction, assume that the vectors $\{(v^a - v^b), (v^b - v^c), (v^c - v^d)\}$ are independent (for all quadruples of distinct a, b, c, d) but that, contrary to 4-independence of $\{v^a\}_{a\in A}$, there are distinct $a, b, c, d \in A$ and $\lambda, \mu, \theta \in \mathbb{R}$ such that

$$v^a = \lambda v^b + \mu v^c + \theta v^d.$$

Summation yields $\sum_{i=1}^n v^a = \lambda \sum_{i=1}^n v^b + \mu \sum_{i=1}^n v^c + \theta \sum_{i=1}^n v^d$. Since v^a, v^b, v^c, v^d are lotteries, we obtain $\lambda + \mu + \theta = 1$. This implies that

$$\lambda(v^a - v^b) + \mu(v^a - v^c) + \theta(v^a - v^d) = 0$$

or

$$(\lambda + \mu + \theta)(v^a - v^b) + (\mu + \theta)(v^b - v^c) + \theta(v^c - v^d) = 0.$$

Not all coefficients are zero, (again, since $\lambda + \mu + \theta = 1$), hence $\{(v^a - v^b), (v^b - v^c), (v^c - v^d)\}$ are dependent, a contradiction.□

Next we mention that Theorem 3 is easily proved also for the case $|A| < 4$. (This is a by-product of the proof in Gilboa-Schmeidler (1997).)

We now turn to the proof of Theorem 1. We first prove sufficiency of the axioms. Assume, then, that A1-A5 hold. Thus A1-A4, A5* hold. Hence, by Theorem 3, there is a representation by $\{v^a\}_{a\in A}$ as in (**) such that, for every distinct a, b, c, d, the vectors $\{v^a - v^b, v^b - v^c, v^c - v^d\}$ are linearly independent. Further, applying A5 for $c = (1, ..., 1)$, one obtains that $\sum_{i\in N} v_i^a$ is independent of a.

Since A is finite, one may define $\beta_i = -\min_{a\in A} v_i^a$ and set $u_i^a = v_i^a + \beta_i$. Thus, $\{u_i^a\}_{a\in A}$ also satisfy (**), $\sum_{i\in N} u_i^a$ is independent of a, and $\min_{a\in A} v_i^a =$

0 for all $i \in N$. Let $c = \sum_{i \in N} u_i^a$ for some (hence, all) $a \in A$. If $c = 0$, $u_i^a = 0$ for all $a \in A$ and $i \in N$. Hence $c > 0$. Define $\alpha = c^{-1}$ and $P_a(i) = \alpha u_i^a$. Thus $P = \{P_a\}_{a \in A}$ are lotteries that satisfy $(*)$. By Proposition 4, P is 4-independent.

To see the converse, assume that $P = \{P_a\}_{a \in A}$ is given, such that $(*)$ holds and P is 4-independent. Define $v_i^a = P_a(i)$ and use Theorem 3 to show that A1-A4 hold. Then observe that, since $\sum_{i \in N} P_a(i) = 1$ for all a, $(*)$ implies that A5 holds as well.

This completes the proof of Theorem 1. □□

Proof of Proposition 2:

Consider the collection of lotteries P defined in the proof of Theorem 1. We claim that it is more extreme than any other collection of lotteries Q satisfying $(*)$. Let Q be such a collection, with $Q \neq P$. By the uniqueness part of Theorem 3, there are $\alpha > 0$ and, for each $i \in N$, $\beta_i \in \mathbb{R}$, such that

$$Q_a(i) = \alpha P_a(i) + \beta_i$$

for every $a \in A$ and $i \in N$.

By construction of P, $\min_{a \in A} P_a(i) = 0$ for every $i \in N$. It follows that $\beta_i \geq 0$ (otherwise, we get $Q_a(i) < 0$ for some $a \in A$ and $i \in N$). Observe that, for all a,

$$1 = \sum_{i \in N} Q_a(i) = \alpha \sum_{i \in N} P_a(i) + \sum_{i \in N} \beta_i = \alpha + \sum_{i \in N} \beta_i$$

If $\beta_i = 0$ for all $i \in N$, $\alpha = 1$ and $P = Q$. Since Q differs from P, there exists $\beta_i > 0$. Hence $\alpha < 1$. Define $R(i) = \beta_i/(1-\alpha) \geq 0$. Observe that $\sum_{i \in N} R(i) = \sum_{i \in N} \beta_i/(1-\alpha) = 1$. Hence R is a lottery such that

$$Q_a(i) = \alpha P_a(i) + \beta_i = \alpha P_a(i) + (1-\alpha)R(i)$$

for every $a \in A$ and $i \in N$. This proves that P is more extreme than Q. □□

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