COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1161

AN ADF COEFFICIENT TEST FOR A UNIT ROOT IN ARMA MODELS OF UNKNOWN ORDER WITH EMPIRICAL APPLICATIONS TO THE U.S. ECONOMY

Zhijie Xiao and Peter C. B. Phillips

September 1997

An ADF Coefficient Test for A Unit Root in ARMA Models of Unknown Order with Empirical Applications to the U.S. Economy*

Zhijie Xiao
Department of Economics
Yale University

and

Peter C.B. Phillips
Cowles Foundation for Research in Economics
Yale University

July 25, 1997

Abstract

This paper proposes an ADF coefficient test for detecting the presence of a unit root in ARMA models of unknown order. Our approach is fully parametric. When the time series has an unknown deterministic trend, we propose a modified version of the ADF coefficient test based on quasi-differencing in the construction of the detrending regression as in Elliot, Rothenberg and Stock (1996). The limit distributions of these test statistics are derived. Empirical applications of these tests for common macroeconomic time series in the US economy are reported and compared with the usual ADF t-test.

*This paper was typed by the authors in Scientific Word 2.5 and computations were performed in Gauss. The first author thanks the Cowles Foundation for support under a C. A. Anderson Fellowship and the second author thanks the NSF for research support under Grant No. SBR 94–22922.

1 Introduction

Tests for a unit root have attracted a considerable amount of work in the last ten years. One important reason is that these tests can help to evaluate the nature of the nonstationarity that many macroeconomic data exhibit. In particular, they help in determining whether the trend is stochastic, deterministic or a combination of both. Following Nelson and Plosser (1982), much empirical research has been done and evidence has accumulated that many macroeconomic variables have structures with a unit root. The literature on testing for a unit root is immense. The most commonly used tests for a unit root are the Dickey–Fuller test and the Phillips Z-tests. The Dickey–Fuller test (1979) is based on the regression of the observed variable on its one-period lagged value, sometimes including an intercept and time trend. In an important extension of Dickey and Fuller (1979), Said and Dickey (1984) show that the Dickey–Fuller t-test for a unit root, which was originally developed for AR representations of known order, remains asymptotically valid for a general ARMA process of unknown order. This t-test is usually called the Augmented Dickey–Fuller (ADF) test. An alternative semiparametric approach to detecting the presence of a unit root in general time series setting was proposed by Phillips (1987) and extended in Phillips and Perron (1988). These tests are known as Phillips Z_α and Z_t tests. The Z-tests allow for a wide class of time series with heterogeneously and serially correlated errors.

The ADF test is a t-test in a long autoregression. Said and Dickey (1984) prove the validity of this test in general time series models provided the lag length in the autoregression increases with the sample size at a rate less than $n^{1/3}$, where $n =$ sample size. No such extension of the Dickey–Fuller coefficient test is recommended in their work, since even as the lag length goes to infinity, the coefficient estimate has a limit distribution that is dependent on nuisance parameters. However, the Z_α test is a coefficient based test with a nonparametric correction which successfully eliminates nuisance parameters. A similar idea can be applied to construct an ADF coefficient based test. In particular, the nuisance parameters can be consistently estimated and the coefficient estimate transformed to eliminate the nuisance parameters asymptotically, providing an ADF coefficient test with the same limit distribution as the original Dickey–Fuller coefficient test and the Z_α test.

A natural way to compare tests is to examine their power in large samples. Monte Carlo studies (see Phillips and Perron, 1988; Schwert, 1989; DeJong, *et al.* 1988, 1992) indicate that unit root tests often have low power against plausible trend stationary alternatives. Generalized least square detrending using quasi-differenced (QD) data was suggested in Elliot, Rothenberg and Stock (1996) to increase the power of unit root tests for models with deterministic trends. An analysis of the efficiency gain from this detrending procedure (which we call QD detrending) and its effects on test efficiency is given in Phillips and Lee (1996). As yet, few empirical applications of QD detrended unit root tests have appeared in the literature.

This paper develops an ADF-type coefficient based unit root test (called ADF_α) for ARMA models of unknown order, with a parametric correction that frees the limit distribution of the test statistic of nuisance parameters. A modified ADF coefficient test based on QD detrending is also developed. The limit distributions of the ADF coefficient test and its QD detrended version are the same as those of the Z_α test and QD detrended Z_α test. Empirical applications of these tests to the post war quarterly U.S. data, the extended Nelson–Plosser data, and stock price data are also reported. We compare the OLS detrended ADF_α test with the QD detrended ADF_α test, and examine the QD detrended ADF_α tests for different choices of c (the quasi-differencing parameter).

The outline of the paper is as follows. Section 2 develops the theory for the ADF coefficient test. The QD detrended ADF coefficient test and its limit theory are given in Section 3. Section 4 reports some empirical applications to a variety of macroeconomic and financial data. Proofs of theorems are given in Section 5. Concerning notation, we use the symbol "$\Rightarrow$" to signify weak convergence of the associated probability measures. Continuous stochastic process such as the Brownian motion $B(r)$ on [0,1] are usually written simply as B and integrals $\int$ are understood to be taken over the interval [0,1].

2 An ADF Coefficient Test for a Unit Root

Consider a time series

$$y_t = \alpha y_{t-1} + u_t \tag{1}$$

satisfying the following conditions:

Assumption A1: *y_t is initialized at $t = 0$ by y_0, an $O_p(1)$ random variable with finite variance.*

Assumption A2: *u_t satisfies the stationary and invertible* ARMA(p, q) *process, $a(L)u_t = b(L)\varepsilon_t$, where $\varepsilon_t = \mathrm{iid}(0, \sigma^2)$, $a(L) = \sum_{j=0}^{p} a_j L^j$, $b(L) = \sum_{j=0}^{q} b_j L^j$, and L is the lag operator.*

Assumption A3: *$n^{-1/2}\sum_{t=1}^{[nr]} u_t \Rightarrow B(r) = BM(\omega^2) = \omega W(r)$, $n^{-1/2}\sum_{t=1}^{[nr]} \varepsilon_t \Rightarrow B_\varepsilon(r) = BM(\sigma^2) = \sigma W(r)$, where $\omega^2 = E(u_1^2)+2\sum_{k=2}^{\infty} E(u_1 u_k) = \sigma^2[b(1)/a(1)]^2 =$ long run variance of u_t, and $W(r)$ is a standard Brownian motion.*

From A2, we get the AR representation of u_t (e.g., Fuller, 1976, Theorem 2.7.2)

$$\varepsilon_t = d(L)u_t = \sum_{j=0}^{\infty} d_j u_{t-j},\ d_0 = 1.$$

Define $\beta(L) = 1 - d(L) = \beta_1 L + \beta_2 L^2 + \cdots$, and then

$$\Delta y_t = a y_{t-1} + \beta_1 u_{t-1} + \beta_2 u_{t-2} + \cdots + \varepsilon_t.$$

The null hypothesis of interest in (1) is $H_0 : \alpha = 1$, or equivalently $H_0 : a = \alpha - 1 = 0$. Under H_0, $u_t = \Delta y_t$, and applying the operator $d(L) = 1 - \beta(L)$ to both sides of this equation we have the null model

$$\Delta y_t = \beta(L)\Delta y_t + \varepsilon_t,$$

or

$$\Delta y_t = a y_{t-1} + \beta_1 \Delta y_{t-1} + \beta_2 \Delta y_{t-2} + \cdots + \varepsilon_t. \tag{2}$$

In place of the infinite AR regression (2), we consider the ADF regression model

$$\Delta y_t = a y_{t-1} + \beta_1 \Delta y_{t-1} + \cdots + \beta_k \Delta y_{t-k} + e_{tk}, \tag{3}$$

where e_{tk} is defined as $\Delta y_t - a y_{t-1} - \beta_1 \Delta y_{t-1} - \cdots - \beta_k \Delta y_{t-k}$. We use Z to denote the matrix of explanatory variables and partition it in the following way: $Z = (y_{-1}, Z_k)$, where y_{-1} is the vector of lagged variables, and Z_k is the matrix of observations of the k lagged difference variables $(\Delta y_{t-1}, ..., \Delta y_{t-k})$. Thus we have the following matrix representation

$$\Delta y = Z\beta + e_k,$$

where $\beta = (a, \beta_1, ..., \beta_k)'$, $e_k = (..., e_{tk}, ...)'$.

We shall be concerned with the limit behavior of the conventional least square regression coefficient $\widehat{a}$ for a in (3) given by

$$\widehat{a} = (y_{-1}' P_z y_{-1})^{-1} y_{-1}' P_z \Delta y,$$

where $P_z = I - Z_k (Z_k' Z_k)^{-1} Z_k'$. The limit distribution of $\widehat{a}$ is given by the following theorem.

Theorem 1: *If $k \to \infty$ as $n \to \infty$, and $k = o(n^{1/3})$, then, under H_0,*

$$n\widehat{a} \Rightarrow \frac{\sigma \int W dW}{\omega \int W^2}.$$

Remark 1: The limit distribution of the regression coefficient $\widehat{a}$ depends on unknown scale parameters ω and σ, and thus the statistic $n\widehat{a}$ can not be used directly for unit root testing. However, ω and σ can be consistently estimated, and there exists a simple transformation of the statistic $n\widehat{a}$ which eliminates the nuisance parameters asymptotically. In particular, $\widehat{\sigma}^2 = \sum \widehat{e}_{tk}^2 / n$ is a consistent estimator of σ^2 , and ω^2 can be consistently estimated by the AR estimator (Berk, 1974) $\widehat{\omega}^2 = \widehat{\sigma}^2 / (1 - \sum \widehat{\beta}_j)^2$. Thus, we define

$$ADF_\alpha = (\widehat{\omega}/\widehat{\sigma}) n\widehat{a}.$$

Under the null hypothesis that $\alpha = 1$, it is apparent that the modified coefficient test statistic

$$ADF_\alpha \Longrightarrow \frac{\int W dW}{\int W^2},$$

the same limit distribution as that of the Phillips Z_α test and that of the original Dickey–Fuller coefficient test.

It is necessary that a statistical test be able to discriminate between the null and alternative in large samples. The next theorem guarantees this property.

Theorem 2: *If y_t is generated by* (1) *with $\alpha < 1$, and $f_{yy}(0) > 0$, where $f_{yy}(\cdot)$ is the spectral density of y_t, then*

$$ADF_\alpha = O_p(n).$$

Remark 2: As the sample size n increases, the test statistic ADF_α diverges faster under H_1 than does the ADF t-ratio statistic. This suggests that the coefficient based statistic is likely to have higher power than the t-ratio statistic in large samples.

3 An Efficiently Detrended ADF Coefficient Test

If we allow for a deterministic trend in the time series y_t , we have the following representation

$$y_t = \gamma' x_t + y_t^s \tag{4}$$
$$y_t^s = \alpha y_{t-1}^s + u_t \tag{5}$$

where x_t is the deterministic trend, and u_t is defined as in A2. The traditional way of removing this deterministic component in unit root tests is to run an OLS regression on an augmented equation. In the present case, this is

$$\Delta y_t = \gamma' x_t + a y_{t-1} + \beta_1 \Delta y_{t-1} + \cdots + \beta_k \Delta y_{t-k} + e_{tk},$$

and we can construct the test statistic ADF_α based on the above regression. In most cases of interest, there exists a scaling matrix D_n and a limit trend function $X(r)$ for which $D_n x_{[nr]} \to X(r)$, in which case the limit distribution for ADF_α is given by the ratio $\int W_X dW / \int W_X^2$, if k increases with n at a rate of $o(n^{1/3})$, where W_X is the detrended Brownian motion

$$W_X(r) = W(r) - \left[\int WX'\right]\left[\int XX'\right]^{-1} X(r),$$

and depends on the limit trend function $X(r)$.

However, the power of unit root tests in the case of deterministic trends can be improved if we perform the detrending regression in a way that is efficient under the alternative (Elliot, *et al.* 1996). For alternatives that are distant from unit root, the Grenander–Rosenblatt theorem implies that OLS detrending as in (4) will be asymptotically efficient, so the gain from efficient detrending occur for local alternatives of the form

$$H_1' : \alpha = 1 + c/n,$$

where c is a fixed constant in such cases. We can estimate the trend coefficient by taking quasi-difference on (4), and running a least square regression of

$$\Delta_c y_t = \gamma' \Delta_c x_t + \Delta_c y_t^s,$$

where Δ_c is the quasi-difference operator $1-L-(c/n)L$. If the fitted trend parameter vector is $\widetilde{\gamma}$, we compute the QD detrended series

$$y_t^* = y_t - \widetilde{\gamma}' x_t,$$

which can now be used in the construction of unit root test just as in the case where there are no deterministic trends to eliminate.

This detrending procedure is sometimes called GLS detrending in the literature (e.g., Elliot, Rothenberg and Stock, 1996). It is perhaps more accurate to describe the procedure as detrending after quasi-differencing (see Phillips and Lee, 1996, and Canjels and Watson, 1997, for recent implementations) since full GLS is not used in the detrending regression, but only quasi-differencing. We therefore refer to the procedure as QD detrending.

To derive the asymptotics for the efficiently detrended ADF_α test, it is convenient to employ the following matrix notation,

$$\begin{aligned}
X' &= (x_1, ..., x_t, ..., x_n), \\
y' &= (y_1, ..., y_t, ..., y_n), \\
\Delta_c X' &= (\Delta_c x_1, ..., \Delta_c x_t, ..., \Delta_c x_n), \\
\Delta_c y' &= (\Delta_c y_1, ..., \Delta_c y_t, ..., \Delta_c y_n).
\end{aligned}$$

Then $\widetilde{\gamma} = (\Delta_c X' \Delta_c X)^{-1} \Delta_c X' \Delta_c y$. Let $A = I - (1 + c/n)H$, where

$$H = \begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \end{bmatrix}.$$

Then $\Delta_c y = Ay$, and $\Delta_c X = AX$. In matrix form, we have

$$y^* = y - X\widetilde{\gamma} = y - X(\Delta_c X' \Delta_c X)^{-1} \Delta_c X' \Delta_c y = [I - X(\Delta_c X' \Delta_c X)^{-1} \Delta_c X' A] y = Q_c y$$

where $Q_c = I - X(\Delta_c X' \Delta_c X)^{-1} \Delta_c X' A$. Since $y = X\gamma + y^s$ and

$$Q_c X = [I - X(\Delta_c X' \Delta_c X)^{-1} \Delta_c X' A] X = 0$$

thus $y^* = Q_c y = Q_c y^s = y^{s*}$.

Assume that the scaling matrix D_n is such that $D_n x_{[nr]} \to X(r)$, and $nD_n \Delta x_{[nr]} \to g(r)$. Let $X_c(r) = g(r) - cX(r)$. Then we have the following asymptotic result for the QD detrended series y_t^*.

Lemma 1: *Under the null* $\alpha = 1$,

$$n^{-1/2} y^*_{[nr]} = n^{-1/2} y^{s*}_{[nr]} \Rightarrow \underline{B}_c(r) = \omega \underline{W}_c(r)$$

where

$$\begin{aligned}\underline{B}_c(r) &= B(r) - X(r)' \left[\int X_c(r) X_c(r)' dr\right]^{-1} \left[\int X_c(r) dB(r) - c \int X_c(r) B(r) dr\right] \\ &= \omega \left\{ W(r) - X(r)' \left[\int X_c(r) X_c(r)' dr\right]^{-1} \left[\int X_c(r) dW(r) - c \int X_c(r) W(r) dr\right] \right\} \\ &= \omega \underline{W}_c(r)\end{aligned}$$

For example, in the case where x_t is the polynomial time trend $(t, ..., t^p)'$, $X(r) = (r, ..., r^p)'$, and $g(r) = (1, 2r, ..., pr^{p-1})'$.

The detrended data y_t^* can be used to construct an ADF_α test for a unit root by running the following regression

$$\Delta y_t^* = a y_{t-1}^* + \beta_1 \Delta y_{t-1}^* + \cdots + \beta_k \Delta y_{t-k}^* + e_{tk}^* \qquad (6)$$

Let $\tilde{a}$ be the estimated coefficient of a in this regression. Then the QD detrended ADF_α statistic is

$$ADF_\alpha^* = (\widehat{\omega}/\widehat{\sigma}) n \tilde{a}.$$

Theorem 3: *Under the null of a unit root, if $k \to \infty$, as $n \to \infty$ and $k = o(n^{1/3})$, then*

$$ADF_\alpha^* \Rightarrow \frac{\int \underline{W}_c(r) dW(r)}{\int \underline{W}_c(r)^2}.$$

Remark 3: The limit distribution of the modified ADF_α test depends on both the trend function and the value of c that is used in the quasi-differencing filter. This limit distribution has the same form as that of a modified semiparametric Z_α test when we use the efficiently detrended y in the construction of Z_α .

Remark 4: We can construct a modified ADF t-ratio test in exactly the same way and the limit distribution for this modified ADF_t statistic is

$$\left[\int \underline{W}_c(r)^2\right]^{-1/2} \int \underline{W}_c(r) dW(r).$$

4 Empirical Applications

4.1 The Extended Nelson–Plosser Data

The ADF_α test and efficient detrending QD prefilter were applied to the fourteen time series of the U.S. economy studied in Nelson and Plosser (1982), and extended by Schotman–Van Dijk (1991). The starting dates for the series vary from 1860 for industrial production and consumer prices through to 1909 for GNP. All series terminate in 1970 in the original Nelson–Plosser data. Schotman and Van Dijk extended all these 14 series to 1988. In their original study, Nelson and Plosser conducted the ADF_t test on these series and could not reject the unit root hypothesis at the 5% level of significance for all of the series except the unemployment rates. Perron (1988) arrived at similar conclusions using Z-tests.

We consider the null hypothesis that the variables are difference stationary ARMA processes versus the trend stationary alternatives. We use three detrending procedures for the ADF_α test:

(T1): OLS detrending

(T2): QD detrending with the choice $c = -10$

(T3): QD detrending with the choice $c = -13.5$

Thus, in the first test, we estimate the following ADF regression

$$\Delta y_t = ay_{t-1} + \beta_1 \Delta y_{t-1} + \cdots + \beta_k \Delta y_{t-k} + \gamma_0 + \gamma_1 t + e_t$$

In the second and third tests, we run ADF regression (6) for the QD detrended data y_t^*. The value $c = -10$ was chosen because the sample sizes of the Nelson–Plosser series are around 100 (80–129) and estimates of autoregressive coefficients in economic time series are often around 0.9, corresponding to $1 + c/n$ for $n = 100$, $c = -10$. Also the c value for which local asymptotic power is 50% is approximately -13.5 for the case of a linear trend (Elliot *et al.*, 1996), so this value of c is another natural choice. To provide a basis for comparison, we also calculated the ADF_t statistics based on these three detrending procedures. We use the BIC criterion of Schwarz (1978) and Rissanen (1978) in selecting the appropriate lag length of the autoregression for all three data sets considered in this paper. The critical values of the ADF_α and ADF_t tests corresponding to different choices of c values were calculated from simulations based on 15,000 replications. Table 1 provides the finite sample critical values in the case of $N = 100$.

Table 2 reports the values of the ADF tests based on OLS detrending. Table 3 and Table 4 give their values under QD detrending for $c = -10$ and $c = -13.5$. The estimated autoregressive coefficients are reported in the columns labelled "$\widehat{\alpha}$". We are interested in testing whether or not the AR coefficient differs from unity. For most of the time series, we can not reject the null of unit root at the 5% level of significance. A few series exhibit values of ADF_α below the 5% level critical values. In particular, the unit root hypothesis is rejected for the unemployment series by all these tests (i.e., all three detrending procedures). For two series, per capita GNP and industrial production, unit roots were rejected in the OLS detrended test, but not rejected in the QD detrending procedure. However, the calculated test statistics are very close to the corresponding critical values. Thus the evidence is marginal for these two series. The ADF_t test gives qualitatively the same results. In conclusion, our results in Tables 2, 3, 4 are generally in accord with the findings in Nelson and Plosser (1982).

Table 1: 5% Level Critical Values ($N = 100$)

	ADF_α test	ADF_t test
$c = -2.5$	–15.79	–2.81
$c = -5$	–17.15	–2.91
$c = -7.5$	–18.05	–2.97
$c = -10$	–18.71	–3.02
$c = -12.5$	–19.25	–3.07
$c = -13.5$	–19.47	–3.09
$c = -15$	–19.91	–3.11
OLS detrending	–20.7	–3.45

Table 2: ADF_α and ADF_t Tests with a Linear Trend (OLS detrending)

Series	$\hat{\alpha}$	ADF_α	ADF_t	Series	ADF_α	ADF_t	$\hat{\alpha}$
CPI	9.986	–5.23	–1.4	Employment	0.854	–19.38	–3.28
GNP Def.	0.967	–6.44	–1.63	GNP/Cap.	0.81	–24.12*	–3.59*
Ind. Prod.	0.818	–25.8*	–3.68*	Interest rate	0.94	–6.01	–1.69
Money	0.936	–18.5	–2.89	Real GNP	0.812	–19.68	–3.05
Nom. GNP	0.938	–8.87	–2.03	Real wage	0.927	–8.49	–1.73
Stock price	0.916	–12.4	–2.42	Unemployment	0.772	–43.55*	–3.94*
Velocity	0.964	–4.62	–1.44	Nominal wage	0.933	–11.56	–2.43

*Values are smaller than the 5% level critical values.

Table 3: ADF_α and ADF_t tests with a linear trend
(QD detrending, $c = -10$)

Series	$\hat{\alpha}$	ADF_α	ADF_t	Series	$\hat{\alpha}$	ADF_α	ADF_t
CPI	0.99	–3.21	–1.04	Employment	0.88	–15.5	–2.76
GNP Def.	0.98	–3.62	–1.13	GNP/Cap.	0.86	–16.74	–2.88
Ind. Prod.	0.87	–17.4	–2.92	Interest Rate	0.95	–5.58	–1.61
Money	0.94	–17.5	–2.87	Real GNP	0.87	–16.85	–2.9
Nom. GNP	0.95	–7.56	–1.85	Real Wage	0.94	–6.96	–1.73
Stock Price	0.95	–6.89	–1.71	Unemployment	0.77	–43.6*	–3.96*
Velocity	0.98	–2.43	–0.93	Nominal wage	0.95	–8.89	–2.04

*Values are smaller than the 5% level critical values.

Table 4: ADF_α and ADF_t Tests with a Linear Trend
(QD detrending, $c = -13.5$)

Series	$\hat{\alpha}$	ADF_α	ADF_t	Series	$\hat{\alpha}$	ADF_α	ADF_t
CPI	0.99	–3.52	–1.07	Employment	0.88	–16.5	–2.86
GNP Def.	0.978	–4.11	–1.19	GNP/Cap.	0.85	–18.6	–3.05
Ind. Prod.	0.863	–19.1	–3.05	Interest Rate	0.94	–5.7	–1.63
Money	0.937	–17.9	–2.89	Real GNP	0.85	–18.5	–3.04
Nom. GNP	0.944	–7.99	–1.89	Real Wage	0.94	–7.56	–1.74
Stock Price	0.946	–7.86	–1.81	UnemployM	0.77	–43.7*	–3.95*
Velocity	0.98	–2.67	–0.97	Nom. Wage	0.94	–9.6	–2.13

*Values are smaller than the 5% level critical values.

4.2 Stock Price Data

We examined the monthly average stock price data from Standard and Poor's series. DeJong *et al.* (1988) conducted various unit root tests on the annual stock price data and they could not reject the unit root hypothesis at 5% level for most of the series. DeJong and Whiteman (1989) revisited the same data set using a flat prior Bayesian analysis and found that trend stationarity is supported by the data. The series we studied here include the S&P 500 composite stock prices, consumer goods stock prices, capital goods stock prices, industrial stock prices, and automobiles stock prices. The consumer goods stock price is calculated from an average of 164 stocks, capital goods price from 101 stocks, industrials stock price from 381 stocks, and automobiles stock price from 3 stocks. We examined the data from January 1980 to December 1988. Each series contains 108 observations. We use the three tests (T1, T2, T3) in Section 4.1 and so the critical values in Table 1 can be used in this section. Our empirical analysis of these stock price series is summarized in Tables 5, 6, and 7.

The OLS detrended ADF_α test can not reject the unit root hypothesis at the 5% level for all of these series except the automobiles stock price. When we examine these series by the QD detrended ADF_α tests, there is no evidence to reject the hypothesis of a unit root at the 5% level, in both $c = -10$ and $c = -13.5$ cases. The ADF_t tests give the same results. Our conclusion from the evidence presented in this section is that the composite stock price series are nonstationary with a unit root. The automobile price series outcome is marginally in favor of rejecting the null hypothesis of a unit root.

Table 5: Tests for the Stock Price Data (OLS detrending)

Series	Estimated AR Coefficient	ADF_α	ADF_t
Composite	0.907	−17.33	−2.95
Capital Goods	0.895	−17.60	−2.94
Consumer Goods	0.903	−17.95	−3.02
Industrials	0.908	−18.05	−3.02
Automobiles	0.846	−30.33*	−3.97*

*Values are smaller than the 5% level critical values.

Table 6: Tests for the Stock Price Data (QD detrending, $c = -10$)

Series	Estimated AR Coefficient	ADF_α	ADF_t
Composite	0.926	−13.75	−2.64
Capital Goods	0.914	−14.27	−2.69
Consumer Goods	0.925	−13.71	−2.61
Industrials	0.927	−14.16	−2.67
Automobiles	0.913	−16.45	−2.81

Table 7: Tests for the Stock Price Data (QD detrending, $c = -13.5$)

Series	Estimated AR Coefficient	ADF_α	ADF_t
Composite	0.922	−14.54	−2.71
Capital Goods	0.910	−15.00	−2.75
Consumer Goods	0.921	−14.65	−2.69
Industrials	0.922	−15.03	−2.74
Automobiles	0.901	−18.97	−3.02

Table 8: 5% Level Finite Sample Critical Values
($N = 200$)

	ADF$_\alpha$ test	ADF$_t$ test
$c = -10$	–17.00	–2.88
$c = -13.5$	–17.60	–2.92
$c = -20$	–18.43	–2.99
$c = -25$	–19.03	–3.05
OLS detrending	–21.20	–3.44

Table 9: OLS detrended Tests on Post War Quarterly U.S. Data

Series	Estimated AR Coefficient	ADF_α	ADF_t
Real GDP	0.97	-8.5	–1.94
Real Investment	0.928	–37.28*	–3.84*
Real Consumption	0.938	–14.77	–3.07
Employment	0.95	–18.58	–3.114

*Values are smaller than the 5% level critical values.

Table 10: QD detrended Tests on Post War Quarterly U.S. Data, $c = -10$

Series	Estimated AR Coefficient	ADF_α	ADF_t
Real GDP	0.98	–3.88	–1.17
Real Investment	0.969	–14.67	–2.339
Real Consumption	0.98	–4.336	–1.4
Employment	0.979	–10.11	–2.199

Table 11: QD detrended Tests on Post War Quarterly U.S. Data, $c = -13.5$

Series	Estimated AR Coefficient	ADF_α	ADF_t
Real GDP	0.98	–4.43	–1.23
Real Investment	0.96	–17.93	–2.55
Real Consumption	0.977	–5.277	–1.55
Employment	0.976	–11.6	–2.36

Table 12: QD detrended Tests on Post War Quarterly U.S. Data, $c = -20$

Series	Estimated AR Coefficient	ADF_α	ADF_t
Real GDP	0.98	-5.1	-1.33
Real Investment	0.956	-21.58*	-2.79
Real Consumption	0.972	-6.528	-1.758
Employment	0.973	-13.26	-2.53

*Values are smaller than the 5% level critical values.

Table 13: QD detrended Tests on Post War Quarterly U.S. Data, $c = -25$

Series	Estimated AR Coefficient	ADF_α	ADF_t
Real GDP	0.98	-5.79	-1.44
Real Investment	0.969	-14.67	-2.34
Real Consumption	0.966	-7.926	-1.98
Employment	0.969	-14.8	-2.7

4.3 Post War Quarterly U.S. Data

In this section, we analyzed some post-war quarterly U.S. macroeconomic time series data. The data set consists of Real GDP, Real Investment, Real Consumption, and Employment. All these variables are from Citibase, over the period 1947:1–1993:4. The number of observations for these time series is 188. Table 8 gives the finite sample critical values for the case of $N = 200$. These critical values are calculated from simulation based on 15,000 replications. We tried the following detrending procedures for both ADF_α and ADF_t tests:

(T1): OLS detrending

(T2): QD detrending with the choice $c = -10$

(T3): QD detrending with the choice $c = -13.5$

(T4): QD detrending with the choice $c = -20$

(T5): QD detrending with the choice $c = -25$

Tables 9, 10, 11, 12, and 13 give the estimated test statistics and coefficients for these five detrending procedures. We can not reject the null of a unit root in all these tests at the 5% level of significance for the consumption series, which, as argued in Hall (1978), should behave as a martingale. Thus, there is no evidence to reject the hypothesis that consumption behaves as a unit root process. We also find support for the hypothesis of a unit root in the series of real GDP, and employment in all these tests. For the series of real investment, the unit root hypothesis is rejected in the OLS detrended ADF_α and ADF_t tests. In QD detrending cases, when we choose $c = -20$, the unit root is rejected in the series of real investment by the ADF_α test, but not by ADF_t test. For the values $c = -10, -13.5, -25$, we can not reject a unit root in any series. These results are generally in agreement with the conclusion of the extended Nelson–Plosser data that many macroeconomic time series are characterized by the presence of a unit root.

5 Proofs

5.1 Proof of Theorem 1

The limit distribution of $\widehat{a}$ can be established in the following steps by using the BN (Beveridge and Nelson, 1981) decomposition for the operators $a(L)$ and $b(L)$. Following the lines of Berk (1974), we use the standard Euclidean norm, $\|x\| = (x'x)^{1/2}$, of a column vector and use the matrix norm $\| B \| = \sup\{\|Bx\| : \|x\| < 1\}$.

Let $G_n = \text{diag}(n^{-1}, n^{-1/2}, ..., n^{-1/2})$, then

$$G_n^{-1}(\widehat{\beta} - \beta) = (G_n Z' Z G_n)^{-1} G_n Z' e_k.$$

If $k = o(n^{1/3})$ and k goes to ∞ with n, then, under the null hypothesis, we have:

(a) $k^{1/2}\|G_n Z' Z G_n - R_n\| \xrightarrow{p} 0$ and $k^{1/2}\|(G_n Z' Z G_n)^{-1} - R_n^{-1}\| \xrightarrow{p} 0$, as $n, k \to \infty$ (Said and Dickey, 1984), where

$$\begin{aligned} R_n &= \text{diag}[n^{-2}(b(1)/a(1))^2 \sum S_{t-1}^2, \Gamma], \\ S_{t-1} &= \sum_{j=1}^{t-1} \varepsilon_j, \\ \Gamma &= [\gamma_{ij}],\ \gamma_{ij} = \gamma(i-j) = E(u_i u_j); \end{aligned}$$

(b) $\|G_n Z' e_k - G_n Z' \varepsilon\| = O_p(1/n), \varepsilon = (\varepsilon_1, ..., \varepsilon_n)'$ (Said and Dickey, 1984);

(c) $\|\widehat{\beta} - \beta\|$ converges in probability to 0 (Said and Dickey, 1984).

Under H_0, we have

$$a(L)y_t = a(L)\sum_{j=1}^{t} u_j + O_p(1) = b(L)\sum_{j=1}^{t} \varepsilon_j + O_p(1)$$

Use the BN decomposition again, giving

$$a(1)y_t = b(1)\sum_{j=1}^{t} \varepsilon_j + O_p(1) \tag{7}$$

the term $O_p(1)$ includes linear combinations of finite numbers of u_t and ε_t. Since $\sum_1^n \varepsilon_j$ is the $I(1)$ component in (7), we get

$$y_t = \frac{b(1)}{a(1)} S_t + O_p(1)$$

From (a), (b) and (c), the limit distribution of $G_n^{-1}(\widehat{\beta} - \beta)$ is the same as that of $R_n^{-1} G_n Z' \varepsilon$. Thus the limit of $n\widehat{a}$ is the same as that of the first element in $R_n^{-1} G_n Z' \varepsilon$, which is

$$(n^{-2} \sum y_{t-1}^2)^{-1} (n^{-1} \sum y_{t-1} \varepsilon_t) = \frac{a(1)}{b(1)} [n^{-2} \sum S_{t-1}^2]^{-1} [n^{-1} \sum S_{t-1} \varepsilon_t]$$

Notice that $n^{-2} \sum S_{t-1}^2 \Rightarrow \int B_\varepsilon^2 = \sigma^2 \int W^2$, $n^{-1} \sum S_{t-1} \varepsilon_t \Rightarrow \int B_\varepsilon dB_\varepsilon = \sigma^2 \int W dW$, and $\omega^2 = 2\pi f_{uu}(0) = \sigma^2 \left[b(1)/a(1)\right]^2$, where f_{uu} is the spectral density of u_t, and thus

$$n\widehat{a} \Rightarrow \frac{\sigma \int W dW}{\omega \int W^2}. \quad \square$$

5.2 Proof of Theorem 2

Under the alternative $H_1 : \alpha < 1$,when $f_{yy}(0) > 0$, y_t has a representation

$$\sum_{j=0}^{\infty} c_j y_{t-j} = e_t, c_0 = 1 \tag{8}$$

where $\{e_t\}$ = orthogonal $(0, \sigma_e^2)$. Following Fuller (1976), we can write (8) as

$$\Delta y_t = (\theta_1 - 1) y_{t-1} + \theta_2 \Delta y_{t-1} + \theta_3 \Delta y_{t-2} + \cdots + e_t$$

where $\theta_i = \sum_{j=i}^{\infty} c_j$ $(i = 2, 3, ...)$ and $\theta_1 = -\sum_{j=1}^{\infty} c_j$. Since y_t is stationary, $\theta_1 - 1 \neq 0$. In the ADF regression, as $k \to \infty$, we find that

$$\begin{aligned}
&\widehat{a} \xrightarrow{p} \theta_1 - 1 \neq 0 \\
&\widehat{\sigma}^2 \xrightarrow{p} \sigma_e^2 > 0 \\
&\widehat{\omega}^2 \xrightarrow{p} 2\pi f_{uu}(0) > 0
\end{aligned}$$

Thus, $ADF_\alpha = n(\widehat{\omega}/\widehat{\sigma})\widehat{a} = nO_p(1) = O_p(n)$. $\square$

5.3 Proof of Lemma 1

We prove the result for the polynomial time trend case. Let

$$D_n = \text{diag}(n^{-1}, ..., n^{-p}), F_n = \text{diag}(1, n^{-1}, ..., n^{-p+1})$$

then

$$\begin{aligned} n^{-1/2}y^{s*}_{[nr]} &= n^{-1/2}y^{s}_{[nr]} - n^{-1/2}x'_{[nr]}(\Delta_c X'\Delta_c X)^{-1}\Delta_c X'Ay^s \\ &= n^{-1/2}y^{s}_{[nr]} - (D_n x'_{[nr]})(n^{-1}F_n\Delta_c X'\Delta_c XF_n)^{-1}F_n\Delta_c X'(n^{-1/2}Ay^s). \end{aligned}$$

Notice that $n^{-1/2}y^s_{[nr]} \Rightarrow B(r)$ and

$$\begin{aligned} & n^{-1}F_n\Delta_c X'\Delta_c XF_n \\ &= n^{-1}F_n[\Delta X'\Delta X - n^{-1}c\Delta X'X_{-1} - n^{-1}cX'_{-1}\Delta X + n^{-2}c^2X'_{-1}X_{-1}]F_n \\ &\Rightarrow \int[g(r)g(r)' - cX(r)g(r)' - cg(r)X(r)' + c^2X(r)X(r)']dr \\ &= \int X_c(r)X_c(r)'dr \end{aligned}$$

and

$$\begin{aligned} & F_n\Delta_c X'n^{-1/2}Ay^s \\ &= n^{-1/2}F_n[\Delta X'\Delta y^s - n^{-1}c\Delta X'y^s_{-1} - n^{-1}cX'_{-1}\Delta y^s + n^{-2}c^2X'_{-1}y^s_{-1}] \\ &\Rightarrow \int[g(r)dB(r) - cX(r)dB(r) - cg(r)B(r) + c^2X(r)B(r)] \\ &= \int X_c(r)dB(r) - c\int X_c(r)B(r)dr. \end{aligned}$$

Thus,

$$\begin{aligned} & n^{-1/2}y^*_{[nr]} \\ &\Rightarrow B(r) - X(r)'\left[\int X_c(r)X_c(r)'dr\right]^{-1}\left(\int X_c(r)dB(r) - c\int X_c(r)B(r)dr\right) \\ &= \underline{B}_c(r). \quad \square \end{aligned}$$

5.4 Proof of Theorem 3

We know that $\tilde{a} = (y^{*\prime}_{-1}P_M y^*_{-1})^{-1}y^{*\prime}_{-1}P_M\Delta y^*$, where $P_M = I - M(M'M)^{-1}M'$, M is the matrix of the k lagged difference variables $(\Delta y^*_{t-1}, ..., \Delta y^*_{t-k})$. We have

$$\begin{aligned} n^{-2}y^{*\prime}_{-1}P_M y^*_{-1} &= n^{-2}y^{*\prime}_{-1}(I - M(M'M)^{-1}M')y^*_{-1} \\ &= n^{-2}y^{*\prime}_{-1}y^*_{-1} - n^{-1}(n^{-1}y^{*\prime}_{-1}M)(n^{-1}M'M)^{-1}(n^{-1}M'y^*_{-1}) \\ &= n^{-2}y^{*\prime}_{-1}y^*_{-1} + o(1) \\ &\Rightarrow \int \underline{B}_c(r)^2 \\ &= \omega^2\int \underline{W}_c(r)^2. \end{aligned}$$

Since $\Delta y_t^* = \Delta y_t - \widetilde{\gamma}'\Delta x_t \Rightarrow u_t$, and u_t is a stationary ARMA process with AR representation $\varepsilon_t = d(L)u_t$. When k $\rightarrow \infty$ as $n \rightarrow \infty$, and $k = o(n^{1/3})$, e_{tk}^* converges to ε_t (as in Said and Dickey, 1984), and thus

$$n^{-1}y_{-1}^{*\prime}P_M\Delta y^* = n^{-1}y_{-1}^{*\prime}\varepsilon + o(1) \Rightarrow \int \underline{B}_c(r)dB_\varepsilon(r) = \omega\sigma \int \underline{W}_c(r)dW(r).$$

It follows that

$$ADF_\alpha^* \Rightarrow \int \underline{W}_c(r)dW(r)/\int \underline{W}_c(r)^2. \quad \square$$

References

[1] Berk, K. N. (1974). "Consistent Autoregressive Spectral Estimates," *Annals of Statistics*, 2, 489–502.

[2] Beveridge, S. and C. R. Nelson (1981). "A New Approach to Decomposition of Economic Time series into Permanent and Transitory Components with particular attention to Measurement of the Business Cycle," *Journal of Monetary Economics,* 7, 151–174.

[3] Canjels, E. and M. W. Watson (1997). "Estimating Deterministic Trends in the Presence of Serially Correlated Error," *Review of Economics and Statistics*, May, 184–200.

[4] DeJong, D. N., J. C. Nankervis, N. E. Savin and C. H. Whiteman (1988). "Integration Versus Trend-Stationarity in Macroeconomic Time Series," Department of Economics, University of Iowa, Iowa City, IA.

[5] DeJong, D. N. and C. H. Whiteman (1989). "The Temporal Stability of Dividends and Stock Prices, Evidence from the Likelihood Function," *American Economic Review.*

[6] DeJong, D. N., J. C. Nankervis, N. E. Savin and C. H. Whiteman (1992). "The Power problems of Unit Root Tests in Time Series with Autoregressive Errors," *Journal of Econometrics*, 53, 323–43.

[7] Dickey, D. A., and W. A. Fuller (1979). "Distribution of Estimators for Autoregressive Time Series with a Unit Root." *Journal of the American Statistical Association*, 74, 427–431.

[8] Elliot, G., Rothenberg, T. J., and J. H. Stock (1996). "Efficient Tests for an Autoregressive Unit Root," *Econometrica* 64, 813–836.

[9] Fuller, W. A. (1976). *Introduction to Statistical Time Series.* New York: Wiley.

[10] Hall, R. E. (1978). "Stochastic Implications of the Life Cycle Permanent Income Hypothesis, Theory and Evidence," *Journal of Political Economy*, 86, 971–988.

[11] Lee, C. C., and P. C. B. Phillips (1994). "Efficiency Gains Using QD over OLS under Nonstationarity," mimeo, Yale University.

[12] Nelson, C. R. and C. I. Plosser (1982). "Trends and Random Walks in Macroeconomic Time Series," *Journal of Monetary Economics*, 10, 139–162.

[13] Perron, P. (1988). "Trends and random Walks in Macroeconomic Time Series: Further Evidence from a New Approach," *Journal of Economic Dynamics and Control*, 12 297–332.

[14] Phillips, P. C. B. (1987), "Time Series Regression with Unit Roots," *Econometrica*, 55, 277–301.

[15] Phillips, P. C. B., and P. Perron (1988). "Testing for a Unit Root in Time Series Regression," *Biometrica*, 75, 335–346.

[16] Phillips, P. C. B. (1995). "Nonstationary Time Series and Cointegration," *Journal of Applied Econometrics*, 10, 87–94.

[17] Phillips, P. C. B. (1995). "A Primer on Unit Root Tests," Yale University.

[18] Phillips, P. C. B. and C. C. Lee (1996). "Efficiency Gains from Quasi-Differencing under Nonstationarity," in P. M. Robinson and M. Rosenblatt (eds), *Essays in Memory of E. J. Hannan* (forthcoming).

[19] Rissanen, J. (1978), "Modeling by Shortest Data Description," *Automatica*, 14, 465–471.

[20] Said, S. E., and D. A. Dickey (1984). "Testing for Unit Roots in Autoregressive-Moving Average Model of Unknown Order," *Biometrika*, 71, 599–608.

[21] Schotman, P. and H. K. Van Dijk (1991). "On the Bayesian Routes to Unit Root," *Journal of Applied Econometrics*, 6, 387–402.

[22] Schwarz, G. (1978). "Estimating the Dimension of a Model," *Annals of Statistics*, 6, 461–464.

[23] Schwert, G. W. (1989). "Tests for Unit Roots: A Monte Carlo Investigation," *Journal of Business and Economic Statistics*, 7, 147–160.