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WARRANTIES, DURABILITY, AND MAINTENANCE: TWO-SIDED MORAL HAZARD IN A CONTINUOUS-TIME MODEL

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Warranties, Durability, and Maintenance: Two-sided Moral Hazard in a Continuous-Time Model*

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ABSTRACT

We consider the provision of an optimal warranty in a continuous-time model with two-sided moral hazard. The optimal warranty must balance the producer's durability incentive and the buyer's maintenance incentive. Too little warranty protection gives the producer too much incentive to produce low durability, while too much warranty protection gives the consumer too much incentive to neglect maintenance. The derived optimal warranty is a "block warranty" that is high for an initial block of time and zero thereafter. The first-best would be available under a very high warranty for a very short time interval, except for the incentive this would create for the consumer to abuse the product to collect the warranty.

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1 Introduction

Product warranties are ubiquitous, at least in part by government mandate. In the United States, the Magnuson-Moss Warranty and Federal Trade Commission Act of 1978 requires that all consumer products sold for more than fifteen dollars be sold with a written warranty.[1] But this legal requirement does little to limit the terms of actual warranty contracts. Nonetheless, nearly all of these contracts share a common overall pattern of a high level of insurance for an initial block of time and no insurance against later breakdowns. We will refer to this generic warranty pattern as a "block warranty." In many cases, the insurance takes the form of a promise by the producer to repair early breakdowns. This paper presents a theoretical model of warranty provision over continuous time. The main result of the paper is that the optimal warranty in this model is a block warranty.

The economic force driving the warranty choice in the model is a desire to resolve a two-sided moral hazard problem. The two-sided moral hazard arises from the assumption that both the consumer and the producer take privately observed actions that affect the failure rate of the product.[2] By producing a less durable product, a producer can save costs, but the decrease in durability may be observable only indirectly through the failure rate. Similarly, the consumer can obtain direct benefits at the expense of increasing failures by neglecting to maintain the product or by abusing the product in various ways. Introducing a warranty has opposite impact on these two problems: having a warranty increases the producer's durability incentives but decreases the consumer's maintenance incentives. The optimal warranty in the second-best solution trades off these two moral hazard problems.

One particularly interesting aspect of consumer purchase of many items with warranties is the timing of the actions over many periods. The producer chooses durability once and for all before the sale is made. After the sale is made, the consumer chooses maintenance in many different periods, and the impact of maintenance on product failure is cumulative.[3] The timing of actions implies an asymmetry between the moral

[1]This paper uses the term "warranty" in the popular (narrow) sense, and attention is limited to promised remedies for failure rather than promised attributes or guarantees of quality or satisfaction or whatever. Legally, almost any statement made by the seller or printed on the package creates an explicit warranty, and there are also many sorts of implicit warranties.

[2]See Cooper and Ross (1985, 1988), Emons (1988), Mann and Wissink (1989), and Priest (1981).

[3]Of course, neglect may also increase the possibility of immediate failure. However, this feature seems less interesting and is not explicit in our model, although it is part of our description of where

hazard of producers and consumers. Once a product is delivered to the consumer, the producer can no longer affect its failure rate, but a consumer selects a maintenance level continuously throughout the life of the product. We find that this asymmetry leads a profit-maximizing producer to offer a block warranty.

If there is no upper bound on the warranty payment, the producer can achieve payoffs arbitrarily close to the first-best by offering a very large warranty for a very short period of time. This sort of warranty will give the producer good incentives because of the threat of a very large loss in the case of an early failure. Furthermore, the consumer's incentives are distorted only during the very short time interval. However, this extreme solution is not very practical, and it is more likely that the producer will not profitably offer an enormous warranty payment. Perhaps the most compelling reason is again moral hazard. As one example, any agents who are skilled at breaking can collect any difference between the warranty payment and the purchase price. Alternatively, if the producer knows how to produce a low-cost unit that will definitely survive for a short period, that also could defeat the extreme warranty. Another problem with the very large warranty over a short period is that it relies heavily on time-of-use being measured well by passage of time. In our formal model, we do not devote any effort to modeling the details of the reasons for a bound on the warranty, and the bound is taken to be an exogenous function of time, which for interpretation we usually think of as the smaller of the repair cost and the approximate residual value of the product. This bound is motivated by the case in which the consumer finds it costless to abuse the product to simulate a failure. In this case, a violation of the bound would give the consumer a money pump.

Our analysis contrasts with several views of optimal warranties to be found in recent literature. The model in Cooper and Ross (1988) is the closest to our analysis. They work in only two periods, and they do not model the effect of maintenance as cumulative, whereas cumulative maintenance over continuous time is an important and sensible ingredient of our model. Also, in their two-period model, the general pattern of maximal constrained warranty followed by no warranty doesn't mean much. Other articles studying two-sided moral hazard, including Cooper and Ross (1985), Emons (1988), Mann and Wissink (1989), and Priest (1981), work with single-period models and cannot hope to make multiperiod predictions.

Warranties can obviously insure risk-averse consumers against product failure, as

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the warranty bound, exogenous in the model, may come from.

Heal (1977) has argued. Insurance motives can play an important role in explaining warranty contracts. But not all consumers are plausibly risk-averse; for example, finance theory suggests that producers should be risk-neutral on gambles uncorrelated with aggregate risks. However, we observe warranties even when it must be a good approximation that consumers are risk-neutral. And, risk-averse consumers would obviously prefer a full warranty to the partial warranty usually observed.

Spence (1977), Grossman (1981) and Lutz (forthcoming) have all focused on how warranties can serve as a signal to consumers of product durability, while there is nothing to signal in our model. In general warranties can indeed be a signal when product durability or cost of durability is randomly generated and known privately by the producer. However, it seems doubtful that signaling alone can replace our model in explaining the observed intertemporal patterns of warranties.

Our model can have important implications for the legal issue of how to interpret implied warranties which say that the producer must repair or replace a product that fails soon after purchase. The law is generally vague about what is a reasonable amount of time over which such implied warranties should be in place. The length of the block warranty in our solution is a natural candidate, and our model suggests on what parameters the optimal warranty should depend and in what direction. Specifically, the specialization of our model with quadratic cost functions admits a closed-form solution and (perhaps more importantly) derivation of useful comparative statics.[4]

Section 2 describes the economic setting, the formal game, and the payoffs. Section 3 contains the definition of equilibrium and the solution of the game. Section 4 gives a closed-form solution for quadratic cost functions. Section 5 closes the paper.

2 The General Model

We consider a risk-neutral monopoly that sells its product to a group of identical risk-neutral consumers. We will talk of a single consumer; in doing so, we are assuming that consumers' contracts are not linked.[5] Before the time of sale, the producer chooses how

[4]The underlying reason for having implied warranties is not contained in our model, and presumably relates to the producer's cost of writing the warranty and the consumer's cost of interpreting it. However, we see no particular reason to think that including those reasons in the model would change the qualitative results in this paper.

[5]In principle, allowing the contracts to be linked would actually imply an improvement in welfare, but additional monitoring and reporting incentive problems would arise. For example, if producers must reimburse all consumers equally based on average failure rates, then the producer's incentives

durable to make the product, how to price the product, and what warranty to include with the product. At point of purchase, the consumer observes the warranty and the price but not the durability. Based on this information, the consumer chooses whether to purchase, and if so, chooses a maintenance plan. Our model treats these events as a two-move game: first the producer chooses durability, price, and warranty at a single decision node, then the consumer chooses a maintenance plan and whether to purchase at a separate decision node for each price and warranty pair. (It simplifies notation to have the consumer choose a maintenance plan even if no purchase is made.)

Before making explicit the formal definitions of the choice variables, consider first the timing of the choices and the names of the choice variables. At time -1, the firm chooses the price p, the warranty $w(\cdot)$, and the frailty ϕ. (Using frailty, the opposite of durability, simplifies notation later.) At time 0, the consumer learns the price p and warranty $w(\cdot)$ but not the frailty ϕ, then chooses the purchase decision $d(p, w(\cdot))$ and the neglect plan $n(t; p, w(\cdot))$. (Using neglect, the opposite of maintenance, simplifies notation later.) In much of the paper, the notation will suppress the dependence of the consumer's decision on p and $w(\cdot)$: d and $n(t)$ are short for $d(p, w(\cdot))$ and $n(t; p, w(\cdot))$, respectively. After purchase, the product has a maximum potential useful life from time 0 until time T. The actual useful life will be shorter if the product fails sometime before T.

Now we turn to more formal definitions of the choice variables. The price $p \in \Re_+$ is the price offered by the producer and which the consumer must pay to buy one unit of the good. The warranty $w : [0, T] \rightarrow \Re_+$ is a measurable function chosen by the producer; $w(t)$ is the amount the producer promises to pay the consumer in the event of a failure at time t. The upper bound $W(\cdot)$ is an exogenous constraint, and we assume that this function itself is measurable and uniformly bounded on $[0, T]$—as we will see later in Theorem 2, some boundedness is required for the problem to have a solution. As a leading case, we can think of $W(t)$ as being the smaller of the repair cost and the approximate residual value of the unit looking forward from t.[6] Without the constraint, agents would have a positive incentive to game the producer by purchasing the product

are maintained without hurting individual incentives. In a sense, inefficiency in our model is caused by the overall budget constraint that can prevent a Groves [1976] mechanism from achieving the first-best. Making the warranty payment to a third party (such as another consumer) unlinks producer and consumer incentives, but might introduce incentives for collusion. It might be interesting to model why it is that we do not see more reimbursement based on average performance; difficulty of monitoring and incentives to under-report when others receive the payments seem like plausible reasons.

[6]The exact residual value is endogenous, but to first approximation we can think of it as fixed.

to break it intentionally and collect the warranty.[7] It is possible to model the details of consumer abuse a number of different ways, for example by having a cost of abuse. Many of these ways can be handled by changing the function $W(\cdot)$. We will denote by $\mathcal{W}$ the set of feasible warranties, i.e., the set of measurable functions from $[0,T]$ to $\Re_+$ (up to modification on a set of measure zero[8]), such that $(\forall t \in [0,T])(w(t) \leq W(t))$. The frailty $\phi \in [0,\Phi]$ is the producer's input to the probability distribution of product failure. The purchase decision $d : \Re_+ \times \mathcal{W} \rightarrow \{0,1\}$ is the consumer's purchase decision: $d(p, w(\cdot))$ is 1 if the consumer will purchase when offered a price p and warranty $w(\cdot)$ and is 0 if not. Consumer neglect $n : [0,T] \times \Re_+ \times \mathcal{W} \rightarrow [0,1]$ is the consumer's input to the probability distribution of product failure, where $n(t; p, w(\cdot))$ gives the consumer's degree of neglect at time t when offered a price p and warranty $w(\cdot)$. In principle, the units used in measuring frailty and neglect are arbitrary. To keep the notation simple we have measured both in units that are closely related to the failure probability. Formally, it would be equivalent to express either or both in terms of money saved; only the particular expressions for concavity, Inada conditions, and so forth would change.

The absolute failure rate[9] at time t is taken to be a linear function of frailty and the cumulative neglect before t. Specifically,

$$\begin{aligned} f(t;\phi,n(\cdot);\lambda) &= \phi + \lambda \int_0^t n(\tau)d\tau \\ &= \phi + \lambda N(t), \end{aligned} \tag{1}$$

[7]One can devise an elaborate story about how such abuse could serve to punish the firm when many failures occur. Here is a sketch of how that would work. (Note: the rest of this footnote assumes the reader already understands the equilibrium derived in Section 3.) Suppose the firm offers a warranty equal to the repair cost on $[0,\epsilon]$, and the consumer plans to break and repair the product k times at time ϵ if (and only if) the product breaks n times on the interval $[0,\epsilon)$. Then from the distribution of a Poisson process, the expected warranty cost is approximately $(n+k)R\epsilon^n\phi^n e^{-\epsilon\phi}/n!$, while the derivative of the warranty cost with respect to ϕ is the cost times n/ϕ. For n large and ϵ small, we can choose k such that the derivative of warranty cost with respect to ϕ is whatever we want while the expected warranty cost is small. Since the derivative is what determines the firm's frailty choice at the margin, this means we can get first-best frailty with little dead-weight cost from the abuses. Furthermore, for ϵ small, the consumer's incentives for neglect are close to first-best.

[8]Changing the warranty $w(\cdot)$ or consumer neglect $n(\cdot)$ on a set of times of measure zero changes nothing. Therefore, we will emulate the informality from probability theory of referring to individual functions when literally speaking we should be referring to an equivalence class of functions that differ pairwise at most on a set of measure zero. For example, if we say there is a unique optimal warranty, we really mean that it is unique up to changes on a set of times of measure zero.

[9]Here absolute means as a proportion of the original population. The relative failure rate at t (the "hazard rate"), which is conditional on survival until t, would be the absolute failure rate divided by the probability of surviving until t.

where $N(t) \equiv \int_0^t n(\tau)d\tau$ is the consumer's cumulative neglect from 0 to t, and $\lambda > 0$ is an exogenous parameter. The probability that the product fails at or before time t is therefore

$$\begin{aligned} F(t;\phi,n(\cdot);\lambda) &= \phi t + \lambda \int_0^t (t-\tau)n(\tau)d\tau \\ &= \phi t + \lambda \mathcal{N}(t), \end{aligned} \tag{2}$$

where $\mathcal{N}(t) \equiv \int_0^t (t-\tau)n(\tau)d\tau$ is the cumulative impact of the consumer's neglect to date. We assume that $\Phi + \lambda T^2/2 \leq 1$ to ensure by (2) that F lies in $[0,1]$ for all $t \in [0,T]$ and all feasible ϕ and $n(\cdot)$.

The producer's only cash inflow is from sales. Cash outflows include warranty payments and production costs. The cost to the producer of producing a unit of frailty ϕ is given exogenously as $c(\phi)$, and the continuous and twice differentiable function $c : [0,\Phi] \rightarrow \Re_+$ is common knowledge. To keep the game simple and to ensure that the producer captures all the surplus in equilibrium, we assume that the production cost is incurred only if the consumer purchases.[10] It is assumed that $c(\cdot)$ satisfies $c' < 0$ (frailer is cheaper), $c'' \geq 0$ (decreasing returns), $c'(0) = -\infty$ (infinite marginal cost to producing perfect durability, an Inada condition), $c'(\Phi) = 0$ (another Inada condition), and $c(\Phi) = 0$ (fixing the origin). The producer's expected payoff is

$$d(p, w(\cdot)) \left[p - \int_0^T w(t) \left(\phi + \lambda \int_0^t n(\tau; p, w(\cdot))d\tau \right) dt - c(\phi) \right]. \tag{3}$$

Conditional on purchase, the producer receives the purchase price p and makes expected warranty payments of $\int_0^T w(t) \left(\phi + \lambda \int_0^t n(\tau; p, w(\cdot))d\tau \right) dt$. The producer incurs the production cost $c(\phi)$ only when there is a purchase.

The consumer's surplus is the net of the value obtained by owning the good while it is working, plus any warranty payment, less maintenance costs. A consumer benefits

[10]The assumption that production costs are conditional on purchase is a simple device for preventing consumers from making threats that allow them to extract surplus, hwich would make the equilibrium indeterminate. Without this assumption or some other structure, the threat of moving to an equilibrium in which the demand is zero and the durability a minimum ($\phi = \Phi$) is a credible one in many cases. Assuming production costs are conditional on purchase may actually be a good approximation in many cases in which there are many consumers and production and sales alternate: if total demand is incorrectly estimated, production is off by a trivial amount. Other, more elaborate devices for extracting all surplus seem common in practice. For example, if the warranty is fixed at the time of manufacture (or packaging) and the any unsold units are later sold, with warranty, to a third party at very low prices, then the manufacturer has an incentive to avoid warranty costs by producing good durability, even if they are not sure it will be sold at the price offered initially. Adding this stage to the game in the paper provides more palatable reason why the producer should receive all the surplus; however, the sharing of surplus is a tangential issue in this model and would not be a worthy focus of the model.

from a working product at a rate ρ per unit time, and pays maintenance costs at the rate $k(n(t))$ per unit time, both in present value terms. We will assume that the consumer pays just as much for maintenance of a unit whether or not it has failed—it is as if the consumer buys a nonrefundable maintenance contract. Provided the overall failure rate is small, this assumption changes payoffs little, but it makes the model much easier to solve because it implies that the impact of neglect is separable across time. The function $k : [0,1] \to \Re_+$ satisfies $k'' > 0$ (convexity), $k'(0) = \infty$ and $k'(1) = -\infty$ (Inada conditions), and $\min_{n\in[0,1]}(k(n)) = 0$ (it is feasible to do no maintenance).[11] Cumulative maintenance costs are given by $K(t) \equiv \int_0^t k(n(\tau))d\tau$. The consumer's expected surplus is

$$d(p, w(\cdot)) \left(\int_0^T (\phi + \lambda N(t))(\rho t + w(t))dt + (1 - \phi T - \lambda \mathcal{N}(T))\rho T - K(T) - p \right) \quad (4)$$

$$= d(p, w(\cdot)) \left(\int_0^T \left[\left(\lambda \int_t^T w(\tau)d\tau - \frac{\rho\lambda}{2}(T-t)^2 \right) n(t; p, w(\cdot)) - k(n(t; p, w(\cdot))) \right] dt \right.$$
$$\left. + \rho T - \frac{\phi\rho T^2}{2} + \phi \int_0^T w(t)dt - p \right). \quad (5)$$

In (4), the factor $d(\cdot)$ is there because consumer surplus is zero in the absence of a purchase. The integral gives the expected benefit and warranty given failure at some $t < T$, and the following term gives the probability of survival until T times the corresponding payoff, and the other term $-K(T)$ gives the total maintenance cost. The expression (5) is derived from (4) by substituting in the definitions of $N(\cdot)$, $\mathcal{N}(\cdot)$, and $K(\cdot)$, rearranging, and changing order of integration.

3 Characterization of Equilibrium

In Section 2, we described the economic choice situation and modeled it as a game with two plays. In this section, we characterize equilibrium in the game, defined to be any Nash equilibrium of the original normal form game, in strategies that survive iterated elimination of weakly dominated strategies. In other words, first we eliminate all inadmissable strategies again and again until all are gone. Then we look for a Nash equilibrium of the *original* game in strategies that survive throughout. Note that any such equilibrium is a Nash equilibrium in the remaining strategies as well, because a

[11] Taking $k'(1) = 0$ is a plausible alternative to $k'(1) = -\infty$. In economic terms, $k'(1) = -\infty$ admits expenditure of effort to increase failure while $k'(1) = 0$ does not. In the context of the model, taking $k'(1) = -\infty$ is the more convenient assumption because it avoids a potentially messy corner at $n(t) = 1$.

best response against all strategies is a best response against the subset remaining.[12] In this section it is shown that the game has an equilibrium and that the producer offers a block warranty in equilibrium.[13]

Let us summarize and review the game. First, the producer chooses a price p in $\Re_+$, a warranty $w \in \mathcal{W}$ (where $\mathcal{W}$ is the set of measurable functions from $[0,T]$ to $\Re_+$ such that $(\forall t \in [0,T])(w(t) \leq W(t))$), and frailty (or implicitly durability) $\phi \in [0,\Phi]$. Then, with knowledge of p and $w(\cdot)$ but not ϕ, the consumer chooses the purchase indicator $d \in \{0,1\}$ and the neglect plan (or implicitly the maintenance plan) n, a measurable function from $[0,T]$ to $[0,1]$. The producer's payoff is (3) and the consumer's payoff is (5). Parameters of the game include the positive numbers T, λ, and ρ, satisfying $\phi + \lambda T^2/2 \leq 1$, the cost functions $c(\cdot)$ and $k(\cdot)$ satisfying the regularity conditions introduced just before (3) and (4), and the warranty bound $W(\cdot)$, a uniformly bounded function from $[0,T]$ to $\Re_+$.

To compute the equilibrium, we must perform the iterated elimination of dominated strategies and then look for equilibrium. To characterize equilibrium, it is convenient to write down a problem (Problem 1 below) that generates the same solutions as the game we are analyzing. The problem has the flavor of a standard agency formulation and makes intuitive sense in those terms. Furthermore, starting from the formal game

[12]In a game with finitely many strategies, this is the same as looking for a Nash equilibrium in the ultimately surviving strategies. However, when there are infinitely many strategies, this is no longer true. For example, consider a producer setting a price to sell a costless unit of a good to a consumer with known reservation price 1. Then the producer chooses $p \in \Re_+$ to maximize dp, the consumer chooses $d : \Re_+ \rightarrow \{0,1\}$ to maximize $1 - dp$. In the first round of iterated elimination, the producer eliminates $p = 0$ and the consumer eliminates $d(p) = 0$ for $p < 1$ and $d(p) = 1$ for $p > 1$. In the second round, the producer eliminates $p > 1$ and $p < 1$ ($p < 1$ is dominated by $(p+1)/2$). All that remains for the producer is $p = 1$ and for the consumer to reply arbitrarily. Because the consumer is indifferent, $p = 1$ followed by $d = 0$ is an equilibrium. The producer would rather play any $p < 1$ than be in this equilibrium. The "problem" is that while each $p < 1$ was eliminated, no strategy that dominated it remains. More pathological examples exist as well, in which the original game has no equilibrium but the game in strategies surviving iterated admissibility has a unique equilibrium. For example, consider the following symmetric game in which players A and B each have strategies x_i in $\{-1\} \bigcup \Re_+$. Player A receives 1 if $x_A = -1$, x_A if $x_A \geq 0$ and $x_B = -1$, and $x_A x_B$ if $x_A \geq 0$ and $x_B \geq 0$. Iterated dominance eliminates all strategies except -1 in one round, and therefore play. However, any number over 1 beats -1 in response to -1 and in fact the original game has no equilibrium. This reinforces the importance of verifying that the equilibrium in the game in surviving strategies is also an equilibrium of the original game.

[13]Like other useful refinements of Nash Equilibrium, this refinement embodies strong implicit assumptions about out-of-equilibrium beliefs, and in some games this refinement can generate results that seem inappropriate. For the particular model in the paper, however, this refinement is simple to implement and generates reasonable results. Given the current state of game theory, it seems difficult to find any stronger justification for a choice of refinement.

makes explicit the assumptions that are usually left implicit.

Here is the "agency problem formulation" we will prove to be equivalent to the equilibrium of the game. In the problem, the producer offers a "block warranty" that equals $W(\cdot)$ at or before some critical time t^* and is 0 afterwards. This warranty is written as $W(t)I_{t\le t^*}$ where $I_{t\le t^*}$ is 1 if $t \le t^*$ and 0 otherwise. Also, x^+ is short for $\max(x,0)$.

Problem 1 *(producer's reduced-form problem) Choose $t^* \in [0,T]$, $n : [0,T] \to [0,1]$, $\phi \in [0,\Phi]$, and $w(t) = W(t)I_{t\le t^*}$ to maximize the total expected surplus*

$$\int_0^T (\phi + \lambda N(t))\rho t dt + (1 - \phi T - \lambda \mathcal{N}(T))\rho T - K(T) - c(\phi) \tag{6}$$

$$= \int_0^T \left(-\frac{\rho\lambda}{2}(T-t)^2 n(t;p,w(\cdot)) - k(n(t;p,w(\cdot))\right) dt$$

$$+\rho T - \frac{\phi\rho T^2}{2} - c(\phi) \tag{7}$$

subject to

$$-c'(\phi) = \int_0^{t^*} W(t)dt, \quad \textit{incentive compatibility of frailty} \tag{8}$$

$$-k'(n(t)) = \frac{\rho\lambda}{2}(T-t)^2 - \lambda \int_t^{t^*} W(\tau)^+ d\tau, \quad \textit{incentive compatibility of neglect} \tag{9}$$

$$K(t) \equiv \int_0^t k(n(\tau))d\tau, \quad \textit{definition of } K(\cdot) \tag{10}$$

$$N(t) \equiv \int_0^t n(\tau)d\tau, \quad \textit{definition of } N(\cdot) \tag{11}$$

and

$$\mathcal{N}(t) \equiv \int_0^t (t-\tau)n(\tau)d\tau. \quad \textit{definition of } \mathcal{N}(\cdot) \tag{12}$$

Theorem 1 *There is a one-to-one correspondence between the set of solutions to Problem 1 and the set of equilibrium paths of our game. In the correspondence, t^* in Problem 1 is associated with the block warranty of the form*

$$w(t) = \begin{cases} W(t) & \textit{for } t \le t^* \\ 0 & \textit{for } t > t^* \end{cases}$$

along the equilibrium path of the game, demand $d = 1$, a price p that extracts all surplus, and the other choice variables are labeled the same in both.

PROOF We want to characterize the equilibrium. Along the way, we will show that the constraints of Problem (1) are satisfied, that the optimal warranty is a block warranty, and that the equilibrium path has the producer extracting all surplus and the consumer purchasing the unit. First we have iterative elimination of weakly dominated strategies. Here is what happens iteration by iteration. To make the proof more understandable, we are describing only the important eliminations in each step; an elimination of all dominated strategies at each step would eliminate some sooner, but would not otherwise change anything along the equilibrium path.[14]

Iteration 1: producer. The important elimination is of all strategies that do not satisfy

$$-c'(\phi) = \int_0^T w(t)dt, \tag{13}$$

which is the first-order condition for optimality of ϕ given $w(\cdot)$ which is obtained by setting the derivative of (3) equal to zero. The convexity of $c(\cdot)$ implies that (13) is necessary and sufficient for an interior optimum and the Inada conditions imply that for each $w(\cdot)$ there is a unique ϕ satisfying (13). Note that measurability and the range of $w(\cdot)$ imply the condition is always well-defined.

Iteration 1: consumer. The important elimination is of all strategies that do not satisfy

$$-k'(n(t)) = \frac{\rho\lambda}{2}(T-t)^2 - \lambda\int_t^T w(t)dt, \tag{14}$$

for all $(p, w(\cdot))$ for which $d(p, w(\cdot)) = 1$. This is the first-order condition for optimal choice of neglect $n(\cdot)$ given $w(\cdot)$ and given $d(p, w(\cdot)) = 1$.[15] Convexity of $k(\cdot)$ implies that (14) is necessary and sufficient for an interior optimum, and our Inada conditions imply that for each $w(\cdot)$ there is a unique $n(\cdot)$ that satisfies (14). Note that measurability and the range of $n(\cdot)$ imply the condition is always well-defined.

Iteration 2: producer. No important eliminations.

Iteration 2: consumer. By now, the consumer can infer ϕ from $w(\cdot)$ using (13), which holds for all remaining strategies. Let $\phi^*(w(\cdot))$ be the unique value of ϕ that solves (13) given $w(\cdot)$. Then the consumer keeps only those strategies for which $d(p, w(\cdot)) = 1$

[14] For example, some firm strategies are known to be unprofitable in iteration 1 (because no opinion about consumer neglect is needed to see that) but we are not eliminating them until iteration 3. However, all such strategies involve gross underpricing and are far from the equilibrium path. Eliminating them later changes nothing.

[15] Recall that when we refer to $n(\cdot)$ we are implicitly referring to an equivalence class defined up to changes on a set of measure zero.

whenever $d(p, w(\cdot))$ multiplies a positive number in (5) and for which $d(p, w(\cdot)) = 0$ whenever $d(p, w(\cdot))$ multiplies a negative number in (5). (When $d(p, w(\cdot))$ multiplies zero in (5), both $d(p, w(\cdot)) = 0$ and $d(p, w(\cdot)) = 1$ remain.) In words, this elimination implies that for remaining strategies (for which p and $w(\cdot)$ imply ϕ and $n(\cdot)$ through (13) and (14)), a purchase is made if purchase would give the consumer positive surplus, no purchase is made if purchase would give the consumer negative surplus, and we don't know yet if a purchase is made if purchase would give the consumer zero surplus.

Iteration 3: producer. There are different important types of eliminations. We will discuss them as if they are sequential even though the eliminations are actually simultaneous.

Iteration 3a: producer. The producer eliminates all warranty and price pairs for which purchase would give the consumer negative surplus, given (13) and (14). These are eliminated because they imply zero demand and therefore zero producer surplus, and because there exist strategies that ensure positive surplus to the producer (take a zero warranty and a price just less than the consumer's reservation price[16]).

Iteration 3b: producer. The producer eliminates all price and warranty pairs that give the consumer positive surplus (under (13) and (14)) or for which the warranty is not a block warranty. In particular, we can dominate any surviving $(p, w(\cdot))$ for which $w(\cdot)$ is not a block warranty, by choosing the block warranty $W(t)I_{t\leq t^*}$ for t^* such that $\int_0^{t^*} W(t)dt = \int_0^T w(t)dt$ and a slightly higher price. This change reduces $\int_t^T w(t)dt$ for some nonnull set of t without increasing it for any t. Therefore, by (14) it decreases neglect towards the neglect plan that maximizes total surplus given ϕ, therefore increasing total surplus by convexity of $k(\cdot)$. Hence, there is some new price that will ensure purchase and extract more surplus than was available under $(p, w(\cdot))$. Now that all remaining strategies have block warranties, we can specialize (13) and (14) to (8) and (9).

Iteration 3c: producer. The producer eliminates all price and warranty pairs that do not give the highest possible total surplus consistent with (8) and (9). The dominance in this is by choosing the warranty that does give the highest surplus and a price that gives the consumer half of the increase in total surplus. This is better for the producer and ensures positive demand (because we already eliminated strategies which gave

[16]To see that the consumer's reservation price is positive, note first that the expected life of the product is positive whatever ϕ and $n(\cdot)$ are. This is true in particular for ϕ satisfying (13) and $n(\cdot)$ always 1, which therefore has positive surplus. Taking instead $n(\cdot)$ satisfying (14) increases surplus and also has positive surplus.

the consumer negative surplus). Now, any remaining warranty is the block warranty corresponding to the t^* in some solution to Problem 1.

Iteration 3d: producer. The producer eliminates all price and warranty pairs that give the consumer any surplus on purchase. Given any $(p, w(\cdot))$ that gives the consumer any surplus, leaving $w(\cdot)$ the same but increasing p to take away half as much surplus still ensures positive demand but is strictly better for the producer.

Iteration 3: consumer. No eliminations.

After these three iterations, all that remains for the producer is to offer the block warranty that corresponds to t^* in some solution of Problem 1, the corresponding ϕ in the solution of Problem 1, and the corresponding price that extracts all surplus. All that remains for the customer is to reply with an arbitrary purchase decision and, if there is purchase, the neglect plan in the solution to Problem 1. All we have left to show is that the consumer must purchase in every equilibrium. But this follows from the fact that the equilibrium must be a Nash equilibrium after iteration 2. At that point, the producer can attain arbitrarily close to the value of Problem 1 by offering the warranty that solves Problem 1 and a slightly lower price. The lower price gives the customer some surplus to ensure purchase.

At this point, we can summarize an equilibrium that satisfies the conditions of this theorem.[17] Take any solution of Problem 1. The producer plays the warranty and frailty in that solution, and the corresponding price that extracts all surplus. The consumer choose $n(\cdot)$ to solve (14) and chooses to purchase if and only if that gives nonnegative surplus given the observed p and $w(\cdot)$, the neglect chosen, and ϕ implied by (14). Given the analysis in this proof, it is trivial to verify that these strategies give a Nash equilibrium of the original game in strategies that survive iterated elimination of all weakly dominated strategies. ∎

We have thus demonstrated that the presence of two-sided moral hazard leads to all-or-nothing warranties of the type commonly observed. A constant level of warranty protection is provided for some time period. Afterwards, the warranty drops to zero.

While the first-best outcome can never be achieved under two-sided moral hazard, we can show that the firm's profit in the second-best outcome approaches its first-best level as the warranty bound increases.

[17]There are typically many equilibria that agree on the equilibrium path but differ elsewhere. These different equilibria may correspond to the same solution of Problem 1.

Theorem 2 *As the warranty bound becomes uniformly large near the beginning, we approach the first-best. More formally, fix $\varepsilon > 0$. As the exogenous upper bound $W(\cdot)$ on the warranty payment increases uniformly without bound on $[0, \varepsilon]$, equilibrium payoffs to the solution of Problem 1 approach the first-best level.*

PROOF Let $[0, \varepsilon]$ be containted in the (relative) neighborhood of 0 in which the warranty payment is increasing uniformly without bound. When the warranty payment is uniformly bigger than at least $\frac{\rho T^2}{2\varepsilon}$, then there exists t^* such that $\int_0^{t^*} W(t)dt = \frac{\rho T^2}{2}$. Suppose then that the firm chooses a warranty that offers a payment of $W(t)$ for breakdowns occurring before time t^*, and no payment for any later breakdown. According to (8), this warranty induces the firm to produce the first-best level of frailty ϕ such that $\frac{\rho T^2}{2} = c'(\phi)$. Therefore, any loss due to the two-sided moral hazard problem will stem from second-best effort on the part of the consumer. However, any deviation from first-best neglect comes before the warranty ends at t^*. As $W \uparrow \infty$ uniformly on $[0, \varepsilon]$, $t^* \downarrow 0$, and it is easy to verify that the impact of the deviation from first-best neglect goes to zero. ∎

4 Solution for a Quadratic Example

Our results suggest that we can derive the optimal warranty for any specification of the two cost functions and the warranty bound. By Theorem 1, we need only solve for the optimal warranty expiration date t^*. This allows us to derive comparative statics on the optimal warranty. Following this procedure when both of the cost functions are quadratic and the warranty bound is a constant, we obtain a closed-form solution for t^* which itself solves a quadratic equation.

Specifically, take the cost functions to be

$$c(\phi) \equiv \gamma(\Phi - \phi)^2$$

and

$$k(n(t)) \equiv \kappa(1 - n(t))^2,$$

and assume a constant warranty bound $W(t) \equiv W$. While this example does not satisfy all of the regularity assumptions we used earlier, it is straightforward to follow the outline of Theorem 1 to show that the following equations give the unique solution along the equilibrium path, provided the answer makes economic sense (that is, provided $(\forall t \in [0,T])(n(t) \geq 0)$, $0 \leq \phi \leq \Phi$, $t^* \leq T$, and $\Phi + \lambda T^2/2 \leq 1$).

$$Wt^* = -c'(\phi) = 2\gamma(\Phi - \phi)$$

$$\phi = \Phi - \frac{Wt^*}{2\gamma}$$

$$c(\phi) = \frac{W^2 t^{*2}}{4\gamma}$$

$$\frac{\rho\lambda}{2}(T-t)^2 - \lambda W(t^* - t)^+ = -k'(n(t)) = 2\kappa(1 - n(t))$$

$$n(t) = 1 - \frac{\rho\lambda}{4\kappa}(T-t)^2 + \frac{\lambda W}{2\kappa}(t^* - t)^+$$

$$k(n(t)) = \frac{\rho^2\lambda^2}{16\kappa}(T-t)^4 - \frac{\rho\lambda^2 W}{4\kappa}(T-t)^2(t^*-t)^+ + \frac{\lambda^2 W^2}{4\kappa}[(t^*-t)^+]^2$$

$$\begin{aligned} K(t) &= \frac{\rho^2\lambda^2}{80\kappa}(T^5 - (T-t)^5) + \frac{\rho\lambda^2 W}{16\kappa}((T - t\wedge t^*)^4 - T^4) \\ &\quad + \frac{\rho\lambda^2 W}{12\kappa}(T-t^*)[T^3 - (T - t\wedge t^*)^3] + \frac{\lambda^2 W^2}{12\kappa}[t^{*3} - (t^* - t\wedge t^*)^3] \end{aligned}$$

$$K(T) = \frac{\rho^2\lambda^2}{80\kappa}T^5 + \frac{\rho\lambda^2 W}{16\kappa}((T-t^*)^4 - T^4) + \frac{\rho\lambda^2 W}{12\kappa}(T-t^*)[T^3 - (T-t^*)^3] + \frac{\lambda^2 W^2}{12\kappa}t^{*3}$$

$$N(t) = t - \frac{\rho\lambda}{12\kappa}(T^3 - (T-t)^3) + \frac{\lambda W}{4\kappa}(t^{*2} - (t^* - t\wedge t^*)^2)$$

$$\mathcal{N}(t) = \frac{1}{2}t^2 - \frac{\rho\lambda}{12\kappa}T^3 t + \frac{\rho\lambda}{48\kappa}(T^4 - (T-t)^4) + \frac{\lambda W}{4\kappa}t^{*2}t - \frac{\lambda W}{12\kappa}(t^{*3} - (t^* - t\wedge t^*)^3)$$

$$\int_0^T tN(t)dt = \frac{T^3}{3} - \frac{3\rho\lambda T^3}{80\kappa} + \frac{\lambda W t^{*2}T^2}{8\kappa} - \frac{\lambda W t^{*4}}{48\kappa}$$

Given these results, the terms of the objective function of Problem 1 that involve t^* are

$$-\frac{\lambda^2 W^2}{12\kappa}t^{*3} - \frac{W^2}{4\gamma}t^{*2} + \frac{\rho W T^2}{4\gamma}t^*.$$

For $t^* \geq 0$, this function is strictly concave (because the first two terms are strictly concave and the third is linear). Furthermore, its derivative is positive at 0 and negative for t^* large. Therefore, there is a unique optimum that is the only positive value of t^* that makes the derivative zero. The derivative is a quadratic, and the condition that it is equal to zero can be written as

$$\frac{\lambda^2\gamma}{\kappa}t^{*2} + 2t^* - \frac{\rho T^2}{W} = 0.$$

We can use the quadratic formula to obtain the exact solution, which is

$$t^* = \frac{\kappa}{\lambda^2\gamma}\left(\sqrt{1 + \left(\frac{\kappa}{\lambda^2\gamma}\right)^{-1}\frac{\rho T^2}{W}} - 1\right).$$

Comparative statics can be obtained from this expression or directly from the first-order condition. We find that t^* is increasing in T, so that the longer the product's useful life, the longer the warranty. Intuitively, a longer product life decreases the severity of the consumer moral hazard problem caused by a warranty of fixed length. This relaxation of the constraint imposed by consumer moral hazard means that the second-best solution calls for a longer warranty and a more durable product. An increase in ρ means that the consumer values a working product more highly. This also reduces the severity of the consumer moral hazard problem, increasing t^*. An increase in the warranty bound W decreases the duration of the warranty, for a similar reason. Higher W increases the incentive provided by a warranty of fixed duration, and this relaxation of the constraint imposed by producer moral hazard results in a second-best solution calling for a shorter warranty and less consumer neglect. Finally, t^* increases in the ratio $\kappa/\lambda^2\gamma$. This ratio can be interpreted as a measure of the *relative* severity of the two moral hazard problems.[18] Increasing κ reduces the severity of the consumer's moral hazard problem, and increasing the parameter γ reduces the severity of the producer's moral hazard problem, almost symmetrically. And, increasing λ has the same effect on the consumer's problem as decreasing κ, because increasing the effect of neglect is essentially the same as reducing the cost of neglect. These arguments form the basis of interpreting monotonicity of t^* in the ratio $\kappa/\lambda^2\gamma$ as dependence of the optimal length of the warranty on the *relative* severity of the two moral hazard problems.

[18] It is not obvious *a priori* how to measure the relative severity of the two moral hazard problems. One reasonable measure is the difference in total surplus between using cost-minimizing neglect and first-best neglect. This measure works fine for the quadratic case and is used implicitly in the text. However, more generally, we should probably develop some sort of measure that works at the margin.

5 Conclusion

Almost all commonly observed warranties are block warranties with one of two forms (or bundles of these warranties for various components). Each warranty either offers repair or replacement for a block of time, or it offers a declining warranty based on time or some other measure of use for a block of time. We have presented a theoretical model in which the optimal warranties that arise endogenously are all block warranties of the sort commonly observed in practice.

As always, a number of avenues remain for future research. Stochastic repair costs may make repair by the producer dominate a cash payment. Stochastic residual value observed privately by the consumer could lead to a trade-off between the warranty offered and the number of cases of abuse to collect on the warranty, and would explain why some tire warranties have declining schedules that are very conservative estimates of the residual value. One generic feature that could be added to the formal model is a strategic analysis of whatever is bounding the warranties. In some cases, this will lead to a simple bound of the form analyzed in this paper, but we believe other cases will lead to a richer model. The game theory for any such extension will present some challenges as well. For example, in the case of modeling abuse by the consumer as a source of the bound, the procedure used here may lead to elimination of all strategies, because every abuse strategy is dominated in the contingency that the firm offers a warranty that offers a money pump.

Bibliography

1. Cooper, Russell, and Thomas W. Ross, "An Intertemporal Model of Warranties," *Canadian Journal of Economics* **21** (February 1988), 72–86.

2. Cooper, Russell, and Thomas W. Ross, "Product Warranties and Double Moral Hazard," *Rand Journal of Economics* **16** (Spring 1985), 103–113.

3. Emons, Winand, "Warranties, Moral Hazard and the Lemons Problem," forthcoming in *Journal of Economic Theory*

4. Grossman, Sanford, "The Informational Role of Warranties and Private Disclosure about Product Quality," *Journal of Law and Economics* **24** (December 1981), 461–484.

5. Groves, Theodore, "Information, incentives, and the internationalization of production externalities," in: S. Lin, ed., *Theory and measurement of economic externalities* (1976), Academic Press, New York.

6. Heal, G.K. "Guarantees and Risk Sharing," *Review of Economic Studies* **44** (1977), 549–560.

7. Lutz, Nancy A. "Warranties as Signals Under Consumer Moral Hazard," *Rand Journal of Economics*, forthcoming Summer 1989.

8. Mann, Duncan P. and Jennifer P. Wissink, "Hidden Actions and Hidden Characteristics in Warranty Markets" *International Journal of Industrial Organization*, forthcoming 1989.

9. Priest, George L., "A Theory of the Consumer Product Warranty," *Yale Law Journal* **90** (May 1981), 1297–1352.

10. Spence, A. Michael, "Consumer Misperceptions, Product Failure and Product Liability," *Review of Economic Studies* **44** (1977), 561–572.