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A STOCHASTIC OVERLAPPING GENERATIONS ECONOMY WITH INHERITANCE

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A Stochastic Overlapping Generations Economy with Inheritance

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Abstract

An overlapping generations model of an exchange economy is considered, with individuals having a finite expected life-span. Conditions concerning birth, death, inheritance and bequests are fully specified. Under such conditions, the existence of stationary Markov equilibrium is established in some generality, and several explicitly solvable examples are treated in detail.

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> . . . let us suppose a whole generation of men to be born on the same day, to attain mature age on the same day, and to die on the same day, leaving a succeeding generation in the moment of attaining their mature age, all together. Let the ripe age be supposed of twenty-one years, and their period of life thirty-four years more, that being the average term given by the bills of mortality to persons of twenty-one years of age. Each successive generation would, in this way, come and go off the stage at a fixed moment, as individuals do now. Then I say, the earth belongs to each of these generations during its course, fully and in its own right. The second generation receives it clear of the debts and incumbrances of the first, the third of the second, and so on. For if the first could charge it with a debt, then the earth would belong to the dead and not to the living generation. Then, no generation can contract debts greater than may be paid during the course of its own existence.
>
> Excerpt of a letter from Thomas Jefferson to James Madison
>
> Paris, September 6, 1789

1 Introduction

The two basic infinite-horizon models of growth and monetary economics, are the *dynasty* or everlasting individual model, and the *overlapping generations model*. The dynasty model has, either explicitly or implicitly, an infinite-horizon separable utility function being optimized, usually with a "natural" discount factor (see for example, Harrod (1948) and Solow (1988)); the interpretation of the utility function can be made in terms of a "dynasty" and the discount can be interpreted as a "coefficient of concern" between successive members of the dynasty. The overlapping generations (OLG) model has individuals who live for a finite length of time and are replaced by new individuals. In general, the birth-and-death process is presented at a high degree of abstraction with a single sex, and there is little modelling of the specific aspects of individual birth and the raising of the young until they are independently economically active. The origin of the overlapping generations model was in the work of Allais (1947) and then Samuelson (1958). In both instances it was introduced in connection with the consideration of aspects of a monetary economy.

Since the 1960's there has been an explosion of OLG models. Much of the literature has been reviewed in the perceptive survey article of Geanakoplos (1987). In this paper we consider the influence of inheritance rules and noneconomic transfers of wealth at both ends of the life-cycle, and examine the differences between a non-stochastic life-span and a stochastic span with the same expected length of life. Our methods are based on earlier work of Karatzas, Shubik & Sudderth (1994), (1997) (referred to as [KSS1] and [KSS2], respectively) and of Geanakoplos, Karatzas, Shubik & Sudderth (1998) (referred to as [GKSS]) for the dynasty model. Here we extend the formal results of [KSS1] to a stochastic OLG model.

2 A Few Crude Facts

No human life-span beyond 130 years has ever been verified. The record verified span reported in the Guiness book of records of 1989 was for Mr. Shigechiyo Izumi of Japan, who was 120 years and 273 days on the day of his death in 1986. The change in life expectancy in the United States even in the last forty years has been considerable, as indicated in Table 1. Using the 1993 statistics we note that the probability of death in the United States for the first year of life is .835%, and it decreases until the age of 10 when it is .14%. It then climbs at a relatively slow but increasing rate until around the age of 60, when the increase becomes noticeably faster until at 85 it has reached 32.4%.

Table 1. Life Expectancy and Death Probabilities by Age

Age	Life expectancy					Death Probability (*1,000)				
	1960	1971	1980	1989	1993	1960	1971	1980	1989	1993
at birth	69.7	71	73.7	75.3	75.5	26.1	19	12.9	9.86	8.35
1	70.6	71.4	73.7	75	75.2	1.74	1.18	0.88	0.69	0.63
2	69.7	70.5	72.7	74.1	74.2	1.04	0.8	0.68	0.52	0.47
3	68.8	69.5	71.8	73.1	73.3	0.83	0.66	0.54	0.4	0.36
4	67.9	68.6	70.8	72.1	72.3	0.67	0.55	0.44	0.33	0.3
5	66.9	67.6	69.8	71.1	71.3	0.65	0.61	0.38	0.29	0.26
6	65.9	66.7	68.9	70.2	70.3	0.55	0.49	0.34	0.26	0.23
7	65	65.7	67.9	69.2	69.3	0.47	0.38	0.3	0.24	0.21
8	64	64.7	66.9	68.2	68.4	0.4	0.31	0.27	0.21	0.19
9	63	63.8	65.9	67.2	67.4	0.36	0.26	0.23	0.18	0.16
10	62.1	62.8	64.9	66.2	66.4	0.35	0.25	0.2	0.16	0.14
11	61.1	61.8	64	65.2	65.4	0.37	0.28	0.2	0.17	0.15
12	60.1	60.8	63	64.3	64.4	0.42	0.36	0.25	0.22	0.2
13	59.1	59.8	62	63.3	63.4	0.49	0.48	0.37	0.32	0.31
14	58.2	58.9	61	62.3	62.4	0.59	0.64	0.53	0.47	0.45
15	57.2	57.9	60	61.3	61.5	0.71	0.83	0.71	0.63	0.62
16	56.2	56.9	59.1	60.4	60.5	0.83	1.01	0.88	0.79	0.77
17	55.3	56	58.1	59.4	59.5	0.94	1.16	1.02	0.91	0.9
18	54.3	55.1	57.2	58.5	58.6	1.02	1.26	1.12	0.99	0.97
19	53.4	54.1	56.3	57.5	57.7	1.08	1.33	1.19	1.03	1
20	52.4	53.2	55.3	56.6	56.7	1.14	1.39	1.25	1.06	1.02
21	51.5	52.3	54.4	55.6	55.8	1.21	1.45	1.32	1.1	1.05
22	50.6	51.3	53.5	54.7	54.8	1.25	1.48	1.36	1.13	1.08
23	49.6	50.4	52.5	53.8	53.9	1.27	1.48	1.37	1.15	1.1
24	48.7	49.5	51.6	52.8	52.9	1.27	1.46	1.36	1.17	1.11
25	47.7	48.6	50.7	51.9	52.1	1.27	1.43	1.34	1.18	1.13
26	46.8	47.6	49.7	50.9	51.	1.28	1.41	1.32	1.2	1.14
27	45.9	46.7	48.8	50	49.2	1.3	1.4	1.31	1.23	1.18
28	44.9	45.8	47.9	49.1	48.3	1.33	1.42	1.31	1.27	1.23
29	44	44.8	46.9	48.1	47.3	1.38	1.47	1.31	1.32	1.31
30	43	43.9	46	47.2	46.4	1.44	1.53	1.33	1.38	1.39
31	42.1	43	45.1	46.2	45.5	1.51	1.6	1.35	1.45	1.47
32	41.2	42	44.1	45.3	44.5	1.6	1.69	1.38	1.51	1.55
33	40.2	41.1	43.2	44.4	43.6	1.7	1.79	1.43	1.59	1.63
34	39.3	40.2	42.2	43.5	42.7	1.82	1.92	1.5	1.66	1.71
35	38.4	39.3	41.3	42.5	41.8	1.97	2.07	1.59	1.75	1.8

Table 1 (continued)

Age	Life expectancy					Death Probability (*1,000)				
	1960	1971	1980	1989	1993	1960	1971	1980	1989	1993
36	3.7	38.3	40.4	41.6	40.8	2.13	2.24	1.69	1.85	1.89
37	36.5	37.4	39.4	40.7	39.9	2.32	2.42	1.81	1.94	1.99
38	35.6	36.5	38.5	39.8	39	2.53	2.63	1.96	2.04	2.08
39	34.7	35.6	37.6	38.8	38.1	2.77	2.86	2.12	2.14	2.18
40	33.8	34.7	36.7	37.9	37.2	3.04	3.11	2.31	2.25	2.3
41	32.9	33.8	35.7	37	36.3	3.34	3.39	2.53	2.38	2.43
42	32	32.9	34.8	36.1	35.4	3.67	3.68	2.77	2.53	2.56
43	31.1	32	33.9	35.2	34.5	4.03	4	3.04	2.71	2.69
44	30.2	31.2	33	34.3	33.6	4.43	4.34	3.35	2.92	2.84
45	29.4	30.3	32.1	33.4	32.7	4.85	4.7	3.68	3.19	3.01
46	28.5	29.4	31.3	32.5	31.8	5.32	5.11	4.04	3.41	3.2
47	27.7	28.6	30.4	31.6	30.9	5.85	5.56	4.44	3.71	3.43
48	26.8	27.7	29.5	30.7	30	6.46	6.07	4.89	4.05	3.7
49	26	26.9	28.7	29.8	29.2	7.13	6.63	5.38	4.43	4
50	25.2	26.1	27.8	28.9	28.3	7.88	7.25	5.91	4.85	4.35
51	24.4	25.3	27	28.1	27.4	8.66	7.91	6.48	5.31	4.73
52	23.6	24.5	26.1	27.2	26.6	9.43	8.61	7.09	5.84	5.14
53	22.8	23.7	25.3	26.3	25.7	10.2	9.35	7.72	6.43	5.61
54	22	22.9	24.5	25.6	24.9	10.9	10.3	8.39	7.08	6.12
55	21.3	22.1	23.7	24.7	24.1	11.7	11	9.1	7.79	6.66
56	20.8	21.4	22.9	23.9	23.3	12.6	11.9	9.88	8.55	7.24
57	19.8	20.6	22.2	23.1	22.5	13.6	12.9	10.7	9.37	7.88
58	19	19.9	21.4	22.3	21.7	14.9	14.1	11.7	10.3	8.59
59	18.3	19.2	20.6	21.6	20.9	16.3	15.4	12.7	11.2	9.34
60	17.6	18.5	19.9	20.8	20.2	17.9	16.9	13.8	12.3	10.14
61	16.9	17.8	17.2	20.1	19.5	19.6	18.4	15	13.4	10.96
62	16.3	17.1	18.5	19.3	18.7	21.3	20	16.3	14.5	11.8
63	15.6	16.4	17.8	18.6	18	23	21.5	17.7	15.7	12.64
64	15	15.8	17.1	17.9	17.6	24.8	23	19.2	16.9	13.48
65	14.3	15.1	16.4	17.2	17.3	26.7	24.6	20.8	18.2	14.35
66	13.7	14.5				28.7	26.3			
67	13.1	13.9				31	28.4			
68	12.5	13.1				33.5	30.9			
69	11.9	12.7				36.4	33.9			
70			13.2	13.9	14			30.9	27.3	19.21
75			10.4	10.9	10.9			45.7	41.3	24.52
80			7.9	8.3	8.3			69	63.7	29.38
85 & over			5.9	6.2	6			1000	1000	324.28

In the United States Federal and State death taxes are high for estates above $600,000, with margins going above 45%, but there are many legal loopholes and thus in comparison to income tax the revenue raised is relatively modest. In 1996 GNP was 7,637.7 billion, the Federal budget was 1,453 billion and income taxes were 1,500 billion with estate taxes accounting for 18 billion.

Table 2 shows an estimate of the percent of family income devoted to taking care of children:

Table 2. Estimated Percentage of Income Expenditure on a Child in Rural Areas[a] (1995)

Age	Income Range 1[b] AV = 21,200	Income Range 2 AV = 45,300	Income Range 3 AV = 85,700
0–2	23.7	15.8	12.6
3–5	24.2	16.2	12.8
6–8	24.9	16.4	12.8
9–11	25.4	16.5	12.8
12–14	29.0	18.1	12.6
15–17	28.6	18.4	14.1

Sources: M. Lino, 1996, Expenditures on Children by Families 1995 Annual Report, USDA, p. 20.

[a]The ranges are less than $34,000; $34,000 – $57,200 and greater than $85,000.

[b]Fewer than 2,500 people outside a Metropolitan Statistical Area.

These figures do not include prenatal expenditures or college and other higher education support for children older than 17.

Table 3 shows disposable personal income and population change in the United States recently.

Table 3. Disposable Income and Population

	Disposable personal income (1992 dollars)	Births (1,000)	Deaths (1,000)	Population (1,000)
1960	8,660	4,307	1,712	179,386
1970	12,022	3,739	1,921	203,849
1980	14,813	2,743	1,988.80	226,546
1990	17,941	4,148	2,148.50	248,143
1995	18,757	3,961	2,286	261,626

Data sources: Statistical Abstract of U.S., 1996, 1980.

3 Modelling Considerations

There are several considerations which differentiate the infinite-horizon, nonstochastic, dynastic equilibrium models, from an overlapping generations economy modelled in terms of a noncooperative game (stochastic or not).

In particular, if we take the human life cycle into account, then we must consider, following Modigliani (1986), the age-span and changes in economic activity of an economic agent over the course of his life. A rough heuristic suggests that an individual is scarcely economic before the age of around 10, and that (except in countries with child labor) children are consumers, not producers, until around the age of 14 or later. The representative individual at the start of the twenty-first century may then be a producer from somewhere between the ages of 14–25 to somewhere around the ages of 60–75.

We limit our investigation to an economy with a fixed quantity Q of a single consumption good and stochastic shares Y of the income derived from its sale, thus avoiding most of the problems concerning the variability of the money supply. Even without this complication, the formulation of a complete process model requires that we indicate how child support, old age support, inheritance, and the possibility of death while in debt, are specified.

3.1 Modelling Specifications

Prior to presenting a formal model we note, then discuss after listing, the modelling choices to be made:[1]

1. The *number of sexes* modelled: one or two?
2. The *life span*: infinite; finite and fixed; finite expectation.
3. *Demographics*: stationary; growing.
4. *Span of economically productive life*: birth to death; adolescence to death; adolescence to retirement.
5. *Intergenerational wealth transfer*: none; rearing young; leaving inheritance; rearing young and leaving inheritance; supporting parent.
6. *Availability of loan market*: none; money market; outside bank.
7. *Bankruptcy rules*: secured lending; fully secured with default penalty.
8. *Availability of insurance*: none; guarantee of income; guarantee of debt.
9. *Treatment of debt at death*: inherited; forgiven; insured.

[1] If we were to include the existence of a public good, then we would also be required to specify its means of allocation. This specification is often politico-economic, and thus requires a specification of the span of politically active life.

10. *Taxes and government subsidies*: none; various.

We discuss these ten items and indicate our modelling choices and the reasons for the choices.

We consider a "unisex" model. The two-sex model of overlapping generations has been highly difficult to develop analytically. As there are still many good questions to be explored that do not require a two-sex model, we have decided to avoid it.

We concentrate on models with a fixed finite life-span, and on models with the much more realistic feature of a stochastic life-span with finite expectation.

For simplicity, we shall confine our investigation to a demographically stationary population.

In the equilibrium existence proofs, any span of economically productive life can be considered. But in specific examples, considerable differences are encountered in the need for insurance and in the need for intergenerational wealth transfer, as age-span changes.

We consider several variants of inter-generational transfer, but observe that with stochastic life-spans, and in the absence of insurance, even totally selfish individuals may leave assets after death. A rule must be specified for their disposal.

In this paper we consider only models without borrowing or lending, as we did in our first paper [KSS1] on immortal agents. We hope to extend our results to models with borrowing and lending, as was done for the case of immortal agents in [KSS2] and [GKSS].

In an economy with stochastic elements, insurance can be provided by government tax and subsidy policies.

If a fully-defined process model of an overlapping generations economy is to be built, it must cover all of the items noted above. The choices made and specified for the model of this paper are now summarized:

1. A unisex model is constructed.

2. Life-span has a finite expectation (no uncertainty is a special case).

3. Demographics are stationary.

4. The span of economically productive life is treated in generality (parametrically).

5. Inter-generational wealth transfer is specified, with some variants and problems noted.

4 A Formal Model

Time in the economy is discrete and runs $t = 0, 1, 2, ...$ There is uncertainty about future endowments, as well as about births and deaths. Both will be modelled by random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

For each $t \geq 0$, I_t is a copy of the unit interval used to parametrize the collection of agents alive at time t. Let $I_{k,t}$ be the subset of I_t corresponding to agents of age k at time t, so that

$$I_t = \bigcup_{k=0}^{\infty} I_{k,t}.$$

We assume that each $I_{k,t}$ is a Borel subset of $[0,1]$.

Let φ_t be a non-atomic probability measure defined on $\mathcal{B}(I_t)$, the Borel subsets of I_t, that corresponds to the "spatial" distribution of agents at time t. Thus

$$\varphi_t(I_t) = \sum_{k=0}^{\infty} \varphi_t(I_{k,t}) = 1$$

and $\varphi_t(I_{k,t})$ is the proportion of agents having age k at time t.

Associated with every agent α are two stochastic processes, representing his age and his wealth in fiat money, respectively. The age-process has the simpler structure, and we shall discuss it first.

4.1 The Age-Process

Let us fix $\alpha \in I_{k,t}$, meaning that agent α has age k at time t. We denote by η_k the probability that α survives to the next period, for every $k \geq 0$. Thus α goes on to $I_{k+1,t+1}$ with probability η_k, and α dies at age k with probability $1 - \eta_k$. More generally, we assume that α survives until period $k + \ell$ with probability $\eta_k \cdot \eta_{k+1} \cdots \eta_{k+\ell-1}$. Using a technique of Feldman & Gilles (1985), we can construct age-processes for the collection of all agents so that these survival probabilities also correspond to the proportions of agents who survive, in the sense that we have

$$\varphi_{t+1}(I_{k+1,t+1}) = \eta_k \cdot \varphi_t(I_{k,t}), \quad \text{for all } k \text{ and } t.$$

We shall assume throughout this paper, that agents are born at the same overall rate at which they die. To simplify notation, whenever an agent α dies, we use the same index α for a newborn agent. With this abuse of notation, we can employ the same index set I for every I_t. Also we can represent the age-process corresponding to an index α as a Markov chain $\{\mathcal{K}_t^\alpha,\ t = 0, 1, ...\}$ with transition probabilities

$$\mathbf{P}[\mathcal{K}_{t+1}^\alpha = k+1 | \mathcal{K}_t^\alpha = k] = \eta_k, \qquad \mathbf{P}[\mathcal{K}_{t+1}^\alpha = 0 | \mathcal{K}_t^\alpha = k] = 1 - \eta_k. \tag{4.1}$$

A necessary and sufficient condition for this chain to have a stationary distribution, is that the expected lifespan of a newborn agent (i.e., the expected return time to zero) be finite. This condition is equivalent to the requirement that the infinite sum

$$\Delta := 1 + \eta_0 + \eta_0\eta_1 + \eta_0\eta_1\eta_2 + \cdots \tag{4.2}$$

be finite. We assume that Δ is finite, and let $\{\nu_k,\ k = 0, 1, ...\}$ be a stationary distribution. Then $\{\nu_k\}$ must satisfy

$$\nu_{k+1} = \eta_k \nu_k, \qquad k = 0, 1, ...$$

hence

$$\nu_{k+1} = \eta_k \eta_{k-1} \ \cdots \ \eta_0 \nu_0, \qquad k = 0, 1, \ldots .$$

Using the normalization $\sum \nu_k = 1$, we conclude that

$$\nu_0 = 1/\Delta, \quad \nu_{k+1} = \eta_k \eta_{k-1} \ \cdots \ \eta_0/\Delta \qquad \text{for} \quad k \geq 0. \tag{4.3}$$

Through the rest of this paper, we shall assume that the age-distribution is the stationary distribution of (4.3): at every time t, the proportion of agents having age k is ν_k. (In other words, for the sake of simplicity, we have ruled out the possibility of a "baby boom" in our model.)

Here are two simple examples.

Example 4.1 (A constant death-rate) *Suppose $\eta_k = \eta$ for all $k = 0, 1, \ldots$, where $0 < \eta < 1$. Then the sum Δ of* (4.2) *equals $(1-\eta)^{-1}$ and the stationary age-distribution of* (4.3) *is the geometric distribution given by $\nu_k = (1-\eta)\eta^k$ for all k.*

Example 4.2 (A nonstochastic life-span of length K) *Suppose $\eta_0 = \eta_1 = \cdots = \eta_K = 1$ and $\eta_k = 0$ for $k > K$. Then $\Delta = (K+1)^{-1}$ and $\nu_0 = \nu_1 = \cdots = \nu_K = (K+1)^{-1}$, $\nu_k = 0$ for $k > K$.*

4.2 The Wealth-Process

For each $\alpha \in I_{k-1,t-1}$ with $t \geq 1$ and $k \geq 1$, the random variable $S^\alpha_{k-1,t-1}$ denotes the wealth in fiat money of an agent α of age $k-1$ at the beginning of period t. The dynamics of the process $\{S^\alpha_{k-1,t-1}\}$ depend on agent α's spending, inheritance, and endowment in each period, as we now explain.

In every period t, each agent α receives a random endowment $Y^\alpha_{k,t}(\omega)$ in units of a nondurable commodity. The distribution λ_k of the random variable $Y^\alpha_{k,t}$ depends on the age k of agent α, but not on the period t. Endowments in different periods are assumed to be independent, but the total endowment of the commodity in each period t for agents of age k, namely

$$Q_k = \int_{I_{k,t}} Y^\alpha_{k,t}(\omega)\varphi_t(d\alpha) \tag{4.4}$$

is taken to be nonrandom, and constant from period to period. (The technique of Feldman & Gilles (1985) gives a simple construction of jointly measurable functions $(\alpha, \omega) \mapsto Y^\alpha_{k,t}(\omega) = Y_{k,t}(\alpha, \omega)$, $k \geq 1$, $t \geq 1$, which are independent for each fixed α but aggregate to a constant as in (4.4).) The total endowment for agents of all ages in period t is just the sum

$$Q = \sum_{k=0}^{\infty} Q_k. \tag{4.5}$$

Consider now an agent α, of age $k-1$ and with wealth $S^\alpha_{k-1,t-1}(\omega)$ at the beginning of period t . The agent first decides on the amount

$$b^\alpha_{k-1,t}(\omega) \in [0, S^\alpha_{k-1,t-1}(\omega)] \tag{4.6}$$

that he will bid in the commodity market. We assume that the bids are jointly measurable in (α, ω), so that a price $p_t(\omega)$ for the commodity can be formed as

$$p_t(\omega) := \frac{1}{Q} \sum_{k=1}^{\infty} \int_{I_{k-1,t-1}} b^{\alpha}_{k-1,t}(\omega)\, \varphi_{t-1}(d\alpha)\ , \tag{4.7}$$

the ratio of total demand over total supply for the commodity. Our agent α then receives his bid's worth $x^{\alpha}_{k,t}(\omega) := b^{\alpha}_{k-1,t}(\omega)/p_t(\omega)$ in units of the commodity, and consumes it immediately, thereby receiving an amount of utility equal to $u_{k-1}(x^{\alpha}_{k,t}(\omega))$. Here $u_{k-1} : [0, \infty) \to [0, \infty)$ is a concave utility function common to all agents of age $k-1$. If agent α is of age 0 at time t_0, then his total utility received over the course of the game is

$$v^{\alpha}(\omega) := \sum_{k=1}^{\infty} (\eta_0 \eta_1 \ \cdots \ \eta_{k-1}) \cdot u_k(x^{\alpha}_{k,t_0+k})\, , \tag{4.8}$$

where $\eta_0 \eta_1 \ \cdots \ \eta_{k-1}$ is the probability that agent α survives to age k.

After the price $p_t(\omega)$ has been formed as in (4.7), agent α receives the value of his endowment in fiat money. This amounts to $p_t(\omega) Y^{\alpha}_{k,t}(\omega)$ and, at this stage, agent α has wealth

$$\tilde{S}^{\alpha}_{k,t}(\omega) = S^{\alpha}_{k-1,t-1}(\omega) - b^{\alpha}_{k-1,t}(\omega) + p_t(\omega) Y^{\alpha}_{k,t}(\omega). \tag{4.9}$$

If the agent survives into the next period (an event that occurs with probability η_{k-1}) , he begins the next period with the wealth (4.9) together with any *inheritance* $Z^{\alpha}_{k,t}(\omega)$ that he may receive in that period. If the agent does not survive, his wealth becomes part of the total legacy. Thus, with probability η_{k-1}, agent α survives with wealth

$$\begin{aligned} S^{\alpha}_{k,t}(\omega) &= S^{\alpha}_{k-1,t-1}(\omega) - b^{\alpha}_{k-1,t}(\omega) + p_t(\omega) Y^{\alpha}_{k,t}(\omega) + Z^{\alpha}_{k,t}(\omega) \\ &= \tilde{S}^{\alpha}_{k,t}(\omega) + Z^{\alpha}_{k,t}(\omega). \end{aligned} \tag{4.10}$$

If the agent α is newborn at the end of period t, then

$$S^{\alpha}_{0,t}(\omega) = Z^{\alpha}_{0,t}(\omega).$$

The *total legacy* in the period is

$$L_t(\omega) = \sum_{k=1}^{\infty} (1 - \eta_{k-1}) \int_{I_{k-1,t-1}} \tilde{S}^{\alpha}_{k,t}(\omega) \varphi_{t-1}(d\alpha), \tag{4.11}$$

which must be equal to the *total inheritance*

$$\sum_{k=0}^{\infty} \int_{I_{k,t}} Z^{\alpha}_{k,t}(\omega) \varphi_t(d\alpha). \tag{4.12}$$

If only the newborn receive an inheritance at any given time (cf. Assumption 5.1 below), then the expression of (4.12) simplifies to $\int_{I_{0,t}} Z^{\alpha}_{0,t}(\omega) \varphi(d\alpha)$. Notice that an agent's survival in any period is independent of his wealth and endowment in that period. This simplifying assumption is, of course, not valid for many real-world economies.

4.3 The Conservation of Wealth

The dynamics of the economy, as described above, have the property that *the total supply of fiat money is conserved from period to period.* To see this, let $W_{t-1}(\omega)$ be the total wealth of all agents at the beginning of period t, namely

$$W_{t-1}(\omega) = \sum_{k=1}^{\infty} \int_{I_{k-1,t-1}} S^{\alpha}_{k-1,t-1}(\omega)\varphi_{t-1}(d\alpha).$$

After bids and income, but before births, deaths or inheritance, an agent α's wealth is $\tilde{S}^{\alpha}_{k,t}(\omega)$ as in (4.9), and the total wealth

$$\tilde{W}_t(\omega) = \sum_{k=1}^{\infty} \int_{I_{k-1,t-1}} \tilde{S}^{\alpha}_{k,t}(\omega)\varphi_{t-1}(d\alpha) \tag{4.13}$$

remains equal to $W_{t-1}(\omega)$, thanks to (4.9), (4.4), (4.5) and (4.7). The total wealth of the living, after births and deaths but before inheritance, is

$$\sum_{k=1}^{\infty} \eta_{k-1} \int_{I_{k-1,t-1}} \tilde{S}^{\alpha}_{k,t}(\omega)\varphi_{t-1}(d\alpha).$$

Added to the total legacy $L_t(\omega)$ of (4.11), this gives total wealth $\tilde{W}_t(\omega)$ as in (4.13) at the end of period t, which is equal to $W_{t-1}(\omega)$.

4.4 The Distribution of Wealth

Because of the stochastic nature of our model, agents of the same age k may hold different amounts of money at the same time t. We denote by $\xi_{k,t}(\cdot,\omega)$ the random measure that corresponds to the wealth distribution for agents of age k at time t. Thus, for Borel subsets A of $[0,\infty)$,

$$\xi_{k,t}(A,\omega) := \varphi_t\left(\{\alpha \in I_{k,t} : S^{\alpha}_{k,t}(\omega) \in A\}\right)$$

is the measure under φ_t of the set of agents α of age k having wealth in the set A. The random measures $\{\xi_{k,t}(\cdot,\omega),\ k = 0, 1, ...\}$, together with the stationary age distribution $\{\nu_k\}$ of (4.3), completely determine the joint distribution $\zeta_t(A,k,\omega)$ of wealth and age at any given time t. That is, for any bounded measurable function $(s,k) \mapsto f(s,k)$ of wealth s and age k, we have

$$\sum_{k=0}^{\infty} \int_A f(s,k)\zeta_t(ds,k,\omega) = \sum_{k=1}^{\infty} \nu_k \int_A f(s,k)\xi_{k,t}(ds,\omega) \tag{4.14}$$

for any Borel subset A of $[0,\infty)$ and any $\omega \in \Omega$.

4.5 The Many-Person Stochastic Game

The model described above can be viewed as a stochastic game with a continuum of players. A strategy π^α for an agent $\alpha \in I$ specifies the bids $b^\alpha_{k,t}$ as random variables satisfying (4.6) and measurable with respect to the σ-field $\mathcal{F}_{k-1}$ generated by all past prices, endowments, wealth, and actions. A collection $\{\pi^\alpha,\ \alpha \in I\}$ of strategies is *admissible*, if the functions $(\alpha, \omega) \mapsto b^\alpha_{k,t}(\omega)$ are $\mathcal{B}(I) \otimes \mathcal{F}_{k-1}$-measurable for all k and t. We will always assume that the collection of strategies played by the agents is admissible, so that the prices are well-defined by (4.7).

Each agent α seeks to maximize the expected value of his total utility given by (4.8). There may be some confusion about the collection of agents, because there are births and deaths in every period. Recall that the age-distribution given by (4.3) is stationary, and we have made the convention that whenever an agent α dies, the index α is reassigned to a newborn agent. By a strategy π^α, we mean, on the one hand, the entire infinite sequence of bids specified by all the agents with index α. On the other hand, each agent seeks only to maximize his own expected utility, not that of other agents, past or present, with the same index. Nevertheless, for simplicity we shall continue to refer to an agent $\alpha \in I$, and mean by this the agent currently alive having index α. When we use the term *optimal* to refer to a strategy π^α, we mean that the strategy π^α is optimal for each of the agents who, in their respective lifetimes, are assigned index α.

A strategy π^α is called *stationary* if, in every period t, the bid b by π^α depends only on the current wealth and age of agent α, together with the price; that is, there is a measurable function $(s, k, p) \mapsto c(s, k, p)$ on $[0, \infty) \times \mathbb{N} \times (0, \infty)$, such that $0 \le c(s, k, p) \le s$ and

$$b^\alpha_{k,t}(\omega) = c(S^\alpha_{k,t-1}(\omega), k, p_{t-1}(\omega)).$$

We are particularly interested in whether the economy of our model admits an equilibrium in which prices and wealth distributions remain constant, while the wealth processes of individual agents fluctuate according to the dynamics described above.

Definition 4.1 *Let $\{\pi^\alpha,\ \alpha \in I\}$ be an admissible collection of stationary strategies, let $p \in (0, \infty)$, and let μ_k be a measure defined on $\mathcal{B}(I_{k,0})$ for every $k = 0, 1, \cdots$. We say that $\{\pi^\alpha\}$, p, and $\{\mu_k\}$ form a* stationary Markov equilibrium *if, when the initial price is $p_0 = p$, the initial wealth distribution for agents of age k is $\xi_{k,0}(\cdot, \omega) = \mu_k$ for all k and ω, and every agent $\alpha \in I$ plays π^α, the following hold:*

(i) $p_t(\omega) = p$ and $\xi_{k,t}(\cdot, \omega) = \mu_k$, for all $k = 1, 2, ...$ and $t = 1, 2, ...$

(ii) for every $\alpha \in I$, π^α is optimal among all strategies for agent α when every other agent $\beta \ne \alpha$ plays π^β.

In the next section, we study the one-person optimization problem faced by a single agent when the many-person game is in stationary equilibrium. The results obtained for the one-person game will enable us to prove the existence of stationary equilibrium for the many-person game in Section 6.

We shall concentrate on two specific inheritance rules, which we call *egalitarian* and *individual inheritance*. For both rules the entire legacy L_t of (4.11) is left to the newborn agents of period t. The egalitarian rule is that the newborn share the legacy equally, while the individual inheritance rule is that the wealth $\tilde{S}^{\alpha}_{k,t}$ of each individual agent α is left to an individual newborn agent — which, for convenience, we assume has the same index α. Our methods could be adapted to handle a variety of other rules.

5 The One-Person Game

Suppose that the many-person model of Section 4 is in stationary Markov equilibrium with a fixed price $p \in (0,\infty)$, and consider an agent with wealth s of age k who seeks to maximize total expected discounted utility over the course of his remaining lifetime. Let $V_k(s) = V(s,k)$ be his optimal reward. (We omit the superscript α in this section.)

Here is the first of several assumptions we shall make — in addition to those already made in Section 4.

Assumption 5.1 *Only the newborn agents receive an inheritance; that is,* $Z_{k,t}(\omega) = 0$ *for* $k \geq 1$ *and all* (t,ω).

Under this assumption, Bellman's equation takes the form

$$V_k(s) = \sup_{0\leq b\leq s} \left[u_k(b/p) + \eta_k \cdot \mathbf{E}V_{k+1}(s-b+pY_{k+1})\right] \tag{5.1}$$

where Y_{k+1} is a generic random variable with distribution λ_{k+1}.

Here are three additional assumptions.

Assumption 5.2 *(i) For each* k*, the utility function* $u_k(\cdot)$ *is strictly concave, nondecreasing, differentiable on* $(0,\infty)$*, with* $u_k(0) = 0$*.*

(ii) Survival probabilities are bounded away from 1: $\eta^* := \sup_k \eta_k < 1$*.*

(iii) Expected endowments are bounded away from infinity: $m := \sup_k \mathbf{E}Y_k < \infty$*.*

The theorem below gives some of the basic properties of the optimal reward function and of an optimal strategy.

Theorem 5.1 *(a) For each* k*, the function* $V_k(\cdot)$ *is concave and increasing.*

(b) There is a unique optimal stationary plan $\pi = \pi_p$ *corresponding to a bid function* $c(s,k) = c(s,k;p)$ *such that* $0 \leq c(s,k) \leq s$ *for all* $s \geq 0$ *and* $k = 0,1,....$

(c) For each k*, the functions* $s \mapsto c(s,k)$ *and* $s \mapsto s - c(s,k)$ *are nondecreasing and continuous in* s*.*

(d) For $s > 0$ *and all* k*,* $c(s,k) > 0$*.*

(e) For all k*, the function* $V_k(\cdot)$ *is differentiable on* $(0,\infty)$

$$V_k'(s) = \frac{1}{p}\, u_k'\left(\frac{c(s,k)}{p}\right), \qquad s > 0.$$

This theorem is a close relative of Theorem 4.2 of [GKSS] and the proof is fairly similar, so we will only sketch the major steps.

5.1 The Proof of Theorem 5.1

First we define the *one-day operator*

$$(Tw)(s,k) := \sup_{0\le b\le s} [u_k(b/p) + \eta_k \cdot \mathbf{E}w(s-b+pY_{k+1})]$$

for measurable functions $w : [0,\infty) \times \mathbf{N} \to [0,\infty)$. Let

$$V^{(1)}(s,k) = (T0)(s,k)$$

and

$$V^{(n+1)}(s,k) = (TV^{(n)})(s,k), \quad n \ge 1.$$

Then $V^{(n)}(s,k)$ corresponds to the optimal n-day return for an agent starting at (s,k). We have also

$$\lim_{n\to\infty} V^{(n)}(s,k) = V(s,k),$$

and thus we can establish properties of V by first proving them for the $V^{(n)}$ and then passing to the limit.

The crucial step is to show that certain properties are preserved by the one-day operator T. Note that in this proposition and below, we write $w'(s,k)$ for $\frac{\partial}{\partial s}w(s,k)$.

Proposition 5.2 *Let $w : [0,\infty) \times \mathbf{N} \to [0,\infty)$, and assume that the function $s \mapsto w(s,k)$ is concave and increasing, differentiable on $(0,\infty)$, with $w'_+(0,k) \le \frac{1}{p}(u_k)'_+(0)$, for every $k \in \mathbf{N}$. Then the same properties hold for Tw.*

The proof of the proposition is similar to, in fact somewhat easier than, that of Proposition 4.1 in [GKSS], so we omit the proof.

Now $V^{(1)}(s,k) = (T0)(s,k) = u_k(s/p)$ obviously satisfies the hypotheses of the proposition. Consequently, so do all the $V^{(n)}$. Theorem 5.1 can now be proved by the arguments of Theorem 4.2 in [GKSS].

We need two additional properties of the optimal bid function $c(s,k)$ of Theorem 5.1. To establish them, we impose some further assumptions.

Assumption 5.3 *(i)* $\alpha := \inf_x u'_0(x) > 0$.
(ii) $u'_0(x) \ge u'_1(x) \ge u'_2(x) \ge \cdots, \quad \forall\, x \in \mathbb{R}$.

Assumption 5.3(i) says that newborn agents have marginal utility for consumption that never falls below a given positive constant α, while (ii) postulates that marginal utility shrinks with age. Notice that because $u_k(0) = 0$ for all k by Assumption 5.2(i), we have $u_k(x) = \int_0^x (u_k)'(y)dy$, $x > 0$, and so

$$u_0(x) \ge u_1(x) \ge u_2(x) \ge \cdots, \quad \forall\, x \in \mathbb{R}.$$

Lemma 5.3 $\lim_{s\to\infty} c(s,0) = \infty$.

Proof Suppose the assertion is false. Then, by Theorem 5.1(c), there exists a finite constant b^* such that $c(s,0) \leq b^*$ for all $s \geq 0$. We shall show, by induction on k, that $c(s,k) \leq b^*, \forall s \geq 0$ holds for every $k = 0, 1, 2, ...$

Suppose that this property holds for k, but not for $k+1$. Then, by Theorem 5.1(c), there is an s_0 such that $c(s, k+1) > b^*$ for all $s \geq s_0$. Consider $s > s_0 + b^*$. Since $c^* := c(s,k) \leq b^* < s$, the function

$$\psi(b) = u_k(b/p) + \eta_k \cdot \mathbf{E} V_{k+1}(s - b + pY_k)$$

attains its maximum over $[0, s]$ at the interior point $c^* \in (0, s)$. Thus,

$$\begin{aligned}
\frac{1}{p} u_k' \left(\frac{b^*}{p} \right) \leq \frac{1}{p} u_k' \left(\frac{c^*}{p} \right) &= \eta_k \cdot \mathbf{E} V_{k+1}'(s - c^* + pY_k) \\
&< V_{k+1}'(s - c^*) \\
&= \frac{1}{p} u_{k+1}' \left(\frac{c(s - c^*, k+1)}{p} \right) \\
&\leq \frac{1}{p} u_k' \left(\frac{c(s - c^*, k+1)}{p} \right) \\
&\leq \frac{1}{p} u_k' \left(\frac{b^*}{p} \right),
\end{aligned}$$

a contradiction. (The next to last inequality is by our assumption that $u_{k+1}'(\cdot) \leq u_k'(\cdot)$, and the last inequality holds because $s - c^* \geq s - b^* \geq s_0$ and so $c(s - c^*, k+1) \geq c(s_0, k+1) \geq b^*$.)

We conclude that $c(s,k) \leq b^*$ for all s and k. But then

$$\begin{aligned}
u_0(s/p) \leq V_0(s) &\leq u_0(b^*/p) + \eta_0 \cdot u_1(b^*/p) + \eta_0 \eta_1 \cdot u_2(b^*/p) + \cdots \\
&\leq u_0(b^*/p)(1 + \eta_0 + \eta_0 \eta_1 + \cdots) \\
&< \infty,
\end{aligned}$$

an impossibility because $u_0(s/p) \to \infty$ as $s \to \infty$. ■

Lemma 5.4 *The function* $s \mapsto s - c(s,0)$ *is bounded.*

Proof Similar to, and simpler than, that of Lemma 4.2(a) in [GKSS]. ■

5.2 The Wealth Process as a Markov Chain: Egalitarian Inheritance

Consider a stochastic process $\{(S_n, \mathcal{K}_n) : n = 0, 1, ...\}$ that corresponds to the wealth and age of an individual agent playing in the one-person game with fixed price p up to the time of his death, and then corresponds to the wealth and age of a second agent born in the period of the first agent's death until the second agent dies, and so on. Each agent in this succession is assumed to play the optimal stationary plan of Theorem 5.1. We also assume in this section that the inheritance of every newborn is the same positive constant ℓ. (In the next section, we shall consider an individual inheritance rule.) The process $\{(S_n, \mathcal{K}_n)\}$ is a Markov chain with dynamics given by

$$(S_{n+1}, \mathcal{K}_{n+1}) = \left\{ \begin{array}{ll} (S_n - c(S_n, \mathcal{K}_n) + pY_{\mathcal{K}_n+1}, \mathcal{K}_n + 1), & \text{with probability } \eta_{\mathcal{K}_n} \\ (\ell, 0), & \text{with probability } 1 - \eta_{\mathcal{K}_n} \end{array} \right\}. \tag{5.2}$$

For the time being the legacy parameter ℓ is arbitrary, but eventually we shall specify it carefully.

Notice that the state $(\ell, 0)$ is a regeneration point for the chain, and let τ be the first hitting time of this state. The following lemma is an easy consequence of Assumption 5.2(ii).

Lemma 5.5 $\mathbf{P}_{(\ell,0)}[\tau > n] \leq (\eta^*)^n$, *for all* $n = 1, 2, \ldots$.

Hence, τ is stochastically dominated by a geometric random variable and has finite moments of all orders. It follows (cf. Asmussen (1987), p. 152) that the Markov chain has a stationary distribution $\zeta_\ell(d(s,k))$ that can be represented by

$$\int f d\zeta_\ell = \frac{1}{\mathbf{E}_{(\ell,0)}\tau} \cdot \mathbf{E}_{(\ell,0)} \left(\sum_{n=0}^{\tau-1} f(S_n, \mathcal{K}_n) \right) \tag{5.3}$$

for measurable $f : [0, \infty) \times \mathbf{N} \to [0, \infty]$.

Under ζ_ℓ the marginal distribution of the age $\mathcal{K}_n$ must be the stationary distribution for the age process, namely, the distribution $\{\nu_k\}$ of (4.3). In fact, the age-process $\{\mathcal{K}_n\}$ does not depend on the parameter ℓ and, in particular, $\mathbf{E}_{(\ell,0)}\tau$ is a positive constant independent of ℓ, and bounded from above by $1/(1 - \eta^*)$.

Lemma 5.6 *Under the stationary distribution ζ_ℓ, the marginal distribution of wealth has a finite mean.*

Proof Recall from Assumption 5.2(iii) that $m = \sup_k \mathbf{E}Y_k < \infty$. Suppose that the chain begins at the regeneration point $(\ell, 0)$. Then for $n < \tau$ we have

$$\begin{aligned} S_0 &= \ell \\ S_1 &= \ell - c(\ell, 0) + pY_0 \leq \ell + pY_0 \\ &\vdots \\ S_{n+1} &= S_n - c(S_n, n) + pY_n \leq \ell + pY_0 + \cdots + pY_n \\ &\vdots \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{E}_{(\ell,0)} \left(\sum_{n=0}^{\tau-1} S_n \right) &= \mathbf{E}_{(\ell,0)} \left(\sum_{n=0}^{\infty} S_n 1_{\{\tau > n\}} \right) \\ &\leq \sum_{n=0}^{\infty} (\ell + npm)(\eta^*)^n < \infty. \end{aligned}$$

The lemma now follows from (5.3) with $f(s,k) = s$. ■

We now want to select the legacy parameter ℓ so that, under the stationary distribution ζ_ℓ, the amount of money inherited by the newborn is equal to the amount bequeathed by the dying in each period. The proportion of newborn agents is ν_0 and each of them inherits the amount ℓ. Thus, in view of (4.9) and (4.11), the desired result is the following:

Lemma 5.7 *There exists $\ell \geq 0$ such that*

$$\nu_0 \ell = \int \{(1-\eta_k)(s - c(s,k) + p\mathbf{E}Y_k)\}\zeta_\ell(d(s,k)).$$

Proof The right-hand side of the desired equality can be written, by (5.3), as

$$\frac{1}{\mathbf{E}_{(\ell,0)}\tau} \cdot \mathbf{E}_{(\ell,0)}\left(\sum_{n=0}^{\tau-1} f(S_n, \mathcal{K}_n)\right) \tag{5.4}$$

where $f(s,k) = (1-\eta_k)(s-c(s,k)+pY_k)$. Consider a coupled family of Markov chains $\{\{(S_n^\ell, \mathcal{K}_n^\ell)\} : 0 \leq \ell < \infty\}$ on a common probability space with initial conditions $(S_0^\ell, \mathcal{K}_0^\ell) = (\ell, 0)$ and dynamics as in (5.2) based on the same income variables $\{Y_n\}$. We can rewrite (5.4) in the form

$$\frac{1}{\mathbf{E}\tau} \cdot \mathbf{E}\left(\sum_{n=0}^{\tau-1} f(S_n^\ell, \mathcal{K}_n^\ell)\right). \tag{5.5}$$

By Lemma 5.2, there is a constant $B \in (0,\infty)$ such that $s - c(s,0) \leq B$ for all s, and an argument similar to that of the previous lemma shows

$$\sum_{n=0}^{\tau-1} f(S_n^\ell, \mathcal{K}_n^\ell) \leq \sum_{n=0}^{\tau-1} (B + pY_0 + \cdots + pY_n) =: Z. \tag{5.6}$$

The final sum Z is an integrable random variable. Also, the expression on the left-hand side in (5.6) is continuous in ℓ for almost every ω in the probability space (i.e., for every ω such that $\tau(\omega) < \infty$). By dominated convergence, the expression in (5.5) is also continuous in ℓ, and is bounded from above by $\mathbf{E}Z/\mathbf{E}\tau$.

A simple continuity argument completes the proof. ■

If we assume that expected endowments are strictly positive ($\mathbf{E}Y_k > 0$ for all k), then the legacy ℓ of Theorem 5.5 will also be strictly positive.

Now fix a value of ℓ as in the previous lemma, and write ζ for ζ_ℓ below.

Lemma 5.8 $p = \frac{1}{Q}\int c(s,k)\zeta(d(s,k))$.

Proof Consider the chain $\{(S_n, \mathcal{K}_n)\}$ starting with the stationary distribution ζ. Let A be the event that the agent alive at time $t = 0$ survives to the next day. Then the equation for S_1 can be written

$$S_1 = (S_0 - c(S_0, \mathcal{K}_0) + pY_{\mathcal{K}_0+1})1_A + \ell 1_{A^c}. \tag{5.7}$$

Notice that

$$\mathbf{E}(\ell 1_{A^c}) = \ell \int (1 - \eta_k)\zeta(d(s,k)) = \ell \sum_{k=0}^{\infty}(1 - \eta_k)\nu_k = \ell\nu_0.$$

Take expectations in (5.7) and use the previous lemma, to get

$$\begin{aligned}
\mathbf{E}S_1 &= \int (s - c(s,k) + p\mathbf{E}Y_k)\eta_k\zeta(d(s,k)) \\
&\quad + \int (s - c(s,k) + p\mathbf{E}Y_k)(1 - \eta_k)\zeta(d(s,k)) \\
&= \int (s - c(s,k) + p\mathbf{E}Y_k)\zeta(d(s,k)) \\
&= \mathbf{E}\{S_0 - c(S_0, \mathcal{K}) + pY_{\mathcal{K}}\}.
\end{aligned}$$

By stationarity and Lemma 5.4, $\mathbf{E}S_1 = \mathbf{E}S_0$, a finite number. Hence,

$$p = \frac{\int c(s,k)\zeta(d(s,k))}{\sum_{n=0}^{\infty} \nu_n \mathbf{E}Y_n} = \frac{\int c(s,k)\zeta(d(s,k))}{Q}.$$

■

5.3 The Wealth Process as a Markov Chain: Individual Inheritance

As in the previous section, we consider a Markov chain $\{(S_n, \mathcal{K}_n) : n = 0, 1, ...\}$ corresponding at each time-period n to the wealth and age of an agent. However, rather than (5.2), the dynamics are given now by

$$(S_{n+1}, \mathcal{K}_{n+1}) = \left\{ \begin{array}{ll} (S_n - c(S_n, \mathcal{K}_n) + pY_{\mathcal{K}_n+1}, \mathcal{K}_n + 1), & \text{with probability } \eta_{\mathcal{K}_n} \\ (S_n - c(S_n, \mathcal{K}_n) + pY_{\mathcal{K}_n+1}, 0), & \text{with probability } 1 - \eta_{\mathcal{K}_n} \end{array} \right\}. \tag{5.8}$$

Thus, the age-process $\{\mathcal{K}_n\}$ is the same as before, but now the wealth of an agent who dies is passed directly on to a newborn agent.

Our main object, as in the previous section, is to show that the Markov chain has a stationary distribution with finite mean. However, unlike that of the previous section, the chain of this section need not have a regeneration point. Consequently, we shall need a different argument.

The first step in the argument is to observe that the chain is a (weak) *Feller chain*. That is, for every bounded, continuous function $f : [0, \infty) \times \mathbf{N} \to \mathbf{R}$, the function

$$(Pf)(s,k) := \mathbf{E}[f(S_1, \mathcal{K}_1)|(S_0, \mathcal{K}_0) = (s,k)] \tag{5.9}$$

is also continuous. This follows from (5.8) and the continuity of the function c (cf. Theorem 5.1(c)).

Lemma 5.9 *There exists a stationary distribution* $\tilde{\zeta}$ *for the Markov chain* $\{(S_n, \mathcal{K}_n)\}$.

Proof By Theorem 12.1.2(i) of Meyn & Tweedie (1993, p. 287), it suffices to show that there is a compact set $C \subseteq [0, \infty) \times \{0, 1, ...\}$ such that

$$\frac{1}{n} \cdot \sum_{i=1}^{n} \mathbf{P}[(S_n, \mathcal{K}_n) \in C | (S_0, \mathcal{K}_0) = (s, k)] \tag{5.10}$$

does not converge to 0, as $n \to \infty$, for some initial state (s, k). To see this, let us recall that the number

$$B_1 =: \sup\{s - c(s, 0) : 0 \leq s < \infty\}. \tag{5.11}$$

is finite by Lemma 5.2. Next, choose B_2 finite such that $\mathbf{P}[Y_1 \leq B_2] > 1/2$, where Y_1 has the distribution of the income variable for an agent of age 0. Then, given $\{\mathcal{K}_n = 0\}$, we have

$$S_{n+1} = S_n - c(S_n, 0) + pY_1 \leq B_1 + pB_2$$

with probability at least 1/2. Also, for $n \geq 1$, the probability of the event that $\{\mathcal{K}_n = 0\}$ is at least $1 - \eta^*$, where η^* is the upper bound on the survival probabilities η_k in Assumption 5.2(ii). Let C be the compact set $\{(s, k) : 0 \leq s \leq B_1 + pB_2,\ k = 0$ or $1\}$. Then

$$\mathbf{P}[(S_n, \mathcal{K}_n) \in C | (S_0, \mathcal{K}_0) = (s, k)\} \geq (1 - \eta^*)/2$$

for every $n \geq 1$ and all (s, k). Thus (5.10) is verified. ■

Here is the analogue of Lemma 5.4 for the case of individual inheritance.

Lemma 5.10 *Under a stationary distribution* $\tilde{\zeta}$, *the marginal distribution of wealth has a finite mean.*

Proof Assume that $(S_0, \mathcal{K}_0)$ has the stationary distribution $\tilde{\zeta}$ so that $(S_n, \mathcal{K}_n)$ also has distribution $\tilde{\zeta}$ for every n. We need to show that $\mathbf{E}(S_0) < \infty$.

Condition on $\mathcal{K}_0$, to obtain

$$\begin{aligned} \mathbf{E}(S_0) = \sum_{n=0}^{\infty} \mathbf{E}[S_0 | \mathcal{K}_0 = n] P[\mathcal{K}_0 = n] &= \sum_{n=0}^{\infty} \mathbf{E}[S_0 | \mathcal{K}_0 = n] \nu_n \\ &\leq \frac{1}{\Delta} \sum_{n=0}^{\infty} \mathbf{E}[S_0 | \mathcal{K}_0 = n] (\eta^*)^n, \end{aligned}$$

where $\{\nu_n\}$ is the stationary age distribution of Section 4.1, Δ is the quantity of (4.2), and η^* is from Assumption 5.2(ii).

By stationarity, $\mathbf{E}[S_0 | \mathcal{K}_0 = n] = \mathbf{E}[S_n | \mathcal{K}_n = n]$ for every n. Also, for $n \geq 1$, $[\mathcal{K}_n = n] \subseteq [\mathcal{K}_{n-1} = n - 1]$. Thus, by Lemma 5.2 and (5.11), we have

$$\begin{aligned} \mathbf{E}[S_0 | \mathcal{K}_0 = 1] &= \mathbf{E}[S_1 | \mathcal{K}_1 = 1] \\ &= \mathbf{E}[S_0 - c(S_0, 0) + pY_1 | \mathcal{K}_1 = 1] \ \leq \ B_1 + pm \end{aligned}$$

and, for $n \geq 2$,

$$\begin{aligned}\mathbf{E}[S_0|\mathcal{K}_0 = n] &= \mathbf{E}[S_n|\mathcal{K}_n = n] \\ &= \mathbf{E}[S_{n-1} - c(S_{n-1}, n-1) + pY_n|\mathcal{K}_n = n] \\ &\leq \mathbf{E}[S_{n-1}|\mathcal{K}_{n-1} = n-1] + pm \\ &= \mathbf{E}[S_0|\mathcal{K}_0 = n-1] + pm\end{aligned}$$

where m is from Assumption 5.2(iii). It follows easily that

$$\mathbf{E}[S_0|\mathcal{K}_0 = n] \leq B_1 + pmn, \quad n \geq 1 \tag{5.12}$$

and thus

$$\sum_{n=1}^{\infty} \mathbf{E}[S_0|\mathcal{K}_0 = n](\eta^*)^n < \infty.$$

It remains for us to check that $\mathbf{E}[S_0|\mathcal{K}_0 = 0] < \infty$. By stationarity

$$\begin{aligned}\mathbf{E}[S_0|\mathcal{K}_0 = 0] &= \mathbf{E}[S_1|\mathcal{K}_1 = 0] \\ &= \sum_{n=0}^{\infty} \mathbf{E}[S_1|\mathcal{K}_0 = n,\ \mathcal{K}_1 = 0] \cdot \mathbf{P}[\mathcal{K}_0 = n|\mathcal{K}_1 = 0].\end{aligned} \tag{5.13}$$

Now calculate

$$\begin{aligned}\mathbf{P}[\mathcal{K}_0 = n|\mathcal{K}_1 = 0] &= \frac{\mathbf{P}[\mathcal{K}_1 = 0|\mathcal{K}_0 = n] \cdot \mathbf{P}[\mathcal{K}_0 = n]}{\mathbf{P}[\mathcal{K}_1 = 0]} \\ &= \frac{(1-\eta_n)\nu_n}{\nu_0} \\ &\leq \frac{(\eta^*)^n}{\nu_0 \Delta}\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\mathbf{E}[S_1|\mathcal{K}_0 = n,\ \mathcal{K}_1 = 0] &= \mathbf{E}[S_0 - c(S_0, n) + pY_{n+1}|\mathcal{K}_0 = n,\ \mathcal{K}_1 = 0] \\ &\leq \mathbf{E}[S_0|\mathcal{K}_0 = n] + pm \\ &\leq B_1 + pm(n+1),\end{aligned} \tag{5.15}$$

where the final inequality is by (5.12). It follows from (5.13)–(5.15) that $\mathbf{E}[S_0|\mathcal{K}_0 = 0] < \infty$. ■

The analogue of Lemma 5.6 holds also.

Lemma 5.11 $p = \frac{1}{Q} \int c(s,k) \tilde{\zeta}(d(s,k))$.

Proof Consider the chain $\{(S_n, \mathcal{K}_n)\}$ with its stationary initial distribution $\tilde{\zeta}$. By (5.8), we have

$$S_1 = S_0 - c(S_0, \mathcal{K}_0) + pY_{\mathcal{K}_0}.$$

Take expectations and use the fact that $\mathbf{E}(S_1) = \mathbf{E}(S_0)$ by stationarity. ■

6 Existence of Stationary Markov Equilibrium

Consider again the many-person game of Section 4 under the assumptions of that section, as well as Assumptions 5.1, 5.2, and 5.3. Fix a possible price $p \in (0,\infty)$ and let $\pi = \pi_p$ be the unique optimal stationary plan of Theorem 5.1(b).

First, suppose that the rule of inheritance is egalitarian and let $\zeta(d(s,k))$ be a stationary distribution for the one-person Markov chain $\{(S_n, \mathcal{K}_n)\}$ that balances inheritance and legacy as in Lemma 5.5. For each $k = 0, 1, ...$, let μ_k be the distribution of wealth among agents of age k under ζ. That is,

$$\mu_k(A) = \zeta(A \times \{k\}), \quad A \in \mathcal{B}([0,\infty)).$$

Theorem 6.1 *Any given price $p \in (0,\infty)$, together with the family of strategies $\{\pi^\alpha = \pi_p,\ \alpha \in I\}$ and wealth distributions $\{\mu_k\}$, form a stationary Markov equilibrium for the many-person game with egalitarian inheritance.*

Proof As mentioned in Section 4, we use the construction of Feldman & Gilles (1985) to obtain the endowment variables $Y^\alpha_{k,t}(\omega) = Y_{k,t}(\alpha,\omega)$ so that

$$(\forall\alpha) \quad Y_{k,0}(\alpha,\cdot), Y_{k,1}(\alpha,\cdot), \cdots \text{ are independent with distributions } \lambda_0, \lambda_1, \cdots \text{ and}$$
$$(\forall\omega) \quad Y_{k,0}(\cdot,\omega), Y_{k,1}(\cdot,\omega), \cdots \text{ are also independent with distributions } \lambda_0, \lambda_1, \cdots.$$

Then the wealth–age process $\{(S^\alpha_n(\omega), \mathcal{K}_n)\}$ satisfying (5.2) has the same dynamics for each fixed ω as for each fixed α. (The age process moves independently in either case.) Thus ζ, being a stationary distribution when α is fixed, is also a stationary distribution for fixed ω.

Assume that the initial price p_0 is p and that ζ is the initial distribution of wealth and age. Then, by Lemma 5.6, the price p_1, formed as in (4.7), remains equal to p. Since ζ is an invariant distribution for the chain, and the price remains fixed, ζ will be the distribution of wealth and age in the next period also. The strategies $\pi^\alpha = \pi_p$ are optimal for individual agents since no one agent can affect the price. ■

Now suppose that there is an individual rule of inheritance and let $\tilde{\zeta}(d(s,k))$ be the stationary distribution for the chain of Section 5.3. Define

$$\tilde{\mu}_k(A) = \tilde{\zeta}(A \times \{k\}), \quad A \in \mathcal{B}([0,\infty)),\ k = 0, 1, ...$$

Theorem 6.2 *Any given price $p \in (0,\infty)$ together with $\{\pi^\alpha = \pi_p,\ \alpha \in I\}$ and $\{\tilde{\mu}_k\}$ form a stationary Markov equilibrium for the many-person game with individual inheritance.*

Proof The same as for Theorem 6.1 using (5.8) and Lemma 5.9. ■

7 Some Examples

In [KSS1] (Example 2.5) we considered a simple one-good economy with infinitely-lived homogeneous agents, each with a utility function

$$u(b) = \begin{Bmatrix} b, & b \leq 1 \\ 1, & b > 1 \end{Bmatrix}$$

and an income variable Y in each period given by

$$\mathbf{P}[Y = 2] = \gamma = 1 - \mathbf{P}[Y = 0], \quad 0 < \gamma < 1/2.$$

Utility was discounted in each period by a factor $\beta \in (0, 1)$, and each agent sought to maximize expected total discounted utility

$$\mathbf{E}\left(\sum_{n=0}^{\infty} \beta^n u(b_{n+1})\right),$$

where b_n is the bid in the nth period and the price of the single good is $p = 1$. The optimal bid for an agent with wealth s was shown to be

$$c(s) = \begin{Bmatrix} s, & s \leq 1 \\ 1, & s > 1 \end{Bmatrix}.$$

That is, the agent bids all of his fortune up to 1 and saves any excess over 1. This leads to a stationary wealth distribution μ given by

$$\mu(0) = \delta(1-\gamma),\ \mu(1) = \delta\gamma,\ \mu(s) = \delta\left(\frac{\gamma}{1-\gamma}\right)^{s-1}, \qquad s = 2, 3, ... \tag{7.1}$$

where $\delta = (1 - 2\gamma)/(1 - \gamma)$.

Here we consider four OLG examples in which agents have the same utility function and income distribution as in this example from [KSS1], but with differing assumptions on life-span and inheritance rules.

Example 7.1 (Geometric life-span and individual inheritance) Assume a constant survival probability $\eta_k = \eta \in (0, 1)$ for all $k = 0, 1, ...$ as in Example 4.1 and an *individual inheritance rule* as explained at the end of Section 4 and studied further in Sections 5 and 6. An individual agent then faces the same optimization problem as in [KSS1], with the discount factor β now replaced by the survival probability η. The optimal strategy for an agent of age k with wealth s remains the same, namely

$$c(s, k) = \begin{Bmatrix} s, & s \leq 1 \\ 1, & s > 1 \end{Bmatrix}. \tag{7.2}$$

The wealth process $\{S_n\}$ of (5.7) has the same dynamics as in [KSS1]. Consequently, in equilibrium the distribution of wealth is again that of (7.1). As in Example 4.1 the age-distribution is the geometric distribution

$$\nu(k) = (1 - \eta)\eta^k, \qquad k = 0, 1, ... \tag{7.3}$$

Since age is independent of wealth, the joint stationary distribution for wealth and age ζ is the product of the marginals

$$\begin{aligned}\zeta(s,k) &= \mu(s)\nu(k) \\ &= \left\{\begin{array}{lll}\delta(1-\gamma)(1-\eta)\eta^k & , & s=0 \\ \delta\gamma(1-\eta)\eta^k & , & s=1 \\ \delta\left(\frac{\gamma}{1-\gamma}\right)^{s-1}(1-\eta)\eta^k & , & s=2,3,...\end{array}\right\}.\end{aligned} \tag{7.4}$$

The fraction of total wealth inherited by the newborn in each period is equal to $1-\eta$, the fraction of the population that dies. The conditional distribution of wealth among the newborn is again (7.1).

Example 7.2 (Geometric life-span with egalitarian inheritance) As in the previous example, assume a constant survival probability $\eta_k = \eta \in (0,1)$, but suppose that inheritance is *egalitarian* as explained at the end of Section 4 and in Section 5.2. At every stage, a fraction $1-\eta$ of the population dies (independently of age and wealth) and is replaced by an equal number of newborn agents, each of whom receives an initial endowment of one unit of money. The equilibrium age distribution ν is still geometric as in (7.3). Also the optimization problem remains the same for each agent and the optimal strategy is again given by (7.2). However, wealth is no longer independent of age. Since every newborn has one unit of money and agents follow the optimal strategy (7.2), their wealth s will increase or decrease by at most one unit in each period when $s \geq 1$. Indeed, it is easy to see that the equilibrium distribution of wealth for agents of age k is concentrated on the set $\{0,1,...,k+1\}$. The equilibrium distribution of wealth μ for the collection of all agents satisfies

$$\begin{aligned}\mu(0) &= (1-\gamma)\eta\mu(0)+(1-\gamma)\eta\mu(1), \quad \mu(1)=(1-\gamma)\eta\mu(2)+(1-\eta), \\ \mu(2) &= \gamma\eta(\mu(0)+\mu(1))+(1-\gamma)\eta\mu(3), \\ \mu(n) &= \gamma\eta\mu(n+1)+(1-\gamma)\eta\mu(n+1), \quad \text{for} \ \ n\geq 3.\end{aligned} \tag{7.5}$$

Notice that all of the newborn are accounted for in the expression for $\mu(1)$. The solution of (7.5) takes the form

$$\mu(n) = \alpha\theta^{n-1} \quad \text{for} \qquad n \geq 3 \tag{7.6}$$

where

$$\theta = \frac{1-\sqrt{1-4\gamma(1-\gamma)\eta^2}}{2(1-\gamma)\eta}$$

and the constant α is determined by (7.5) and the condition $\sum\mu(n) = 1$. It is easy to check that $\theta < \gamma/(1-\gamma)$. Thus, as expected, the wealth distribution (7.6) for egalitarian inheritance has a smaller tail than the wealth distribution (7.1) for individual inheritance. In other words, the proportion of very wealthy agents is smaller than in (7.1). However, if we let $\eta \to 1$ or, equivalently, expected life-span approach infinity, then the wealth distribution (7.6) approaches that of (7.1).

Example 7.3 (Nonstochastic life-span with individual inheritance) Assume that every agent lives for exactly K periods and then dies (as in Example 4.2), and that an *individual inheritance rule* is used. Although a newborn agent now faces a dynamic programming problem with a finite horizon, it is straightforward to show that the same strategy of (7.2) remains optimal. (There are other optimal strategies. For example, an agent in the last period of his life could spend all of his wealth on consumption and leave nothing to his replacement among the newborn. Indeed, this would be uniquely optimal if the utility function u did not saturate but were strictly increasing on $[0, \infty)$. In the present case, an agent earns no further utility after spending one unit and we will assume each agent follows (7.2).) The dynamics of the wealth process are the same under (7.2) as in Example 2.5 of [KSS1] and the stationary distribution of wealth μ is given by (7.1) as it was in Example 7.1. The age-distribution is uniform on $\{0, 1, ..., K\}$ and given by $\nu_k = (K+1)^{-1}$ as in Example 4.2. Furthermore, age and wealth are independent in equilibrium with joint distribution

$$\zeta(s,k) = (K+1)^{-1}\mu(s)$$

for $s = 0, 1, ...;\ k = 0, 1, ..., K$.

Example 7.4 (Nonstochastic life-span with egalitarian inheritance) As in the previous example each agent is assumed to have life-span K, but inheritance is now taken to be egalitarian. The strategy of (7.2) is optimal and we assume that every agent follows it. It appears that there is no simple formula for the equilibrium wealth distribution as a function of the parameters γ and K. However, we know that every newborn agent has the same wealth, say ℓ, and an agent's wealth can grow to at most $\ell + K$ during the course of his lifetime. Hence, wealth is uniformly bounded in equilibrium. Here is the joint equilibrium distribution ζ for wealth and age in the very special case when $\gamma = 1/4$ and $K = 1$ (two generations): Half of the population is newborn and all of these have wealth 3/4 so that

$$\zeta(3/4, 0) = 1/2;$$

of which 3/4 have wealth 0 and 1/4 have wealth 2, so that

$$\zeta(0,1) = 3/8, \quad \zeta(2,1) = 1/8.$$

It can be shown that if we let the life-span K tend to infinity in this example, then the marginal distribution of wealth in equilibrium converges to the distribution μ of (7.1). In fact, the wealth distribution for all four examples is equal or close to (7.1) for moderate values of γ and expected life-span equal to 80 or so.

8 The Life-Cycle, Loans and Insurance

We have confined our remarks in this note to the specification of the birth and death processes and bequests and inheritances. These are bare minimal requirements in order to define fully a process model and be able to demonstrate existence of a stationary equilibrium. An immediate extension of direct economic interest is to consider aspects of the individual life-cycle, including the flow of resources to the pre-productive young and to the post-productive old. Consideration of the life cycle calls for multiperiod borrowing and lending, and creates a need for money and credit independent of transactions needs. Insurance and pension schemes emerge as potentially efficient ways to handle the uncertainties and gaps in timing between earning and expenditures. These additional factors are topics for future research.

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