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A BOUND ON THE PROPORTION OF PURE STRATEGY EQUILIBRIA IN GENERIC GAMES

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In a generic finite normal form game with $2\alpha + 1$ Nash equilibria, at least α of the equilibria are nondegenerate mixed strategy equilibria (that is, they involve randomization by some players).

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1. INTRODUCTION

Studying a few examples of finite n-person games leads one to believe that most games that have multiple pure strategy Nash equilibria also have nondegenerate mixed strategy Nash equilibria. Our purpose is to provide a formal statement and proof of this observation. In fact we will establish the following stronger result: given any finite n-person game Γ, for a generic set of payoffs, if Γ has $\alpha \geq 1$ Nash equilibria in pure strategies, then the total number of Nash equilibria of Γ is at least $2\alpha - 1$. In other words, at least $\alpha - 1$ equilibria involve some randomization by some players.

Our approach relies on a technique that characterizes Nash equilibria as the fixed points of a continuous function g. It is known that the sum of the Leftchetz indices of the fixed points of a Leftchetz function equals +1 (see Guillemin and Pollack (1974)). We establish our main result by showing that generically g is a Leftchetz function and that pure strategy Nash equilibria are fixed points of g with Leftchetz index +1.

The standard proof of the existence of Nash equilibrium (see Nash (1951)) uses the Kakutani fixed point theorem on the product of the best response correspondences. However, the line of argument described above necessitates that equilibria be viewed as the fixed points of a (*single-valued*) function. Hence we are led to the variational inequality (or generalized equation) approach.

The necessary and sufficient conditions for a Nash equilibrium in a 2-person game are equivalent to a linear complementarity problem. Lemke and Howson (1964) have developed a numerical algorithm, similar to the simplex method, to solve this problem. More recently, Wilson (1990) has refined this algorithm to compute stable equilibria. The linear complementarity problem is an example of a variational inequality. Robinson (1979, 1983) has introduced the notion of a generalized equation as an alternative representation for a variational inequality to facilitate sensitivity analysis. A number of authors have studied special cases of variational inequalities. Kehoe's analysis of production economies

(1980, 1982, 1983) and Reinoza's study of generalized equations (1979, 1983) utilize index theory ideas, similar to the ones we use here, to investigate existence, uniqueness and stability (continuity properties) of solutions. Stacchetti (1987) provides a unified analysis of a wide class of variational inequalities.

Section 2 first expresses Nash equilibrium as the solution of a variational inequality, and then uses a construction of Hartman and Stampacchia (1966) to obtain a characterization in terms of the fixed points of a *function*. After summarizing a number of relevant results from Leftchetz index theory and specializing them to games, Section 3 presents our main result: for regular games, the number of equilibria in pure strategies can exceed the number of mixed strategy equilibria by at most one. In section 4 we show that the set of regular games is generic, in a strong sense. Section 5 concludes briefly.

2. CHARACTERIZATION OF EQUILIBRIA

The conditions characterizing a Nash Equilibrium (NE) of an n-person game are a natural extension of the necessary and sufficient conditions introduced by Lemke and Howson (1964) for 2-player games. Without loss of generality we restrict attention throughout the paper to 3-player games. We consider the 3-player case to emphasize that the methods used here do not depend on the linearity of payoffs in opponents' strategies.

Consider a (finite) 3-player game $\Gamma = (A,B,C)$ with payoff matrices $A = [a_{ijk}]$, $B = [b_{ijk}]$ and $C = [c_{ijk}]$: a_{ijk} is player 1's payoff when he chooses the pure strategy $i \in S_1 := \{1,...,m\}$ while players 2 and 3 choose pure strategies $j \in S_2 := \{1,...,n\}$ and $k \in S_3 := \{1,...,p\}$, respectively. Similarly, b_{ijk} is player 2's payoff and c_{ijk} is player 3's payoff. The set of strategies for player 1 is defined by

$$S^m := \{\lambda \in \Re^m \mid \lambda \geq 0, e^T\lambda = 1\} ,$$

where $e^T := (1,...,1)$. Below, e will always denote a column vector with all its entries equal to 1; we will not specify its dimension since it will be clear from the context. The sets of strategies S^n and S^p for players 2 and 3, respectively, are similarly defined. Let $K := S^m \times S^n \times S^p$. Note that K is a convex compact set.

Definition: $\lambda \in S^m$ is a *pure strategy* for player 1 if λ is an extreme point of S^m. (That is, $\lambda = e_i$ for some $i \in S_1$. Here e_i denotes a vector with component i equal to 1 and all other components equal to 0.) $\lambda \in S^m$ is a *mixed strategy* if it is not a pure strategy. An NE $(\lambda,\mu,\gamma) \in K$ is a *pure strategy NE* if it is an extreme point of K; otherwise, (λ,μ,γ) is a *mixed strategy NE*. Our use of the term "strategy" will include both pure and mixed strategies.

Definition: Let $C \subset \Re^l$ be a convex set and $x \in C$. The *normal cone to C at x* is the cone

$$N_C(x) := \{q \in \Re^l \mid \langle q,c-x\rangle \le 0 \quad \text{for all } c \in C\} ;$$

when $x \notin C$, $N_C(x) := \emptyset$.

Lemma 1: Let $\Psi \in \Re^{r\times l}$, $b \in \Re^r$, $\Phi \in \Re^{s\times l}$, and $d \in \Re^s$, and suppose $C \subset \Re^l$ is the convex set defined by

$$C = \{z \in \Re^l \mid \Psi z = b \text{ and } \Phi z \le d\}.$$

Assume $z \in C$ and let $J := \{\alpha \mid \sum_{\beta=1}^{l} \phi_{\alpha\beta}\, z_\beta = d_\alpha\}$. Then

$$N_C(z) = \{q \in \Re^l \mid q = \Psi^T v + \Phi_J^T w \quad \text{for some } v \in \Re^r \text{ and } w \in \Re_+^{|J|}\} ,$$

where Φ_J is the submatrix of Φ comprised of the rows $\alpha \in J$.

Definition: For $\lambda \in \Re^m$, $\mu \in \Re^n$ and $\gamma \in \Re^l$, we denote by $A(\mu,\gamma) \in \Re^m$, $B(\mu,\gamma) \in \Re^n$ and $C(\gamma,\mu) \in \Re^p$ the payoff vectors for players 1, 2 and 3, respectively, whose components are

$$A_i(\mu,\gamma) = \sum_{j,k} a_{ijk}\, \mu_j \gamma_k \qquad i \in S_1 ,$$

$$B_j(\lambda,\gamma) = \sum_{i,k} b_{ijk}\, \lambda_i \gamma_k \qquad j \in S_2 ,$$

$$C_k(\lambda,\mu) = \sum_{i,j} c_{ijk}\, \lambda_i \mu_j \qquad k \in S_3 .$$

$A_i(\mu,\gamma)$, for example, is the payoff for player 1 if he chooses strategy i when players 2 and 3 choose the profile of strategies $(\mu,\gamma) \in S^n \times S^p$.

Lemma 2: (λ,μ,γ) is an NE with value $v^T = (v_1,v_2,v_3)$ iff

(i) $A(\mu,\gamma) + x = v_1 e$, $x \geq 0$, $\lambda \in S^m$ and $\lambda^T x = 0$,

(ii) $B(\lambda,\gamma) + y = v_2 e$, $y \geq 0$, $\mu \in S^n$ and $\mu^T y = 0$, and

(iii) $C(\lambda,\mu) + z = v_3 e$, $z \geq 0$, $\gamma \in S^p$ and $\gamma^T z = 0$.

Condition (i) of Lemma 2, for example, requires that player 1 give strictly positive weight to a pure strategy only if it is a best response to the profile (μ,γ). The proof of the Lemma is immediate, and is omitted.

Corollary 1: (λ,μ,γ) is an NE iff

(i) $A(\mu,\gamma) \in N_{S^m}(\lambda)$,

(ii) $B(\lambda,\gamma) \in N_{S^n}(\mu)$,

(iii) $C(\lambda,\mu) \in N_{S^p}(\gamma)$.

Since $N_{S^m}(\lambda) \times N_{S^n}(\mu) \times N_{S^p}(\gamma) = N_K\begin{pmatrix}\lambda\\ \mu\\ \gamma\end{pmatrix}$, the conditions in Corollary 1 can also be written

$$\begin{pmatrix}A(\mu,\gamma)\\ B(\lambda,\gamma)\\ C(\lambda,\mu)\end{pmatrix} \in N_K\begin{pmatrix}\lambda\\ \mu\\ \gamma\end{pmatrix}.$$

For any closed convex set $C \subset \Re^l$ and $z \in \Re^l$, $P_C(z)$ will denote the projection of z onto C. A well known characterization result states that $P_C(z)$ is the unique point that satisfies

$$<z - P_C(z), c - P_C(z)> \leq 0 \quad \text{for all} \quad c \in C.$$

Let $f : \Re^l \to \Re^l$ and define $g : C \to C$ by $g(z) := P_C(z + f(z)), z \in C$. It is easy to check that z^* is a fixed point of g iff $f(z^*) \in N_C(z^*)$.

Definition: Let $f : \Re^{m+n+p} \to \Re^{m+n+p}$ be the function

$$f\begin{pmatrix}\lambda\\ \mu\\ \gamma\end{pmatrix} = \begin{pmatrix}A(\mu,\gamma)\\ B(\lambda,\gamma)\\ C(\lambda,\mu)\end{pmatrix},$$

and $g : \Re^{m+n+p} \to K$ be the function

$$g\begin{pmatrix}\lambda\\ \mu\\ \gamma\end{pmatrix} = P_K\left(\begin{pmatrix}\lambda\\ \mu\\ \gamma\end{pmatrix} + f\begin{pmatrix}\lambda\\ \mu\\ \gamma\end{pmatrix}\right).$$

Corollary 2: $(\lambda,\mu,\gamma) \in K$ is an NE iff it is a fixed point of g.

The idea behind Corollary 2 can be expressed as follows. Consider any convex set K; the point $z \in K$ is a local maximizer of F in K iff the projection onto K of the point obtained by moving away from z along the gradient of F at z is again z. That is, $P_K(z + F'(z)) = z$. The construction preceding Corollary 2 utilizes this observation to simultaneously solve the interdependent maximization problems of the players in order to obtain a Nash equilibrium.

3. NUMBER OF PURE STRATEGY EQUILIBRIA

The *tangent space* to K is the $m + n + p - 3$ dimensional subspace in $\Re^{m+n+p}$ defined by

$$T := \{(\lambda,\mu,\gamma) \in \Re^{m+n+p} \mid \Lambda \begin{pmatrix} \lambda \\ \mu \\ \gamma \end{pmatrix} = 0\} ,$$

where

$$\Lambda := \begin{bmatrix} e^T & 0 & 0 \\ 0 & e^T & 0 \\ 0 & 0 & e^T \end{bmatrix} \in \Re^{3\times(m+n+p)} .$$

Suppose $h : \Re^{n+m+p} \to K$ is a continuous function. A fixed point z of h is a *Leftchetz fixed point* if h is continuously differentiable in a neighborhood of z and $I - h'(z) : T \to T$ is an isomorphism. Further, h is a *Leftchetz map* if each of its fixed points is a Leftchetz point. If h is a Leftchetz map, it admits only a finite number of fixed points and its *Leftchetz number* can be computed by

$$L(h) = \sum_{z=h(z)} i(h;z), \text{ where}$$

$$i(h;z) = \operatorname{sgn} \det \begin{bmatrix} I-h'(z) & \Lambda^T \\ -\Lambda & 0 \end{bmatrix}$$

is the *index* of h at z. Because the Leftchetz number is a homotopy invariant, $L(h) = L(\hat{h})$ for any $\hat{h}$ homotopic to h, and since K is convex, all maps $h : K \to K$ are in the same homotopy class. In particular, if $\bar{z} \in K$ and $\bar{h} : K \to K$ is the constant map $\bar{h}(z) = \bar{z}$, $z \in K$, then $\bar{h}$ is a Leftchetz map, $\bar{z}$ is its only fixed point, and

$$L(\bar{h}) = i(\bar{h};\bar{z}) = \operatorname{sgn}\det\begin{bmatrix} I & \Lambda^T \\ -\Lambda & 0 \end{bmatrix} = \operatorname{sgn}\det\begin{bmatrix} I & \Lambda^T \\ 0 & \Lambda\Lambda^T \end{bmatrix} = \operatorname{sgn}\det \Lambda\Lambda^T = 1 ,$$

because $\Lambda\Lambda^T = \begin{bmatrix} m & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & p \end{bmatrix}$ and $\det \Lambda\Lambda^T = mnp$.

Definition: An NE $z = (\lambda,\mu,\gamma)$ of Γ is *strongly nondegenerate* if

(i) for each player, every best response is used with positive probability.

(ii) $\det (I - g'(z)) \neq 0$.

The game Γ is *regular* if each of its NE is strongly nondegenerate; otherwise, Γ is said to be *singular*.

Condition (i) guarantees that $g'(z)$ exists for each NE z of Γ. Suppose g is differentiable at $z \in K$. Since $g'(z)\,\xi \in T$ for all $\xi \in \Re^{n+m+p}$, condition (ii) is equivalent to requiring that $I - g'(z) : T \to T$ be an isomorphism. This is also equivalent to

(ii') $\det\begin{bmatrix} I-g'(z) & \Lambda^T \\ -\Lambda & 0 \end{bmatrix} \neq 0$.

Theorem 1: Suppose Γ is regular and $\bar{z} = (\bar{\lambda},\bar{\mu},\bar{\gamma}) \in K$ is a *pure* NE. Then $i(g;\bar{z}) = 1$.

Proof:

Without loss of generality assume that $A > 0$, $B > 0$, $C > 0$, and $\bar{z} = (\bar{\lambda},\bar{\mu},\bar{\gamma}) = (e_1,e_1,e_1)$. One can show (see for example, Stacchetti (1987)) that

$$g'(\bar{z}) = (I - XYX^T)(I + f'(\bar{z})),$$

where $X, Y \in \Re^{(m+n+p)\times(m+n+p)}$ are defined as follows:

$$X = \begin{bmatrix} e & 0 & 0 & I_m^* & 0 & 0 \\ 0 & e & 0 & 0 & I_n^* & 0 \\ 0 & 0 & e & 0 & 0 & I_p^* \end{bmatrix},$$

$$Y = (X^T X)^{-1},$$

and $I_m^* \in \Re^{m\times(m-1)}$ is an $m\times m$ identity matrix with the first column deleted. Since X is invertible, $XYX^T = I$ and $g'(\bar{z}) = 0$. This result can be explained more intuitively as follows. Since $\bar{z}$ is a pure NE, $\bar{z}$ is an extreme point of K and $\operatorname{int} N_K(\bar{z}) \neq \emptyset$. Condition (i) of strong nondegeneracy then implies that $f(\bar{z}) \in \operatorname{int} N_K(\bar{z})$. Hence for all z in a neighborhood of $\bar{z}$, $z + f(z) \in \bar{z} + N_K(\bar{z})$, and therefore $g(z) = \bar{z}$. That is, g is constant on a neighborhood of $\bar{z}$ and $g'(z)$ must be 0. Thus,

$$i(g;\bar{z}) = \operatorname{sgn} \det \begin{bmatrix} I & \Lambda^T \\ -\Lambda & 0 \end{bmatrix} = 1$$

Q.E.D.

Corollary 3: Assume Γ is regular and has $\alpha \geq 1$ pure NE. Then Γ must have at least $2\alpha-1$ NE in total.

Proof:

$$1 = L(g) = \sum_{z=g(z)} i(g;z) = \alpha + \sum_{\substack{z=g(z) \\ z \text{ is not pure}}} i(g;z)$$

and since $i(g;z)$ is either +1 or −1 for each NE, there must be at least $(\alpha-1)$ mixed strategy NE z with $i(g;z) = -1$. Q.E.D.

Observe that Corollary 3 does not state that if a game has α pure NE then it must have exactly $2\alpha - 1$ equilibria. While it is the case that each pure strategy equilibrium must have Leftchetz index +1, mixed strategy equilibria can have Leftchetz indices +1 *or* −1. To see this consider any game with a unique NE and no pure strategy NE (such as the "matching pennies" game Γ_1 below). It follows from the discussion above that the unique equilibrium must have index +1. On the other hand the "battle of the sexes" game Γ_2 below has 3 equilibria, 2 of which are pure. Thus, in this case it follows that the mixed strategy equilibrium must have Leftchetz index −1. In general it is quite possible to have a game with 1 pure strategy NE (Leftchetz index +1 by the above theorem), two mixed strategy NE each with Leftchetz indices +1 and two other mixed strategy NE with Leftchetz indices −1. However, the bound is tight, in the sense that for each positive integer α, it is possible to construct a generic game having exactly α pure strategy NE and $\alpha - 1$ mixed strategy NE.

Γ_1

	x	y
a	1,0	0,1
b	0,1	1,0

Γ_2

	x	y
a	2,1	0,0
b	0,0	1,2

Figure 1

4. GENERIC GAMES

In this section we show that the set of regular games is generic. More precisely, we show that the set of regular games is open and its complement has measure 0.

Let $\Pi := (\Re^{m \times n \times p})^3$, and with abuse of notation, let $f : (\Re^m \times \Re^n \times \Re^p) \times \Pi \to \Re^m \times \Re^n \times \Re^p$ denote the function

$$f(\lambda,\mu,\gamma;A,B,C) := \begin{pmatrix} A(\mu,\gamma) \\ B(\lambda,\gamma) \\ C(\lambda,\mu) \end{pmatrix},$$

and $f_\pi(\lambda,\mu,\gamma;\pi)$ denote the Jacobian of f with respect to π. We have

$$\frac{\partial f_\alpha}{\partial a_{ijk}} = \begin{cases} 0 & \text{if } \alpha \neq i \\ \mu_j\gamma_k & \text{if } \alpha = i \end{cases}, \quad \frac{\partial f_\alpha}{\partial b_{ijk}} = \frac{\partial f_\alpha}{\partial c_{ijk}} = 0 \ \forall i,j,k .$$

$$\frac{\partial f_{m+\alpha}}{\partial b_{ijk}} = \begin{cases} 0 & \text{if } \alpha \neq j \\ \lambda_i\gamma_k & \text{if } \alpha = j \end{cases}, \quad \frac{\partial f_{m+\alpha}}{\partial a_{ijk}} = \frac{\partial f_{m+\alpha}}{\partial c_{ijk}} = 0 \ \forall i,j,k .$$

$$\frac{\partial f_{m+n+\alpha}}{\partial c_{ijk}} = \begin{cases} 0 & \text{if } \alpha \neq k \\ \lambda_i\mu_j & \text{if } \alpha = k \end{cases}, \quad \frac{\partial f_{m+n+\alpha}}{\partial a_{ijk}} = \frac{\partial f_{m+n+\alpha}}{\partial b_{ijk}} = 0 \ \forall i,j,k .$$

For each $(\lambda,\mu,\gamma) \in K$, there exist pairs (j,k), (i,k) and (i,j) such that $\mu_j\gamma_k \neq 0$, $\lambda_i\gamma_k \neq 0$ and $\lambda_i\mu_j \neq 0$. It is easy then to see that the rows of the Jacobian $f_\pi(\lambda,\mu,\gamma;\pi)$ are linearly independent, and $f_\pi(\lambda,\mu,\gamma,\pi)$ is of rank $n+m+p$ for each $(\lambda,\mu,\gamma) \in K$.

A Sard Theorem (see Theorem 3 in Stacchetti (1987) and Theorem 4.1 in Reinoza (1983)[1]) establishes the following result.

Theorem 2: The set $\Pi_0 \subset \Pi$ of regular games $\Gamma = (A,B,C)$ is open and its complement has Lebesgue measure 0 in Π.

5. CONCLUSION

This paper provides a strong restriction on the set of Nash equilibria of a large class of games. In particular we establish that concentrating on pure strategy equilibria in games with multiple equilibria typically entails a loss. Furthermore, our result gives a lower bound on the total number of equilibria as a function of the number of pure strategy equilibria (which are easy to compute). This should be useful when trying to identify the entire set of Nash equilibria. Finally, our results illustrate the usefulness of the generalized equations approach for analyzing finite games.

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[1] Reinoza (1983) does not prove that Π_0 is open. This is a consequence of the Implicit Function Theorem (Theorem 2) in Stacchetti (1987).

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