COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1279

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than mere acknowledgment by a writer that he has access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

THE EFFECT OF BIDDERS' ASYMMETRIES ON EXPECTED REVENUE IN AUCTIONS

Estelle Cantillon

October 2000

The Effect of Bidders' Asymmetries on Expected Revenue in Auctions*

Estelle Cantillon†

First draft: April 1999
This version: October 2000

Abstract

Bidders' asymmetries are widespread in auction markets. Yet, their impact on behavior and, ultimately, revenue and profits is still not well understood. In this paper, I define a natural benchmark auction environment to which to compare any private value auction with asymmetrically distributed valuations. I show that the expected revenue from the benchmark auction always dominates that from the asymmetric auction, both in the first price auction and the second price auction. These results formalize and make transparent the idea that competition is reduced by bidders' asymmetries. The paper also contributes to a better understanding of competition and the nature of rents in auction markets. Anonymity of the allocation mechanism seems to be an important factor.

Keywords: Auctions, asymmetries, anonymous mechanisms, benchmark, reduced competition.

---

*I am particularly indebted to Eric Maskin and John Riley for discussion and suggestions on an earlier draft. I have also benefited from the comments of Isabelle Brocas, Wouter Dessein, Mathias Dewatripont, Peter Eso, Paul Klemperer, Patrick Legros, Martin Pesendorfer and Al Roth, as well as seminar audiences at Harvard, LSE, MEDS, Nuffield College, Tilburg and Yale. Huagang Li and John Riley kindly provided me with the source codes of their Bidcomp2 program. Financial support from the Belgian Fonds National de la Recherche Scientifique is gratefully acknowledged.

†Cowles Foundation (Yale University), Harvard Business School and CEPR. Correspondence: Department of Economics, Yale University, P.O.Box 208268, New Haven CT 06520-8268. Phone: (203) 432 3590, Fax: (203) 432 5779, Email: Estelle.Cantillon@yale.edu

1 Introduction

Known ex-ante asymmetries among bidders are widespread in auction markets. For instance, firms with a toehold in the target firm are favored in takeover battles and this advantage is usually understood by all potential buyers. In arts auctions, bidders' tastes are known to be quite idiosyncratic. Asymmetries among bidders have also been recently documented in procurement markets, with sectors ranging from the public works (Bajari, 1998) to the procurement of school milk (Porter and Zona, 1999, Pesendorfer, 2000). In all these auctions, there was one or several firms with a clear comparative advantage over the others. Last but not least, the competition in defense procurement also tends to be unequal.

In this paper, I focus on private value auctions, that is, auctions where at the time when bidders submit their bids, they know how much they value the object they are bidding for. There have been recent advances in our understanding of how these auctions work when bidders' distributions of valuations differ. Most importantly, we now know that an equilibrium exists under quite general conditions in the sealed bid first price auction (Lebrun, 1996, Maskin and Riley, 2000a, and Athey, 1999) and understand under what conditions it is unique (Maskin and Riley, 1996).[1]

Nevertheless, the effect of asymmetries on the auctioneer's expected revenue is still not well understood. Maskin and Riley (2000b) have shown that the revenue ranking between the second price auction and the first price auction depends generally on the kind of asymmetries among bidders. In addition, we know that, in the presence of asymmetries, the first price auction is inefficient (it fails to allocate the object to the highest valuer) and that both the first price auction and the second price auction are generally suboptimal (they fail to maximize the seller's expected revenue). However, these results shed little light on the impact of bidders' asymmetries within a single institution: for instance, the first price or the second price auction.

In this paper, I am interested in understanding how (common knowledge) ex-ante differences in bidders' distributions of valuations affect their behavior and, in turn, expected revenue and profits. To do so, I define a benchmark auction environment to which to compare any auction with asymmetric bidders. An important property of my benchmark is that the expected value of

[1] In the second price auction, existence and uniqueness of the equilibrium do not depend on the distributional assumption, so asymmetries do not introduce any difficulty.

the highest valuation among bidders is the same as in the original auction. In other words, I shall be comparing two auction environments for which the potential social surplus is the same.[2] The key difference is that, while bidders' distributions may differ in the original auction, they are identical in the benchmark auction.

A priori, it is unclear how asymmetries affect the auctioneer. In the two auction formats I consider (the first price auction and the second price auction), the auctioneer is the residual claimant of bidders' strategic interactions. It is the competition among bidders that determines the winning price, and this is the auctioneer's revenue.[3] When bidders' valuations are asymmetrically distributed, bidders' strategic adjustment to these asymmetries will for sure affect the distribution of social surplus among bidders and the auctioneer. How so is less clear. In particular, Maskin and Riley (2000b) have shown that "strong" bidders, that is, bidders who are more likely to have a high valuation for the object, are better off under a second price auction rather than a first price auction. This suggests that bidders' attempts to take advantage of their favorable positions might be self-defeating in the first price auction. This could benefit the auctioneer. On the other hand, our economic intuition suggests that asymmetries among market participants reduce the competitive pressure they face, and that this should hurt the auctioneer.

My main results (theorem 1 and propositions 1 and 2) are that asymmetries hurt the auctioneer in the first price auction (FPA) and the second price auction (SPA). Indeed, in both institutions, the expected revenue from the benchmark auction dominates that from the original (asymmetric) auction. A corollary for the second price auction is that bidders always gain *in the aggregate* from asymmetries. By contrast, the first price auction is inefficient in the presence of asymmetries and so both bidders *and* the auctioneer can lose from asymmetries. I provide one such example.

These results are interesting on two counts. First, they provide some insights on how these specific auction markets work. This is especially useful in the case of the FPA where the lack of analytical solutions has slowed down our understanding. Second, auctions can also be seen as a paradigm of how

---

[2] Remember that auctions are allocation mechanisms (they provide rules to allocate an object among several bidders) and that social surplus is maximized when the object goes to the bidder with the highest valuation.

[3] In that sense, the auctioneer is pretty much in the same position as consumers in an oligopoly market.

markets in general work. From that perspective, the results hightlight the decrease in the toughness of competition that market heterogeneity induces.

The results also shed light on the respective strengths of the equilibrium versus optimal mechanism design approaches to auctions. Since the seminal contributions of Myerson (1981) and Riley and Samuelson (1981), auctions have been very fruitfully studied from an optimal mechanism design perspective.[4]

A message of this paper is that, in the presence of asymmetries, the optimal mechanism design approach might not be as good a substitute to study specific auction formats. Indeed, optimal auctions and auction rules like the FPA or SPA differ in one important respect: the ability in the former to discriminate among bidders. As long as bidders are symmetric, this difference is unimportant because the optimal auctioneer does not want to treat bidders differently anyway. However, when bidders are asymmetric, optimal mechanism design becomes a less useful guide. A systematic study of equilibrium behavior, though admittedly much less elegant, might be needed to understand how these markets work. In particular, the fact that asymmetries are harmful could not have been anticipated simply using a mechanism design perspective. In section 5, I provide an example where the optimal auction yields more revenue in the presence of asymmetries than under the benchmark (symmetric) configuration.

The structure of the paper is as follows. Section 2 describes the model and introduces the benchmark auction environment. Equilibrium behavior in the first price and second price auctions is very different, and so, I deal with these institutions in turn. Section 3 studies the effect of asymmetries in the second price auction. Section 4 deals with the first price auction. In section 5, I first show that asymmetries do not need to hurt the optimal auctioneer. I then elaborate on the nature of bidders' rents in an auction. Bidders' private information is a central element in the analysis of auctions. However, its importance depends on the competitive situation of bidders. This contrasts with the optimal auction where bidders' profits are entirely driven by informational rents.

[4]Klemperer's (1999) recent survey of the literature is a good place to realize the contribution of mechanism design to auction theory.

2 A symmetric benchmark

I consider a simple two-bidder private value auction environment. There is one object for sale through a sealed bid first price auction or a second price auction. Bidders' valuations are independently distributed according to the continuously differentiable cumulative distribution functions F_i with support on $[\underline{v}_i, \overline{v}_i]$, $i = 1, 2$. Valuations are private information but their distributions are common knowledge. Without loss of generality, I assume throughout that $0 \leq \underline{v}_1 \leq \underline{v}_2$. Bidders are risk neutral.

With these assumptions and given the selling procedure, an auction environment is fully characterized as soon as we define the distributions of bidders' valuations, (F_1, F_2). I refer to the pair of cumulative distribution functions, (F_1, F_2), as a *configuration*.

In this paper, I want to understand how asymmetries affect the outcome in the first price and second price auctions. One way of doing this is to compare the outcome in an asymmetric auction with that of a symmetric auction which, somehow, we consider a natural point of comparison.

What properties should this benchmark have? At this point, it is useful to remember that an auction is an allocation mechanism. In the private value environment that we consider, the highest level of social surplus (efficiency) is achieved when the object is allocated to the bidder with the highest valuation. A property that seems reasonable if we want to draw any meaningful conclusion on whether asymmetries are a good thing or not for the auctioneer is that the expected potential social surplus ("the size of the pie") is the same in both the original (asymmetric) auction environment and the benchmark environment. Indeed, without this condition, we would need to compare the *ratios* of expected revenue to social surplus but these are not invariant to cardinal changes to the environment.

To construct the benchmark, I go further and impose the condition that, not only the expected potential social surplus is identical in both environments, but that the *distribution* of this surplus is also the same in both cases. The motivation for this additional condition is again the fact that auctions are allocation mechanisms. Imposing identical distributions of potential social surplus means that the distribution of the first order statistics is the same in both cases. In other words, suppose that v is the highest realization from (F_1, F_2) with probability p. Then v will also be the highest realization in the benchmark environment with probability p. As a result, both auctions are similar from an allocation perspective.

This second condition uniquely defines the benchmark. Indeed, the cumulative distribution function of the highest realization from (F_1, F_2) is given by $F_1(v)F_2(v)$. Similarly, the cumulative distribution function of the highest realization from a symmetric configuration (F, F) is given by $F(v)F(v)$. Equating these two expressions, we have:

Definition 1: Given two cumulative distributions F_1 and F_2 with support on $[\underline{v}_1, \overline{v}_1]$ and $[\underline{v}_2, \overline{v}_2]$ respectively, their *symmetric benchmark*, F, is defined, for all v, by:

$$F(v) = \sqrt{F_1(v)F_2(v)}$$

F has support on $[\underline{v}, \overline{v}]$ where $\underline{v} = \max\{\underline{v}_1, \underline{v}_2\}$ and $\overline{v} = \max\{\overline{v}_1, \overline{v}_2\}$.

Because the choice of a proper benchmark takes on an important role in determining whether asymmetries hurt or benefit the auctioneer, it is worthwhile to discuss what the benchmark of definition 1 does and does not. First of all, notice that the benchmark treats bidders' distributions symmetrically and that it straightforwardly generalizes to any number of bidders.

Second, by construction, the benchmark preserves the distribution of the first order statistics and the expected value of the potential surplus, two properties that, I have argued, are desirable: the first because auctions are fundamentally about orders, the second because it guarantees consistent comparisons.

By contrast, the support of the benchmark distribution need not include any of the supports of the original distributions and therefore the range of valuations under the benchmark might differ from the range of valuations in the original configuration. In addition, when the supports of the original distributions differ, the probability distribution function derived from the benchmark is discontinuous.

In my view, these are unimportant features. Theorem 2 in the next section provides further support for the benchmark by analyzing the class of averages of F_1 and F_2 of the form $(\frac{1}{2}F_1(v)^\alpha + \frac{1}{2}F_2(v)^\alpha)^{\frac{1}{\alpha}}$. For $\alpha \neq 0$, these averages have the same range of valuations as the original configuration and their probability distribution functions are continuous. However, it is shown that the benchmark of definition 1 (which corresponds to $\alpha = 0$) is the only average in that class that preserves the *expected* value of potential social surplus. Figures 1 and 2 illustrate some of these properties.

[insert figure 1 here]

[insert figure 2 here]

3 The effects of asymmetries in the SPA

I start by considering how asymmetries affect revenue and profits in the second price auction. In the SPA, the winner is the bidder who places the highest bid and he pays the value of the second highest bid. It is well known that bidding one's own valuation is a dominant strategy in this setting.[5] Given this, we have:

Theorem 1: *Consider any configuration of bidders, (F_1, F_2). The expected revenue from the second price auction for the symmetric benchmark, $R^s(F, F)$, is always greater than that for the asymmetric configuration (F_1, F_2).*

Proof. Denote by $v_{(2)}$, the expected value of the second order statistics for (F, F). Similarly, $v_{(2)}^{1,2}$ denotes the expected value of the second order statistics for (F_1, F_2). Since the winner in the SPA is always the bidder who has the highest valuation and since bidders bid their true valuations at equilibrium, the auctioneer gets the expected value of the second highest valuation, i.e., $R^s(F, F) = v_{(2)}$ and $R^s(F_1, F_2) = v_{(2)}^{1,2}$.

By definition,

$$v_{(2)} = 2\int_{\underline{v}}^{\overline{v}} v(1 - F(v))dF(v) = 2\int_{\underline{v}}^{\overline{v}} vdF(v) - v_{(1)} \tag{1}$$

where $v_{(1)}$ denotes the expected value of the first order statistics.

[5] This might not be the unique equilibrium when the supports of valuations differ. However, it is straightforward to show that all equilibria generate the same revenue.

Without loss of generality, let $\underline{v}_1 \leq \underline{v}_2$.

$$
\begin{aligned}
v_{(2)}^{1,2} &= \int_{\underline{v}_1}^{\overline{v}_1} v(1-F_2(v))dF_1(v) + \int_{\underline{v}_2}^{\overline{v}_2} v(1-F_1(v))dF_2(v) \\
&= \int_{\min\{\underline{v}_1,\underline{v}_2\}}^{\max\{\overline{v}_1,\overline{v}_2\}} v(1-F_2(v))dF_1(v) + \int_{\min\{\underline{v}_1,\underline{v}_2\}}^{\max\{\overline{v}_1,\overline{v}_2\}} v(1-F_1(v))dF_2(v) \\
&= \int_{\underline{v}_1}^{\overline{v}} vdF_1(v) + \int_{\underline{v}_1}^{\overline{v}} vdF_2(v) - \int_{\underline{v}_1}^{\overline{v}} vd[F_1(v)F_2(v)] \\
&= \int_{\underline{v}_1}^{\overline{v}} vd[F_1(v)+F_2(v)] - v_{(1)}^{1,2} \qquad (2)
\end{aligned}
$$

By construction, $v_{(1)} = v_{(1)}^{1,2}$. Hence, subtracting (2) from (1), we get:

$$
\begin{aligned}
v_{(2)} - v_{(2)}^{1,2} &= 2\int_{\underline{v}}^{\overline{v}} vdF(v) - \int_{\underline{v}_1}^{\overline{v}} vd[F_1(v)+F_2(v)] \\
&= 2\int_{\underline{v}_1}^{\overline{v}} vdF(v) - \int_{\underline{v}_1}^{\overline{v}} vd[F_1(v)+F_2(v)] \text{ (change of integration bounds)} \\
&= -2\int_{\underline{v}_1}^{\overline{v}} F(v)dv + \int_{\underline{v}_1}^{\overline{v}} [F_1(v)+F_2(v)]dv \text{ (integration by parts)} \\
&= \int_{\underline{v}_1}^{\overline{v}} (\sqrt{F_1(v)} - \sqrt{F_2(v)})^2 dv > 0 \ \blacksquare
\end{aligned}
$$

Why do asymmetries hurt the auctioneer in the second price auction? Because bidders' strategies are unaltered by asymmetries (bidding one's own valuation remains a dominant strategy), the origin of this effect is purely statistical. Intuitively, the benchmark auction was constructed such that the expected value of the highest draw from (F, F) and from (F_1, F_2) is the same. Since we expect the two draws from (F_1, F_2) to be more "noisy", the expected value of the lowest draw must be lower in that case, that is, $v_{(2)}^{1,2} < v_{(2)}$. Another way of thinking about this is the following. By construction,

$$
\frac{F'}{F} = \frac{1}{2}[\frac{F_1'}{F_1} + \frac{F_2'}{F_2}] \qquad (3)
$$

Now, suppose that for some v^*, $\frac{F_1'(v^*)}{F_1(v^*)} > \frac{F_2'(v^*)}{F_2(v^*)}$. Since $F_1'(v^*)F_2(v^*) > F_2'(v^*)F_1(v^*)$, this means that bidder 1 is more likely to win at v^*. Moreover, by (3),

$\frac{F_2'(v^*)}{F_2(v^*)} < \frac{F'(v^*)}{F(v^*)}$. Therefore, when bidder 1 wins, her expected payment is likely to be lower than under the benchmark. This boosts expected revenue in the benchmark auction relative to the asymmetric auction.[6]

As plausible as these explanations sound, the result of theorem 1 actually holds for a larger class of "averages", where the intuition developed in the previous paragraph is not as straightforward.

Theorem 2: *Let* $F_\alpha(v) = (\frac{1}{2}F_1(v)^\alpha + \frac{1}{2}F_2(v)^\alpha)^{\frac{1}{\alpha}}$. *Denote by* $v_{(1)}^\alpha$ *the expected value of the first order statistics of* (F_α, F_α), *and by* $v_{(2)}^\alpha$, *the expected value of the second order statistics. Then, for any asymmetric configuration* (F_1, F_2),
(1) $v_{(1)}^\alpha$ *is strictly decreasing in* α;
(2) $v_{(2)}^\alpha$ *is strictly decreasing in* α;
(3) $v_{(2)}^\alpha > v_{(2)}^{1,2}$ *for* $\alpha \leq 1$.

Notice that F_α is the constant elasticity of substitution (CES) equivalent for (F_1, F_2). In particular, when $\alpha = 0$, $F_\alpha(v) = F_1(v)^{\frac{1}{2}} F_2(v)^{\frac{1}{2}}$, that is, the symmetric benchmark. When $\alpha = 1$, $F_\alpha(v) = \frac{1}{2}F_1(v) + \frac{1}{2}F_2(v)$, the arithmetic average.

Theorem 2 is interesting in two respects. First, it provides further motivation for the benchmark since it shows that, in the class of CES-like averages, our symmetric benchmark is the only one that preserves the *expected* value of the potential social surplus. At the same time, theorem 2 suggests that the result about asymmetries hurting the auctioneer in the SPA is actually fairly robust to the choice of the benchmark. Indeed, for any convex CES average of (F_1, F_2), expected revenue in the benchmark auction is higher (point (3)). Moreover, if we focus on the share of potential surplus that the auctioneer is able to capture (the $v_{(2)}/v_{(1)}$ ratios), then asymmetries also hurt the auctioneer for $\alpha \in [0, 1]$ (combining points (1) and (3)).

Proof. Claim 1: $v_{(1)}^\alpha$ is strictly decreasing in α.

Note first that F_α is increasing in $F_1(v)$ and $F_2(v)$ and that $F_\alpha(v) = F_{\alpha'}(v)$ for $\alpha \neq \alpha'$ if and only if $F_1(v) = F_2(v)$. Moreover, $F_{\alpha'}$ is a monotonic transformation of F_α at $F_1(v) = F_2(v)$ for $\alpha' < \alpha$ (or, in

[6] Arguably, if bidder 2 won in the asymmetric auction with v^*, his expected payment would be quite high and actually higher than if he won in the symmetric benchmark auction with the same valuation. However, this is an unlikely event by assumption and so the driving force in favor of the symmetric benchmark is the one mentioned in the text.

other words, $F_{\alpha'}$ is more convex than F_α). Therefore, F_α is increasing in α. Because the cumulative distribution function of the first order statistics is equal to $F_\alpha(v)^2$, $v^\alpha_{(1)}$ strictly decreasing in α follows directly.

Claim 2: $v^\alpha_{(2)}$ is strictly decreasing in α.

Let $S_\alpha(v)$ be the cumulative distribution of the second order statistics for (F_α, F_α). We want to show that $\frac{d}{d\alpha}S_\alpha(v) > 0$. By definition, $S_\alpha(v) = 2(1 - F_\alpha(v))F_\alpha(v) + F_\alpha(v)^2 = 2F_\alpha(v) - F_\alpha(v)^2$. Hence, $\frac{d}{d\alpha}S_\alpha(v) = 2(1 - F_\alpha(v))\frac{d}{d\alpha}F_\alpha(v) > 0$ by claim 1 (except at points where $F_1 = F_2$ where the derivative is nul), and $v^\alpha_{(2)}$ is decreasing in α.

Claim 3: $v^\alpha_{(2)} > v^{1,2}_{(2)}$ *for* $\alpha \leq 1$.

Given claim 2, we only need to show that this holds when $\alpha = 1$ in order to prove that $v^\alpha_{(2)} > v^{1,2}_{(2)}$ for all $\alpha \leq 1$. The idea is very similar to the proof of theorem 1, except that when $\alpha = 1, (F_\alpha, F_\alpha)$ and (F_1, F_2) have now in common their means instead of the expected value of their first order statistics.

$$\begin{aligned} v^{\alpha=1}_{(2)} &= 2\int_{\underline{v}_1}^{\overline{v}} v(1 - F_\alpha(v))dF_\alpha(v) = 2\int_{\underline{v}_1}^{\overline{v}} vdF_\alpha(v) - 2\int_{\underline{v}_1}^{\overline{v}} vF_\alpha(v)dF_\alpha(v) \\ &= \int_{\underline{v}_1}^{\overline{v}} vd[F_1(v) + F_2(v)] - v^{\alpha=1}_{(1)} \qquad (4) \end{aligned}$$

By claim 1, $v^{\alpha=1}_{(1)} < v^{\alpha=0}_{(1)} = v^{1,2}_{(1)}$, and so the result follows directly by comparing (4) with (2) ■

What about bidders? How do they fare under asymmetries? Because the SPA is efficient, we have the following direct consequence of theorem 1:

Corollary 1: *In the second price auction, bidders' ex-ante aggregate payoffs from the asymmetric auction* (F_1, F_2) *always dominate that from the symmetric benchmark.*

Proof. The result follows directly from theorem 1 and the fact that the second price auction is always efficient in our environment. The expected value of social surplus is the same under both configurations and equal to $v_{(1)} = v^{1,2}_{(1)}$. Bidders' ex-ante expected payoffs are equal to $v_{(1)} - v_{(2)}$ in the symmetric benchmark, and to $v^{1,2}_{(1)} - v^{1,2}_{(2)}$ under (F_1, F_2). $v_{(1)} - v_{(2)} < v^{1,2}_{(1)} - v^{1,2}_{(2)}$ follows from theorem 1 ■

Corollary 1 is an aggregate statement. However, we can be more precise for the second price auction:

Theorem 3: *Let μ_i denote the average valuation of bidder i. In the SPA, the ex-ante expected payoff of bidder i, U_i^s, is equal to*

$$U_i^s = \frac{v_{(1)}^{1,2} - v_{(2)}^{1,2}}{2} + \frac{\mu_i - \mu_j}{2} \text{ for } j \neq i \tag{5}$$

Proof. Given the efficiency of the SPA, we know that

$$U_1^s + U_2^s = v_{(1)}^{1,2} - v_{(2)}^{1,2} \tag{6}$$

Claim: $U_1^s = \int_{\min\{\underline{v}_1,\underline{v}_2\}}^{\max\{\overline{v}_1,\overline{v}_2\}} F_2(v)(1-F_1(v))dv$

Without loss of generality, let $\underline{v}_1 \leq \underline{v}_2$. By definition,

$$\begin{aligned}
U_1^s &= \int_{\underline{v}_2}^{\overline{v}_1}\int_{\underline{v}_2}^{v}(v-x)dF_2(x)dF_1(v) \\
&= -(1-F_1(v))\int_{\underline{v}_2}^{v}(v-x)dF_2(x)\Big|_{\underline{v}_2}^{\overline{v}_1} + \int_{\underline{v}_2}^{\overline{v}_1}(1-F_1(v))\int_{\underline{v}_2}^{v}dF_2(x)dv \\
&\qquad \text{(integration by parts)} \\
&= \int_{\underline{v}_2}^{\overline{v}_1}(1-F_1(v))F_2(v)dv \\
&= \int_{\min\{\underline{v}_1,\underline{v}_2\}}^{\max\{\overline{v}_1,\overline{v}_2\}}(1-F_1(v))F_2(v)dv.
\end{aligned}$$

Similarly, we can show that $U_2^s = \int_{\min\{\underline{v}_1,\underline{v}_2\}}^{\max\{\overline{v}_1,\overline{v}_2\}} F_1(v)(1-F_2(v))dv$, therefore,

$$\begin{aligned}
U_1^s - U_2^s &= \int_{\min\{\underline{v}_1,\underline{v}_2\}}^{\max\{\overline{v}_1,\overline{v}_2\}} F_2(v)dv - \int_{\min\{\underline{v}_1,\underline{v}_2\}}^{\max\{\overline{v}_1,\overline{v}_2\}} F_1(v)dv \\
&= \int vdF_1(v) - \int vdF_2(v) \text{ (integration by parts)} \\
&= \mu_1 - \mu_2 \qquad (7)
\end{aligned}$$

Putting (6) and (7) together, we get claim (5) ■

Given theorem 3, it is easy to generate examples where *both* bidders benefit from the asymmetries. In particular, consider the case where bidders' distributions have the same mean valuation but differ in terms of variance. Since $v^{1,2}_{(1)} - v^{1,2}_{(2)} > v_{(1)} - v_{(2)}$, both bidders gain from the asymmetries.

4 The effects of asymmetries in the FPA

In this section, I turn to the first price auction. In the FPA, the winner is the bidder who places the highest bid and he pays his own bid. Formally, if bidder i has valuation v_i and wins the auction by submitting a bid b, his resulting payoff is equal to $u_i(v_i, b) = v_i - b$ (and zero otherwise). An equilibrium in this Bayesian game is a pair of bidding functions $b_i : [\underline{v}_i, \bar{v}_i] \longrightarrow \mathbb{R}_+$, $i = 1, 2$. For the analysis, it is convenient to look at the inverse bid functions, $\phi_i : \mathbb{R}_+ \longrightarrow [\underline{v}_i, \bar{v}_i]$, $i = 1, 2$. Maskin and Riley (1996 and 2000a) have shown that there exists a unique equilibrium in this environment.[7] The equilibrium inverse bid functions have support on $[\underline{b}, \bar{b}]$ and are solutions to the following system of differential equations:

$$\frac{F_2'(\phi_2(b))\phi_2'(b)}{F_2(\phi_2(b))} = \frac{1}{\phi_1(b) - b} \tag{8}$$

$$\frac{F_1'(\phi_1(b))\phi_1'(b)}{F_1(\phi_1(b))} = \frac{1}{\phi_2(b) - b} \tag{9}$$

subject to some boundary conditions. First, the minimum equilibrium winning bid, $\underline{b}$, is uniquely determined by the following lemma (adapted from Maskin and Riley, 1996):

Lemma 1: Lower bound to the equilibrium distribution of winning bids: *If* $\underline{v}_1 = \underline{v}_2$, *then* $\underline{b} = \underline{v}_1 = \underline{v}_2$. *If* $\underline{v}_1 < \underline{v}_2$ *then* $\underline{b}$ *solves* $\max\{\arg\max_b (\underline{v}_2 - b)F_1(b)\} \in (\underline{v}_1, \underline{v}_2)$. Moreover, we must have that $\phi_i(\underline{b}) = \underline{v}_i$ when $\underline{v}_1 = \underline{v}_2$ and $\phi_1(\underline{b}) = \underline{b}$ and $\phi_2(\underline{b}) = \underline{v}_2$ when $\underline{v}_1 < \underline{v}_2$.

Second, the maximum equilibrium bid, $\bar{b}$, is common to both bidders and endogenously determined by the following condition: $F_i(\phi_i(\bar{b})) = 1$. It can be shown that the equilibrium inverse bid functions are strictly increasing and continuously differentiable on their support.

[7] If one bidder's support of valuations is very far off the other bidder's support, the equilibrium is degenerate: At equilibrium, the "strong" bidder outbids the highest possible valuation of the "weak" bidder and wins all the time. I shall ignore this case.

To gain intuition for why the solutions (ϕ_1, ϕ_2) are indeed equilibrium inverse bid functions, it suffices to realize that equations (8) and (9) are the first order conditions of the pseudo-concave optimization problem for bidder i with valuation v_i:

$$\max_{b \geq 0} (v_i - b) F_j(\phi_j(b)) \qquad i \neq j \tag{10}$$

There are three things to point out about the equilibrium in the FPA. First, there is in general no analytical solution to the equilibrium in the FPA when bidders are asymmetric.

Second, notice that in the presence of ex-ante asymmetries, bidders will, in general, bid differently. This can be seen from bidders' FOCs (8) and (9). Suppose that bidders bid identically at v. Turning to the inverse bid functions, we must have that $v = \phi_1(b) = \phi_2(b)$ for some b. However, different distributions will, in general, have different likelihood ratios, $\frac{F_1'(v)}{F_1(v)} \neq \frac{F_2'(v)}{F_2(v)}$. Therefore (8) and (9) cannot be satisfied at the same time. Because bidders submitting the same bid will usually not have the same valuation for the object, the allocation in the asymmetric first price auction is generically inefficient.

Third, due to the winner-take-all nature of the first price auction, there is some kind of "downward bias" in the way bidders adjust to asymmetries. Indeed, when deciding how much to bid, a bidder takes into account the distribution of bids of his opponent only to the extent that his own bid is the highest (refer to (10) if needed).

I now turn to examples to illustrate the competitive pressure that a more equal distribution of high realizations among bidders puts on bidding behavior.

Example 1: Suppose that bidders' valuations are distributed uniformly over $[0, 1]$ (for bidder 1) and $[1, 2]$ (for bidder 2) respectively. Then, bidders never bid more than 1 in equilibrium (by submitting a bid of 1, bidder 2 wins for sure, so he has no incentives to bid any higher - bidder 1 does not bid more than her valuation at equilibrium). Therefore, $R^f(F_1, F_2) \leq 1$.[8]

[8] Actually, it can be shown that the equilibrium is not degenerate and that the distribution of equilibrium winning bids has support on [0.5, 0.875], so $R^f(F_1, F_2) < 0.875$.

On the other hand, the benchmark distribution has support on $[1, 2]$ with cumulative distribution $F(v) = \sqrt{v-1}$. Because this auction satisfies all the conditions of the Revenue Equivalence Theorem, we can appeal to this result and conclude that $R^f(F, F) = v_{(2)} > 1.$ In turn, this implies that $R^f(F, F) > R^f(F_1, F_2)$.

Example 1 is clearly extreme because, under (F_1, F_2), the highest valuation is always bidder 2's. Knowing this, bidder 2 is able to shade his bid significantly, and this hurts the auctioneer. By contrast, under the symmetric benchmark (F, F), both bidders are as likely to have the highest valuation and this keeps them on their toes.

However, the intuition generalizes to less extreme cases of asymmetries as example 2 illustrates.

Example 2: Suppose F_1' is uniform on $[0, v_1]$ and F_2' is uniform on $[0, v_2]$. Let $v_1 < v_2$. Griesmer, Levitan and Shubik (1967) have shown that the equilibrium inverse bid functions for this configuration are:

$$\phi_1(b) = \frac{2b}{1 + Cb^2} \tag{11}$$

$$\phi_2(b) = \frac{2b}{1 - Cb^2} \tag{12}$$

with $C = \frac{v_2^2 - v_1^2}{v_1^2 v_2^2} > 0$. In particular, notice that $\phi_2(b) > \phi_1(b)$: Bidder 2, who can be seen as the "strong" bidder, is partially insulated from competition (when he has a valuation in $(v_1, v_2]$, he knows that he has the highest realization of the two).[9] Therefore, he is able to shade his bid more at equilibrium.

Now, on $[0, v_1]$, $F(v) = \frac{v}{\sqrt{v_1 v_2}}$, the cumulative distribution function of a uniform. Hence, using the well-known solution for the symmetric FPA, $b(v) = \frac{1}{F(v)} \int_0^v xF'(x)dx = \frac{v}{2}$ on $[0, v_1]$ or, in terms of inverse bid function, $\phi(b) = 2b$ for $b \in [0, \frac{v_1}{2}]$.

Let $G(b)$ and $G^*(b)$ be the cumulative distribution function of bids under (F_1, F_2) and (F, F) respectively. We have $G(b) = F_1(\phi_1(b))F_2(\phi_2(b))$

[9] And, in any case, for any realization v, he knows that he is *more likely* to have the highest valuation.

$= \frac{1}{v_1 v_2} \frac{4b^2}{(1-C^2 b^4)}$ and $G^*(b) = F_1(\phi(b))F_2(\phi(b)) = \frac{4b^2}{v_1 v_2}$ on $[0, \frac{v_1}{2}]$. Comparing these expressions, we see that $G^*(b) < G(b)$ on $[0, \frac{v_1}{2}]$, i.e. the distribution of bids in the benchmark auction first order stochastically dominates that of the asymmetric auction over that interval. As proposition 2 demonstrates, this relationship continues to hold for $b > \frac{v_1}{2}$ and this implies again that $R^f(F_1, F_2) < R^f(F, F)$.

Though the supports of valuations in example 2 are overlapping, there is still a clear notion of who the "strongest" or most eager bidder is. The following lemma provides a characterization of equilibrium bidding for these situations.

Lemma 2: *Suppose that* $\frac{F_i'}{F_i} > \frac{F_j'}{F_j}$ *for all* v *for which both ratios are defined (in particular, this means that* $F_i < F_j$ *on the interior of their common support – bidder* i *is the most eager bidder). Then, the equilibrium under configuration* (F_i, F_j) *is such that:*
(a) $\phi_i(b) > \phi_j(b)$ *for all* b *on the interior of their supports (the "strong" bidder bids less aggressively for every realization); and*
(b) $F_i(\phi_i(b)) < F_j(\phi_j(b))$ *for all* b *on the interior of their supports (i.e. the "strong" bidder continues to be more likely to win).*

Proof. Lemma 2 has been proved under various degrees of generality (see, e.g. Maskin and Riley, 2000b for the two bidder case under slightly more general assumptions) ■

Lemma 2 provides a rough intuition for what happens in example 2. Because bidder 1 bids more aggressively and bidder 2 bids less aggressively than under the symmetric benchmark, there is *a priori* no reason why the revenue in the benchmark configuration should be higher. However, because bidder 2 remains the most likely bidder to win the auction, his less aggressive behavior is the force that dominates.

With these examples in mind, we have:

Conjecture: *Consider any configuration of bidders,* (F_1, F_2). *The expected revenue from the first price auction in the symmetric benchmark,* $R^f(F, F)$, *is always greater than that for the asymmetric configuration* (F_1, F_2).

Numerical simulations under a wide range of distributional assumptions were used to test, and confirm, the conjecture. Unfortunately, and as has become usual in the asymmetric auction literature, some distributional restriction is required to get analytical results.

In the remainder of this section, I prove the conjecture under two kinds of distributional restrictions. First, I consider a particular form of bidders' asymmetry that is generated by collusion or merger among bidders. Suppose that α_1 ex-ante identical bidders (with distribution of valuations $H(v)$) decide to merge or collude. If the merger produces no diseconomies nor economies of scale,[10] the distribution of valuations of the new entity is the distribution of the highest valuation among the α_1 realizations, that is $H(v)^{\alpha_1}$.

Proposition 1 covers this type of asymmetries in a slightly stronger version than the conjecture.

Proposition 1: *Suppose that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ for $\alpha_i, \beta_i \in \mathbb{N}$. Consider two configurations of bidders. In the α-configuration, bidders' distributions are $F_1(v) = H(v)^{\alpha_1}$ and $F_2(v) = H(v)^{\alpha_2}$ where $H(v)$ is a continuously differentiable cumulative distribution function with strictly positive density on $[\underline{v}, \overline{v}]$. In the β-configuration, $F_1(v) = H(v)^{\beta_1}$ and $F_2(v) = H(v)^{\beta_2}$. Let $\alpha_1 \geq \alpha_2$. Then, if $\beta_1 > \alpha_1$, the expected revenue from the β-configuration, $R^f(\beta)$ is lower than that from the α-configuration, $R^f(\alpha)$.*

Notice that when $\alpha_1 = \alpha_2$, the α-configuration is the symmetric benchmark (F, F) and so the claim corresponds to that of the conjecture. When $\alpha_1 > \alpha_2$, both configurations are asymmetric. However, the β-configuration is "more asymmetric" than the α-configuration. Proposition 1 then tells us that expected revenue decreases as asymmetries increase.

Denote by (ϕ_1, ϕ_2) the equilibrium in the α-configuration, and by $(\widetilde{\phi}_1, \widetilde{\phi}_2)$, the equilibrium in the β-configuration. Let $G_\alpha(b) = H(\phi_1(b))^{\alpha_1} H(\phi_2(b))^{\alpha_2}$ and $G_\beta(b) = H(\widetilde{\phi}_1(b))^{\beta_1} H(\widetilde{\phi}_2(b))^{\beta_2}$, that is, $G_\alpha(b)$ and $G_\beta(b)$ are the cumulative distributions of bids under the α- and β-configurations respectively. With these notations,

$$\begin{aligned} R^f(\alpha) &= \int b dG_\alpha(b) \\ R^f(\beta) &= \int b dG_\beta(b) \end{aligned}$$

[10] Or if transfers are possible within the ring and so the cartel is of the "strong" variety (McAfee and McMillan, 1992).

A sufficient condition for $R^f(\alpha) > R^f(\beta)$ is that $G_\alpha(b) < G_\beta(b)$ in the interior of their common support (first order stochastic dominance). By lemma 1, we know that the minimum bid, $\underline{b}$, is equal to $\underline{v}$ in both configurations. We can also show that $G_\alpha(b) < G_\beta(b)$ close to $\underline{b}$.

To show that this relationship continues to hold for all b, we use bidders' first order conditions to prove that

$$G_\alpha(b) = G_\beta(b) \Rightarrow \frac{G'_\alpha(b)}{G_\alpha(b)} < \frac{G'_\beta(b)}{G_\beta(b)} \qquad \text{(COND1)}$$

This allows us to rule out any crossing of G_α and G_β to the right of $\underline{b}$ and we conclude that $R^f(\alpha) > R^f(\beta)$. A detailed proof of proposition 1 can be found in the appendix.

For the second distributional restriction, I consider uniform distributions:

Proposition 2: *Consider any asymmetric configuration of bidders* (F_1, F_2) *and its symmetric benchmark* (F, F), *where* F_1 *and* F_2 *are c.d.f. of uniform distributions (some restrictions apply when the supports are nested). Then* $R^f(F, F) > R^f(F_1, F_2)$.

Again, I only provide an outline of the proof here and refer the interested reader to the appendix. Let $G(b)$ be the cumulative distribution function of the winning bids under the asymmetric configuration (with support on $[\underline{b}, \overline{b}]$) and let $G^*(b)$ denote the cumulative distribution function of bids under the symmetric benchmark (with support on $[\underline{b}^*, \overline{b}^*]$). It is easily shown that $\underline{b} \leq \underline{b}^*$ (using lemma 1) and that $G(b) > G^*(b)$ close to $\underline{b}^*$.

When bidders' supports of valuations are non-nested (i.e. $\underline{v}_1 \leq \underline{v}_2 \leq \overline{v}_1 \leq \overline{v}_2$), we can again use bidders' FOCs to derive a condition equivalent to (COND1), that is,

$$\frac{G^{*\prime}(b)}{G^*(b)} < \frac{G'(b)}{G(b)} \text{ whenever } G^*(b) = G(b) \qquad \text{(COND2)}$$

This is used to conclude the proof.

When the supports of distributions are nested (i.e. $\underline{v}_1 < \underline{v}_2 < \overline{v}_2 < \overline{v}_1$), the proof becomes more involved because, a priori, both $\frac{G'(b)}{G(b)} > \frac{G^{*\prime}(b)}{G^*(b)}$ and $\frac{G'(b)}{G(b)} < \frac{G^{*\prime}(b)}{G^*(b)}$ are compatible with $G(b) = G^*(b)$. Therefore we cannot rule out a crossing.[11] At this stage, it is useful to realize that only bidders' first order conditions are used to derive (COND2). As is well known, a system of differential equations admits a family of solutions. In that sense, (COND2) captures the properties of any pair of solutions to the systems of differential equations for the original and the benchmark auctions, *irrespective of the boundary conditions*. Therefore, it does not fully account for equilibrium behavior in the two underlying auctions.

To deal with this difficulty, a separate appendix derives additional conditions that equilibrium behavior imposes on bidding functions when the supports are nested.[12] These are then used to show that $R^f(F, F) > R^f(F_1, F_2)$ must hold for $\overline{v}_2$ either close enough to $\underline{v}_2$ or close enough to $\overline{v}_1$.

To summarize, we have now shown that asymmetries hurt the auctioneer in the FPA when bidders' asymmetries are of the kind generated by mergers or collusion, or when bidders' valuations are uniformly distributed. These two distributional restrictions are meant to be illustrative. In particular, the conjecture is easily proved for the cases where an analytical solution to the equilibrium in the asymmetric FPA does exist.[13] In addition, other distributional restrictions lend themselves to an analysis along the same lines as those of propositions 1 and 2.[14]

We can now turn to bidders. In section 3, we saw that bidders benefited from asymmetries in the SPA. Does the equivalent result hold for the FPA? Not necessarily. The reason is that the FPA is inefficient and therefore the expected social surplus in the asymmetric configuration is less than in the symmetric benchmark (remember by construction, the *potential* social

[11] The reason why nested supports are different from non-nested supports in that respect is that, in the case of non-nested supports, there exists a relationship of stochastic dominance between bidders (bidder 1 is the least eager bidder). From lemma 2, we then have $\phi_1 < \phi_2$. This, in turn, is critical when deriving (COND1) or (COND2).

[12] This appendix is available from the author upon request.

[13] See Plum (1992) and Jofre-Bonet and Pesendorfer (1999) for specific examples. For configurations (F_1, F_2) with $F_1(v) = v^{\frac{1}{\alpha-1}}$ and $F_2(v) = \frac{2v}{\alpha}$, $\phi_1(b) = 2b$ and $\phi_2(b) = \alpha b$ at equilibrium (thanks are due to Paul Klemperer for suggesting this class of asymmetry). More generally, looking back at the system of differential equations (8) and (9), other analytical solutions can be derived by imposing, say, a function for $\phi_1(b)$, generating $F_2(\phi_2(b)$ by integration and working back the conditions on F_2 and F_1 to provide a consistent system of differential equations.

[14] For example, consider the class of configurations (F_1, F_2) with $F_2 = \gamma F_1$, $\gamma \in (0, 1)$. (I thank John Riley for suggesting this example).

surplus is identical in both cases).

The next example provides a case where bidders are made worse off by asymmetries. By contrast, the numerical results reported by Marshall, Meurer, Richard and Stromquist (1994, tables III and V) seem to indicate that bidders benefit from asymmetries for the kind of heterogeneity examined in proposition 1. Intuitively, inefficiencies are unlikely to be very big when the supports of distributions are common to both bidders. Therefore, if the auctioneer loses from asymmetries, it is quite likely that bidders benefit from them.

Example 3 Let bidders' distributions be uniformly distributed over $[0,1]$ and $[0,x]$ with $x > 1$. Equilibrium inverse bid functions are given by (11) and (12) with $C = \frac{x^2-1}{x^2} > 0$. Bidder 1's ex-ante expected payoff is equal to

$$\begin{aligned} U_1(x) &= \int_0^{\overline{b}} (\phi_1(b) - b) F_2(\phi_2(b)) dF_1(\phi_1(b)) \\ &= \frac{4}{x} \int_0^{\overline{b}} \frac{b^2(1 - Cb^2)}{(1 + Cb^2)^3} db \end{aligned}$$

and similarly,

$$U_2(x) = \frac{4}{x} \int_0^{\overline{b}} \frac{b^2(1 + Cb^2)}{(1 - Cb^2)^3} db$$

where $\overline{b} = \frac{x}{1+x}$. We consider how bidders' aggregate payoff changes as x changes.

$$\begin{aligned} \frac{d}{dx}[U_1(x) + U_2(x)] &= -\frac{1}{x}[U_1(x) + U_2(x)] + \frac{8x}{(1+x)^4} + \\ & \frac{4}{x} \int_0^{\frac{x}{1+x}} b^4 db [\frac{-1}{(1 + Cb^2)^3} - \frac{3(1 - Cb^2)}{(1 + Cb^2)^4} + \\ & \frac{1}{(1 - Cb^2)^3} + \frac{3(1 + Cb^2)}{(1 - Cb^2)^4}] \frac{dC}{dx} \\ &= \frac{1}{6} \text{ when evaluated at } x = 1 \text{ (the last term cancels out).} \end{aligned}$$

Moving to the second derivative, evaluated at $x = 1$, we find

$$\frac{d^2}{dx^2}[U_1(x) + U_2(x)] = \frac{13}{42} \tag{13}$$

Turning to the symmetric benchmark, we have:

$$F(v) = \begin{cases} \frac{v}{\sqrt{x}} & v \in [0,1] \\ \frac{\sqrt{v}}{\sqrt{x}} & v > 1 \end{cases} \quad \text{and } f(v) = \begin{cases} \frac{1}{\sqrt{x}} & v \in [0,1] \\ \frac{1}{2\sqrt{xv}} & v > 1 \end{cases}$$

Denote bidders' ex-ante expected payoff by $\overline{U}_i(x)$. By the Revenue Equivalence Theorem, $\overline{U}_1(x) + \overline{U}_2(x) = v_{(1)} - v_{(2)}$.

$$\begin{aligned} v_{(1)} &= 2\int_0^x vF(v)dF(v) = x - \int_0^x F(v)^2 dv \\ &= \frac{3x^2+1}{6x} \\ v_{(2)} &= 2\int_0^x v(1-F(v))dF(v) \\ &= \frac{2}{x}\int_0^1 v(\sqrt{x}-v)dv + \frac{1}{x}\int_1^x \sqrt{v}(\sqrt{x}-\sqrt{v})dv \\ &= \frac{2\sqrt{x}-1+x^2}{6x} \end{aligned}$$

Therefore $\overline{U}_1(x) + \overline{U}_2(x) = \frac{x^2+1-\sqrt{x}}{3x}$. Differentiating and evaluating at $x = 1$,

$$\begin{aligned} \frac{d}{dx}[\overline{U}_1(x) + \overline{U}_2(x)] &= \frac{1}{3x^2}[x^2 + \frac{1}{2}\sqrt{x} - 1] \\ &= \frac{1}{6} \text{ at } x = 1 \\ \frac{d^2}{dx^2}[\overline{U}_1(x) + \overline{U}_2(x)] &= \frac{2}{3}\frac{1}{x^3} - \frac{1}{4}\frac{1}{x^{\frac{5}{2}}} = \frac{5}{12} \text{ at } x = 1 \end{aligned}$$

Comparing with (13), we conclude that for $x > 1$ close to 1, bidders are better off under the symmetric benchmark i.e. $\overline{U}_1(x) + \overline{U}_2(x) > U_1(x) + U_2(x)$.

5 On the nature of rents in auctions

Are the results derived in the previous sections likely to hold for any auction format? A good starting point to investigate this question is to look at how the optimal auction performs in these environments.[15] However, asymmetries do not necessarily hurt the "optimal" auctioneer as example 4 illustrates.

Example 4: Suppose that F_1' is uniform on $[0,1]$ and F_2' is uniform on $[0,x]$ for $x>1$. Following Myerson (1981), we first compute bidders' virtual valuations:

$$\begin{aligned} J_1(v_1) &= v_1 - \frac{1-F_1(v_1)}{F_1'(v_1)} = 2v_1 - 1 \\ J_2(v_2) &= 2v_2 - x \end{aligned}$$

It is easy to check that $J_1(v_1) \geq J_2(v_2)$ when $v_2 \leq v_1 + \frac{x-1}{2}$. Therefore,

$$\begin{aligned} R^{opt}(F_1,F_2) &= \frac{1}{x}\int_0^1 dv_1 \int_0^x dv_2 \max\{J_1(v_1), J_2(v_2)\} \\ &= \frac{1}{x}\int_0^1 dv_1 [\int_0^{\frac{x-1}{2}+v_1} dv_2(2v_1-1) + \int_{\frac{x-1}{2}+v_1}^{x} dv_2(2v_2-x)] \\ &= \frac{1}{x}\int_0^1 [v_1(x-1) - \frac{x-1}{2} + 2v_1^2 - v_1 \\ &\quad -\frac{(x-1)^2}{4} - v_1^2 - (x-1)v_1 + \frac{x(x-1)}{2} + xv_1]dv_1 \\ &= \frac{1}{x}\int_0^1 (v_1^2 + \frac{(x-1)^2}{4} + (x-1)v_1)dv_1 \\ &= \frac{3x^2+1}{12x} \end{aligned} \tag{14}$$

Computing $R^{opt}(F,F)$ is more difficult because the problem is not regular in the sense of Myerson (1981) (virtual valuations are not increasing everywhere in the symmetric benchmark). In the appendix, I derive an upper bound to $R^{opt}(F,F)$ when $x=4$ and find that $R^{opt}(F,F) < 0.9784 < R^{opt}(F_1,F_2) = \frac{49}{48}$ (by (14)).

[15] In this section, I abstract from reservation prices. The reason for this assumption is to place the "optimal auction" (or rather, in this case, the constrained optimal auction) on the same playing field as the FPA and SPA as analyzed in the previous sections. This restriction is inessential for the insights developed here. In particular, we would just need to shift the supports of the distributions in example 4 to the right so that even the optimal auction does not require a reservation price (alternatively, consider that the auctioneer has a negative value for the object).

Example 4 is a useful reminder that the optimal auction operates very differently from standard ("real life") auctions. In the optimal auction, bidders' treatment is differentiated and informational rent extraction is the driving force. In example 4, both bidders in the symmetric benchmark have a larger support for their valuations than under the asymmetric configuration, and, as a result, they are able to extract higher informational rents.

By contrast, the FPA and SPA are anonymous mechanisms (bidders are treated equally) and bidders' competition is the driving element.[16] Informational rents are not as costly to the auctioneer as an unmatched competitive position. When bidder 2 has a valuation between 1 and x in example 4, he knows that he has an absolute comparative advantage over his opponent, and this allows him to take a larger profit margin.

As it is widely thought that incomplete information is an essential element in auctions, it might useful to elaborate further on this last point. First, consider the SPA. In the private value SPA, a bidder wins if and only he has the highest valuation and, in that case, his payoff is equal to the difference between his valuation and the second highest valuation (that is, his payoff corresponds to his *comparative advantage*). Informational incompleteness plays no role in the SPA. Bidders' rents are competitive rents. At the other extreme, bidders' rents in the optimal auction can be termed as purely informational.

The FPA lies somewhere in-between. Informational incompleteness is an important element of the strategic environment in the FPA (it does affect bidding behavior) and bidders' rents are partly informational.

To illustrate, consider an extreme example. Suppose bidder 1 has valuations distributed over $[\underline{v}, \overline{v}]$ according to the cumulative distribution function F. Bidder 2 has (known) valuation $v^* \in (\underline{v}, \overline{v})$. We claim that bidder 2's ex-ante expected payoff in the FPA is less than in the SPA (in other words, he earns less than his expected comparative advantage). Consider first bidder

[16] A useful example here is the following: Suppose that bidder 1's valuation is distributed over some interval $[\underline{v}, \overline{v}]$ according to $F(v)$, and that bidder 2 has valuation $v_2 \in (\underline{v}, \overline{v})$. It can be shown that bidder 2's ex-ante expected profit from both the SPA and the FPA is strictly positive, whereas it is zero under the optimal auction (bidder 2 has no private information).

2's ex-ante expected payoff in the SPA:

$$U_2^s = \int_{\underline{v}}^{v^*} (v^* - v) dF(v) = \int_{\underline{v}}^{v^*} F(v) dv \text{ (integ. by parts)}$$

In the equilibrium of the FPA, bidder 2 mixes over an interval $[\underline{b}, \overline{b}]$ with $\underline{v} < \underline{b} < \overline{b} < v^*$. For bidder 2 to be willing to mix, he must be indifferent between any b on $[\underline{b}, \overline{b}]$, so his ex-ante expected payoff is equal to $U_2^f = (v^* - \underline{b})F(\underline{b})$. Drawing F as a function of v in a graph, it is easy to check that $U_2^f < U_2^s$.

For a slightly more general example, consider 2 distributions that are single peaked and symmetric around their means. Suppose that both distributions have the same mean, but bidder 2's distribution has a lower variance.[17] By theorem 3, we know that bidders' expected comparative advantages are equal. Numerical simulations for the FPA indicate that (1) the bidder with the lower variance has a lower ex-ante expected payoff than the other bidder, and (2) he earns less than his expected comparative advantage.

At this point, it is tempting to reinterpret corollary 1 as suggesting that bidders' *competitive* rents are always reduced by symmetry. It is also interesting to note the analogy between example 3 and example 4. In example 3, bidders are better off under the symmetric benchmark for the FPA. In example 4, the optimal auctioneer is worse off under the symmetric benchmark. It is probably not a coincidence that both examples involve the same kind of distributional asymmetry.

6 Concluding remarks

In this paper, I have sought to understand how ex-ante differences in the distributions of bidders' valuations affect revenue and profits. I have shown that, holding the distribution of potential social surplus equal, asymmetries reduce expected revenue, both in the first price and in the second price auction. In other words, in both cases, asymmetries reduce the share of social surplus that the auctioneer is able to capture. Auctions are decentralized allocation

[17] Variances are a very imperfect way to capture the relevant privateness of a bidder's distribution of valuations in a FPA since private information is worthier when one's valuation is high than when it is low.

mechanisms and the outcome is ultimately driven by bidders' strategic interactions. In that sense, the results formalize the idea that asymmetries reduce the competitive pressure on bidders.

Common decentralized auction rules and the optimal auction differ in one important respect: the first are anonymous mechanisms while in the optimal auction, the auctioneer can treat bidders differently. Informational rent extraction is a key element in the optimal auction. By contrast, the auctioneer in the FPA or SPA needs to rely more on competitive forces. Symmetry increases competition and this effect is at the basis of our result. However, symmetry does not necessarily decrease informational rents and we saw that the optimal auctioneer might actually suffer from a greater symmetry among participants. A conjecture is that the results derived in this paper apply more generally to any anonymous competitive allocation mechanism.

I have made several assumptions in my analysis. First, I have assumed private values. A reason for this is that I wanted to focus on how asymmetries affect competition. With common values, there is an additional element of inference that affect bidders' behavior. At the very extreme, in the pure mineral oil model where bidders receive a signal about a common underlying value v, it is unclear what the appropriate benchmark is to study asymmetries in bidders' distributions of signals.

Second, I have assumed that bidders are risk neutral. Though the comparison between expected revenues remains legitimate as long as the seller is risk neutral, the argument for the benchmark and in particular for comparing auctions with the same expected social surplus (defined as the expected value of the first order statistics) is obviously weakened. Nevertheless, it is useful to note that the behavior in the SPA is unaltered by risk attitudes and therefore theorems 1 and 2 continue to hold.[18]

Third, I have assumed that valuations are distributed independently. This is clearly a simplification and this assumption should be relaxed in the future.

[18] Allowing for risk aversion, we might also want to ask a different but related question: that of the effect of heterogeneities in risk attitudes on expected revenue. Marshall et al. (1994) note that the kind of distributional heterogeneity studied in proposition 1 can be reinterpreted as an environment with symmetrically distributed valuations but different risk attitudes across bidders. Generalizing wildly from this example, we might want to conjecture that asymmetries in risk attitudes also hurt the auctioneer in the first price auction.

Fourth, I have ignored the possibility of a reserve price. As long as reserve prices are held constant across configurations, this assumption is unimportant. One way to view reserve prices is that they limit the range of valuations over which asymmetries matter. However, as long as the distributions of valuations above the reserve price differ, the intuition, and results, continue to hold. This is easy to check for the SPA (the same proofs, slightly amended for the possibility of a mass point at the reserve price, go through). In the FPA, notice that reserve prices only affect the boundary conditions of the system of differential equations that characterize the equilibrium. Therefore, the key condition, $G^* = G \Rightarrow \frac{G^{*\prime}}{G^*} < \frac{G'}{G}$, continues to hold and this is all we need.

Last but not least, the analysis has focused on the (simpler) two-bidder case. This is also when we can expect strategic interactions to be most important in bidders' decisions and, therefore, the two-bidder case is a natural starting point to study the effect of asymmetries on equilibrium behavior in auctions. Intuitively, with $N > 2$ bidders, bidder 1 and bidder 2 share $N-2$ opponents. As N grows large, the common element in the environment they face tends to dominate, and, as a consequence, bidders' optimal strategies become less differentiated. Nevertheless, the intuition developed in the previous sections for why asymmetries reduce expected revenue continues to hold and we expect the results to generalize to $N > 2$.[19]

References

[1] Athey, Susan (1999), Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information, forthcoming *Econometrica*.

[2] Bajari, Patrick (1998), A Structural Econometric Model of the First Price Sealed Bid Auction with Asymmetric Bidders, Harvard University manuscript.

[19] For the second price auction, Li and Riley (1999) provide a proof of theorem 1 for any number of bidders with a common lower bound to valuations (though this assumption is not important).

[3] Griesmer, James H., Richard E. Levitan and Martin Shubik (1967), Toward a Study of Bidding Processes, Part IV - Games with unknown Costs, *Naval Research Logistics Quarterly,* 14, 415-33.

[4] Jofre-Bonet, Mireia and Martin Pesendorfer (1999), Bidding Behavior in a Repeated Procurement Auction, Yale manuscript.

[5] Klemperer, Paul (1999), Auction Theory: A Guide to the Literature, *Journal of Economic Surveys,* 13(3), 227-286.

[6] Lebrun, Bernard (1996), Existence of an Equilibrium in First Price Auctions, *Economic Theory,* 7, 421-443.

[7] Li, Huagang, and John G. Riley (1999), Auction Choice, UCLA manuscript.

[8] McAfee, R. Preston and John McMillan (1992), Bidding Rings, *American Economic Review*, 82(3), 579-599.

[9] Maskin, Eric and John G. Riley (1996), Uniqueness in Sealed High Bid Auctions, Harvard University manuscript.

[10] Maskin, Eric and John G. Riley (2000a), Equilibrium in Sealed High Bid Auctions, *Review of Economic Studies*, 67, 439-454

[11] Maskin, Eric and John G. Riley (2000b), Asymmetric Auctions, *Review of Economic Studies*, 67, 413-438.

[12] Marshall, Robert C., Michael J. Meurer, Jean-Francois Richard and Walter Stromquist (1994), Numerical Analysis of Asymmetric First Price Auctions, *Games and Economic Behavior,* 7, 193-220.

[13] Myerson, Roger B. (1981), Optimal Auction Design, *Mathematics of Operations Research,* 6(1), 58-73.

[14] Riley, John G. and William F. Samuelson (1981), Optimal Auctions, *The American Economic Review,* 71(3), 381-392.

[15] Porter, Robert H. and J. Douglas Zona (1999), Ohio School Milk Markets: An Analysis of Bidding, *RAND Journal of Economics*, 30(2), 263-288.

[16] Pesendorfer, Martin (2000), A Study of Collusion in First-Price Auctions, *Review of Economic Studies*, 67, 381-411.

[17] Plum, M. (1992), Characterization and Computation of Nash-equilibria for Auctions with Incomplete Information, *International Journal of Game Theory*, 20, 393-418.

7 Appendix

7.1 Proof of proposition 1

Proposition 1: *Suppose that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ for $\alpha_i, \beta_i \in \mathbb{N}$. Consider two configurations of bidders. In the α-configuration, bidders' distributions are $F_1(v) = H(v)^{\alpha_1}$ and $F_2(v) = H(v)^{\alpha_2}$ where $H(v)$ is a continuously differentiable cumulative distribution function with strictly positive density on $[\underline{v}, \overline{v}]$. In the β-configuration, $F_1(v) = H(v)^{\beta_1}$ and $F_2(v) = H(v)^{\beta_2}$. Let $\alpha_1 \geq \alpha_2$. Then, if $\beta_1 > \alpha_1$, the expected revenue from the β-configuration, $R^f(\beta)$ is lower than that from the α-configuration, $R^f(\alpha)$.*

Proof. Denote by (ϕ_1, ϕ_2) the equilibrium in the α-configuration and by $(\widetilde{\phi}_1, \widetilde{\phi}_2)$ the equilibrium in the β-configuration. Let $G_\alpha(b)$ be the cumulative distribution function of bids in the α-configuration. Define $G_\beta(b)$ similarly. By lemma 1, we know that the minimum winning bid is $\underline{b} = \underline{v}$. For future reference, bidders' FOCs in the α-configuration read:

$$\frac{\alpha_2 H'(\phi_2(b))\phi_2'(b)}{H(\phi_2(b))} = \frac{1}{\phi_1(b) - b} \tag{15}$$

$$\frac{\alpha_1 H'(\phi_1(b))\phi_1'(b)}{H(\phi_1(b))} = \frac{1}{\phi_2(b) - b} \tag{16}$$

and similarly for the FOCs in the β-configuration.

Step 1: $G_\alpha(b) < G_\beta(b)$ for b close to $\underline{b}$.

At $\underline{b}$, $G_\alpha(\underline{b}) = G_\beta(\underline{b}) = 0$. We want to show that $\lim_{b \downarrow \underline{b}} \frac{G_\beta(b)}{G_\alpha(b)} > 1$. By definition,

$$\frac{G_\beta(b)}{G_\alpha(b)} = \frac{H(\widetilde{\phi}_1(b))^{\beta_1} H(\widetilde{\phi}_2(b))^{\beta_2}}{H(\phi_1(b))^{\alpha_1} H(\phi_2(b))^{\alpha_2}}$$

By successive applications of l'Hôpital's rule (since $H(\phi_i(\underline{b})) = H(\widetilde{\phi}_i(\underline{b})) = H(\underline{v}) = 0$ for $i = 1, 2$), we get:

$$\begin{aligned}
\lim_{b \downarrow \underline{b}} \frac{G_\beta(b)}{G_\alpha(b)} &= \frac{\frac{(\beta_1+\beta_2)!}{\beta_1!\beta_2!}\beta_1!\beta_2![H'(\widetilde{\phi}_1(\underline{b}))]^{\beta_1}[H'(\widetilde{\phi}_2(\underline{b}))]^{\beta_2}[\widetilde{\phi}_1'(\underline{b})]^{\beta_1}[\widetilde{\phi}_2'(\underline{b})]^{\beta_2}}{\frac{(\alpha_1+\alpha_2)!}{\alpha_1!\alpha_2!}\alpha_1!\alpha_2![H'(\phi_1(\underline{b}))]^{\alpha_1}[H'(\phi_2(\underline{b}))]^{\alpha_2}[\phi_1'(\underline{b})]^{\alpha_1}[\phi_2'(\underline{b})]^{\alpha_2}} \\
&= \frac{[\widetilde{\phi}_1'(\underline{b})]^{\beta_1}[\widetilde{\phi}_2'(\underline{b})]^{\beta_2}}{[\phi_1'(\underline{b})]^{\alpha_1}[\phi_2'(\underline{b})]^{\alpha_2}} \text{ since } \phi_1(\underline{b}) = \phi_2(\underline{b}) = \widetilde{\phi}_1(\underline{b}) = \widetilde{\phi}_2(\underline{b})
\end{aligned} \tag{17}$$

Claim 1: $\phi_i'(\underline{b}) = \frac{1}{\alpha_j} + 1$, $\widetilde{\phi}_i'(\underline{b}) = \frac{1}{\beta_j} + 1$ with $i \neq j$.

Proof: We prove the claim for ϕ_1' only. The other proofs are analogous. Using bidder 1's FOC (15) and solving for $\phi_2'(b)$, we get:

$$\phi_2'(b) = \frac{H(\phi_2(b))}{\alpha_2(\phi_1(b) - b)H'(\phi_2(b))}$$

When b tends to $\underline{b}$, both the numerator and the denominator of this expression go to zero. Applying l'Hôpital's rule,

$$\begin{aligned} \phi_2'(\underline{b}) &= \lim_{b \downarrow \underline{b}} \frac{H'(\phi_2(b))\phi_2'(b)}{\alpha_2 H'(\phi_2(b))(\phi_1'(b) - 1) + \alpha_2(\phi_1(b) - b)H''(\phi_2(b))\phi_2'(b)} \\ &= \frac{\phi_2'(\underline{b})}{\alpha_2(\phi_1'(\underline{b}) - 1)} \end{aligned}$$

Solving for ϕ_1', we get the claim.$\diamond$

Consequence of claim 1: Expression (17) reduces to:

$$\lim_{b \downarrow \underline{b}} \frac{G_\beta(b)}{G_\alpha(b)} = \frac{(\frac{1}{\beta_1} + 1)^{\beta_2}(\frac{1}{\beta_2} + 1)^{\beta_1}}{(\frac{1}{\alpha_1} + 1)^{\alpha_2}(\frac{1}{\alpha_2} + 1)^{\alpha_1}}$$

We want to show that this expression is greater than 1.

Claim 2: For all $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ and $\beta_1 > \alpha_1 \geq \alpha_2 > \beta_2$,

$$(\frac{1}{\beta_1} + 1)^{\beta_2}(\frac{1}{\beta_2} + 1)^{\beta_1} > (\frac{1}{\alpha_1} + 1)^{\alpha_2}(\frac{1}{\alpha_2} + 1)^{\alpha_1} \tag{18}$$

Proof: Let $f(x, y) = y \ln(\frac{1}{x} + 1) + x \ln(\frac{1}{y} + 1)$. Taking the total differential yields:

$$df(x, y) = [\ln(\frac{1}{y} + 1) - \frac{y}{(1 + x)x}]dx + [\ln(\frac{1}{x} + 1) - \frac{x}{(1 + y)y}]dy$$

Setting $dx = -dy > 0$, we get:

$$df(x, y) = [\underbrace{\ln(\frac{1}{y} + 1) - \ln(\frac{1}{x} + 1)}_{>0 \text{ for } x>y} + \underbrace{\frac{x}{(1 + y)y} - \frac{y}{(1 + x)x}}_{>0 \text{ for } x>y}]dx$$

This proves the claim (18) for the case when $\alpha_1 > \alpha_2$. When $\alpha_1 = \alpha_2$, we use the convexity of $\ln(\frac{1}{x}+1)$ in x (for $x > 0$). Specifically, let $L = \frac{\beta_2}{\beta_1+\beta_2}(\frac{1}{\beta_1}+1) + \frac{\beta_1}{\beta_1+\beta_2}(\frac{1}{\beta_2}+1)$. Because $\ln(\frac{1}{x}+1)$ is convex,

$$\frac{\beta_2}{\beta_1+\beta_2}\ln(\frac{1}{\beta_1}+1) + \frac{\beta_1}{\beta_1+\beta_2}\ln(\frac{1}{\beta_2}+1) > \ln L$$

Now, $L = \frac{\beta_2^2(1+\beta_1)+\beta_1^2(1+\beta_2)}{(\beta_1+\beta_2)\beta_1\beta_2} = \frac{\beta_2^2+\beta_2^2\beta_1+\beta_1^2+\beta_1^2\beta_2}{(\beta_1+\beta_2)\beta_1\beta_2} = \frac{\beta_1\beta_2(\beta_1+\beta_2)}{(\beta_1+\beta_2)\beta_1\beta_2} + \frac{\beta_1^2+\beta_2^2}{(\beta_1+\beta_2)\beta_1\beta_2} > 1+\frac{2}{\beta_1+\beta_2}$ since $\beta_1^2+\beta_2^2 > 2\beta_1\beta_2$. Therefore $\ln L > \ln(1+\frac{2}{\beta_1+\beta_2}) = \ln(1+\frac{1}{\alpha_i})$.$\diamond$

Step 2: $G_\alpha(b) = G_\beta(b) \Rightarrow \frac{G'_\alpha}{G_\alpha} < \frac{G'_\beta}{G_\beta}$

Summing up bidders' FOCs and using the definition of G_α and G_β, we get:

$$\frac{G'_\alpha(b)}{G_\alpha(b)} = \frac{1}{\phi_1(b)-b} + \frac{1}{\phi_2(b)-b} \tag{19}$$

$$\frac{G'_\beta(b)}{G_\beta(b)} = \frac{1}{\widetilde{\phi}_1(b)-b} + \frac{1}{\widetilde{\phi}_2(b)-b} \tag{20}$$

We first derive restrictions that equilibrium behavior and the fact that $G_\alpha = G_\beta$ impose on the relationship between ϕ_1, ϕ_2, $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$. First, note that since $\alpha_1 \geq \alpha_2$ and $\beta_1 > \beta_2$, lemma 2 implies that

$$\phi_1(b) \geq \phi_2(b) \tag{21}$$

$$\widetilde{\phi}_1(b) > \widetilde{\phi}_2(b) \tag{22}$$

Second, $G_\alpha(b) = G_\beta(b)$ means that

$$H(\phi_1(b))^{\alpha_1}H(\phi_2(b))^{\alpha_2} = H(\widetilde{\phi}_1(b))^{\beta_1}H(\widetilde{\phi}_2(b))^{\beta_2}$$

Let $\lambda = \frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\widetilde{\lambda} = \frac{\beta_1}{\beta_1+\beta_2}$. This expression can be rewritten as

$$\lambda\ln H(\phi_1(b))+(1-\lambda)\ln H(\phi_2(b)) = \widetilde{\lambda}\ln H(\widetilde{\phi}_1(b))+(1-\widetilde{\lambda})\ln H(\widetilde{\phi}_2(b)) \tag{23}$$

with $\lambda < \widetilde{\lambda}$.

Finally, we can prove the following claim:

Claim 3: There cannot be any value of bid $\widehat{b}$ for which $\phi_2(\widehat{b}) \leq \widetilde{\phi}_2(\widehat{b}) < \widetilde{\phi}_1(\widehat{b}) \leq \phi_1(\widehat{b})$.

Proof: We first claim that if there exists such a $\widehat{b}$, then

$$\phi_2(b) < \widetilde{\phi}_2(b) < \widetilde{\phi}_1(b) < \phi_1(b) \text{ for all } b > \widehat{b} \tag{24}$$

Towards a contradiction, suppose that, starting from $\widehat{b}$ onwards, $\widetilde{\phi}_2$ is the first one to leave the "bounds" and that ϕ_2 and $\widetilde{\phi}_2$ cross at $b \geq \widehat{b}$. Then, we must have $\phi_2(b) = \widetilde{\phi}_2(b)$ and $\phi_2'(b) \geq \widetilde{\phi}_2'(b)$ and so

$$\alpha_2 \frac{H'(\phi_2)}{H(\phi_2)} \phi_2' > \beta_2 \frac{H'(\widetilde{\phi}_2)}{H(\widetilde{\phi}_2)} \widetilde{\phi}_2' \text{ since } \alpha_2 > \beta_2$$

Using bidder 1's FOC (15) and its equivalent for the β-configuration, we conclude that $\phi_1(b) < \widetilde{\phi}_1(b)$, a contradiction. Alternatively, suppose that $\widetilde{\phi}_1$ first hits ϕ_1 at b. We must have $\phi_1(b) = \widetilde{\phi}_1(b)$ and $\phi_1'(b) \leq \widetilde{\phi}_1'(b)$. So

$$\alpha_1 \frac{H'(\phi_1)}{H(\phi_1)} \phi_1' < \beta_1 \frac{H'(\widetilde{\phi}_1)}{H(\widetilde{\phi}_1)} \widetilde{\phi}_1' \text{ since } \alpha_1 < \beta_1$$

Using bidder 2's FOC (16) and its equivalent for the β-configuration, we conclude that $\widetilde{\phi}_2(b) > \phi_2(b)$, again a contradiction. Hence, (24) must hold.

We are now ready to reach a contradiction. From the discussion in section 2, we know that at equilibrium, the maximum bid is common to both bidders. Let's denote them by $\overline{b}_\alpha$ (for the maximum bid in the α-configuration) and $\overline{b}_\beta$ respectively. In addition, since the upper bound to the distributions of valuations, $\overline{v}$, is common in both configurations, we have $\phi_1(\overline{b}_\alpha) = \phi_2(\overline{b}_\alpha) = \widetilde{\phi}_1(\overline{b}_\beta) = \widetilde{\phi}_2(\overline{b}_\beta) = \overline{v}$. This is impossible if (24) holds.$\diamond$

Conditions (21) to (23) together with claim 3 imply that at any crossing of G_α and G_β only two configurations of bidding behavior are possible:

$$\phi_1(b) \geq \widetilde{\phi}_1(b) \geq \phi_2(b) \geq \widetilde{\phi}_2(b) \tag{25}$$

$$\widetilde{\phi}_1(b) \geq \phi_1(b) \geq \phi_2(b) \geq \widetilde{\phi}_2(b) \tag{26}$$

where (22) and the fact that $G_\alpha(b) = G_\beta(b)$ imply that at least one inequality is strict (two in the case of (25)).

We can now easily show that $\frac{G_\alpha'}{G_\alpha} < \frac{G_\beta'}{G_\beta}$ must hold at any crossing. Referring back to (19) and (20), the claim follows trivially when (25) holds. In the second case, the claims follows from the convexity of $\frac{1}{x-b}$ in x. This concludes step 2.

Steps 1 and 2 together imply that $G_\alpha(b) < G_\beta(b)$ for all b on the interior of their common support. Therefore, $R^f(\beta) < R^f(\alpha)$. ■

7.2 Proof of proposition 2 (the uniform case).

Proposition 2: *Consider any asymmetric configuration of uniform distributions* (F_1, F_2) *and its symmetric benchmark* (F, F) *(when the supports are nested, some restrictions apply). Then* $R^f(F, F) > R^f(F_1, F_2)$.

Proof. Let $\underline{v}_1 \leq \underline{v}_2$. Let (ϕ_1, ϕ_2) denote the equilibrium under (F_1, F_2) (with support on $[\underline{b}, \overline{b}]$) and let (ϕ, ϕ) denote the equilibrium in the benchmark auction (with support on $[\underline{b}^*, \overline{b}^*]$). Let $G(b) = F_1(\phi_1(b))F_2(\phi_2(b))$ and $G^*(b) = F(\phi(b))^2$. Given lemma 1 (when $\underline{v}_1 < \underline{v}_2$) and example 2 (when $\underline{v}_1 = \underline{v}_2$), we know that $\underline{b} \leq \underline{b}^*$ and that $G(b) > G^*(b)$ holds in a neighborhood to the right of $\underline{b}^*$.

Strategy of the proof: The proof considers two scenarios in turn: (1) When the supports for valuations are non nested, there exists a stochastic dominance relationship between the two distributions and so $\phi_1 < \phi_2$ holds. In turn, this allows us to prove that

$$G(b) = G^*(b) \Rightarrow \frac{G'(b)}{G(b)} > \frac{G^{*\prime}(b)}{G^*(b)} \tag{27}$$

Therefore, $G^*(b) < G(b)$ everywhere and the claim holds.

(2) When the supports are nested, ϕ_1 can be lower or greater than ϕ_2 so a claim along the lines of (27) is no longer available. Instead, we find (a) conditions on $\overline{v}_2$ such that $G^*(b) < G(b)$ for all b (claims 3-5), and (b) conditions on $\overline{v}_2$ such that $\overline{b} < R(F, F)$. Because the proof in this case tends to be more involved without adding any new insight, it is reported in a separate appendix.

Part I: Non-nested supports: $\underline{v}_1 \leq \underline{v}_2 \leq \overline{v}_1 \leq \overline{v}_2$.

Claim 1: For all b in $(\underline{b}, \overline{b})$, $\phi_1(b) < \phi_2(b)$.

Proof: When $\underline{v}_1 < \underline{v}_2$, this follows from lemma 2. When $\underline{v}_1 = \underline{v}_2$, this follows from the explicit solutions (see example 2). $\diamond$

Claim 2: $G(b) = G^*(b)$ for $b > \underline{b}^* \Rightarrow \frac{G'(b)}{G(b)} > \frac{G^{*\prime}(b)}{G^*(b)}$.

Proof: Adding bidders' FOCs, G and G^* satisfy:

$$\begin{aligned} \frac{G'(b)}{G(b)} &= \frac{1}{\phi_1(b) - b} + \frac{1}{\phi_2(b) - b} \\ \frac{G^{*\prime}(b)}{G^*(b)} &= \frac{2}{\phi(b) - b} \end{aligned}$$

on their respective supports. The following change of variables turns out to be useful. Let $d_1(b) = \phi_1(b) - b$, $d_2(b) = \phi_2(b) - b$, $d(b) = \phi(b) - b$, $\underline{u}_1(b) = b - \underline{v}_1$ and $\underline{u}_2(b) = b - \underline{v}_2$ (Notice that $\underline{u}_1$ and $\underline{u}_2$ are both strictly positive at equilibrium). The condition $\frac{G^{*\prime}}{G^*} = \frac{G'}{G}$ can be rewritten as

$$d = \frac{2d_1 d_2}{(d_1 + d_2)} \tag{28}$$

(where the arguments have been dropped for simplicity). Similarly, $G(b) = G^*(b)$ means that $F_1(\phi_1(b))F_2(\phi_2(b)) = F(\phi(b))^2 = F_1(\phi(b))F_2(\phi(b)$, that is,

$$(d_1 + \underline{u}_1)(d_2 + \underline{u}_2) = (d + \underline{u}_1)(d + \underline{u}_2)$$

Solving for d, we get:

$$d = \frac{1}{2}[\sqrt{(\underline{u}_1 + \underline{u}_2)^2 + 4(d_1 d_2 + \underline{u}_1 d_2 + \underline{u}_2 d_1)} - (\underline{u}_1 + \underline{u}_2)] \in (d_1, d_2) \tag{29}$$

Next, we solve for the locus of points in the (u_1, u_2) space such that both (28) and (29) hold. This yields the following expression:

$$u_1 = u_2 \frac{d_1}{d_2} - \frac{d_1(d_2 - d_1)}{d_1 + d_2} \tag{30}$$

This is represented in figure 3. By claim 1, $\phi_1(b) < \phi_2(b)$, so $d_1 < d_2$ and the locus describes a line *above* the 45^0 line. By construction $G^* = G$ and $\frac{G^{*\prime}}{G^*} = \frac{G'}{G}$ on that line.

[insert figure 3 here]

We now claim that $G = G^*$ implies $\frac{G^{*\prime}}{G^*} < \frac{G'}{G}$ to the right of this line. To see this, just consider one such point, the origin (0,0), and suppose that (29) holds that is $d = \sqrt{d_1 d_2}$. Substituting back into (28), it is easy to check that $d > \frac{2d_1 d_2}{(d_1+d_2)}$ so $\frac{G^{*\prime}}{G^*} < \frac{G'}{G}$ must hold.

This allows us to conclude the proof. By assumption, $\underline{v}_1 \leq \underline{v}_2$, so $\underline{u}_1 \geq \underline{u}_2$ and we are always to the right of the locus. Therefore, at any potential crossing of G with G^*, $\frac{G^{*\prime}}{G^*} < \frac{G'}{G}$ is satisfied.◇

Since $G(b) > G^*(b)$ close to $\underline{b}^*$, claim 2 implies that this relationship holds for all b on their common support. $R^f(F_1, F_2) < R^f(F, F)$ follows. ■

7.3 Derivation of an upper bound to $R^{opt}(F, F)$ in example 3

Suppose F_1' is uniform over $[0, 1]$ and F_2' is uniform over $[0, 4]$. Then:

$$F(v) = \begin{cases} \frac{v}{2} & v \in [0,1] \\ \frac{\sqrt{v}}{2} & v > 1 \end{cases} \quad \text{and } F'(v) = \begin{cases} \frac{1}{2} & v \in [0,1] \\ \frac{1}{4\sqrt{v}} & v > 1 \end{cases}$$

Turning to virtual valuations, we have:

$$J(v) = \begin{cases} 2v - 2 & \text{for } v \in [0,1] \\ 3v - 4\sqrt{v} & \text{for } v > 1 \end{cases}$$

It is easy to check that $J'(v) > 0$ on $[0, 1)$ and on $(1, 4]$. However,

$$\lim_{v \uparrow 1} J(v) = 0 > \lim_{v \downarrow 1} J(v) = -1$$

This means that the problem is not regular in the sense of Myerson (1981) and the optimal auction requires bunching over an interval of valuations. Consider the following expression:

$$R = \int_0^4 dv_1 f(v_1) \int_0^4 dv_2 f(v_2) \max\{J(v_1), J(v_2)\} \tag{31}$$

This expression would correspond to the expected revenue from the optimal auction *if* the auction were regular. Given that it is not, (31) *overestimates* the expected revenue from the optimal auction: $R^{opt}(F, F) < R$.

Claim: $R < 0.9784$.

Proof. Because of the symmetry of the situation, R can be rewritten as

$$R = 2 \int_0^4 f(v) J(v) \text{proba}(J(\widehat{v}) < J(v)) dv \tag{32}$$

where $\text{proba}(J(\widehat{v}) < J(v))$ stands for the probability that a valuation draw from F has a virtual valuation lower than $J(v)$.[20] Define v^* such $J(v^*) = 3v^* - 4\sqrt{v^*} = 0$. We have $v^* = \frac{16}{9}$, and $J(.)$ monotonically increasing over $[\frac{16}{9}, 4]$. Therefore, $\text{proba}(J(\widehat{v}) < J(v)) = F(v)$ for all $v \geq v^*$. Similarly, $J(.)$ is increasing over $[0, 0.5]$ and $J(v) \leq J(v')$ for all $v \leq 0.5 \leq v'$. Therefore,

[20] Notice that, since $J(.)$ is not monotonic, $\text{proba}(J(\widehat{v}) < J(v))$ is not equal to $F(v)$.

proba$(J(\widehat{v}) < J(v)) = F(v)$ for $v \leq 0.5$. Finally, notice that $J(v) \leq 0$ for all $v < v^*$. We can now rewrite (32) as:

$$
\begin{aligned}
R &= \int_0^{0.5} (v-1)v dv + 2\int_{0.5}^{\frac{16}{9}} f(v)J(v)\text{proba}(J(\widehat{v}) < J(v))dv \\
&\quad + \frac{1}{4}\int_{\frac{16}{9}}^{4} (3v - 4\sqrt{v})dv \\
&< \int_0^{0.5} (v-1)v dv + \frac{1}{4}\int_{\frac{16}{9}}^{4} (3v - 4\sqrt{v})dv \\
&= \left.\frac{v^3}{3} - \frac{v^2}{2}\right|_0^{0.5} + \left.\frac{1}{4}(\frac{3v^2}{2} - \frac{8v^{\frac{3}{2}}}{3})\right|_{\frac{16}{9}}^{4} = 0.9784 \blacksquare
\end{aligned}
$$

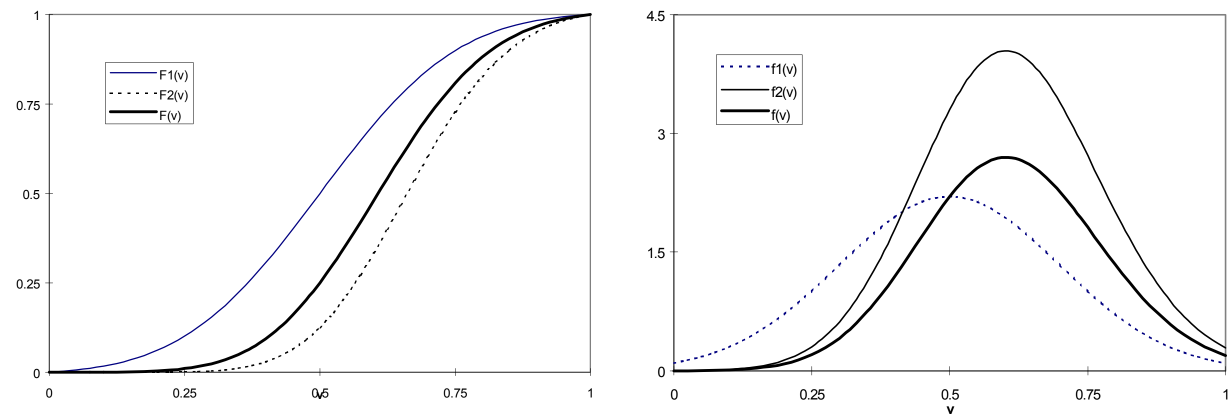

Figure 1: F_1 is the c.d.f of a normal N(0.5,0.2) truncated on [0,1]. $F_2(v) = F_1(v)^3$. Left panel shows the c.d.f. Right panel shows the derived p.d.f.

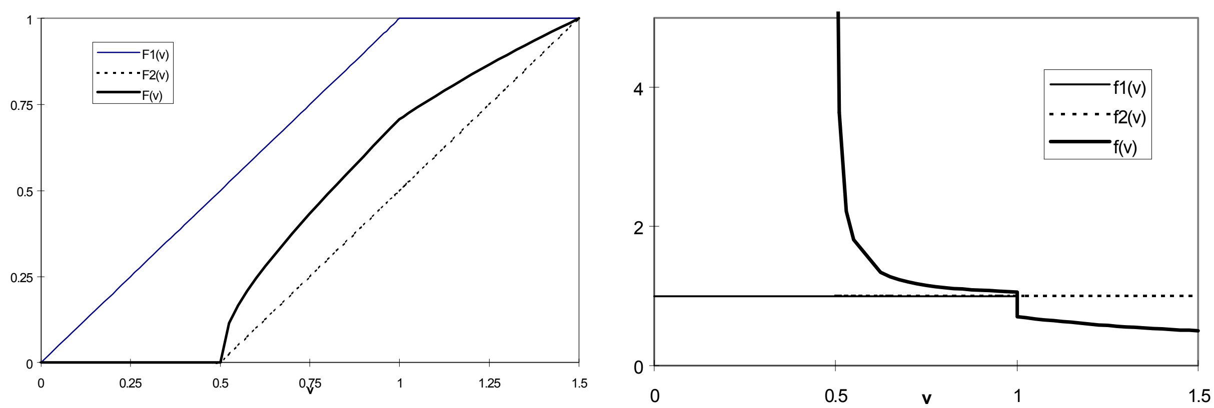

Figure 2: $F_1(v) = v$ and $F_2(v) = v - 0.5$. Again, left panel shows the c.d.f and the right panel represents the associated p.d.f.

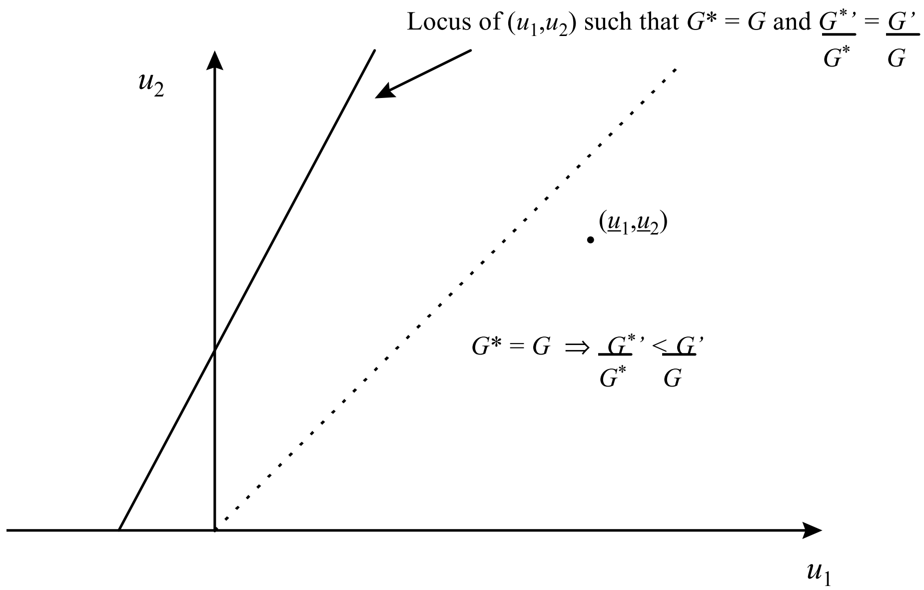
Locus of (u_1,u_2) such that $G^* = G$ and $\frac{G^{*\prime}}{G^*} = \frac{G'}{G}$
u_2
$(\underline{u}_1,\underline{u}_2)$
$G^* = G \Rightarrow \frac{G^{*\prime}}{G^*} < \frac{G'}{G}$
u_1

Figure 3