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THE NONCOOPERATIVE EQUILIBRIA OF A TRADING ECONOMY WITH COMPLETE MARKETS AND CONSISTENT PRICES

by

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1. INTRODUCTION

An exchange economy with complete markets is described and a general theorem for the existence of active Nash equilibria is proved. It is further shown that under replication of traders, these equilibria approach competitive equilibria of the economy.

The model under discussion here was first proposed by L. Shapley and represents one of two[1] possible generalizations of the "single money" model described in Dubey and Shubik [3]. It has the pleasant feature that it yields consistent prices.

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[1]See Amir, Sahi, Shubik and Yao [1] for the other generalization of [3].

2. DESCRIPTION OF THE MODEL

Let $I_n = \{1, 2, \ldots, n\}$ and $I_m = \{1, 2, \ldots, m\}$ be the sets of traders and commodities, respectively; where both m and n are at least 2. We shall use superscripts α and β for traders, and subscripts i and j for comodities.

We assume that each trader α has a nonnegative initial endowment $a_i^\alpha \geq 0$ of each commodity i. The traders' utility functions u^α are assumed to be concave, increasing, and continuous from the nonnegative orthant R_+^m to R_+. Assume that there are at least two traders with positive initial endowments and utility functions which are continuously differentiable in the interior of R_+^m. Further, assume[2] that for these traders, the level sets of their utility functions through their initial endowments are completely contained in the interior of R_+^m.

For convenience, let us fix units such that

$$\sum_\alpha a_i^\alpha = 1 \quad \text{for all } i . \tag{0}$$

A bid by trader α is an $m \times m$ matrix $B^\alpha = (b_{ij}^\alpha)$ such that

$$b_{ij}^\alpha \geq 0 \qquad i, j \in I_m \tag{1}$$

and

$$\sum_i b_{ji}^\alpha \leq a_j^\alpha \qquad j \in I_m . \tag{2}$$

The i-th column of the matrix B^α is to be thought of as a vector of commodities that trader α offers in exchange for commodity i.

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[2]This assumption is satisfied, for instance, by Cobb-Douglas utility functions.

The <u>strategy set</u> of α is the set of all matrices B^{α} satisfying (1) and (2), and is denoted by S^{α}.

Write $S = S^1 \times \ldots \times S^{\alpha} \times \ldots \times S^n$. Then S is compact and convex and a point $B \in S$ represents an n-tuple of bids - one by each trader.

Let Γ be the game in which $B = (B^1, \ldots, B^n) \in S$ has the outcome determined in the following steps. First, we define the aggregate bid matrix $\bar{B}$ to be

$$\bar{b}_{ij} = \sum_{\alpha \in I_n} b^{\alpha}_{ij} \ . \tag{3}$$

<u>Definition 1</u>: Given an n-tuple of bids B in S, we say that a price vector p is <u>market-clearing</u>[3] (for B) if

$$p > 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i \bar{b}_{ij} = p_j \left(\sum_{i=1}^{m} \bar{b}_{ji} \right) \quad \text{all} \quad j \in I_m \ . \tag{4}$$

Such a price vector need not always exist; and even if it does exist it need not be unique. Its existence and uniqueness (up to a positive scalar multiple) depend, in a crucial way, upon the location of zero entries in $\bar{B}$. The relevant notion is

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[3] The prices are best thought of as "measures of relative worth" of the different commodities. Then (4) says that the cumulative worth of the aggregate commodity bundle being offered for commodity j is equal to the worth of the total amount of commodity j in the market.

Definition 2: A nonnegative square matrix A is said to be irreducible[4] if for every pair $i \neq j$, there is a positive integer $k = k(i,j)$ such that $a_{ij}^{(k)} > 0$; where $a_{ij}^{(k)}$ denotes the ij-th entry of the k-th power A^k of A .

We also need a related notion:

Definition 3: A nonnegative $\ell \times \ell$ matrix A is said to be completely reducible if there is a partition $J_1, \ldots, J_t$ of $\{1, \ldots, \ell\}$, such that

a) for each $s = 1, \ldots, t$, the $|J_s| \times |J_s|$ submatrix $A(J_s)$ of A (with rows and columns in J_s) is irreducible;

b) if $s \neq s'$ and $i \in J_s$ and $j \in J_{s'}$, then $a_{ij} = 0$.

In other words, a matrix is completely reducible if and only if (after a permutation of indices) it can be written in block-diagonal form such that each diagonal block is irreducible.

Lemma 1: Let $B \in S$ be an n-tuple of bids, and let $\bar{B}$ be the aggregate bid matrix as in (3). Then B has a market-clearing price vector if and only if $\bar{B}$ is completely reducible; this price vector is unique (up to a scalar multiple) if and only if $\bar{B}$ is irreducible.

We defer all proofs to a later section.

It will be of interest to us to be able to compute all possible market-clearing price vectors corresponding to a given B in S . We proceed as follows.

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[4]This is different from Definition 1.6 in Seneta [4] only in that the 1×1 matrix 0 is irreducible according to our definition but not irreducible according to [4].

Let $\Delta(\bar{B})$ be the diagonal matrix of row sums of $\bar{B}$, and write

$$\tilde{B} = \Delta(\bar{B}) - \bar{B} .$$

Then (4) may be rewritten succinctly as

$$p > 0 \quad \text{and} \quad p\tilde{B} = 0 . \tag{5}$$

Suppose $\bar{B}$ is completely reducible, and let $J_1, \ldots, J_t$ be as in Definition 3. For each $s = 1, \ldots, t$, let $\bar{B}(J_s)$ and $\tilde{B}(J_s)$ be the $|J_s| \times |J_s|$-submatrices of $\bar{B}$ and $\tilde{B}$ obtained by taking rows and columns in J_s. Given an m-vector p, we shall write $p(J_s)$ for the $|J_s|$-subvector obtained by taking the components in J_s.

The following statement is completely clear. For purposes of reference we will call it

Remark 1: If $\bar{B}$ is completely reducible and p satisfies (5), then

$$p(J_s) > 0 \quad \text{and} \quad p(J_s)\tilde{B}(J_s) = 0 .$$

Conversely, if for *each* s there is a vector q^s such that

$$q^s > 0 \quad \text{and} \quad q^s\tilde{B}(J_s) = 0$$

then there is a p satisfying (5) such that

$$p(J_s) = q^s , \quad s = 1, \ldots, t .$$

In view of Remark 1, it suffices to find market-clearing price vectors for *irreducible* matrices. This is the content of the next lemma.

Lemma 2: Suppose A is an irreducible $k \times k$ matrix. Let $\Delta(A)$ be the diagonal matrix of row sums of A, and let $\tilde{A} = \Delta(A) - A$. If $k \geq 2$, let $p_i = \tilde{A}_{i1}$ (where $\tilde{A}_{ij}$ is the cofactor of ij-th entry of $\tilde{A}$); if $k = 1$, let $p_1 = 1$; then $p = (p_1, \ldots, p_k)$ satisfies

$$p > 0 \quad \text{and} \quad p\tilde{A} = 0 .$$

Conversely, if q satisfies

$$q > 0 \quad \text{and} \quad q\tilde{A} = 0$$

then there is a positive scalar λ such that $q = \lambda p$.

This will also be proved in the next section.

Continuing with the description of Γ, the final holding by α as a result of the bids B is x^α, where

$$x_j^\alpha = \begin{cases} x_j^\alpha(p) = a_j^\alpha - \sum_i b_{ji}^\alpha + \sum_i b_{ij}^\alpha (p_i/p_j) & \text{if } p \text{ satisfies (4)} \\ a_j^\alpha & \text{if } p \text{ does not exist} \end{cases} \tag{6}$$

Lemma 1 and Remark 1 have an easy corollary, which we call

Remark 2: Suppose p, q are positive market-clearing prices corresponding to the bids B, then $x_j^\alpha(p) = x_j^\alpha(q)$ for all α, j.

In other words, the choice of a market-clearing price does not affect the final holdings.

Finally, the payoff to trader α is given by

$$\Pi^\alpha(B) = u^\alpha(x^\alpha) .$$

A <u>Nash Equilibrium</u> (or N.E.) of Γ is a pair (B,p) satisfying (4), with $B = (B^1, B^2, \ldots, B^n) \in S$ such that for each trader α in I_n

$$\Pi^\alpha(B^1, \ldots, B^n) = \sup_{T \in S^\alpha} \{\Pi^\alpha(B^1, \ldots, B^{\alpha-1}, T, B^\alpha, \ldots, B^n)\}$$

For later purposes, we shall also need to consider the k-fold replication ${}^k\Gamma$ of the game Γ. This is the game in which each player is replaced by k copies of himself, all with the same endowments and utility functions.

We will use $I_{n\times k}$ to denote the set of traders in ${}^k\Gamma$. Note that I_n may be regarded as the set of <u>trader types</u> for ${}^k\Gamma$. When considering ${}^k\Gamma$, the letters α and β will be used for typical elements in <u>both</u> I_n and $I_{n\times k}$. This will lead to no confusion, since the meaning will be clear from the context.

For ${}^k\Gamma$, (3) becomes

$$\bar{b}_{ij} = \sum_{\alpha \in I_{n\times k}} b^\alpha_{ij} \tag{7}$$

and (0) becomes

$$\sum_{\alpha \in I_{n\times k}} a^\alpha_i = k\Big(\sum_{\alpha \in I_n} a^\alpha_i\Big) = k .$$

A <u>type-symmetric Nash Equilibrium</u> (T.S.N.E.) of ${}^k\Gamma$ is an N.E. of ${}^k\Gamma$ such that traders of the same type use the same strategies.

3. THE MODIFIED GAME AND ACTIVE EQUILIBRIA

Observe that T.S.N.E.'s of ${}^k\Gamma$ exist for trivial reasons. For example the n-tuple of strategies in which no trader bids anything is clearly an N.E. with any $p > 0$.

However, as in Dubey and Shubik [3], we wish to prove that ${}^k\Gamma$ has nonpathological T.S.N.E.'s which converge to competitive equilibria as k approaches infinity.

Definition 4: A T.S.N.E. (B,p) of ${}^k\Gamma$ is said to be active[5] if $\bar{B}$ is irreducible (see Definition 2).

The main result of this section is

Theorem 1: For each k, ${}^k\Gamma$ has an active T.S.N.E. (B,p). Moreover, there is a constant $\eta > 0$ (independent of k and B), such that if p is normalized by requiring $\sum_i p_i = 1$, then $p_i \geq \eta$ for all i in I_m.

The proof is in several steps. We start with the proofs of Lemmas 1 and 2 which were deferred from the previous section. These are easy consequences of well-known facts about nonnegative matrices.

Proof of Lemma 2: Clearly $\det(\tilde{A})$ is zero ($\tilde{A}\cdot 1^t = 0$, where 1^t is the column vector of all 1's). So if p is as in the statement of the Lemma, then, by elementary linear algebra, $p\cdot\tilde{A} = 0$.

For $k = 1$, the rest of the Lemma is trivially true; so let us assume $k \geq 2$. In this case A can have no non-zero row (or column). Let

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[5] Since the $m \times m$ zero matrix is completely reducible, but not irreducible for $m \geq 2$ (as we have assumed in Section 2), it is clear that this definition excludes the trivial equilibrium. Moreover, Theorem 2 in Section 4 shows that this is the "right" notion.

$T = \Delta(A)^{-1}A$ and let c_{ij} be the ij-th cofactor of $(I-T)$, then $\tilde{A}_{ij} > 0$ if and only if $c_{ij} > 0$. By definition T is row-stochastic; so Theorem 2.3 of Seneta [4] (applied to T^t) implies that $c_{ij} > 0$ for all i and j ; in particular $p > 0$. Finally, $q\tilde{A} = 0$ is equivalent to $q = qT$; and by Perron-Frobenius Theory (Theorem 1.5 of Seneta [4]) 1 is an eigenvalue of T with multiplicity 1. This proves the uniqueness of p .

Q.E.D.

Proof of Lemma 1: In view of Lemma 2 and Remark 1, it only remains to prove that if there is vector p such that $p > 0$ and $p\bar{B} = 0$, then $\bar{B}$ is completely reducible.

With this in view, consider the matrices $A = I+\bar{B}$ and $T = \Delta(A)^{-1}A$. Then T is row stochastic and p satisfies $pT = T$. Also, $\bar{B}$ is completely reducible if and only if T is completely reducible.

A row-stochastic matrix may be thought of as the transition matrix of a Markov chain (see Section 4.1 of Seneta [4]); and it will be useful to think of T in these terms.

As in Section 1.2 of Seneta [4], we say that i leads to j if $t_{ij}^{(k)} > 0$ (for some $k = k(i,j)$) and write $i \to j$. If i does not lead to j , then we write $i \not\to j$. If $i \to j$ and $j \to i$, we say that i and j communicate, and write $i \leftrightarrow j$. An index i is said to be inessential if there is a j such that $i \to j$ but $j \not\to i$; otherwise i is said to be essential.

If i is essential and $i \to j$, then $i \leftrightarrow j$. So the essential states may be partitioned into equivalence classes such that all states belonging to a single class communicate, but cannot lead to a state outside the class.

It suffices to show that the existence of a $p > 0$ such that $pT = p$ implies that there are no inessential states.

Let us normalize p so that $\Sigma p_i = 1$; then p can be interpreted as a steady-state probability distribution for T . Let J_+ be the essential states and let J_- be the inessential states; then if $j \in J_-$, there is a $j \in J_+$ such that $t_{ij}^{(m)} > 0$. (Clearly $t_{ij}^{(k)} > 0$ for some essential j , with $k = k(i,j) \leq m$. Since $t_{ii} > 0$, we must have $t_{ij}^{(m)} > 0$.) In other words, if the process starts in J_- , there is a positive probability $\pi = \min_{i \in J_-} \max_{j \in J_+} (t_{ij}^{(m)})$ that it is in J_+ after m steps. Since, once the process leaves J_- it never returns, the probability that the process is still in J_- after ℓm steps is less than $(1-\pi)^{\ell}$ which approaches zero as ℓ tends to infinity. Consequently, any steady-state distribution must assign zero probability to all inessential states. Since $p > 0$, there can be no inessential states. Q.E.D.

As in Dubey and Shubik [3], it is convenient to consider slight perturbations of the game Γ .

Definition 5: Given $\varepsilon > 0$, we define the game $\Gamma(\varepsilon)$ as in Section 2, except that (3) is replaced by

$$\bar{b}_{ij} = \sum_{\alpha \in I_n} b_{ij}^{\alpha} + \varepsilon . \tag{8}$$

The interpretation is that some outside agency places fixed bids of ε for each pair (i,j) . This does not change the strategy sets of the various players, but does affect the prices, the final holdings and the payoffs.

The next step in the proof of Theorem 1 is to prove existence of N.E.'s for $\Gamma(\varepsilon)$. First, note that for $\Gamma(\varepsilon)$ with $\varepsilon > 0$, $\bar{B}$ is always irreducible, so the prices may be computed as in Lemma 2. Also, if α changes his bids along the diagonal of B^α , it does not affect the prices or the payoffs of any of the traders. The upshot is that we may restrict all the traders' strategy sets by requiring

$$\sum_i b^\alpha_{ji} = a^\alpha_j \tag{9}$$

without changing the game $\Gamma(\varepsilon)$ in any essential way.

The next remark is more subtle. For fixed bids B^β by traders other than α , define the matrix D by

$$d_{ij} = (\sum_{\beta\neq\alpha} b^\beta_{ij}) + \varepsilon = \bar{b}_{ij} - b^\alpha_{ij} \ . \tag{10}$$

Then (4) can be rewritten as

$$\sum_i p_i(d_{ij} + b^\alpha_{ij}) = \sum_i p_j(d_{ji} + b^\alpha_{ji})$$

or $$-\sum_i b^\alpha_{ji} + \sum_i b^\alpha_{ij}(p_i/p_j) = \sum_i d_{ji} - \sum_i d_{ij}(p_i/p_j) \ .$$

And substituting in (6), we obtain

$$x^\alpha_j = a^\alpha_j + \sum_i d_{ji} - \sum_i d_{ij}(p_i/p_j) \ . \tag{11}$$

In other words, the final holding by α depends on α's bid only through its effect on the prices!

So let us consider the possible prices that arise as α varies his bid in S^α .

First of all, notice that (0) and (9) imply that if $\bar{B}$ is the aggregate bid matrix (see (8)) at any B in S , then each row of $\bar{B}$ sums to $1+m\varepsilon$.

So (11) may be rewritten as

$$x_j^\alpha = (1+m\varepsilon) - \sum_i d_{ij}(p_i/p_j) \quad . \tag{12}$$

Let us write

$$C = (1+m\varepsilon)^{-1}D \quad \text{and} \quad A = (1+m\varepsilon)^{-1}\bar{B} \quad . \tag{13}$$

Then A is row stochastic, $A \geq C$ and (12) becomes

$$(1+m\varepsilon)^{-1}x_j^\alpha = 1 - \sum_i c_{ij}(p_i/p_j) \quad . \tag{14}$$

The next lemma is crucial. In fact, it is more or less the heart of the argument.

<u>Lemma 3</u>: In the game $\Gamma(\varepsilon)$, fix bids by all traders except α ; and let C be as in (13). Let us write P for the set of all positive multiples of price vectors that arise as α varies his bid in S^α . Then,

$$P = \{p > 0 : p \geq pC\} \quad .$$

Proof: By (13), we may write

$$P = \{p > 0 : \exists A \text{ row stochastic}; A \geq C; p = pA\} .$$

Clearly if $p \in P$, then

$$p = pA = p(A-C) + pC \geq pC .$$

Conversely, suppose $p > 0$ and $p \geq pC$. If 1^t is the column vector of all ones, then the row substochasticity of C may be expressed as

$$C1^t \leq 1^t .$$

Let $$v^t = 1^t - C1^t \geq 0 \tag{15}$$

and $$w = p - pC \geq 0 . \tag{16}$$

Observe that

$$pv^t = p1^t - pC1^t = w1^t .$$

Let $$\lambda = pv^t = w1^t . \tag{17}$$

Since each $a_i^\alpha > 0$, we must have $v^t = 1^t - c1^t > 0$; and so $\lambda > 0$. Now let A be given by

$$a_{ij} = c_{ij} + \frac{1}{\lambda} v_i w_j .$$

Then it is easily checked that A is row stochastic and $pA = p$. Consequently $p \in P$. Q.E.D.

In view of (14) above, Lemma 3 has an immediate implication for α's final holding set.

Lemma 4: In the setting of Lemma 3, let H be the set of possible final holdings attainable by α. Then,

(a) If $p, q \in P$, so does r where $r_i = (p_i q_i)^{1/2}$, $i = 1, \ldots, m$.

(b) If $x, y \in H$, there is a point $z \in H$ such that $z \geq \frac{1}{2}(x+y)$.

(c) There is a unique $\tilde{x}$ in H which maximizes α's utility function u^α.

(d) The price $\tilde{p}$ in P which corresponds to $\tilde{x}$ is uniquely determined up to a scalar.

(e) The set of strategies leading to $\tilde{x}$ is a compact, convex subset of S^α.

Proof: For part (a); note that by Lemma 3, $p_j \geq \sum_i p_i c_{ij}$, $q_j \geq \sum_i q_i c_{ij}$.

So by the Cauchy-Schwartz inequality,

$$\begin{aligned} r_j &= (p_j q_j)^{1/2} \\ &\geq (\sum_i p_i c_{ij})^{1/2} (\sum_i q_i c_{ij})^{1/2} \\ &\geq \sum_i (p_i c_{ij})^{1/2} (q_i c_{ij})^{1/2} \\ &= \sum_i (p_i q_i)^{1/2} c_{ij} = \sum_i r_i c_{ij} . \end{aligned}$$

So by Lemma 3, $r \in P$. This proves (a).

Next, let p and q be the prices at two strategies which yield x

and y, respectively. By part (a), there is a strategy which achieves the price vector r, where $r_j = (p_j q_j)^{1/2}$ for all j. Then if z is the final holding obtained by this strategy; we have by (14)

$$
\begin{aligned}
z_j &= \left[1 - \sum_i c_{ij}(r_i/r_j)\right](1+m\varepsilon) \\
&= \left[1 - \sum_i c_{ij}(p_i/p_j)^{1/2}(q_i/q_j)^{1/2}\right](1+m\varepsilon) \\
&\geq \left[1 - \sum_i c_{ij}\left(\frac{1}{2}(p_i/p_j) + \frac{1}{2}(q_i/q_j)\right)\right](1+m\varepsilon) \qquad (18) \\
&= \left[\frac{1}{2}\left(1 - \sum_j c_{ij}(p_i/p_j)\right) + \frac{1}{2}\left(1 - \sum_j c_{ij}(q_i/q_j)\right)\right](1+m\varepsilon) \\
&= \frac{1}{2}(x_j + y_j) \ .
\end{aligned}
$$

Where (18) is a consequence of the A.M.-G.M. inequality. This proves part (b).

Since u^α is concave and increasing, (c) and (d) follow from the <u>strict</u> A.M.-G.M. inequality in (18).

For (e), the set in question is the set of all strategies yielding stochastic matrices A such that $\tilde{p}A = \tilde{p}$. But this set is clearly closed and convex. Since S^α is compact, this proves part (e). Q.E.D.

We can now prove the existence of N.E.'s for $\Gamma(\varepsilon)$.

<u>Lemma 5</u>: For each $\varepsilon > 0$, $\Gamma(\varepsilon)$ has an N.E.

Proof: Let $S = S^1 \times \ldots \times S^\alpha \times \ldots \times S^n$ as before. Given bids by all the traders except α, define the "best response set" of α to be the set of strategies in S^α which maximizes α's payoff. By Lemma 4, the "best response set" is compact, convex and non-empty. Thus, if Φ is the correspondence: $S \to 2^S$ given by

$$\Phi(B^1, \ldots, B^n) = \{(T^1, \ldots, T^n) : T^\alpha \text{ is in } \alpha\text{'s "best response set" with respect to } (B^1, \ldots, B^{\alpha-1}, B^{\alpha+1}, \ldots, B^n)\},$$

then Φ is upper semi-continuous by Berge [2] (P. 116). Also, by Lemma 4, the image of each point is compact, convex and non-empty. Thus, by Kakutani's fixed point theorem, there is a point B in S such that $B \in \Phi(B)$. Such a B is easily seen to be an N.E. Q.E.D.

Since $\varepsilon > 0$, the matrix $\bar{B}$ as defined by (8) is clearly irreducible for all B. However, we wish to examine the equilibria for $\Gamma(\varepsilon)$ as $\varepsilon \to 0$, and the limiting aggregate bid matrices (even if they exist) need not be irreducible. So we need a slight strengthening of Lemma 5.

Definition 6: For $\delta > 0$, a strategy B^α in S^α will be called <u>δ-positive for α</u> if, for each $J \subsetneq I_m$,

$$\sum_{i \notin J} \sum_{j \in J} (b^\alpha_{ij} + b^\alpha_{ji}) \geq \delta \ . \tag{19}$$

An n-tuple $B = (B^1, \ldots, B^n)$ will be called <u>δ-positive</u>, if B^α is δ-positive for <u>each trader</u> α <u>with positive endowments</u> of all commodities. (Recall that there are at least <u>two</u> such traders.)

The concept of δ-positivity is just right for our purposes. As an illustration we have

Remark 3: If B is δ-positive and $\bar{B}$ (as defined by (3)) is completely reducible, then $\bar{B}$ is irreducible.

To see this we need only apply (19) with $J = J_1$ (where $J_1, \ldots, J_t$ are as in Definition 3).

On the other hand, we can easily prove the following

Lemma 6: For each trader α in I_m, let

$$\delta(\alpha) = (1/m)\min_i a_i^\alpha . \tag{20}$$

If $\delta(\alpha) > 0$, then α has a $\delta(\alpha)$-positive "best response" to any choice of strategies by the remaining players.

Proof: Let D and C be as in (10) and (13), and let p be the "best response" price as in Lemma 4(d). Then to prove Lemma 6, we need to find A such that A is row-stochastic, $A \geq C$, $p = pA$ and

$$\sum_{i \notin J} \sum_{j \in J} (a_{ij} + a_{ji}) \geq \delta_1 \quad \text{where} \quad \delta_1 = (1+m\varepsilon)^{-1}\delta(\alpha) . \tag{21}$$

In view of Lemma 3 it suffices to find a substochastic matrix E such that $E \geq C$, $p \geq pE$ and for all $J \subset I_m$

$$\sum_{i \notin J} \sum_{j \in J} (e_{ij} + e_{ji}) \geq \delta_1 . \tag{22}$$

(Given E, we can apply Lemma 3 with E instead of C and obtain $A \geq E \geq C$ satisfying our requirements.)

To obtain such a matrix, note first that if v is as in (15), then (10), (13), (20) and (21) imply that

$$V_i \geq m\delta_1 \quad \text{for all} \quad i \ .$$

So if w is as in (16), then (17) gives

$$\sum_i w_i = \sum_i p_i v_i \geq (\sum_i p_i) m\delta_1 \ .$$

In particular, we can choose an index j_0 such that

$$w_{j_0} \geq (\sum_i p_i)\delta_1 \ . \tag{23}$$

Define E by

$$e_{ij} = \begin{cases} c_{ij} & \text{if} \quad j \neq j_0 \\ c_{ij} + \delta_1 & \text{if} \quad j = j_0 \end{cases} .$$

In other words, E is obtained by adding δ_1 to each entry in the j_0-th column of C .

Then (22) clearly holds, and we only need to show that

$$p - pE \geq 0 \ . \tag{24}$$

The only change from (16) is in the j_0-th component, which has now become

$$p_{j_0} - \sum_i p_i c_{ij_0} - (\sum_i p_i)\delta_1$$

$$= w_{j_0} - (\sum_i p_i)\delta_1 \ .$$

So (24) follows from (23). Q.E.D.

Let

$$\delta = \min\{\delta(\alpha) : \delta(\alpha) > 0\} , \tag{25}$$

and write $S(\delta)$ for the set of δ-positive n-tuples in S. (See (20) and Definition 6.)

Then $S(\delta)$ is a nonempty, compact and convex subset of S.

We can now obtain the desired strengthening of Lemma 5.

Lemma 7: Let δ be given by (25) then, for each $\varepsilon > 0$, $\Gamma(\varepsilon)$ has a δ-positive N.E.

Proof: Let Φ be defined as in the proof of Lemma 5. Consider the modification Φ' given by $\Phi'(B) = \Phi(B) \cap S(\delta)$.

Then by Lemma 6, $\Phi'(B)$ is compact, convex and non-empty for each B. Since Φ' is clearly u.s.c., it has a fixed point which is easily seen to be a δ-positive T.S.N.E. Q.E.D.

In Section 2 we described the game ${}^k\Gamma$ (k-fold replication of Γ). A natural analogue of Definition 5 is

Definition 7: Given $\varepsilon > 0$, we define the game ${}^k\Gamma(\varepsilon)$ as in Section 2 except that (7) is replaced by

$$\bar{b}_{ij} = \sum_{\alpha \in I_{n\times k}} b^{\alpha}_{ij} + k\varepsilon . \tag{26}$$

As in Dubey and Shubik [3], Lemma 7 may be refined to

Lemma 8: With δ as in (25); ${}^{k}\Gamma(\varepsilon)$ has a δ-positive T.S.N.E. for each k and each ε .

Proof: In the argument proving Lemma 7, let S^* be the set of type-symmetric strategies. Define

$$\Phi^* : S^* \to 2^{S^*}$$

by

$$\Phi^*(B) = \Phi'(B) \cap S^* .$$

Now S^* is compact and convex (in fact $S^* \approx S$). Furthermore (since in a type symmetric situation, players of the same type face the same optimization problem) $\Phi'(B) \cap S^* \neq \phi$. So $\Phi^*(B)$ is compact, convex and not empty. Since Φ^* is clearly u.s.c., Kakutani's theorem yields a fixed point of Φ^* which is easily seen to be a δ-positive T.S.N.E. of ${}^{k}\Gamma(\varepsilon)$.

Q.E.D.

The next step is to show uniform positivity of prices at various T.S.N.E.'s of the various ${}^{k}\Gamma(\varepsilon)$'s .

Lemma 9: There is a constant $\eta > 0$, such that: for all ε less than 1, and all k , if p is the price vector at any T.S.N.E. of any ${}^{k}\Gamma(\varepsilon)$, normalized so that $\sum_i p_i = 1$, then

$$p_i \geq \eta , \quad i \in I_m . \tag{27}$$

Proof: Let α and β be two traders who satisfy the stronger assumption in Section 2. First, notice that if x is an N.E. final holding by a trader of type α, we must have $u^\alpha(x) \geq u^\alpha(a^\alpha)$. So by the assumptions on u^α,

$$x_i > 0 , \quad i = 1, 2, \ldots, m . \tag{28}$$

Next, if $\Box = \{x : 0 \leq x_i \leq m+1\}$; then at any type-symmetric final holding of ${}^k\Gamma(\varepsilon)$ (with $\varepsilon \leq 1$), each trader's holding lies in $\Box$.

Let $H(\alpha) = \Box \cap \{x : u^\alpha(x) \geq u^\alpha(a^\alpha)\}$. Then $H(\alpha)$ is a compact subset of the interior of R^m_+ which contains all possible T.S.N.E. holdings by a trader of type α. Similarly define $H(\beta)$.

$$\text{Let} \quad M = 2 \max_{i,j} \left\{ \frac{\partial_i u^\alpha(x)}{\partial_j u^\alpha(x)}, \frac{\partial_i u^\beta(y)}{\partial_j u^\beta(y)} : x \in H(\alpha),\ y \in H(\beta) \right\}$$

(where $\partial_i f = \frac{\partial f}{\partial x_i}$). We will show that we can choose $\eta = 1/(mM^{m-1})$.

Suppose not. Then for some $\varepsilon > 0$, ${}^k\Gamma(\varepsilon)$ has a T.S.N.E. (B,p) with some $p_i < \eta$. We may assume without loss of generality that

$$p_1 \geq p_2 \geq \ldots \geq p_m .$$

Since the prices are normalized, we must have $p_1 \geq 1/m$, and so

$$p_m/p_1 \leq p_i/p_1 < \eta/(1/m) = 1/M^{m-1} .$$

Then there must be an index ℓ such that

$$p_\ell/p_{\ell+1} > M . \tag{29}$$

Let $\bar{B}$ be the aggregate bid matrix as in (26) and consider the quantities

$$v_1 = \sum_{i\le \ell} \sum_{j\ge \ell+1} \bar{b}_{ij} p_i \quad \text{and} \quad v_2 = \sum_{i\le \ell} \sum_{j\ge \ell+1} \bar{b}_{ji} p_j \ .$$

Since p is market-clearing, we have

$$\sum_{i=1}^{m} p_i \bar{b}_{ij} = \sum_{i=1}^{m} p_j \bar{b}_{ji} \quad \text{all} \quad j \in I_m \ .$$

Summing this over j in $\{\ell+1, \ldots, m\}$ and canceling common terms, we obtain $v_1 = v_2$. If v denotes the common value of v_1 and v_2 , then at least one of the following inequalities is true:

$$\sum_{i\le \ell} \Big(\sum_{j\ge \ell+1} b^{\alpha}_{ij} \Big) p_i \le \frac{1}{2} v \quad \text{or} \quad \sum_{i\le \ell} \Big(\sum_{j\ge \ell+1} b^{\beta}_{ij} \Big) p_i \le \frac{1}{2} v \ .$$

For the sake of definiteness, assume the one with α .

Let $D = \bar{B} - B^{\alpha}$, then using (11), we have for all i

$$p_i x_i = p_i \Big(a^{\alpha}_i + \sum_j d_{ij}\Big) - \sum_j p_i d_{ji} \ .$$

But by (28), $x_i > 0$, and since $a^{\alpha}_i + \sum_j d_{ij} = \sum_j \bar{b}_{ij}$ (by (9)). We get

$$p_i \Big(\sum_j \bar{b}_{ij}\Big) > \sum_j p_j d_{ji} \quad \text{for all} \quad i \ .$$

Conversely, if q is any price vector which satisfies the above inequalities in place of p , then Lemma 3 implies that α can achieve the prices q by a suitable strategy. In particular, if $\lambda > 0$ is sufficiently small, then α can achieve the prices

$$q = (p_1, \ldots, p_\ell, (1+\lambda)p_{\ell+1}, \ldots, (1+\lambda)p_m) .$$

We compute the change in the final holdings of α if he changes the prices from p to q . This is given by

$$\Delta x_i = \sum_{t\geq\ell+1} d_{ti}(p_t/p_i) - \sum_{t\geq\ell+1} d_{ti}(1+\lambda)(p_t/p_i)$$

$$= -\lambda \sum_{t\geq\ell+1} d_{ti}(p_t/p_i) \quad \text{for} \quad i \in \{1, \ldots, \ell\}$$

and
$$\Delta x_j = \sum_{s\leq\ell} d_{sj}(p_s/p_j) - \sum_{s\leq\ell} d_{sj}(p_s/(1+\lambda)p_j)$$

$$= \frac{\lambda}{1+\lambda} \sum_{s\leq\ell} d_{sj}(p_s/p_j) \quad \text{for} \quad j \in \{\ell+1, \ldots, m\} .$$

Thus, the change in the utility of α is

$$\Delta u^\alpha = \left(\frac{\lambda}{1+\lambda}\right) \sum_{j\geq\ell+1} \left(\sum_{s\leq\ell} d_{sj}(p_s/p_j)\right)\partial_j u^\alpha(x)$$

$$- \lambda \sum_{i\leq\ell} \sum_{t\geq\ell+1} d_{ti}(p_t/p_i)\partial_i u^\alpha(x) + o(\lambda)$$

Let $\Omega = \max_i \{\partial_i u^\alpha(x)\}$, $\omega = \min_j \{\partial_j u^\alpha(x)\}$. Then

$$\Delta u^\alpha \geq \frac{\lambda}{1+\lambda} \frac{\omega}{p_{\ell+1}} \sum_{s\leq\ell} \left(\sum_{j\geq\ell+1} d_{sj}\right)p_s - \lambda \frac{\Omega}{p_\ell} \sum_{i\leq\ell} \left(\sum_{t\geq\ell+1} d_{ti}p_t\right) + o(\lambda) .$$

Now by assumption,

$$\sum_{s\leq\ell} \sum_{j\geq\ell+1} b^\alpha_{sj}p_s \leq \frac{1}{2}v$$

so
$$\sum_{s\leq\ell} \sum_{j\geq\ell+1} d_{sj}p_s \geq \frac{1}{2}v$$

and since $\sum_{i\leq\ell} \sum_{t\geq\ell+1} d_{ti}p_t \leq v$, we get

$$\Delta u^{\alpha} \geq \frac{\lambda\omega v}{p_{\ell+1}}\left(\frac{1}{2(1+\lambda)} - \frac{\Omega}{\omega}\left(\frac{p_{\ell+1}}{p_{\ell}}\right)\right) + o(\lambda) \ .$$

Since x is an N.E. holding, the first term must be negative, so

$$\frac{p_{\ell}}{p_{\ell+1}} \leq \frac{\omega}{2\Omega(1+\lambda)} \leq \frac{M}{1+\lambda} \ .$$

Taking λ sufficiently small, we obtain a contradiction to (29). This completes the proof. Q.E.D.

Theorem 1 is an easy consequence.

<u>Proof of Theorem 1</u>: Fix k and consider the games ${}^{k}\Gamma(\ell^{-1})$ for $\ell \in N$. By Lemma 8, we can find for each ℓ a δ-positive T.S.N.E. $B(\ell)$ with normalized prices $p(\ell)$. Since $B(\ell)$ and $p(\ell)$ range in compact sets, we may assume (passing to a subsequence if necessary) that $B(\ell)$ and $p(\ell)$ converge to B and p .

Let $\bar{B}$ and $\bar{B}(\ell)$ be the aggregate bid matrices (as in (7), and in (26) with $\varepsilon = \ell^{-1}$), and define $\tilde{B}$ and $\tilde{B}(\ell)$ as in (5).

Then

$$p(\ell)\tilde{B}(\ell) = 0$$

and since $p(\ell) \to p$, $\tilde{B}(\ell) \to \tilde{B}$, we get

$$p\tilde{B} = 0 \ .$$

Also, by Lemma 9, $p_i \geq \eta$ for all i . So Lemma 1 applies and it follows that $\bar{B}$ must be completely reducible. Since each $B(\ell)$ is δ-positive, B must be δ-positive and so, by Remark 3, $\bar{B}$ is irreducible.

It remains only to show that (B,p) is a T.S.N.E. for ${}^{k}T$. To see this, consider a trader α in $I_{n\times k}$. Let $x^{\alpha}(\ell)$ and x^{α} be the final holdings by α at $B(\ell)$ and B. Let $B'(\ell)$ and B' be the new nk-tuples after α switches to a strategy T at $B(\ell)$ and B'. We divide the argumnt into two cases.

Case 1: $\bar{B}'$ is completely reducible. Clearly $\bar{B}'(\ell)$ is irreducible; and by Remark 3, so is $\bar{B}'$! By Lemma 2, market clearing prices exist at $B'(\ell)$ and B'; and if we call them $p'(\ell)$ and p', then $p'(\ell) \to p'$. Consequently, if $x^{\alpha}(T,\ell)$ and $x^{\alpha}(T)$ are the final holdings by α at $B'(\ell)$ and B', then

$$x^{\alpha}(T,\ell) \to x^{\alpha}(T) \quad \text{as} \quad \ell \to \infty .$$

Also $\quad x^{\alpha}(\ell) \to x^{\alpha} \quad$ as $\quad \ell \to \infty$.

Since $B(\ell)$ is an N.E., we must have

$$u^{\alpha}(x^{\alpha}(\ell)) \geq u^{\alpha}(x^{\alpha}(T,\ell)) .$$

Letting $\ell \to \infty$, we obtain

$$u^{\alpha}(x^{\alpha}) \geq u^{\alpha}(x^{\alpha}(T)) .$$

Case 2: If $\bar{B}'$ is not completely reducible, then $x^{\alpha}(T) = a^{\alpha}$ so $u^{\alpha}(x^{\alpha}(T)) = u^{\alpha}(a^{\alpha}) \leq u^{\alpha}(x^{\alpha}(\ell)) \to u^{\alpha}(x^{\alpha})$.

This shows that (B,p) is an N.E. The rest of the Theorem is clear.

Q.E.D.

We have actually proved a stronger result than Theorem 1. For later use we will call it

Corollary 1: If δ is as in (25), then for each k, ${}^k\Gamma$ has a δ-positive, active T.S.N.E.

4. CONVERGENCE OF ACTIVE EQUILIBRIA

Consider now the sequence of games ${}^k\Gamma$ as $k \to \infty$. By Theorem 1, each ${}^k\Gamma$ has an active T.S.N.E. $B(k)$ with normalized prices $p(k)$. Since $B(k)$ is type-symmetric we may view it as an element in $S = S^1 \times \ldots \times S^n$ (rather than in $\underset{\alpha \in I_{n\times k}}{\times} S^\alpha$). Now $B(k)$ and $p(k)$ range in fixed compact sets and, passing to a subsequence, we may ensure that they converge.

We wish to examine the nature of these limits (as in Dubey and Shubik [3]).

Given a price vector $p > 0$, we define the budget set of a trader of type α to be the set

$$BS^\alpha(p) = \{x \in R^m_+ : p \cdot x = p \cdot a^\alpha\}$$

A competitive equilibrium for Γ is a price p together with allocations x^α, $\alpha = 1, \ldots, n$ such that for each α,

$$u^\alpha(x^\alpha) = \max\{u^\alpha(x) : x \in BS^\alpha(p)\} .$$

Given prices $p > 0$ and a bid B^α by α, we define the competitive outcome of (B^α, p) to be the allocation

$$x^\alpha_j = a^\alpha_j - \sum_i b^\alpha_{ji} + \sum_i b^\alpha_{ij}(p_i/p_j) . \tag{30}$$

THEOREM 2: If $\{(B(k)); k \in N\}$ is any sequence of δ-positive, active T.S.N.E.'s with normalized prices $\{p(k)\}$ (see Corollary 1), then $\{(B(k), p(k))\}$ has a limit point.

If (B,p) is any such limit point then $p_i \geq \eta$ for all i (where η is as in Theorem 1.)

If $\{(B(k_\nu), p(k_\nu))\}$ is any subsequence converging to (B,p) ; and if x^α are the competitive outcomes of (B,p) and $x^\alpha(k_\nu)$ are the final holdings at $B(k_\nu)$, then

(a) $x^\alpha(k_\nu) \to x^\alpha$ as $\nu \to \infty$

(b) (x^α, p) is a competitive equilibrium for Γ .[6]

Proof: The existence of a limit point was discussed at the beginning of this section; and if (B,p) is as described, then $p_i = \lim_\nu p_i(k_\nu) \geq \eta > 0$ (by Theorem 1). Statement (a) follows by comparing (30) with (6) (for $B^\alpha(\nu)$ and $p(\nu)$, as $\nu \to \infty$). It remains only to prove (b).

Changing notation, let us write $B(\nu)$, $p(\nu)$ and $x^\alpha(\nu)$ for $B(k_\nu)$, $p(k_\nu)$ and $x^\alpha(k_\nu)$. Let $B^\alpha(\nu)$ and B^α denote the α component of $B(\nu)$ and B (for α in I_n), and set

$$A(\nu) = \sum_{\alpha \in I_n} B^\alpha(\nu)$$

$$A = \sum_{\alpha \in I_n} B^\alpha .$$

Then by (0) and (9), $A(\nu)$ is stochastic; and if $\bar{B}(\nu)$ is the aggregate bid matrix at $B(k_\nu)$ (as in (7)) then

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[6]This should be compared with the corresponding result in [3], where additional conditions are needed on the amount of money and its distribution.

$$\bar{B}(\nu) = k_\nu A(\nu) \ . \tag{31}$$

Consequently (4) becomes

$$p(\nu) = p(\nu)A(\nu) \ . \tag{32}$$

As $\nu \to \infty$, $p(\nu) \to p$ and $A(\nu) \to A$. So A is stochastic and

$$p = pA \ . \tag{33}$$

By Lemma 1, A must be completely reducible; and since each $B(\nu)$ is δ-positive, B must be δ-positive. It follows from Remark 3 that A is irreducible.

Let $\bar{q}(\nu)$, $q(\nu)$ and q be computed for $\bar{B}(\nu)$, $A(\nu)$ and A as in Lemma 2. Then by (31)

$$\bar{q}(\nu) = (k_\nu)^{m-1} q(\nu) \ . \tag{34}$$

Also, (32) and (33) imply that $q(\nu)$ and q are positive multiples of $p(\nu)$ and p . Since $A(\nu) \to A$, we get

$$q(\nu) \to q \quad \text{as} \quad \nu \to \infty \ . \tag{35}$$

Now suppose a <u>single</u> trader in $I_{n \times k_\nu}$ changes his strategy to T . Denote by $B(T,\nu)$ the resulting nk_ν-tuple of strategies. Let $\bar{B}(T,\nu)$ be the new aggregate bid matrix, and let

$$A(T,\nu) = (1/k_\nu)\bar{B}(T,\nu) \ . \tag{36}$$

If $\bar{q}(T,\nu)$ and $q(T,\nu)$ are computed from $\bar{B}(T,\nu)$ and $A(T,\nu)$ as in Lemma 2, then

$$\bar{q}(T,\nu) = (k_\nu)^{m-1} q(T,\nu) \quad . \tag{37}$$

Furthermore, (31) and (36) imply that all entries in $(A(T,\nu) - A(\nu))$ are $O(1/k_\nu)$. Consequently

$$|q(T,\nu) - q(\nu)| \to 0 \quad \text{as} \quad \nu \to \infty \; . \tag{38}$$

Suppose now that (b) does not hold. Then there is a player of type α and an allocation y in $BS^\alpha(p)$, such that

$$u^\alpha(y) > u^\alpha(x^\alpha) \quad . \tag{39}$$

Since commodities are freely exchangeable, any allocation in $BS^\alpha(p)$ can be achieved by α as the competitive outcome of a bid. Let T be such a bid corresponding to y , then (30) gives

$$y_j = a_j^\alpha - \sum_i t_{ji} + \sum_i t_{ij}(p_i/p_j) \quad . \tag{40}$$

The final holding resulting from T in $\Gamma(k_\nu)$ is

$$x_j^\alpha(T,\nu) = a_j^\alpha - \sum_i t_{ji} + \sum_i t_{ij}(\bar{q}_i(T,\nu)/\bar{q}_j(T,\nu)) \tag{41}$$

where $\bar{q}(T,\nu)$ is as in (37).

Since $\bar{q}(T,\nu)$ and p are multiples of $q(T,\nu)$ and q , we may rewrite (40) and (41) as

$$y_j = a_j^\alpha - \sum_i t_{ji} + \sum_i t_{ij}(q_i/q_j) \tag{42}$$

and
$$x_j^\alpha(T,\nu) = a_j^\alpha - \sum_i t_{ji} + \sum_i t_{ij}(q_i(T,\nu)/q_j(T,\nu)) \ . \tag{43}$$

By (35) and (38) it follows that

$$x_j^\alpha(T,\nu) \to y_j \quad \text{as} \quad \nu \to \infty \ . \tag{44}$$

Since $x^\alpha(\nu)$ is an N.E. final holding, we must have

$$u^\alpha(x^\alpha(\nu)) \geq u^\alpha(x_j^\alpha(T,\nu)) \ . \tag{45}$$

Letting $\nu \to \infty$ in (45) and using (44) and part (a) of this Theorem, we get

$$u^\alpha(x^\alpha) \geq u^\alpha(y)$$

which contradicts (39). This completes the proof of Theorem 2. Q.E.D.

There is an "easy converse" to Theorem 2. Consider the game Γ with a continuum $(I = [0,1])$ of players (no assumptions of "finite-typeness").

<u>Theorem 3</u>: If (x,p) is a competitive equilibrium for Γ . Then there is a Nash equilibrium (B,p) with x as the final allocation.

(In the theorem x and B are integrable functions on I with values in R_+^m and nonnegative $m \times m$ matrices, respectively.)

<u>Proof</u>: The proof involves a simple back-calculation to compute a measurable selection of bids B ; which achieve the final holdings x , at the prices p . Then (B,p) is easily seen to be an N.E. We omit the details. Q.E.D.

A more interesting question is whether <u>every</u> competitive equilibrium is the limit (in the sense of Theorem 2) of T.S.N.E.'s of the games ${}^{k}\Gamma$. We leave this as an open problem (possibly for a future paper).

REFERENCES

[1] Amir, R.; S. Sahi, M. Shubik and S. Yao. "A strategic market game with complete markets," CFDP No. 813R, September 18, 1987.

[2] Berge, C. Topological Spaces. New York: Macmillan, 1963.

[3] Dubey, P. and M. Shubik. "A theory of money and financial institutions 28. The non-cooperative equilibria of a closed trading economy with market supply and bidding strategies," Journal of Economic Theory, Vol. 17, No. 1, February 1978.

[4] Seneta, E. Non-Negative Matrices. New York: Halsted Press (John Wiley), 1973.