COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 956

THE HYBRID SOLUTIONS OF AN *N*-PERSON GAME

Jingang Zhao

September 1990

THE HYBRID SOLUTIONS OF AN N-PERSON GAME

By Jingang Zhao*

Economics Department
Yale University

Revised September 1990

We introduce a solution concept intermediate between the cooperative and non-cooperative solutions of an n-person game in normal form. Consider a partition p of the players, with each $s \in p$ a coalition. A joint strategy $x = \{x_S \mid s \in p\}$ is a *hybrid solution* for the partition p if, for each $s \in p$, x_S is a core solution of the corresponding parametric subgame, where this game is played by the players in s and is parameterized by x_{-S}, the strategies played by all outside players. This assumes that players behave cooperatively within each coalition and competitively across coalitions. Sufficient conditions are given for a general n-person game to have hybrid solutions for any partition.

1. INTRODUCTION. The solutions to a general n-person game are of two kinds: non-cooperative and cooperative. The primary non-cooperative solution concept is the concept of Nash equilibrium, and the major cooperative solution concept is the core. The definition of the core most commonly seen in the literature and textbooks is Aumman's α-core (1961) for games without sidepayments; its general existence was first proved by Scarf (1967). A core solution is obtained if and only if no coalition can guarantee a higher payoff for each of its members by choosing another strategy. In this sense, no coalition has any incentive to "block" such an outcome.

---

* I am indebted to Professor Herbert E. Scarf for his insights, encouragement and innumerable discussions. I am also indebted to Tatsuro Ichiishi, Alvin K. Klevorick, David Roth, Martin Shubik, Donald Smythe and Lin Zhou for their many valuable comments and suggestions. I thank Truman Bewley, Pradeep Dubey, John Geanakoplos, Jean-Michel Grandmont, David Pearce and Rajiv Vohra for their valuable discussions. I also thank the Associate Editor and two anonymous referees of *Games and Economic Behavior* for their helpful comments and suggestions. The support of the 1990-91 Carl Arvid Anderson Prize Fellowship from the Cowles Foundation for Research in Economics at Yale University is also appreciated.

These two solution concepts are shown here to be *two special cases* of a more general solution concept---the hybrid solution concept. The hybrid solution concept assumes that the players are partitioned into coalitions and that they will cooperate within each coalition but compete (in the Nash sense) between coalitions[1]. The set of Nash equilibria and the core are then, respectively, simply the hybrid solutions based on the finest and the coarsest partitions of players.

One important caveat must be kept in mind, however. The hybrid solution concept takes the partition of the game, the coalitional structure, as given. It is as important to understand the formation of a particular partition as to explain the behavior once the partition has been formed. We shall leave this important issue aside, however, and concentrate only on the mathematical concept and an existence theorem.

In the next section of this paper we shall provide a formal definition of a hybrid solution to an n-person game. In Section 3 we shall provide sufficient conditions for the existence of hybrid solutions for any partition of the players in an n-person normal form game. We shall conclude the paper with some remarks in Section 4.

2. DEFINITION OF THE HYBRID SOLUTIONS. Now we turn to the definition of hybrid solutions. A general n-person game in normal form is defined as $\Gamma = \{N, X^i, u^i\}$, where $N = \{1, 2, ..., n\}$ is the set of players. For each $i \in N$, X^i is player i's strategy set, which is assumed to be a nonempty subset in some finite-dimensional Euclidean space ($X^i \subset R^{m_i}$); u^i: $X = \Pi_{i=1}^n X^i \rightarrow R$ is player i's real payoff function.

Let $\mathcal{N}$ denote the set of all nonempty subsets of N. For each coalition $s \in \mathcal{N}$, let $/s/$ denote the number of elements in s, and R^s denote the $/s/$-dimensional Euclidean space whose coordinates have as subscripts the members in s. For any $x = \{x^1, ..., x^n\} \in X$, $u = \{u^1, ..., u^n\} \in R^n$, where $x^i = \{x_1^i, x_2^i, ..., x_{m_i}^i\} \in X^i$, let $x_s = \{x^i / i \in s\} \in X_s = \Pi_{i \in s} X^i$ be the strategies of coalition s; $x_{-s} = x_{N \setminus S} = \{x^i / i \notin s\} \in X_{-s} = \Pi_{i \notin s} X^i$ be the strategies of the players not in the coalition s (or in the complementary coalition $N\backslash s$); $u_s = \{u^i / \ i \in s\} \in \mathbf{R}^S$ and $u_{-s} = u_{N\backslash s} = \{u^i / \ i \notin s\} \in \mathbf{R}^{-S}$ be the projections of u on $\mathbf{R}^S$ and $\mathbf{R}^{-S}$ respectively; $\bar{u}_s : X_s \rightarrow \mathbf{R}^S$ be the worst vector payoffs to s and be defined as $\bar{u}_s(x_s) = \{ \min_{x_{-s} \in X_{-s}} u^i(x_s, x_{-s}) / \ i \in s\} \in \mathbf{R}^S$ for each $x_s \in X_s$. Without loss of generality, we can assume $\bar{u}_s(x_s) >> 0$ for all $s \in \mathcal{N}$ and $x_s \in X_s$. For any two vectors $a, b \in \mathbf{R}^n$, $a \geq b \Leftrightarrow a_i \geq b_i$, all i; $a > b \Leftrightarrow a \geq b$ and $a \neq b$; $a >> b \Leftrightarrow a_i > b_i$, all i.

It is customary to define cooperative games in the characteristic form, $\Gamma = \{N, V^S\}$, where $N = \{1, 2, ..., n\}$ is the set of players, and V^S is the set of achievable payoff vectors for each coalition $s \in \mathcal{N}$. Although this is, in many cases, more general than the normal form, it does not present players' strategy spaces explicitly. Moreover, the description of a game is often tedious ----for an n-person game, the achievable payoff vectors need to be described by $2^n - 1$ sets. For reasons that will become clear later, we shall introduce the following notion of a general cooperative game:

Definition 1: *A* ***general cooperative game*** *is defined as* $\Gamma = \{N, X_s, u_s\}$, *where* $N = \{1, 2, ..., n\}$ *is the set of players. For each coalition* $s \in \mathcal{N}$, X_s *is the strategy set of s (which is assumed to be a nonempty subset in some finite-dimensional Euclidean space), and* $u_s : X_s \rightarrow \mathbf{R}^S$ *is the vector payoff function for coalition s, with the i-th component equal to the payoff of its i-th member.*

Note that A game in characteristic form can be transformed to a general cooperative game easily. To see this, let $X_s = V^S$ be the strategy set for each coalition s; and let $u_s : X_s \rightarrow \mathbf{R}^S$ be defined by $u_s(x_s) = t^* x_s$ for each $x_s \in X_s$, where $t^* = \max\{t / \ t x_s \in V^S\}$.

Compared to the definition of a normal form game, our definition of a general cooperative game is still very cumbersome----$2^n - 1$ sets and functions need to be described. But our new definition is more explicit about the players' strategies, and it is more general than the characteristic form in the following respect. For each coalition $s \in \mathcal{N}$ in our general cooperative game, we put no restrictions on the dimension of X_s or on the properties of u_s. We impose no requirements of the form: For $s_1, s_2 \in \mathcal{N}$, $s = s_1 \cup s_2 \in \mathcal{N}$, $X_{s_1} \cup X_{s_2} \subset X_s$, $(u_{s_1}, u_{s_2}) \leq u_s$. Thus cooperation does not necessarily increase welfare. Players who cooperate might be better off or worse off depending on the particular features of the game.

A sufficient condition for the existence of the core is that the game should be balanced. Let $e = \{1, 1, ..., 1\} \in \boldsymbol{R}^n$ be a vector of one's. For each coalition $s \in \mathcal{N}$, $|s|$ is the number of elements in s. Let χ_s denote the characteristic vector for s, and let h_s denote the vector in $\boldsymbol{R}^n$ whose i^{th} component is $1 / |s|$ if $i \in s$ and 0 otherwise. Clearly $\chi_s = |s|\, h_s$, and $\chi_N = e = \{1, 1, ..., 1\}$. A subset of $\mathcal{N}$ or a collection of coalitions $\theta = \{s_1, s_2, ..., s_k\}$ is balanced if and only if there are nonnegative numbers w_s for each $s \in \theta$ such that $\Sigma_{s \in \theta} w_s \chi_s = \Sigma_{i=1}^{k} w_{s_i} \chi_{s_i} = e$. Then a balanced game is defined as:

Definition 2: *A general cooperative game $\Gamma = \{N, X_s, u_s\}$ is **balanced** if for any $y \in \boldsymbol{R}^n$ and any balanced collection of coalitions θ such that for each $s \in \theta$, there exist $x_s \in X_s$ satisfying $u_s(x_s) \geq y_s$, then there exist $x \in X_N = X$ such that $u(x) \geq y$.*

Here, for simplicity, we have written x_N, X_N and $u_N(x_N)$ from the grand coalition respectively as x, X and $u(x)$.

Let the general cooperative game derived from an n-person normal form game $\Gamma = \{N, X^i, u^i\}$ be given as $\Gamma^* = \{N, X_s, \bar{u}_s\}$, where $X_s = \Pi_{i \in s} X^i$, the strategy set of coalition s, and $\bar{u}_s : X_s \rightarrow \boldsymbol{R}^S$, the worst vector payoff function for s, have been defined earlier. It has been verified by Scarf(1971) that Γ^* is balanced if each X^i is a closed nonempty convex subset in $\boldsymbol{R}^{m_i}$ and each $u^i(x)$ is continuous and quasiconcave in x.

We are now ready to define the solutions to a game. We shall define the cooperative solution in Definition 3, the noncooperative solution in Definition 4 and the hybrid solution in Definition 5.

Definition 3: *A joint strategy* $\bar{x} \in X_N = X$ *is a* ***core solution*** *of the general cooperative game* $\Gamma = \{N, X_S, u_S\}$ *if for each coalition* $s \in \mathcal{N}$, $\Omega_S(\bar{x}) = \{ x_s \in X_S / u_S(x_S) >> u(\bar{x})_S \} = \varnothing$.

Similarly, a vector $y \in \boldsymbol{R}^n$ is in the core of Γ if it is feasible and if for each coalition $s \in \mathcal{N}$, $\Omega_S(y) = \{x_s \in X_S / u_S(x_S) >> y_S\} = \varnothing$. By feasible we mean that there exist $x \in X$ such that $u(x) = u_N(x) \geq y$. Thus a joint strategy x is a core solution if and only if $u(x)$ is in the core.

Definition 3*: *A joint strategy* $\bar{x} = \{\bar{x}^1, \bar{x}^2, ..., \bar{x}^n\} \in X$ *is an* ***α-core solution*** *of a normal form game* $\Gamma = \{N, X^i, u^i\}$, *if* $\bar{x}$ *is a core solution of the corresponding general cooperative game* $\Gamma^* = \{N, X_S, \bar{u}_S\}$, *where the strategy set* X_s *and the worst vector payoff* $\bar{u}_S$ *have been defined earlier for each coalition* $s \in \mathcal{N}$.

Definition 4: *A joint strategy* $\bar{x} = \{\bar{x}^1, \bar{x}^2, ..., \bar{x}^n\} \in X$ *is a* ***Nash equilibrium*** *of the normal form game* $\Gamma = \{N, X^i, u^i\}$, *if each* $\bar{x}^i$ *is a best response of player i to* $\bar{x}_{-i}$ *in the sense that* $u^i(\bar{x}) = u^i(\bar{x}^i, \bar{x}_{-i}) \geq u^i(x^i, \bar{x}_{-i})$ *for all* $x^i \in X^i$.

For each partition $p = \{s_1, s_2, ..., s_k\}$ of players in a normal form game, let $/p/ = k$ denote the number of coalitions in p. Then p will induce $/p/ = k$ parametric subgames: $\Gamma(x_{-s_i}) = \{s_i, X^j, u^j\} = \{s_i, \{X^j, u^j(x_{s_i}, x_{-s_i}) / j \in s_i\}\}, i = 1, ..., /p/$. Each $\Gamma(x_{-s_i})$ has $/s_i/$ players and is parameterized by the complementary strategies x_{-s_i}. Thus for the fixed x_{-s_i}, $\Gamma(x_{-s_i})$ is a normal form game.

Definition 5: *For each partition* $p = \{s_1, s_2, \ldots, s_k\}$ *of players in a normal form game, the corresponding* **hybrid solution** *is a joint strategy* $\bar{x} = \{\bar{x}_{s_1}, \bar{x}_{s_2}, \ldots, \bar{x}_{s_k}\} = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\}$ *such that each* $\bar{x}_{s_i} \in X_{s_i}$ *is a best response to all other coalitions' strategy* $\bar{x}_{-s_i}$ *in the sense that* $\bar{x}_{s_i}$ *is an* α*-core solution of* $\Gamma(\bar{x}_{-s_i})$.

Comparing Definitions 3*, 4 and 5, one can see that the Nash equilibrium and the core are two special hybrid solutions. The hybrid solution for the finest partition is a Nash equilibrium; and our hybrid solution is the α-core for the coarsest partition that consists of the grand coalition alone. The hybrid solutions are more general because of the coexistence of competition and cooperation, which captures the omnipresent situation in which a group of people behave collectively to compete with other groups.

3. THE EXISTENCE THEOREM. We shall first restate the Scarf core existence theorem (1967) for our general cooperative game. This is stated as:

Theorem 1(Scarf 1967a): *An n-person general cooperative game* $\Gamma = \{N, X_s, u_s\}$ *has at least one core solution if: (1)* Γ *is balanced, and (2) for each coalition* $s \in \mathcal{N}$, u_s *is continuous and quasiconcave on* X_s, *and* X_s *is a convex and compact subset in some finite-dimensional Euclidean space.*

Note that the cooperative game derived from a normal form game is balanced if the payoffs are continuous and quasiconcave, and each X^i is a nonempty convex and compact subset (Scarf 1971). In our general cooperative game, the quasi–concavity and continuity of payoff functions does not suffice for balancedness. Thus both conditions (1) and (2) are imposed to guarantee the existence of a core solution. Though it may be possible to weaken condition (2), I do not consider it here since it is not our major concern.

Proof of Theorem 1. This is a direct consequence of Theorem 3.

Many alternative proofs of the Scarf core existence theorem, or alternative methods of proving it, have appeared in the literature (Shapley, 1973; Ichiishi, 1981; Keiding and Thorlund-Pertersen, 1987; Vohra, 1987; Vohra and Shapley, 1988). All these share two common features: the game is given in characteristic form, and the proofs employ a particular version of the fixed point theorem. (Our proof in Theorem 3 is no exception.) For example, Scarf's original proof utilizes an extension of the Sperner theorem(1928) due to Scarf(1967b), Shapley's uses the *K-K-M-S* theorem that is a generalization of the *K-K-M* theorem(1929) due to Shapley(1973), Keiding and Thorlund-Pertersen's uses the *K-K-M* theorem directly, Vohra's uses the more familiar Kakutani (1941) fixed point theorem, and Ichiishi(1981) and Vohra and Shapley(1988) both apply the Fan coincidence theorem (1969). As a result, these proofs end up with a fixed point (or an intersection point) that is a core payoff vector. Since the general cooperative game contains strategies, our proof derived from Theorem 3 provides both a core payoff vector and a core solution.

Theorem 2: *For a general n-person game in normal form* $\Gamma = \{ N, X^i, u^i \}$, *the set of hybrid solutions corresponding to a given partition* $p = \{ s_1, s_2, \ldots, s_k \}$ *of* N *is nonempty if* Γ *satisfies: (1) for each player i,* X^i *is a closed bounded convex subset in* $\mathbf{R}^{m_i}$, *(2) for each coalition* $s_i \in p$, $u^j(x)$ *for all* $j \in s_i$ *are continuous in* $x = (x_{s_i}, x_{-s_i})$ *and quasiconcave in* x_{s_i}.

That is, if all the strategy sets are closed, bounded and convex, the payoffs of each coalition are quasiconcave in the coalition's own strategies and are continuous in all strategies, then there exists at least one joint strategy $\bar{x} = \{\bar{x}_{s_1}, \bar{x}_{s_2}, ..., \bar{x}_{s_k}\}$ such that each $\bar{x}_{s_i}$ is an α-core solution of $\Gamma(\bar{x}_{-s_i})$.

Proof of Theorem 2. This is also a consequence of Theorem 3. This theorem can also be proved by applying the subtle technique of Ichiishi (1981). To do this, one needs to define the space of societies, the corresponding strategy set and payoff functions for each coalition s, step by step, in the Ichiishi context[2]. Under the conditions of our Theorem 2, one can verify that the conditions of the Ichiishi Lemma are met, and thus the conclusion holds.

Next we shall state and prove a more general theorem that leads to Theorem 1 and Theorem 2 directly. Before we describe the general model, let us consider first a special case where only two games are involved. Let A, B be two sets of players, $\mathcal{A}$ and $\mathcal{B}$ be the sets of all non-empty subsets of A and B respectively, and for each $s \in \mathcal{A}$ and each $t \in \mathcal{B}$, X_s and Y_t be the strategy sets of s and t respectively. Let the first general cooperative game be given as: $\Gamma_A(y_B) = \{A, X_s, u_s(x_s, y_B)\}$, $y_B \in Y_B$, where for each coalition $s \in \mathcal{A}$, its vector payoff function $u_s(x_s, y_B)$ depends on its strategy x_s and is parameterized by the strategies of the grand coalition B. Similarly, the second game is given as $\Gamma_B(x_A) = \{B, Y_t, v_t(y_t, x_A)\}$ for each $x_A \in X_A$, which is parameterized by the strategies of the grand coalition A. The hybrid solution for a partition with two coalitions is involved exactly in two such intertwined games.

In general we shall consider k general cooperative games $\{\Gamma_1, \Gamma_2, ..., \Gamma_k\}$ that are intertwined together similarly as above. For each $i = 1, ..., k$, $\Gamma_i = \Gamma_i(x_{-N_i}) = \{N_i, X_s, u_s(x_s, x_{-N_i})\}$ is parameterized by $x_{-N_i} = \{x_{N_1}, ..., x_{N_{i-1}}, x_{N_{i+1}}, ..., x_{N_k}\}$ through the payoff functions. There are $|N_i|$ players in Γ_i; for each coalition $s \in \mathcal{N}_i$ ($\mathcal{N}_i$ is the set of non-empty subsets of N_i) in $\Gamma(x_{-N_i})$, the joint strategies of s are $x_s \in X_s$, and its vector payoff is $u_s(x_s, x_{-N_i})$, which depends on the joint strategies x_s and the parameters

x_{-N_i}. Let $X = \Pi_{i=1}^{k} X_{N_i} = X_{N_1} \times \ldots \times X_{N_k}$, and for each $i = 1, 2, \ldots, k$, let $X_{-N_i} = X_{N_1} \times \ldots \times X_{N_{i-1}} \times X_{N_{i+1}} \times \ldots \times X_{N_k} = \Pi_{j \neq i} X_{N_j}$.

Theorem 3: *Consider a collection of general cooperative games as above. Suppose the intertwined games satisfy: (1) each $\Gamma_i = \Gamma_i(x_{-N_i})$ is balanced for any fixed parameters x_{-N_i}, and (2) for all coalitions $s \in \mathcal{N}_i$, all N_i, $i=1, \ldots, K$, $u_s(x_s, x_{-N_i})$ is continuous in (x_s, x_{-N_i}) and quasiconcave in x_s, and X_s is a nonempty convex and compact subset in some finite-dimensional Euclidean space. Then there exist $\bar{x} = \{\bar{x}_{N_1}, \bar{x}_{N_2}, \ldots, \bar{x}_{N_k}\}$ such that each $\bar{x}_{N_i}$ is a core solution of $\Gamma_i(\bar{x}_{-N_i})$.*

Since there is no partition of players considered here, this theorem might be considered a coexistence or coincidence theorem. If k=1, Theorem 3 becomes the Scarf core existence theorem(Theorem 1); if $|N_i| = 1$ for all $i = 1, \ldots, k$, our theorem becomes the Existence theorem of Nash Equilibrium; and if for all i, N_i equals the coalition s_i in Theorem 2, and $\Gamma_i(x_{-N_i})$ is the general cooperative game derived from the sub-parametric normal form game $\{s_i, X^j, u^j(x_{s_i}, x_{-s_i})\}$, then Theorem 3 becomes Theorem 2.

Our proof appeals to the Kakutani fixed point theorem, and the solution of Theorem 3 emerges as the fixed point. This approach is elementary in that it suggests an algorithm for obtaining a core solution of a normal form game. To see this, let the game in the following Lemmas 1-3 be derived from a normal form game. By working through the mathematical programming problems in (1) and (4), we can easily define a map satisfying the conditions of the Kakutani fixed point theorem. Then with the available algorithms for finding a fixed point, one can approximate a core solution. On the other hand, if the game is transformed into characteristic form, defining the same map will require additional steps.

Before we start to prove the theorem, we shall first prove three Lemmas involving a parametric game. If we omit the parameters, the three Lemmas will form a proof for the Scarf core existence (Theorem 1).

Consider a parametric general cooperative game $\Gamma(y) = \{N, X_s, u_s(x_s,y)\}$, where $y \in X^y$ are the parameters and X^y is a closed bounded nonempty set in some finite-dimensional Euclidean space. For each $s \in \mathcal{N}$, $u_s: X_s \times X^y \rightarrow \boldsymbol{R}^S$ is coalition s' vector payoff function, with the i-th component equal to the payoff of its i-th member. Without loss of generality, we can assume that there is a constant c such that $0 << u_s(x_s, y) << c\, e_s$ for all $s \in \mathcal{N}$, $x_s \in X_s$ and $y \in X^y$, where $e_s = \{1, 1, ..., 1\} \in \boldsymbol{R}^S$. Let $\boldsymbol{R}^n_+ = \{ x \in \boldsymbol{R}^n / x_i \geq 0, \text{ all } i \}$ denote the nonnegative quadrant in $\boldsymbol{R}^n$, $\Delta = \Delta^n = \Delta^n_+ = \{ \lambda \in R^n_+ / \Sigma^n_{i=1} \lambda_i = 1\}$ denote the n-1 closed simplex.

To relate this to the proof of Theorem 3, we can take $\Gamma(y)$ as $\Gamma_1(x_{-N_1})$, the first parametric game in the collection of general cooperative games in our theorem, assuming $|N_1| > 1$. The set of players in $\Gamma(y)$ is then replaced by N_1, and $x_{-N_1} = \{x_{N_2}, \ldots, x_{N_k}\}$, the product of the joint strategies of $\{\Gamma_2, ..., \Gamma_k\}$ in the collection, are the corresponding parameter y.

Now let us return to our parametric game $\Gamma(y)$. For each coalition $s \in \mathcal{N}$, define a function $t_s: \Delta \times X^y \rightarrow \boldsymbol{R}$ by

(1) $$t_s = t_s(\alpha, y) = \underset{s.t.\ t>0;\ x_s \in X_s}{Max}\ t$$

(2) $$u_s(x_s, y) - t\, \alpha_s \geq 0$$

(3) $$c\, e_{-s} - t\, \alpha_{-s} \geq 0$$

for each $(\alpha, y) \in \Delta \times X^y$; where $\alpha = (\alpha_s, \alpha_{-s}) \in \Delta$, $e = (e_s, e_{-s}) = (1, 1, ..., 1) \in \boldsymbol{R}^n$, $c \in \boldsymbol{R}$ is a constant such that $0 < u^i_s(x_s, y) < c$ for all $i \in s$, $x_s \in X_s$, $y \in X^y$, and all $s \in \mathcal{N}$.

Define a function $\bar{t}: \Delta \times X^y \rightarrow \boldsymbol{R}$ by

(4) $$\bar{t} = \bar{t}\,(\alpha, y) = \underset{s \in \mathcal{N}}{Max}\ t_s(\alpha, y).$$

The following three lemmas will be based on Assumption I defined by

Assumption I: (1) For each $s \in \mathcal{N}$, X_s is a closed, bounded, nonempty and convex set, (2) for each $s \in \mathcal{N}$, $u_s(x_s,y)$ is quasi-concave in x_s and continuous in (x_s,y).

Lemma 1: If the parametric game $\Gamma(y) = \{N, X_s, u_s(x_s,y)\}$ satisfies Assumption I, then the following three claims hold:

(I). The function $f: \Delta \times X^y \rightarrow \boldsymbol{R}^n$ defined by

(5) $$f(\alpha,y) = \bar{t}\ \alpha = \bar{t}(\alpha,y)\ \alpha$$

is continuous in (α,y).

(II). The map $\theta: \Delta \times X^y \rightarrow 2^{\mathcal{N}}$ defined by

(6) $$\theta(\alpha,y) = \{s \in \mathcal{N} / t_s(\alpha,y) = \bar{t}(\alpha,y)\}$$

is nonempty and has a closed graph.

(III). For each value $\{f_1, f_2, \ldots, f_n\}$ of the function f defined in (5), let

(7) $$I(f) = \{ i \in N / f_i = \bar{t}\ \alpha_i = 0 \}, \quad J(f) = \{ j \in N / f_j = \bar{t}\ \alpha_j = c \}$$

where c is the constant such that $0 < u^i_s(x_s, y) < c$ for all $i \in s$, $x_s \in X_s$, $y \in X^y$, and $s \in \mathcal{N}$. Then $I(f) \neq \varnothing$ implies $J(f) \neq \varnothing$.

Proof of Lemma 1. (I)[3] We focus on the particular parametric mathematical programming problem (1), where $s \in \mathcal{N}$, $(\alpha,y) \in \Delta \times X^y$ are fixed. Since all the constraint functions are concave in (x_s, t) and are continuous in both (x_s, t) and (α,y), the feasible choice set is a closed bounded convex set, the objective function is concave and continuous in (x_s, t), it follows, by standard results regarding the stability of mathematical programming (For example, see Fiacco, 1983), that the extreme value function $t_s = t_s(\alpha,y)$ is continuous. Thus $\bar{t} = \bar{t}(\alpha,y) = \underset{s \in \mathcal{N}}{Max}\ t_s$ is continuous. Since the product of two continuous functions are continuous, $f(\alpha,y) = \bar{t}\ \alpha = \bar{t}(\alpha,y)\alpha$ is also continuous.

(II). The definition $\bar{t} = \bar{t}(\alpha,y) = Max\{t_s / s \in \mathcal{N}\}$ leads directly to the nonemptiness of $\theta(\alpha,y)$. To show that it has a closed graph, consider any sequence $\{\alpha_n, y_n\} \subset \Delta \times X^y$, $\{s_n\} \subset \mathcal{N}$, satisfying $s_n \in \theta(\alpha_n, y_n)$, $\{\alpha_n, y_n\} \rightarrow (\bar{\alpha},\bar{y}) \in \Delta \times X^y$, and $\{s_n\} \rightarrow \bar{s} \in \mathcal{N}$ as $n \rightarrow \infty$. Since the range of θ is finite, there exists an $\bar{n} > 0$ such that for all $n > \bar{n}$, $s_n = \bar{s}$ is constant. Thus $\bar{t}(\alpha_n, y_n) = t_{\bar{s}}(\alpha_n, y_n)$ for all $n > \bar{n}$. The continuity of $\bar{t}(\alpha, y)$ and $t_{\bar{s}}(\alpha, y)$ leads to $\bar{t}(\bar{\alpha}, \bar{y}) = t_{\bar{s}}(\bar{\alpha}, \bar{y})$. Hence $\bar{s} \in \theta(\bar{\alpha}, \bar{y})$ and θ has a closed graph.

(III). Assume, by way of contradiction, that $J(f) = \{j \in N / f_j = \bar{t}\ \alpha_j = c\} = \emptyset$. Let $s = I(f) = \{i \in N / f_i = \bar{t}\ \alpha_i = 0\} \in \mathcal{N}$. It is apparent that $\alpha_s = \{\alpha_i / i \in s\} = 0$ since our choice of the constant c can guarantee $\bar{t} > 0$. Thus for the mathematical programming problem (1) defined by this particular s, the constraints (2) and (3) become: $u_s^i(x_s, y) \geq 0$, for all $i \in s$; $c - t\ \alpha_i \geq 0$, for all $i \notin s$. Then $J(f) = \emptyset$ implies that $c - \bar{t}\,\alpha_i > 0$ for all $i \notin s$. This in turn implies that $t_s > \bar{t}$, which is absurd. Thus $J(f)$ is nonempty, and we have proved Lemma 1. **Q.E.D.**

For each $s \in \mathcal{N}$, h_s is the vector of $\boldsymbol{R}^n$ whose i^{th} component is $\frac{1}{|s|}$ if $i \in s$ and 0 otherwise. Clearly $h_s \in \Delta$. It is easy to verify that $\theta = \{s_i / s_i \in \mathcal{N}\} \in 2^{\mathcal{N}}$ is balanced if and only if there exists a positive number $b > 0$ such that

(8) $$be_N = b\,(1, 1, \ldots, 1) \in Co\{\, h_s / s \in \theta \,\}$$

where $Co\{A\}$ is the convex hull for any set A.

Define a continuous function $g: \Delta \times \Delta \to \Delta$ by

(9) $$g(\alpha,\beta) = \left\{ \frac{\alpha_1 + (\beta_1 - \frac{1}{n})^+}{1 + \sum_{i=1}^{n} (\beta_i - \frac{1}{n})^+} \quad \frac{\alpha_2 + (\beta_2 - \frac{1}{n})^+}{1 + \sum_{i=1}^{n} (\beta_i - \frac{1}{n})^+} \quad \cdots\cdots \quad \frac{\alpha_n + (\beta_n - \frac{1}{n})^+}{1 + \sum_{i=1}^{n} (\beta_i - \frac{1}{n})^+} \right\}$$

for each $(\alpha,\beta) \in \Delta \times \Delta$, where $r^+ = max\{r, 0\}$ for any real number. Define also a map σ : $\Delta \times X^y \to 2^{\Delta}$ by

(10) $$\sigma(\alpha, y) = Co\{\, h_s / s \in \theta(\alpha, y) \}$$

where $\theta(\alpha, y)$ is defined in (6).

Lemma 2 [Vohra, 1987]: Under Assumption I, the map π: $\Delta \times \Delta \times X^y \to 2^{\Delta \times \Delta}$ defined by

(11) $$\pi\,(\alpha, \beta, y) = \{\, g(\alpha,\beta) \,\} \times \sigma(\alpha, y)$$

satisfies:

(I). It has a closed bounded nonempty convex value and has a closed graph.

(II). For a fixed y, the map $\pi : \Delta\times\Delta\times X^y \to 2^{\Delta\times\Delta}$ is reduced to a map π' : $\Delta\times\Delta\to 2^{\Delta\times\Delta}$ by $\pi'(\alpha,\beta) = \pi(\alpha,\beta,y)$. Then π' has a fixed point $(\bar{\alpha}, \bar{\beta})$ (that is, there exist $(\bar{\alpha}, \bar{\beta})\in\Delta\times\Delta$ such that $(\bar{\alpha}, \bar{\beta})\in \pi'(\bar{\alpha}, \bar{\beta}) = \pi(\bar{\alpha}, \bar{\beta},y))$; and at this fixed point, $\theta(\bar{\alpha},y) = \{s\in\mathcal{N} / t_s = \bar{t}(\bar{\alpha},y)\}$ is a balanced set; where t_s, $\bar{t}$ and $\theta(\bar{\alpha},y)$ are given by (1), (4) and (6) respectively.

Proof of Lemma 2. (I) For each (α,β,y), $g(\alpha,\beta)$ is continuous, $\sigma(\alpha,y)$ is certainly a closed bounded nonempty convex set by properties of the convex hull and by the nonemptiness of $\theta(\alpha,y)$. Thus $\pi(\alpha,\beta,y)$ has a closed bounded nonempty convex value. To show that $\pi(\alpha,\beta,y) = \{g(\alpha,\beta)\}\times\sigma(\alpha,y)$ has a closed graph, we need only to show the closedness of $\sigma(\alpha,y)$, since g is continuous. Now consider any sequence $\{\alpha_n,y_n\}\subset\Delta\times X^y$, $\{z_n\}\subset\Delta$, satisfying $z_n\in\sigma(\alpha_n,y_n)$, $(\alpha_n,y_n)\to(\bar{\alpha}, \bar{y})\in\Delta\times X^y$, and $\{z_n\}\to\bar{z}\in\Delta$ as $n\to\infty$. Since $2^{\mathcal{N}}$, the range of θ, is finite, there exist a $\bar{\theta}\in 2^{\mathcal{N}}$ and a subsequence $\{\alpha_{n_i}, y_{n_i}\}$ of (α_n,y_n) such that $\bar{\theta} = \theta(\alpha_{n_i}, y_{n_i})$. Letting $n_i\to\infty$, we get $\bar{\theta}\subset\theta(\bar{\alpha}, \bar{y})$ by the closedness of θ (see Lemma 1, (II)). Since $z_{n_i}\in$ $\sigma(\alpha_{n_i},y_{n_i}) = Co\{h_s/s\in\theta(\alpha_{n_i},y_{n_i})\} = Co\{h_s/s\in\bar{\theta}\}$, $\{z_{n_i}\}\to\bar{z}$ as $n_i\to\infty$, and $Co\{h_s/ s\in\bar{\theta}\}$ is closed, we get $\bar{z}\in Co\{h_s / s\in\bar{\theta}\}\subset Co\{h_s/ s\in\theta(\bar{\alpha}, \bar{y})\} =$ $\sigma(\bar{\alpha}, \bar{y})$. Thus σ has a closed graph.

(II). It is now clear that for a fixed y, the map $\pi : \Delta\times\Delta\to 2^{\Delta\times\Delta}$ given by (11) is nonempty, closed, bounded and convex valued, and has a closed graph. Thus by the Kakutani fixed point theorem, there exists a fixed point $(\bar{\alpha}, \bar{\beta})\in\pi'(\bar{\alpha}, \bar{\beta}) = \pi(\bar{\alpha}, \bar{\beta},y)$ such that

$$(12)\quad \bar{\alpha} = g(\bar{\alpha}, \bar{\beta}) = \left\{\frac{\bar{\alpha}_1+(\bar{\beta}_1-\frac{1}{n})^+}{1+\Sigma_{i=1}^n(\bar{\beta}_i-\frac{1}{n})^+}\ \frac{\bar{\alpha}_2+(\bar{\beta}_2-\frac{1}{n})^+}{1+\Sigma_{i=1}^n(\bar{\beta}_i-\frac{1}{n})^+}\ \cdots\ \frac{\bar{\alpha}_n+(\bar{\beta}_n-\frac{1}{n})^+}{1+\Sigma_{i=1}^n(\bar{\beta}_i-\frac{1}{n})^+}\right\}$$

and

(13) $\overline{\beta} = \Sigma_{s\in \theta(\overline{\alpha},y)} \{ \lambda_s h_s\}$;

where $\lambda_s \geq 0$ and $\Sigma_{s\in \theta(\overline{\alpha},y)} \{ \lambda_s\} = 1$. (12) is equivalent to

(14) $\overline{\alpha}_j \{ \Sigma_{i=1}^{n} (\overline{\beta}_i - \frac{1}{n})^+\} = (\overline{\beta}_j - \frac{1}{n})^+$

for $j = 1, 2, ..., n$. We shall prove that

(15) $\overline{\beta}_j = \frac{1}{n}$

for j=1, 2, ..., n. For (15) to hold, it suffices to show that $(\overline{\beta}_i - \frac{1}{n})^+ = 0$ for all j. Since "$(\overline{\beta}_i - \frac{1}{n})^+ = 0$ for all j" if and only if "$\overline{\beta}_j \leq \frac{1}{n}$ for all j", the equality follows from $1 = \Sigma_{j=1}^{n} \overline{\beta}_j \leq \Sigma_{j=1}^{n} \frac{1}{n} = 1$. Assume by way of contradiction that $J^{**} = \{j\in N/ \ \overline{\beta}_j > \frac{1}{n}\} \neq \emptyset$.

Then from (14) we get

(16) $I^*(\overline{\alpha}) = \{i \in N/ \ \overline{\alpha}_i = 0\} = \{i \in N/ \ \overline{\beta}_i \leq \frac{1}{n}\}$ and

$J^*(\overline{\alpha}) = \{j \in N/ \ \overline{\alpha}_j > 0\} = \{j \in N/ \ \overline{\beta}_j > \frac{1}{n}\} = J^{**}$.

It follows from (7), (16) and $\overline{\tau} > 0$ that $I(f) = I^*(\overline{\alpha})$ for $f = f(\overline{\alpha},y) = \overline{\tau}\ \overline{\alpha}$. By $\Sigma_{j=1}^{n} \overline{\beta}_j = \Sigma_{j=1}^{n} \frac{1}{n} = 1$ we get $I(f) = I^*(\overline{\alpha}) \neq \emptyset$. Since for every $j\in J^*(\overline{\alpha})$, $\overline{\beta}_j > \frac{1}{n}$, and by (1) and (13), we can find $s\in \theta(\overline{\alpha},y)$ and $x_s \in X_s$ such that $j\in s$ and $0 < f_j(\overline{\alpha}, y) = \overline{\tau}\ \overline{\alpha}_j \leq u_s^j(x_s,y) < c$. This implies $J(f) = \emptyset$ and contradicts part (III) of Lemma 1. Thus $J^{**} = \emptyset$ and (15) holds. Now rewriting (13) as $(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}) = \Sigma_{s\in \theta(\overline{\alpha},y)}\{\lambda_s h_s\}$, and letting $e = e_N = (1, 1, ..., 1)\in \mathbf{R}^n$, we have

(17) $$\frac{1}{n} e_N = \frac{1}{n} (1, 1, \ldots, 1) = \Sigma_{s \in \theta(\bar{\alpha},y)} \{ \lambda_s h_s \}.$$

It follows from (8) that $\theta(\bar{\alpha},y)$ is a balanced set. This proves Lemma 2. **Q.E.D.**

Lemma 3: Let $x = x_N \in X_N = X$ be the joint strategies of all n players; $u = u_N(x_N) = \{u^1, \ldots, u^n\} \in \boldsymbol{R}^n$ be their payoffs, and $t_N = t_N(\alpha,y)$ be given by (1). Then under Assumption I, the map $\delta: \Delta \times X^y \to 2^X$ defined by

(18) $$\delta(\alpha,y) = \{x \in X \;/\; u(x,y) - t_N(\alpha,y)\alpha \geq 0\}$$

for each $(\alpha,y) \in \Delta \times X^y$ satisfies:

(I). It has a closed bounded nonempty convex value and has a closed graph.

(II). If $\theta(\alpha,y)$ is a balanced family and $\Gamma(y)$ is a balanced game for each y, then each $x \in \delta(\alpha,y)$ is a core solution of $\Gamma(y)$.

Proof of Lemma 3. (I) is obvious by the continuity of $u(x,y)$ and $t_N(\alpha,y)\alpha$ and by the concavity of $u(x,y)$ in x. To prove part (II), we shall show first that $t_N(\alpha,y) = \tau(\alpha,y)$, if $\theta(\alpha,y)$ is a balanced family and if $\Gamma(y)$ is balanced.

To see this, recall that $\theta(\alpha,y) = \{s \in \mathcal{N} / \; t_s = \tau\}$ and for each $s \in \theta(\alpha,y)$, there exists $x_s \in X_s$ such that $u_s(x_s,y) \geq \tau\, \alpha_s$. Since our game is balanced and $\theta(\alpha,y)$ is a balanced family, by Definition 1 there exists $x \in X_N = X$ such that $u(x,y) \geq f(\alpha,y) = \tau(\alpha,y)\alpha$. Thus $t_N(\alpha,y) \geq \tau(\alpha,y)$. The equality follows from (4).

Next we claim that $f(\alpha,y) = \tau(\alpha,y)\alpha$ is in the core of $\Gamma(y)$. Assuming by way of contradiction that $f = f(\alpha,y)$ is not in the core, then there exists $s \in \mathcal{N}$, $x_s \in X_s$ such that

(19) $$u_s(x_s,y) >> \tau\, \alpha_s.$$

By our choice of the constant c and the fact that $u(x, y) \geq f(\alpha, y) = \tau(\alpha, y)\, \alpha$ for some $x \in X_N = X$ (shown above), we have

(20) $$c\, e_{-s} >> u(x,y)_{-s} \geq \tau\, \alpha_{-s}.$$

Then for this particular s, the corresponding t_s defined by (1) satisfies $t_s > \bar{t}$, which is impossible. Thus $f(\alpha,y) = \bar{t}(\alpha,y)\alpha$ must be in the core of $\Gamma(y)$.

Now for each $x \in \delta(\alpha, y)$, $u(x, y) \geq t_N \alpha = \bar{t}(\alpha, y)\alpha$. Since $f(\alpha, y) = \bar{t}(\alpha,y)\alpha$ is in the core, $u(x,y)$ is also in the core and x is a core solution of $\Gamma(y)$. Lemma 3 is then proved. **Q.E.D.**

With Lemmas 1-3, we are now ready to prove our main theorem.

Proof of Theorem 3. Recall that we have k parametric cooperative games $\Gamma(x_{-N_i})$, which are intertwined together by the joint strategies of each component game. For each $\Gamma(x_{-N_i}) = \{N_i, x_s, u_s(x_s, x_{-N_i})\}$, $i = 1, 2, ..., k$, the number of players is $|N_i|$. For each coalition $s \in \mathcal{N}_i$ in $\Gamma(x_{-N_i})$, its strategy set is X_s, and its vector payoff is $u_s(x_s, x_{-N_i})$, which is parameterized by $x_{-N_i} = \{x_{N_1}, ..., x_{N_{i-1}}, x_{N_{i+1}}, ..., x_{N_k}\}$.

For each $i = 1, 2, ..., k$, let $X_{-N_i} = X_{N_1} \times ... \times X_{N_{i-1}} \times X_{N_{i+1}} \times ... \times X_{N_k} = \Pi_{j \neq i} X_{N_j}$. Now applying our Lemmas 1-3 to each $\Gamma(x_{-N_i})$, $i=1, 2, ..., k$, we can define the following maps, which are the counterparts of (9), (10), (11) and (18) respectively:

$$(21) \qquad \begin{aligned} g_i &: \Delta^{|Ni|} \times \Delta^{|Ni|} \rightarrow \Delta^{|Ni|} \\ \sigma_i &: \Delta^{|Ni|} \times X_{-N_i} \rightarrow 2^{\Delta^{|Ni|}} \\ \pi_i &: \Delta^{|Ni|} \times \Delta^{|Ni|} \times X_{-N_i} \rightarrow 2^{\Delta^{|Ni|} \times \Delta^{|Ni|}} \end{aligned}$$

and

$$\delta_i: \Delta^{|Ni|} \times X_{-N_i} \rightarrow 2^{X_{Ni}}.$$

Where $\pi_i(\alpha_i, \beta_i, x_{-N_i}) = \{g_i(\alpha_i, \beta_i)\} \times \sigma_i(\alpha_i, x_{-N_i})$ for each $(\alpha_i, \beta_i, x_{-N_i}) \in \Delta^{|Ni|} \times \Delta^{|Ni|} \times X_{-N_i}$. Obviously (21) satisfy all the assumptions made through Lemmas 1-3.

Define a set Ω by

$$(22) \qquad \Omega = \{\Delta^{|N1|} \times \Delta^{|N1|}\} \times \{\Delta^{|N2|} \times \Delta^{|N2|}\} \times \cdots \times \{\Delta^{|Nk|} \times \Delta^{|Nk|}\} \times X,$$

where $X = \Pi_{i=1}^{k} X_{N_i}$ is the product of the joint strategy sets for all $\Gamma(x_{-N_i})$. Define a map $\phi : \Omega \rightarrow 2^{\Omega}$ by

(23) $\phi\,(\alpha_1, \beta_1; \alpha_2, \beta_2; \cdots; \alpha_k, \beta_k; x) =$

$$\pi_1(\alpha_1,\beta_1,x_{-N_1}) \times\cdots\times \pi_k(\alpha_k,\beta_k,x_{-N_k}) \times \{\delta_1(\alpha_1,x_{-N_1}) \times\cdots\times \delta_k(\alpha_k,x_{N_{-k}})\}$$

for each $(\alpha_i, \beta_i) \in \Delta^{|Ni|} \times \Delta^{|Ni|}$, $i=1, 2, \ldots, k$ and $x=\{x_{N_1}, \ldots, x_{N_k}\} \in X$. Clearly ϕ is defined on a closed bounded convex set, has a nonempty closed convex value and has a closed graph. Thus by the Kakutani fixed point theorem, ϕ has at least one fixed point $(\bar{\alpha}_i,\bar{\beta}_i)$, $i=1, 2, \ldots, k$, and $\bar{x} =\{\bar{x}_{N_1}, \bar{x}_{N_2}, \ldots, \bar{x}_{N_k}\}$, such that $(\bar{\alpha}_i,\bar{\beta}_i) \in \pi_i(\bar{\alpha}_i,\bar{\beta}, \bar{x}_{-N_i})$, $\bar{x}_{N_i} \in \delta_i(\bar{\alpha}_i, \bar{x}_{-N_i})$ for $i=1, 2, \ldots, k$. It follows from our Lemmas 2 and 3 that each $\bar{x}_{N_i}$ is a core solution of $\Gamma(\bar{x}_{-N_i})$. Our theorem is finally proved. **Q.E.D.**

4. CONCLUDING REMARKS. We have proved an existence theorem for hybrid solutions for games without sidepayments. For an n-person game with sidepayments that satisfies the conditions of our theorem, it is easy to verify the existence of hybrid solutions without using the subtle techniques developed earlier. Consider each of the k subgames $\Gamma\,(x_{-s_i})$ induced by a partition $p = \{s_1, s_2, \ldots, s_k\}$. Because each of the $\Gamma\,(x_{-s_i})$ is played with sidepayments and satisfies the conditions of our theorem, the set of its core solutions $\delta_i\,(x_{-s_i})$ is nonempty, closed and convex, and has a closed graph with x_{-s_i}. Thus the map $\{\delta_1(x_{-s_1}) \times\cdots\times \delta_k(x_{-s_k})\}$ from X to 2^X satisfies the conditions of Kakutani's fixed point theorem and has a fixed point which is a hybrid solution.

For readers who have persisted to the end of our proof, it should be clear that Theorem 2 can be extended to generalized normal form games. By replacing each strategy set X^i by $X^i(x_{-s_i})$ and imposing some other mild conditions in addition to the conditions of our theorem, we can guarantee that all the maps defined through Lemmas 1-3 have the same properties, and thus the same proof will apply.

The hybrid solution concept can also be used in Multiple Objective Games (Zhao 1989). A multiple objective game (*MOG*) differs from a normal form game only in that each player has a vector payoff. The partition $p = \{s_1, s_2, \ldots, s_k\}$ of a scalar normal form game can be associated with a k-person *MOG* $\Gamma(x_{-s_i})$ (the same notation is used both for a game and for a *MOG*), where each player i has a vector payoff $u_{s_i}(x_s) = \{u^j(x_{s_i}, x_{-s_i})/ j \in s_i\}$. A hybrid solution corresponding to the partition p is then a particular non-cooperative solution for the k-person *MOG*.

There are many fields in which the hybrid solution concept may find applications. In sociology people are commonly partitioned into groups such as the family, a village, a state, a club or religious groups; in political science members of legislative bodies are often divided into two or more parties; in labor economics the owners, managers and employees of a firm are usually three separate interest groups; and in industrial economics firms selling a homogeneous product may be divided into a variety of competing groups, thus allowing for a much wider set of alliances and rivalries than in the Cournot and monopolistic equilibrium models. All of these cases imply the coexistence of cooperation and competition, thus the resulting outcome should be close to that implied by the hybrid solution concept.

Judging from the widespread applications of the Nash equilibrium and the core solution concepts, we can imagine a much wider range of applications for the hybrid solution concept. Wherever there exist conflicting interests among groups of individuals, the hybrid solutions should be able to find a niche.

1. It is common to see such coalitional structures in economic and social life. The ruling game between NATO and WTO (North Atlantic Treaty Organization, Warsaw Treaty Organization) is an example with two coalitions. Good examples rarely exist in industrial organization because of antitrust regulation, but many industries are close to the hybrid structure. For example, the auto industry is close to a two-coalition-structure, where the American car makers, in searching for protective polices and strategies, are competing with the Japanese car makers.

Theoretically, our hybrid solution concept shows that *in a simple one-period-one-commodity market, an array of intermediate solutions exist between the cooperative and non-cooperative division.* This is different from other intermediate solutions such as Chamberlin's monopolistic competition(1950), Robinson's inperfect competition(1950), and the semi-cooperation in Shubik and Thompson's game of economic survival(1959). For more discussions, see Chapter 2 in Shubik (1959).

2. To see this, let $F = \{p\} = \{\{s_1, s_2, \ldots, s_k\}\}$ contain the single partition p given in our theorem. For each $s \in \mathcal{N}$, $X_s = \prod_{i \in s} X^i$ is the strategy set of s (it is a constant in the Ichiishi context.). If $s \subset s_i$ for some i, let

$$u_s = \bar{u}_s(x_s, x_{-s_i}) = \{ \min_{x_{s_i \setminus s} \in X_{s_i \setminus s}} u^i(x_s, x_{s_i \setminus s}, x_{-s_i}) / i \in s\} \in \boldsymbol{R}^s$$

be the vector payoffs of s, where $x_{s_i} = (x_s, x_{s_i \setminus s})$ and $x_{-s_i} = \prod_{j \neq i} x_{s_j}$. If there is no s_i in the partition such that $s \subset s_i$ (or equivalently, there exist s_i and s_j, $i \neq j$ such that $s \cap s_i \neq \emptyset$, $s \cap s_j \neq \emptyset$), let $u_s \in \boldsymbol{R}^s$ be defined as $u_s = \{-\infty, \ldots, -\infty\}$. Then under the conditions of our Theorem 2, one can verify that Assumptions 1-5 in the Ichiishi Lemma(1981) are satisfied.

3. For those who prefer the game in characteristic form and who are familiar with the techniques involved, it can be seen that the same function $f(\alpha,y)$ is defined both in the present case and in the characteristic form. As suggested by Professor Ichiishi, let

$$V_s(y) = \bigcup_{x_s} \{u \in \boldsymbol{R}^n / \forall i \in s, u_i \leq u^i(s)(x_s, y)\},$$

$$W(y) = \{ u \in \bigcup_{s \in \mathcal{N}} V_s(y) / \forall j \in N, u_j \leq c\}.$$

Then the function $f: \Delta \times X^y \to \boldsymbol{R}^n$ defined by $f(\alpha,y) \in \partial W(y), f(\alpha,y) = t\,\alpha\ (t>0)$ is the same as that in (5), where c is the constant given earlier and $\partial W(y)$ is the boundary of $W(y)$. See Ichiishi (1988b) and Vohra (1987).

REFERENCES

[1]. Aumann, R. J. (1961) . The core of a cooperative game without side payments, *Trans. Amer. Math. Soc.*, Vol. 98, 539-552.

[2]. Chamberlin, E. H. (1950). *The Theory of Monopolistic Competition.* Cambridge: Harvard University Press.

[3]. Fan, K. (1969). Extensions of two fixed point theorems of F. E. Browder. *Mathematische Zeitschrift*, Vol. 112, 234-240.

[4]. Fiacco, A. V. (1983). *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*. New York: Academic Press.

[5]. Ichiishi, T. (1981). A social coalitional equilibrium existence lemma. *Econometrica*, Vol. 49, 369-377.

[6]. Ichiishi, T. (1988a). Theorems on closed coverings of a simplex that are related to cooperative game theory. *Working paper*, Ohio State University, March 1988.

[7]. Ichiishi, T. (1988b). Alternative version of Shapley's theorem on closed coverings of a simplex. *Proceedings of the American Mathematical Society*, Vol. 104, No. 3, 759-763.

[8]. Kakutani, S. (1941). A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, Vol. 8, 457-459.

[9]. Keiding, H., and L. Thorlund-Petersen (1987). The core of a cooperative game without side payments. *Journal of Optimization Theory and Applications,* Vol. 54, 273-288.

[10]. Knaster, B., C. Kuratowski and S. Mazurkiewicz (1929). Ein beweis des fixpunktsatzes fur n-demensionale simplexe. *Fundamenta Mathematicae*, Vol. 14, 132-137.

[11]. Robinson, J. (1950). *The Theory of Imperfect Competition.* London: Macmillan.

[12]. Scarf, H. (1967a). The core of an n-person game. *Econometrica*, Vol. 35, 50- 69.

[13]. Scarf, H. (1967b). The approximation of fixed points of a continuous mapping. *SIAM J. Appl. Math.*, Vol. 15, 1328-1342.

[14]. Scarf, H.(1971). On the existence of a cooperative solution for a general class of n-person games. *Journal of Economic Theory*, Vol. 3, No. 2, 169-181.

[15]. Shapley, L. S. (1973). On balanced games without side payments, in T. C. Hu and S. M. Robinson (eds.), *Mathematical Programming*, New York: Academic Press, 261-290.

[16]. Shapley, L. S., and R. Vohra (1988). On Kakutani's fixed point theorem, the K-K-M-S theorem and the core of a balanced game. *Mimeo.*, University of California at Los Angeles and Brown University.

[17]. Shubik, M. (1959). *Strategy and Market Structure*. New York: John Wiley & Sons, Inc..

[18]. Shubik, M., and G. L. Thompson(1959). Games of economic survival. *Naval Logistics Research Quarterly*, Vol. 6, 111-123.

[19]. Sperner, E. (1928). Neuer beweis fur die invarianz der dimensionszahl und des gebietes, *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universitat*, Vol. 6, 265-272.

[20]. Vohra, R.(1987). On Scarf's Theorem on the non-emptiness of the core: a direct proof through Kakutani's fixed point theorem. *Working paper*, Brown University, January 1987.

[21]. Zhao, J. (1989). The equilibria of multiple objective games. *Working paper*, Economics Department, Yale University(Nov. 1989).