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We present versions of the two fundamental welfare theorems of economics for exchange economies with a countable number of agents and an infinite dimensional commodity space. These results are then specialized to the overlapping generations model.

1. INTRODUCTION

In exchange economies, having a finite number of agents and a finite number of commodities, there are two celebrated welfare theorems. The first welfare theorem states that every competitive allocation is Pareto optimal. This theorem fails to be true in exchange economies with a countable number of agents, as was first observed by P. Samuelson in his classic paper on the Consumption Loan Model [15].

The second welfare theorem states that every Pareto optimal allocation is a competitive allocation subject to transfers. This theorem was extended to economies with a finite number of agents and infinite dimensional commodity spaces, whose positive cone has a non-empty interior, by G. Debreu in a seminal

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work [6]. Recently, A. Mas-Colell [11] generalized Debreu's results to infinite dimensional commodity spaces, where the positive cone may have empty interior.

In this paper, we address the relationship between the welfare and market mechanisms in exchange economies with a countable number of agents and an infinite dimensional space of commodities. We are interested in what sense competitive equilibria in overlapping generations models on an infinite dimensional Riesz space are optimal.

For the first welfare theorem, a definitive answer has been given for the notion of Pareto optimality by Y. Balasko and K. Shell [5], where agents' consumption sets lie in finite dimensional spaces and agents' characteristics are "smooth." Of course, in our more general setting, we are unable to invoke arguments depending on the existence of smooth demand functions derived from utility maximization subject to a budget constraint.

Y. Balasko and K. Shell [5] were able to prove analogues of both welfare theorems using a notion of optimality which is much weaker than Pareto optimality. They called this notion *weak Pareto optimality*. Since weak Pareto optimality has already been defined in general equilibrium analysis, we shall follow B. Peleg and M. E. Yaari [14] and term this concept Malinvaud optimality after the fundamental work of E. Malinvaud [8,9].

An allocation is *Malinvaud optimal* if no finite set of agents can make at least one of their group better off and no one else in the group worse off by using the consumption bundles assigned to them in the given allocation. Following G. Debreu [6], we define a *valuation equilibrium* as an allocation for which there exists a non-zero price such that each individual's consumption bundle is maximal in the budget set, where the income is the value of his consumption at the given price.

Given these two notions, the main result of this paper simply asserts that an irreducible allocation is Malinvaud optimal if and only if it is a valuation equilibrium. This result is first shown for general exchange economies with a countable number of agents and then specialized to overlapping generations models. In particular, it is demonstrated how the concept of properness can be used to establish the price supportability of Malinvaud optimal allocations.

In an economy with a countable number of agents, the main obstacle in establishing these results is to prove that a Malinvaud optimal allocation can be price supported. In order to accomplish this, we introduce new ideas and techniques which are of independent interest in their own right and may be applicable to other economic situations as well.

2. MATHEMATICAL PRELIMINARIES

This paper will utilize the theory of Riesz spaces. For detail accounts of the theory of Riesz spaces see the books [3, 4, 7, 16, 19]. The material in section 3 of [2] will be used freely. A few basic facts about Riesz spaces are briefly discussed below.

A Riesz space is a partially ordered (real) vector space which is in addition a lattice, i.e., with the extra property that finite sets have suprema (least upper bounds) and infima (greatest lower bounds). The supremum and infimum of two elements x and y will be denoted by $x \vee y$ and $x \wedge y$, respectively. That is,

$$x \vee y = \sup\{x, y\} \quad \text{and} \quad x \wedge y = \inf\{x, y\}.$$

If x is an element in a Riesz space, then the elements

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x),$$

are referred to as the positive part, the negative part and the absolute value of x, respectively.

For the rest of the discussion in this section the letter E will denote a Riesz space. The positive cone of E will be denoted by E^+, i.e.,

$$E^+ = \{ x \in E\colon\ x \geq 0 \}.$$

The Riesz space E is said to be *Dedekind complete* whenever every non-empty subset of E which is bounded from above has a supremum.

A vector subspace A of E is said to be an *ideal* whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. Every element x belongs to a smallest ideal A_x, called the *principal ideal generated by* x. We have

$$A_x = \{ y \in E\colon\ \exists\ \lambda > 0 \text{ with } |y| \leq \lambda |x| \}.$$

More generally, every non-empty subset S of E is contained in a smallest ideal, called the ideal generated by S. For a countable subset $\{ x_1, x_2, \dots \}$ of E, the ideal generated by $\{ x_1, x_2, \dots \}$ will be denoted by A. If A_n denotes the ideal generated by $\hat{x}_n := \sum_{i=1}^{n} x_i$, then $A_n \subseteq A_{n+1}$ holds for each n and

$$A = \bigcup_{n=1}^{\infty} A_n.$$

The ideal A enjoys some remarkable algebraic and topological properties that will be employed in our study. For details regarding the properties of the ideal A we refer the reader to section 3 of [2].

The sets of the form

$$[x, y] := \{ z \in E\colon\ x \leq z \leq y \},$$

where $x \leq y$, are called the order intervals of E. A linear functional $f\colon E \longrightarrow \mathcal{R}$ is said to be *order bounded* whenever f carries order intervals onto bounded subsets $\mathcal{R}$. The vector space of all order bounded linear functionals on E is called the *order dual* of E and is denoted by $E^\sim$. Under the ordering $f \geq g$ whenever $f(x) \geq g(x)$ for each $x \in E^+$, the order dual $E^\sim$ is a Dedekind complete Riesz space. Its lattice operations are given by

$$\begin{aligned} f \vee g\,(x) &= \sup\{ f(y) + g(z)\colon\ \ y, z \in E^+ \quad \text{and} \quad y + z = x \}, \\ f \wedge g\,(x) &= \inf\{ f(y) + g(z)\colon\ \ y, z \in E^+ \quad \text{and} \quad y + z = x \}, \end{aligned}$$

for each $x \in E^+$.

The symbol $x_\alpha \uparrow$ means that the net $\{ x_\alpha \}$ satisfies $x_\alpha \geq x_\beta$ whenever $\alpha \geq \beta$. The notation $x_\alpha \uparrow x$ means $x_\alpha \uparrow$ and $x = \sup\{ x_\alpha \}$ both hold. The meanings of $x_\alpha \downarrow$ and $x_\alpha \downarrow x$ are similar. If $\{ x_n \}$ is a sequence of E^+ such that the sequence of partial sums $\{ \sum_{i=1}^{n} x_i\colon\ n = 1, 2, \dots \}$ has a supremum x in E (i.e., $\sum_{i=1}^{n} x_i \uparrow_n x$) then the element x will be denoted by $\sum_{i=1}^{\infty} x_i$. That is, we shall write

$$\sum_{i=1}^{\infty} x_i := \sup\{ \sum_{i=1}^{n} x_i\colon\ n = 1, 2, \dots \}.$$

A net $\{ x_\alpha \}$ is *order convergent to* x (in symbols, $x_\alpha \xrightarrow{o} x$) whenever there exists another net $\{ y_\alpha \}$ with the same indexed set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each α. A linear functional $f\colon E \longrightarrow \mathcal{R}$ is said to be *order continuous* whenever $x_\alpha \xrightarrow{o} 0$ implies $f(x_\alpha) \longrightarrow 0$ in $\mathcal{R}$. The set of all order continuous linear functionals on E is denoted by $E_n^\sim$ and is referred to as the order continuous dual of E. The order continuous dual $E_n^\sim$ of E is always an ideal of the order dual $E^\sim$.

A Riesz space E is said to be a *normal Riesz space* whenever

a) E is Dedekind complete, and
b) $E_n^\sim$ separates the points of E, i.e., for each $x \neq 0$ there exists some $f \in E_n^\sim$ satisfying $f(x) \neq 0$.

An order continuous function $u: E^+ \longrightarrow \mathcal{R}$ (i.e., $x_\alpha \xrightarrow{o} x$ in E^+ implies $u(x_\alpha) \longrightarrow u(x)$ in $\mathcal{R}$) is called a *myopic utility function*. Myopic utility functions were introduced for the first time in [2].

Regarding normal Riesz spaces and myopic utility functions we have the following result which will be used in our study. For a proof see [2, Theorem 4.8].

Theorem 2.1. *Assume that E is a normal Riesz space, $\{x_n\}$ is a sequence of some order interval [0,a], and x is a $\sigma(E, E_n^\sim)$-accumulation point of $\{x_n\}$. Then for every monotone, quasi-concave and myopic utility function $u: E^+ \longrightarrow \mathcal{R}$ we have*

$$u(x) \geq \liminf_{n\to\infty} u(x_n).$$

The commodity-price duality in our economic model will be described by a Riesz dual system. A *Riesz dual system* $\langle E, E' \rangle$ is a Riesz space E together with an ideal E' of $E^\sim$ that separates the points of E such that the duality is the natural one, i.e., $\langle x, x' \rangle = x'(x)$ holds for all $x \in E$ and all $x' \in E'$. Riesz dual systems were introduced in Economics for the first time by C. D. Aliprantis and D. J. Brown in [1].

3. THE ECONOMIC MODEL

The following five properties will characterize the economic model of our study.

1. The commodity-price duality is described by a Riesz dual system $\langle E, E' \rangle$; E is the commodity space and E' is the price space. In accordance with the economic tradition, the value of the bundle $x \in E$ at prices $p \in E'$ will be denoted by $p \cdot x$, i.e., $p \cdot x := \langle x, p \rangle$.
2. There is a countable number of consumers indexed by i; the set of consumers will be assumed to be the set of natural numbers $\mathcal{N} = \{1, 2, \ldots\}$.
3. Each consumer has E^+ as his consumption set.
4. Each consumer i has an initial endowment $\omega_i > 0$. The total endowment ω is defined by

$$\omega = \sum_{i=1}^{\infty} \omega_i = \sup\{\sum_{i=1}^{n} \omega_i \colon n = 1, 2, \ldots\},$$

   where the supremum is assumed to exist in E.
5. The preference $\succeq_i$ of each consumer i is represented by a quasi-concave, monotone and myopic utility function u_i.

Definition 3.1. *A pure exchange economy (or simply an economy) $\mathcal{E}$ is a triplet*

$$\mathcal{E} = \big(\langle E, E' \rangle, \{\omega_i \colon i \in \mathcal{N}\}, \{\succeq_i \colon i \in \mathcal{N}\}\big),$$

where the components of $\mathcal{E}$ satisfy properties (1) through (5) above.

From now on we shall assume that $\mathcal{E}$ is a fixed economy. An ***allocation*** for the economy $\mathcal{E}$ (or simply an ***allocation***) is a sequence $(x_1, x_2, \ldots)$ such that $x_i \geq 0$ for each i and

$$\sum_{i=1}^{\infty} x_i := \sup\{\sum_{i=1}^{n} x_i \colon n = 1, 2, \ldots\} = \omega.$$

Definition 3.2. *An allocation* $(x_1, x_2, \ldots)$ *is said to be:*

1) **Pareto optimal**, *whenever there is no other allocation* $(y_1, y_2, \ldots)$ *satisfying* $y_i \succeq_i x_i$ *for all* i *and* $y_i \succ_i x_i$ *for at least one* i; *and*
2) **Malinvaud optimal**, *whenever there is no other allocation* $(y_1, y_2, \ldots)$ *satisfying*
   a) $y_i = x_i$ *for all but a finite number of* i,
   b) $y_i \succeq_i x_i$ *for all* i, *and*
   c) $y_i \succ_i x_i$ *for at least one* i.

Every Pareto optimal allocation is clearly Malinvaud optimal. It is easy to demonstrate that Pareto optimal allocations exist. For instance, if one consumer (say the first) has a strictly monotone preference, then the allocation $(\omega, 0, 0, \ldots)$ is obviously Pareto optimal. However, it is considerably more difficult to show that individually rational Pareto optimal allocations exist. (Recall that an allocation $(x_1, x_2, \ldots)$ satisfying $x_i \succeq_i \omega_i$ for all i is known as an ***individually rational allocation***.)

The next result gives a condition that guarantees the existence of individually rational Pareto optimal allocations.

Theorem 3.3. *If the commodity space of an economy is a normal Riesz space, then the economy has individually rational Pareto optimal allocations.*

Proof. We know that the order interval $[0, \omega]$ is $\sigma(E, E_n^\sim)$-compact. Therefore, if we equip $[0, \omega]$ with $\sigma(E, E_n^\sim)$, then by Tychonoff's classical compactness theorem the product topological space $[0, \omega]^{\mathcal{N}}$ is compact.

Now let

$$\mathcal{A} = \{ (x_1, x_2, \ldots) \colon\ x_i \geq 0 \quad \text{and} \quad x_i \succeq_i \omega_i \quad \text{for all } i \quad \text{and} \quad \sum_{i=1}^{\infty} x_i = \omega \},$$

i.e., $\mathcal{A}$ is the non-empty set of all individually rational allocations. Next, choose constants $\lambda_i > 0\,(i \in \mathcal{N})$ such that $\sum_{i=1}^{\infty} \lambda_i u_i(\omega) < \infty$, and then define a social welfare function $U \colon \mathcal{A} \longrightarrow \mathcal{R}$ by

$$U(x_1, x_2, \ldots) = \sum_{i=1}^{\infty} \lambda_i u_i(x_i).$$

Put $s = \sup\{ U(x_1, x_2, \ldots) \colon\ (x_1, x_2, \ldots) \in \mathcal{A} \} < \infty$. Clearly, any allocation $(y_1, y_2, \ldots) \in \mathcal{A}$ which satisfies $s = U(y_1, y_2, \ldots)$ is automatically an individually rational Pareto optimal allocation. The rest of the proof is devoted to proving that there exists some $(y_1, y_2, \ldots) \in \mathcal{A}$ with $s = U(y_1, y_2, \ldots)$.

To this end, for each n pick $(x_1^n, x_2^n, \ldots) \in \mathcal{A}$ with

$$U(x_1^n, x_2^n, \ldots) > s - \tfrac{1}{n}.$$

Next, note that for each fixed i, the sequence $\{ u_i(x_i^n) \colon\ n = 1, 2, \ldots \}$ is bounded in $\mathcal{R}$. Thus, by an easy "diagonal" argument, we can construct a subsequence of $\{ (x_1^n, x_2^n, \ldots) \}$ (which we shall denote by $\{ (x_1^n, x_2^n, \ldots) \}$ again) such that $\lim_{n \to \infty} u_i(x_i^n)$ exists in $\mathcal{R}$ for each i. Since $\{ (x_1^n, x_2^n, \ldots) \}$ is a sequence of $[0, \omega]^{\mathcal{N}}$, it follows that $\{ (x_1^n, x_2^n, \ldots) \}$ has an accumulation point $(x_1, x_2, \ldots) \in [0, \omega]^{\mathcal{N}}$. Clearly, each $x_i \in [0, \omega]$ is a $\sigma(E, E_n^\sim)$-accumulation point of the sequence $\{ x_i^n \colon\ n = 1, 2, \ldots \}$. The latter implies

$$\sum_{i=1}^{\infty} x_i \leq \omega,$$

and moreover, in view of $u_i(x_i^n) \geq u_i(\omega_i)$ for all i and all n, it follows from Theorem 2.1 that

$$u_i(x_i) \geq \liminf_{n \to \infty} u_i(x_i^n) = \lim_{n \to \infty} u_i(x_i^n) \geq u_i(\omega_i),$$

i.e., $x_i \succeq_i \omega_i$ holds for all $i = 1, 2, \ldots$.

Now let $\varepsilon > 0$. Fix some k with $\sum_{i=k}^{\infty} \lambda_i u_i(\omega) < \varepsilon$ and then note that

$$\sum_{i=1}^{k} \lambda_i u_i(x_i^n) > s - \tfrac{1}{n} - \varepsilon \qquad (\star)$$

holds for all n. Thus, from $(\star)$ and Theorem 2.1, we see that

$$\begin{aligned}\sum_{i=1}^{\infty} \lambda_i u_i(x_i) &\geq \sum_{i=1}^{k} \lambda_i u_i(x_i) - \varepsilon \\ &\geq \sum_{i=1}^{k} \lambda_i \big(\inf_{n \in \mathcal{N}} u_i(x_i^n)\big) - \varepsilon \\ &= \lim_{n\to\infty} \Big(\sum_{i=1}^{k} \lambda_i u_i(x_i^n)\Big) - \varepsilon \\ &\geq (s - \varepsilon) - \varepsilon = s - 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the latter implies $\sum_{i=1}^{\infty} \lambda_i u_i(x_i) \geq s$.

Finally, define $(y_1, y_2, \ldots)$ by $y_1 = x_1 + \omega - \sum_{i=1}^{\infty} x_i$ and $y_i = x_i$ for $i \geq 2$, and note that $(y_1, y_2, \ldots)$ is an individually rational allocation. Now observe that $U(y_1, y_2, \ldots) = \sum_{i=1}^{\infty} \lambda_i u_i(y_i) = s$ holds, and conclude from the latter that $(y_1, y_2, \ldots)$ is also Pareto optimal. The proof of the theorem is now complete. ∎

4. THE FUNDAMENTAL THEOREMS OF WELFARE ECONOMICS

In this section we shall study decentralization properties of allocations. Let $(x_1, x_2, \ldots)$ be a fixed allocation for an economy $\mathcal{E}$. The ideal generated in E by the countable set $\{x_1, x_2, \ldots\}$ will be denoted by A. By A_k we shall denote the ideal generated by the finite set $\{x_1, \ldots, x_k\}$. The ideal A_k is, of course, the same as the ideal generated by the element $\hat{x}_k := x_1 + \cdots + x_k$. We have $A_k \subseteq A_{k+1}$ for each k and

$$A = \bigcup_{k=1}^{\infty} A_k.$$

The $\|\cdot\|_\infty$-norm on A_k is the lattice norm defined by

$$\|x\|_\infty = \inf\{\lambda > 0 \colon |x| \leq \lambda \hat{x}_k\}, \quad x \in A_k.$$

The norm dual of $(A_k, \|\cdot\|_\infty)$ will be designated by A_k'.

Under the $\|\cdot\|_\infty$-norm, A_k' is a Banach lattice (in fact, an AM-space with unit $\hat{x}_k$). The inductive limit topology ξ generated by the sequence $\{A_n\}$ on A is a locally convex-solid topology such that $(A, \xi)' = A^{\sim}$ holds; see [2, Section 3]. Therefore, $\langle A, A^{\sim}\rangle$ is a Riesz dual system.

We start our discussion with the notion of valuation equilibrium introduced by G. Debreu in [6].

Definition 4.1. (Debreu) *Let F be an ideal of E containing A. An allocation $(x_1, x_2, \ldots)$ is said to be a* **valuation equilibrium** *whenever there exists a non-zero price $p \in F^{\sim}$ (referred to as a* **supporting**

price) *such that* $x \in F^+$ *and* $p \cdot x_i \geq p \cdot x$ *imply* $x_i \succeq_i x$, *i.e., whenever for each* i *the bundle* x_i *is a maximal element in the set* $\{x \in F^+ \colon\ p \cdot x \leq p \cdot x_i\}$.

The requirement that the ideal F of E satisfies $A \subseteq F \subseteq E$ indicates that A and E are the smallest and largest ideals respectively, on which an allocation can be supported as a valuation equilibrium. The main purpose of this paper is to present conditions under which a given allocation is a valuation equilibrium either with respect to A or with respect to E.

Recall that a commodity bundle $v > 0$ is said to be **strongly desirable** for a preference relation $\succeq$ on E^+ whenever $x + \alpha v \succ x$ holds for all $x \in E^+$ and all $\alpha > 0$. It should be noted that if in an economy every consumer has a strongly desirable commodity, then

a) *Every Walrasian equilibrium is a valuation equilibrium; and*
b) *Any price supporting a valuation equilibrium is a positive price.*

To see (a), let $(x_1, x_2, \ldots)$ be a Walrasian equilibrium with respect to a Riesz dual system $\langle F, F^{\sim}\rangle$, where F is an ideal of E containing A. Pick a non-zero price $p \in F^{\sim}$ such that each x_i is a maximal bundle in the ith consumer's budget set $B_i(p) = \{x \in F^+ \colon\ p \cdot x \leq p \cdot \omega_i\}$ and let $0 < v_i \in F$ be a strongly desirable commodity for consumer i. If $p \cdot x_i < p \cdot \omega_i$ holds, then pick some $\alpha > 0$ with $p \cdot (x_i + \alpha v_i) < p \cdot \omega_i$ and note that $x_i + \alpha v_i \succ_i x_i$ violates the maximality property of x_i in $B_i(p)$. Hence, $p \cdot x_i = p \cdot \omega_i$ holds for each i, and from this we infer that $(x_1, x_2, \ldots)$ is a valuation equilibrium supported by the price p. For (b) assume that a price $p \in F^{\sim}$ supports a valuation equilibrium $(x_1, x_2, \ldots)$ and let $x \in F^+$. From $x_i + \alpha v_i + x \succ_i x_i$, we see that $p \cdot x_i + \alpha p \cdot v_i + p \cdot x_i = p \cdot (x_i + \alpha v_i + x) > p \cdot x_i$ holds for all $\alpha > 0$, and from this we infer that $p \cdot x \geq 0$ holds, i.e., $p \geq 0$.

Now the first fundamental theorem of welfare economics takes the following form.

Theorem 4.2. *If in an economy every consumer has a strongly desirable commodity, then every valuation equilibrium is Malinvaud optimal.*

Proof. Let $(x_1, x_2, \ldots)$ be a valuation equilibrium supported by a price $p \in F^{\sim}$ (where F is an ideal of E containing A) and let $0 < v_i \in F^{\sim}$ be a strongly desirable commodity for consumer i. Assume by way of contradiction that $(x_1, x_2, \ldots)$ is not Malinvaud optimal. Then there exists another allocation $(y_1, y_2, \ldots)$ and a finite subset B of $\mathcal{N}$ such that

1. $y_i = x_i$ holds for all $i \notin B$;
2. $y_i \succeq_i x_i$ holds for all $i \in B$; and
3. $y_i \succ_i x_i$ holds for at least one $i \in B$.

An easy argument shows that $\sum_{i \in B} y_i = \sum_{i \in B} x_i$ holds, and so

$$\sum_{i \in B} p \cdot y_i = \sum_{i \in B} p \cdot x_i. \tag{$\star$}$$

On the other hand, $y_i \succeq_i x_i$ implies $y_i + \alpha v_i \succ_i x_i$ for all i, and so by the supportability of the price p, we see that $p \cdot y_i + \alpha p \cdot v_i > p \cdot x_i$ for all $\alpha > 0$. Hence, $p \cdot y_i \geq p \cdot x_i$ holds for each i, and so from ($\star$) we infer that $p \cdot y_i = p \cdot x_i$ holds for all $i \in B$. Now fix some $j \in B$ with $y_j \succ_j x_j$. By the supportability of p we have

$$p \cdot y_j > p \cdot x_j = p \cdot y_j,$$

which is impossible, and the desired conclusion follows. ■

As mentioned before, the objective of this paper is to study supportability properties of allocations. We continue our discussion with the notion of weak irreducibility. If B is a subset of $\mathcal{N}$ with at least two elements and $\{x_i \colon\ i \in B\} \subseteq E^+$, then we shall say that the assignment $\{x_i \colon\ i \in B\}$ is **weakly irreducible over** B whenever for any partition of B into two non-empty subsets B_1 and B_2 there exist some $j \in B_1$, a finite non-empty subset B_3 of B_2 and some $0 < z \leq \sum_{i \in B_3} x_i$ such that

$$x_j + z \succ_j x_j.$$

It should be noted that if $\{x_i: i \in B\}$ is weakly irreducible over B, then $x_i > 0$ holds for all $i \in B$.

We continue with the notion of irreducibility for allocations. The concept originated with the works of L. W. McKenzie [12,13] and it was modified by C. A. Wilson [17]. The definition below is a generalized version of Wilson's definition.

Definition 4.3. *An allocation $(x_1, x_2, \ldots)$ is said to be* **finitely irreducible** *whenever for any non-empty finite subset B of $\mathcal{N}$ there exists another finite subset C of $\mathcal{N}$ containing B such that $\{x_i: i \in C\}$ is weakly irreducible over C.*

It should be noted that if an allocation $(x_1, x_2, \ldots)$ satisfies $x_i > 0$ for each i and all preferences are strictly monotone (i.e., $x > y \geq 0$ implies $x \succ_i y$ for each i), then the allocation $(x_1, x_2, \ldots)$ is automatically finitely irreducible. Thus, finite irreducibility is weaker than the strict monotonicity of preferences.

Definition 4.4. *An allocation $(x_1, x_2, \ldots)$ is said to be a* **basic allocation** *whenever for each n the bundle $\hat{x}_n = \sum_{i=1}^{n} x_i$ is strongly desirable by consumers $1, \ldots, n$, i.e., whenever $x + \alpha \hat{x}_n \succ_i x$ holds for all $x \in E^+$ all $\alpha > 0$ and all $1 \leq i \leq n$.*

We now come to the main objective of this section; namely, to state and prove for economies with a countable number of agents the following version of the second fundamental theorem of welfare economics.

Theorem 4.5. *Every basic finitely irreducible Malinvaud optimal allocation is a valuation equilibrium that can be supported on A by an order continuous price of $A^{\sim}$.*

The proof of this theorem is quite involved and it will be accomplished by a series of steps in the form of lemmas. For the proof we shall construct a sequence of prices that support the given allocation "locally". An "asymptotic limit" of such a sequence will allow us to obtain an extended price that supports the allocation.

We start the discussion about the proof of Theorem 4.5 with one more definition.

Definition 4.6. *A* **fundamental sequence of prices** *for an allocation $(x_1, x_2, \ldots)$ is a sequence $(p_1, p_2, \ldots)$ such that:*

a) *$0 < p_k \in A_k'$ holds for each k; and*

b) *For each $k \geq m$ the price p_k supports $(x_1, \ldots, x_m)$ on A_k, i.e., $x \in A_k^+$ and $x \succeq_i x_i$ for some $1 \leq i \leq m$ imply $p_k \cdot x \geq p_k \cdot x_i$.*

Our first goal is to establish that basic finitely irreducible Malinvaud optimal allocations admit fundamental sequences of prices.

Lemma 4.7. *Let $(x_1, x_2, \ldots)$ be a basic finitely irreducible Malinvaud optimal allocation. Then for every non-empty finite subset B of $\mathcal{N}$ there exists a price $p > 0$ on the ideal generated by $\{x_i: i \in B\}$ such that for each $i \in B$ we have*

a) *$p \cdot x_i > 0$; and*

b) *$x \succeq_i x_i$ in the ideal generated by $\{x_i: i \in B\}$ implies $p \cdot x \geq p \cdot x_i$.*

Proof. Assume that $(x_1, x_2, \ldots)$ is a basic finitely irreducible Malinvaud optimal allocation and let B be a finite non-empty subset of $\mathcal{N}$. Put $k = \max B$ and then select a finite subset C of $\mathcal{N}$ such that $\{1, \ldots, k\} \subseteq C$ and $\{x_i: i \in C\}$ is weakly irreducible over C. Let I denote the ideal generated by $\{x_i: i \in C\}$. Also, for each $i \in C$ consider the non-empty convex set

$$G_i = \{x \in I^+: x \succeq_i x_i\},$$

and let

$$G = \sum_{i \in C} (G_i - x_i).$$

If $K = \mathrm{Int}(I^+)$ with respect to the $\|\cdot\|_\infty$-norm of I, then $K \neq \emptyset$, and we claim that $G \cap (-K) = \emptyset$. To see this, assume by way of contradiction that $G \cap (-K) \neq \emptyset$. Then there exists some $e \in K$ with $-e \in G$. Pick $y_i \in G_i$ ($i \in C$) such that $-e = \sum_{i \in C}(y_i - x_i)$, and note that

$$\sum_{i \in C}(y_i + \tfrac{1}{m}e) = \sum_{i \in C} x_i,$$

where m denotes the number of elements of C. Since $e \in \mathrm{Int}(I^+)$, there exists some $\alpha > 0$ such that $e \geq \alpha\hat{x}$, where $\hat{x} = \sum_{i \in C} x_i$. Clearly, $\hat{x} \geq \hat{x}_k = \sum_{i=1}^k x_i$. Now put

$$z_i = y_i + \tfrac{1}{m}e \quad \text{for} \quad i \in C \quad \text{and} \quad z_i = x_i \quad \text{for} \quad i \notin C,$$

and note that $(z_1, z_2, \ldots)$ is an allocation. In addition, for $1 \leq i \leq k$ we have

$$z_i = y_i + \tfrac{1}{m}e \succeq_i y_i + \tfrac{\alpha}{m}\hat{x}_k \succ_i y_i \succeq_i x_i,$$

and $z_i \succeq_i x_i$ for $i > k$, which contradicts the Malinvaud optimality of $(x_1, x_2, \ldots)$. Hence $G \cap (-K) = \emptyset$.

Since G and $-K$ are both non-empty convex sets and $-K$ is $\|\cdot\|_\infty$-open, it follows from the classical separation theorem (see, for example, [4, Theorem 9.10, p. 136]) that there exists some $0 < p \in I'$ such that $g \in G$ implies $p \cdot g \geq 0$. Since $p \cdot \hat{x} > 0$ must hold, we infer that $p \cdot x_i > 0$ must also hold for some $i \in C$. We claim that $p \cdot x_i > 0$ holds for all $i \in C$.

To see this, assume by way of contradiction that $p \cdot x_i = 0$ for some $i \in C$. Then the two sets

$$C_1 = \{i \in C\colon\ p \cdot x_i > 0\} \quad \text{and} \quad C_2 = \{i \in C\colon\ p \cdot x_i = 0\}$$

are both non-empty. Thus, by the weak irreducibility of $\{x_i\colon\ i \in C\}$ over C there exists some $t \in C_1$ such that

$$y = x_t + \sum_{i \in C_2} x_i \succ_t x_t.$$

Clearly, $p \cdot y = p \cdot x_t > 0$. Now by the order continuity of the utility functions there exists some $0 < \delta < 1$ with $\delta y \succ_t x_t$. From $\delta y - x_t \in G$, it follows that

$$p \cdot y > \delta p \cdot y = p \cdot (\delta y) \geq p \cdot x_t = p \cdot y,$$

which is impossible. Hence, $p \cdot x_i > 0$ holds for all $i \in C$.

Finally, note that the restriction of p to the ideal generated by $\{x_i\colon\ i \in B\}$ satisfies the desired properties. ■

An immediate consequence of the preceding lemma is the following result.

Lemma 4.8. *If $(x_1, x_2, \ldots)$ is a basic finitely irreducible Malinvaud optimal allocation, then $(x_1, x_2, \ldots)$ admits a fundamental sequence of prices $(p_1, p_2, \ldots)$ such that*

$$p_k \cdot x_i > 0$$

holds for all k and all i with $k \geq i$.

Proof. Let $(x_1, x_2, \ldots)$ be a basic finitely irreducible Malinvaud optimal allocation and let k be fixed. By Lemma 4.7 there exists a price $0 < p_k \in A_k'$ such that $p \cdot x_i > 0$ holds for all $1 \leq i \leq k$ and $x \succeq_i x_i$ in A_k^+ implies $p \cdot x \geq p \cdot x_i$. Now the sequence $(p_1, p_2, \ldots)$ satisfies the desired properties. ■

The next lemma presents a growth estimate for a fundamental sequence and is the analogue of C. A. Wilson's Lemma 3 in [17].

Lemma 4.9. *Let $(x_1, x_2, \ldots)$ be a finitely irreducible allocation and let $(p_1, p_2, \ldots)$ be a fundamental sequence of prices for the allocation such that $p_k \cdot x_i > 0$ holds for all k and all i with $k \geq i$. Then for each fixed pair of natural numbers ℓ and m there exists some constant $M > 0$ (depending only upon ℓ and m) such that*

$$0 < p_k \cdot x_\ell \leq M\, p_k \cdot x_m$$

holds for all $k \geq \max\{\ell, m\}$.

Proof. Let $(x_1, x_2, \ldots)$ be a finitely irreducible allocation and let $(p_1, p_2, \ldots)$ be a fundamental sequence of prices satisfying $p_k \cdot x_i > 0$ for $k \geq i$. Let ℓ and m be fixed and suppose by way of contradiction that our claim is not true. That is, assume that $\liminf_{n \geq t}(\frac{p_n \cdot x_m}{p_n \cdot x_\ell}) = 0$, where $t = \max\{\ell, m\}$. By passing to a subsequence (if necessary) we can suppose without loss of generality that

$$\lim_{n\to\infty} \frac{p_n \cdot x_m}{p_n \cdot x_\ell} = 0.$$

Next pick a finite subset C of $\mathcal{N}$ such that $\{\ell, m\} \subseteq C$ and with $\{x_i\colon\ i \in C\}$ weakly irreducible over C. Put

$$C_1 = \{i \in C\colon\ \limsup_{n\to\infty} \frac{p_n \cdot x_i}{p_n \cdot x_\ell} = 0\} \quad \text{and} \quad C_2 = \{i \in C\colon\ \limsup_{n\to\infty} \frac{p_n \cdot x_i}{p_n \cdot x_\ell} > 0\}.$$

Clearly, $C = C_1 \cup C_2$, $\ell \in C_2$ and $m \in C_1$. Now by the weak irreducibility of $\{x_i\colon\ i \in C\}$, there exists some $i \in C_2$ such that

$$x_i + \sum_{j \in C_1} x_j \succ_i x_i.$$

Since the utility function u_i is myopic, there exists some $0 < \delta < 1$ with

$$\delta x_i + \sum_{j \in C_1} x_j \succ_i x_i.$$

Therefore, by the supportability of p_n, we see that

$$\delta p_n \cdot x_i + p_n \cdot \Big(\sum_{j \in C_1} x_j\Big) \geq p_n \cdot x_i$$

holds for all sufficiently large n.

Thus, we have

$$p_n \cdot \Big(\sum_{j \in C_1} x_j\Big) \geq (1 - \delta) p_n \cdot x_i,$$

and hence

$$\sum_{j \in C_1} \frac{p_n \cdot x_j}{p_n \cdot x_\ell} \geq (1 - \delta) \frac{p_n \cdot x_i}{p_n \cdot x_\ell}.$$

holds for all sufficiently large n. Consequently,

$$\limsup_{n\to\infty} \Big(\sum_{j \in C_1} \frac{p_n \cdot x_i}{p_n \cdot x_\ell}\Big) \geq (1 - \delta) \limsup_{n\to\infty} \frac{p_n \cdot x_i}{p_n \cdot x_\ell},$$

which in view of

$$0 \leq \limsup_{n\to\infty} \Big(\sum_{j \in C_1} \frac{p_n \cdot x_j}{p_n \cdot x_\ell}\Big) \leq \sum_{j \in C_1} \limsup_{n\to\infty} \frac{p_n \cdot x_j}{p_n \cdot x_\ell} = 0,$$

implies $\limsup_{n\to\infty} \frac{p_n \cdot x_i}{p_n \cdot x_\ell} = 0$, contrary to $i \in C_2$. The proof of the lemma is now complete. ∎

Now consider a fundamental sequence $(p_1, p_2, \ldots)$ of prices for an allocation $(x_1, x_2, \ldots)$. Then for $k \leq \ell$ we have $A_k \subseteq A_\ell$ (where, of course, A_k is also an ideal of A_ℓ). Thus, if $x \in A_k$, then $p_\ell \cdot x$ is well defined. In other words, for each $\ell \geq k$ the price p_ℓ defines a positive linear functional on A_k. This observation will be used in the next lemma.

Lemma 4.10. *Let $(x_1, x_2, \ldots)$ be a basic finitely irreducible allocation and let $(p_1, p_2, \ldots)$ be a fundamental sequence of prices for the allocation such that for some m we have $p_k \cdot x_m = 1$ for all $k \geq m$. Assume also that $p_k \cdot x_i > 0$ holds for all $k \geq i$.*

If $y_\alpha \downarrow 0$ holds in A_n for some $n \geq m$, then the net $\{y_\alpha\}$ converges to zero uniformly on $\{p_k\colon k \geq n\}$. In particular, p_k restricted to A_n is order continuous for each $k \geq n$.

Proof. Fix $n \geq m$ and let $y_\alpha \downarrow 0$ hold in A_n. Without loss of generality we can assume that $0 \leq y_\alpha \leq \hat{x}_n$ holds for all α, where $\hat{x}_n = \sum_{i=1}^n x_i$. Keep in mind that by our hypothesis, the bundle $\hat{x}_n$ is strongly desirable by each consumer $1 \leq i \leq n$, i.e., we have $x_i + \delta \hat{x}_n \succ_i x_i$ holds for all $\delta > 0$ and all $1 \leq i \leq n$.

Now let $\varepsilon > 0$. Lemma 4.9 applied to the pairs (ℓ, m) for $\ell = 1, \ldots, n$ guarantess the existence of some $M > 0$ satisfying $p_k \cdot \hat{x}_n \leq M$ for all $k \geq n$. Put $\delta = \frac{\varepsilon}{Mn}$. Note that for each $1 \leq i \leq n$ we have $x_i + \delta \hat{x}_n - y_\alpha \wedge x_i \geq 0$. This, coupled with $x_i + \delta \hat{x}_n \succ_i x_i$, the fact that $y_\alpha \wedge x_i \downarrow_\alpha 0$ and the order continuity of the utility functions, implies the existence of some index α_0 with $x_i + \delta \hat{x}_n - y_\alpha \wedge x_i \succ_i x_i$ for all $\alpha \geq \alpha_0$ and all $1 \leq i \leq n$. Since for each $k \geq n$ the price p_k supports $(x_1, \ldots, x_k)$ on A_k, it follows that $p_k \cdot (x_i + \delta \hat{x}_n - y_\alpha \wedge x_i) \geq p_k \cdot x_i$ holds for all $\alpha \geq \alpha_0$ and all $1 \leq i \leq n$. Therefore, for all $\alpha \geq \alpha_0$ and all $1 \leq i \leq n$ we have

$$p_k \cdot (y_\alpha \wedge x_i) \leq \delta p_k \cdot \hat{x}_n \leq \delta \cdot M.$$

Now by using the lattice inequality $y_\alpha = y_\alpha \wedge (\sum_{i=1}^n x_i) \leq \sum_{i=1}^n y_\alpha \wedge x_i$, we see that

$$0 \leq p_k \cdot y_\alpha \leq \sum_{i=1}^n p_k \cdot (y_\alpha \wedge x_i) \leq \sum_{i=1}^n \delta \cdot M \leq \delta \cdot M \cdot n = \varepsilon$$

holds for all $\alpha \geq \alpha_0$ and all $k \geq n$, and the desired conclusion follows. ■

The next lemma tells us that if $(p_1, p_2, \ldots)$ is a fundamental sequence of prices, then for each k the set $\{p_n\colon n \geq k\}$ is a weakly compact subset of A'_k.

Lemma 4.11. *Let $(x_1, x_2, \ldots)$ be a basic finitely irreducible allocation and let $(p_1, p_2, \ldots)$ be a fundamental sequence of prices for the allocation satisfying $p_k \cdot x_m = 1$ for all $k \geq m$ and $p_k \cdot x_i > 0$ for $k \geq i$.*

Then for each $n \geq m$ the set of prices $\{p_k\colon k \geq n\}$ (where each p_k is considered restricted to A_n) is a relatively weakly compact subset of A'_n.

Proof. Let $n \geq m$ be fixed. By Grothendieck's classical compactness theorem [4, Theorem 13.10, p. 208] it suffices to show that the set $\{p_k\colon k \geq n\}$ (as a subset of A'_n) is norm bounded and every disjoint sequence of $[0, \hat{x}_n]$ converges uniformly to zero on $\{p_k\colon k \geq n\}$.

To see that the set $\{p_k\colon k \geq n\}$ is norm bounded on A_n note first that

$$\|p_k\| = p_k \cdot \hat{x}_n = \sum_{i=1}^n p_k \cdot x_i,$$

and then apply Lemma 4.9 to the pairs (ℓ, m) for $\ell = 1, \ldots, n$.

Now let $\{y_m\}$ be a disjoint sequence of $[0, \hat{x}_n]$. For each m put $z_m = \sum_{i=1}^m y_i = \bigvee_{i=1}^m y_i \leq \hat{x}_n$. Since E is Dedekind complete, there exists some $z \in [0, \hat{x}_n]$ with $z_m \uparrow z$. Next, note that

$$0 \leq y_m = z_m - z_{m-1} \leq z - z_{m-1} = \zeta_m \downarrow 0.$$

By Lemma 4.10, the sequence $\{\zeta_m\}$ converges to zero uniformly on $\{p_k\colon\ k \geq n\}$, and from

$$0 \leq p_k \cdot y_m \leq p_k \cdot \zeta_m \ (k \geq n),$$

we infer that $\{y_m\}$ converges likewise uniformly to zero on $\{p_k\colon\ k \geq n\}$. The proof of the lemma is now complete. ■

Let $(p_1, p_2, \ldots)$ be a fundamental sequence of prices for an allocation $(x_1, x_2, \ldots)$ and let (as usual) A denote the ideal generated by $\{x_1, x_2 \ldots\}$. A non-zero positive linear functional $p\colon A \longrightarrow \mathcal{R}$ will be called an **asymptotic limit** for the sequence $\{p_n\}$ whenever there exists a subsequence $\{p_{k_n}\}$ of $\{p_n\}$ such that for each $y \in A$ we have

$$p \cdot y = \lim_{n \to \infty} p_{k_n} \cdot y,$$

where, of course, the value $p_{k_n} \cdot y$ need not be defined for a finite number of n.

The next lemma guarantees the existence of asymptotic limits.

Lemma 4.12. *Let $(x_1, x_2, \ldots)$ be a basic finitely irreducible allocation and let $(p_1, p_2, \ldots)$ be a fundamental sequence of prices for the allocation. If $p_k \cdot x_i > 0$ holds for $k \geq i$ and for some fixed m we have $p_k \cdot x_m = 1$ for all $k \geq m$, then the sequence $\{p_n\}$ has an asymptotic limit.*

Proof. Assume that the allocation $(x_1, x_2, \ldots)$ and $(p_1, p_2, \ldots)$ satisfy the hypotheses of the lemma. The desired subsequence of prices will be constructed by a diagonal process using induction. To do this, we shall construct subsequences $\{p_n^\ell\}$, $\ell = 0, 1, 2, \ldots$, of $\{p_n\}$ such that:

a) The sequence $\{p_n^0\}$ is a subsequence of $\{p_n\}$; and
b) For each $\ell = 1, 2, \ldots$ the sequence $\{p_n^\ell\}$ converges pointwise on the ideal $A_{k_{\ell-1}}$, where $p_{k_{\ell-1}} = p_1^{\ell-1}$.

Start by letting $p_n^0 = p_{n+m-1}$ for each n and $k_0 = m$. Then the sequence $\{p_n^0\}$ considered restricted to A_m forms (by Lemma 4.11) a relatively weakly compact subset of A_m'. Thus, there exists a subsequence $\{p_n^1\}$ of $\{p_n^0\}$ such that $\lim p_n^1 \cdot y$ exists in $\mathcal{R}$ for each $y \in A_m = A_{k_0}$. Now for the inductive step, assume that a subsequence $\{p_n^\ell\}$ of $\{p_n^{\ell-1}\}$ has been chosen such that $\lim p_n^\ell \cdot y$ exists in $\mathcal{R}$ for each $y \in A_{k_{\ell-1}}$, where $p_{k_{\ell-1}} = p_1^{\ell-1}$. Then the sequence $\{p_n^\ell\}$ considered restricted to A_{k_ℓ}, where $p_{k_\ell} = p_1^\ell$, forms (by Lemma 4.11) a relatively weakly compact subset of A_{k_ℓ}', and so there exists a subsequence $\{p_n^{\ell+1}\}$ of $\{p_n^\ell\}$ such that $\lim p_n^{\ell+1} \cdot y$ exists in $\mathcal{R}$ for each $y \in A_{k_\ell}$. The induction is now complete.

Next, consider the subsequence $\{p_{k_n}\}$ of $\{p_n\}$, where $p_{k_n} = p_n^n$. An easy argument shows that $\lim p_{k_n} \cdot y$ exists in $\mathcal{R}$ for each $y \in A$. Therefore, if for each $y \in A$ we put

$$p \cdot y = \lim_{n \to \infty} p_{k_n} \cdot y,$$

then p defines a positive linear functional on A. From $p_{k_n} \cdot x_m = 1$ for all n, we see that $p \cdot x_m = 1$, and so p is a non-zero asymptotic limit of $\{p_n\}$. ■

Now the proof of Theorem 4.5 is as follows. By Lemma 4.8, there exists a fundamental sequence $(p_1, p_2, \ldots)$ of prices for $(x_1, x_2, \ldots)$ such that $p_k \cdot x_i > 0$ holds for all $k \geq i$. Replacing each p_k by $p_k / p_k \cdot x_1$, we can assume that $p_k \cdot x_1 = 1$ holds for all k. By Lemma 4.12 there exists an asymptotic limit p for the sequence $(p_1, p_2, \ldots)$. Applying Lemma 4.9 with $\ell = 1$, we see that for each fixed m the sequence $\{p_k \cdot x_m\colon\ k = 1, 2, \ldots\}$ is eventually bounded away from zero, and so $p \cdot x_m > 0$ holds for all m. Now we claim that p supports the allocation $(x_1, x_2, \ldots)$ on A.

To see that x_i is a maximal element in the set $\{x \in A^+\colon\ p \cdot x \leq p \cdot x_i\}$, let some $x \in A^+$ satisfy $p \cdot x \leq p \cdot x_i$, and assume by way of contraction that $x \succ_i x_i$ holds. Then there exists some k such that $x \succ_i x_i$ holds in A_n^+ for all $n \geq k$, and so

$$p \cdot x = \lim_{n \to \infty} p_{k_n} \cdot x \geq \lim_{n \to \infty} p_{k_n} \cdot x_i = p \cdot x_i > 0.$$

Next pick some $0 < \delta < 1$ satisfying $\delta x \succ_i x_i$ (the order continuity of the utility function u_i guarantees that such a δ always exists), and note that by the above argument we have

$$p \cdot x > \delta p \cdot x = p \cdot (\delta x) \geq p \cdot x_i,$$

contrary to $p \cdot x \leq p \cdot x_i$. Consequently, x_i is a maximal element in the set $\{x \in A^+ \colon\ p \cdot x \leq p \cdot x_i\}$, as desired.

Next, we establish that p is order continuous on A. To see this, it suffices to prove that p is order continuous on each A_k. Indeed, by Lemma 4.10, we know that on each A_k the price p is the pointwise limit of a sequence of order continuous linear functionals, and so p is likewise order continuous on each A_k; see [4, Exercise 14, p. 214].

Finally, we close the section with the presentation of the following combination of the two fundamental theorems of welfare economics (Theorems 4.2 and 4.5).

Theorem 4.13. *A basic finitely irreducible allocation is a valuation equilibrium if and only if it is Malinvaud optimal.*

5. THE OVERLAPPING GENERATIONS MODEL

The objective of this section is to establish the basic welfare theorems for overlapping generations models by applying the results of the previous section. In particular, we shall establish that every Malinvaud optimal allocation can be decentralized by a price.

The index t will denote the time period of our overlapping generations model. The commodity-price duality at each period t will be described by a Riesz dual system $\langle E_t, E_t' \rangle$. Therefore, we have a sequence $(\langle E_1, E_1' \rangle, \langle E_2, E_2' \rangle \ldots)$ of Riesz dual systems each member of which describes the commodity-price duality at the specific time period. We shall write

$$\mathbf{E} = E_1 \times E_2 \times \cdots \qquad \text{and} \qquad \mathbf{E}' = E_1' \times E_2' \cdots .$$

To simplify matters, we shall assume that

a) only one consumer is born in each period, and
b) each consumer lives two periods.

Thus, consumer t is born at period t and lives all his life in periods t and $t+1$. Each consumer trades and has tastes for commodities only during his life-time span. We assume that consumer t has an initial endowment $0 < \omega_t^t \in E_t$ at period t and an initial endowment $0 < \omega_t^{t+1} \in E_{t+1}$ at period $t+1$ and nothing else in any other periods. Consequently, his total initial endowment ω_t is the vector

$$\omega_t = (0, \ldots, 0, \omega_t^t, \omega_t^{t+1}, 0, 0, \ldots) \in \mathbf{E},$$

where ω_t^t and ω_t^{t+1} occupy positions t and $t+1$, respectively. In addition, we shall assume that the "father" of consumer 1 (i.e., consumer 0) is present in the model at period 1. He will be designated as consumer 0 and his initial endowment will be given by the vector

$$\omega_0 = (\omega_0^1, 0, 0, \ldots),$$

where $0 < \omega_0^1 \in E_1$. Thus, the total endowment of the overlapping generations model is given by the vector

$$\omega = \sum_{t=0}^{\infty} \omega_t = (\omega_0^1 + \omega_1^1, \omega_1^2 + \omega_2^2, \omega_2^3 + \omega_3^3, \ldots) \in \mathbf{E}.$$

If $\theta_t = \omega_{t-1}^t + \omega_t^t$, $t = 1, 2, \ldots$, then $\omega = (\theta_1, \theta_2, \ldots)$. The ideal generated by θ_t in E_t will be denoted by Θ_t. That is,

$$\Theta_t = \{x \in E_t \colon \exists \quad \lambda > 0 \quad \text{with} \quad |x| \leq \lambda \theta_t\}.$$

The ideal generated in $\mathbf{E}$ by $(\theta_1, \ldots, \theta_n, 0, 0, \ldots)$ will be denoted by $\mathbf{A}_n$. Clearly,

$$\mathbf{A}_n = \Theta_1 \times \cdots \times \Theta_n \times \mathbf{0} \times \mathbf{0} \cdots ,$$

where $\mathbf{0} = \{0\}$. It should be noted that $\mathbf{A}_n \subseteq \mathbf{A}_{n+1}$ holds for all n. If $\mathbf{A}$ denotes the ideal generated in $\mathbf{E}$ by the sequence $\{\omega_n\}$, then

$$\mathbf{A} = \bigcup_{n=1}^{\infty} \mathbf{A}_n .$$

If ξ denotes the inductive limit topology on $\mathbf{A}$ generated by the sequence $\{\mathbf{A}_n\}$, then the topological dual of $(\mathbf{A}, \xi)$ coincides with the order dual of $\mathbf{A}$. Moreover, we have

$$\mathbf{A}^{\sim} = \Theta_1' \times \Theta_2' \times \cdots ;$$

see [2, Theorem 7.2].

The vectors of the form

$$\mathbf{x}_t = (0, \ldots, 0, x_t^t, x_t^{t+1}, 0, 0, \ldots),$$

where $x_t^t \in E_t^+$ and $x_t^{t+1} \in E_{t+1}^+$ represent the commodity bundles for consumer t during his life time. Each consumer t maximizes a utility function u_t defined on his commodity space, i.e., u_t is a function from $E_t^+ \times E_{t+1}^+$ into $\mathcal{R}$. The value of u_t at the commodity bundle $\mathbf{x}_t = (0, \ldots, x_t^t, x_t^{t+1}, 0, 0, \ldots)$ will be denoted by $u_t(x_t^t, x_t^{t+1})$. The utility functions will be assumed to satisfy the following additional property:

Each u_t is strictly monotone on $E_t^+ \times E_{t+1}^+$, that is, $(x, y) > (x_1, y_1)$ in $E_t^+ \times E_{t+1}^+$ implies $u_t(x, y) > u_t(x_1, y_1)$.

The case $t = 0$ is a special case. The utility function u_0 is a function of one variable defined on E_1^+. It is also assumed to satisfy the above property.

Now assume that $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is an allocation for the overlapping generations model, where

$$\mathbf{x}_0 = (x_0^1, 0, 0, \ldots) \quad \text{and} \quad \mathbf{x}_t = (0, \ldots, 0, x_t^t, x_t^{t+1}, 0, 0, \ldots), \quad t \geq 1.$$

Furthermore, assume that

$$x_0^1 > 0, \quad x_t^t > 0 \quad \text{and} \quad x_t^{t+1} > 0 \quad \text{for} \quad t \geq 1.$$

A moment's thought reveals that

a) $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is a basic allocation, and
b) $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is finitely irreducible.

(Claim (a) follows from the strong monotonicity of each utility function u_t on $E_t^+ \times E_{t+1}^+$. For (b) note that if B is a non-empty finite subset of consumers and $k = \max B$, then the set $C = \{1, \ldots, k\}$ satisfies $B \subseteq C$ and $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is weakly irreducible over C.)

Definition 5.1. *An allocation $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is said to be* **admissible** *for our overlapping generations model whenever*

$$x_0^1 > 0, \quad x_t^t > 0 \quad \textit{and} \quad x_t^{t+1} > 0 \quad \textit{for} \quad t \geq 1$$

hold.

From the above dsicussion and Theorems 4.2 and 4.5 the following results should be immediate.

Theorem 5.2. *Every admissible valuation equilibrium in an overlapping generations model is Malinvaud optimal with respect to the Riesz dual system* $\langle \mathbf{A}, \mathbf{A}^{\sim} \rangle$.

Theorem 5.3. *Every admissible Malinvaud optimal allocation in an overlapping generations model is a valuation equilibrium that can be supported on* $\mathbf{A}$ *by an order continuous price of* $\mathbf{A}^{\sim}$.

We now turn our attention to overlapping generations models with proper preferences. Proper preferences were introduced by A. Mas-Colell [10]; see also [2, Section 4]. Let us say that the preference $\succeq_t$ induced by u_t is *uniformly proper* whenever there exist locally convex-solid topologies on E_t and E_{t+1} consistent with the dualities $\langle E_t, E_t' \rangle$ and $\langle E_{t+1}, E_{t+1}' \rangle$ such that each $\succeq_t$ is uniformly proper with respect to the product topology on $E_t \times E_{t+1}$. Equivalently, u_t is uniformly proper if and only if it is uniformly proper for the Mackey topology $\tau(E_t \times E_{t+1}, E_t' \times E_{t+1}')$. The preference $\succeq_0$ is uniformly proper whenever it is uniformly proper on E_1. Also, let us say that the overlapping generations model is **proper** whenever

a) Each preference $\succeq_t$ $(t = 0, 1, 2, \ldots)$ is uniformly proper; and
b) Each $\theta_t = \omega_{t-1}^t + \omega_t^t$ is a strictly positive element of E_t for each $t \geq 1$. (Recall that θ_t is strictly positive whenever $q \cdot \theta_t > 0$ holds for all $0 < q \in E_t'$.)

Next consider the Riesz space

$$\phi_{\mathbf{E}} = \{ \mathbf{y} = (y_1, y_2, \ldots) \in \mathbf{E}\colon \exists \; k \;\text{ with }\; y_i = 0 \;\; \forall \;\; i > k \}.$$

Clearly, $\phi_{\mathbf{E}}$ is an ideal of $\mathbf{E}$ containing $\mathbf{A}$, and moreover, under the duality

$$\mathbf{p} \cdot \mathbf{y} = \sum_{i=1}^{\infty} p_i \cdot y_i,$$

the dual system $\langle \phi_{\mathbf{E}}, \mathbf{E}' \rangle$ is a Riesz dual system.

For proper overlapping generations models Theorem 5.3 can be improved as follows.

Theorem 5.4. *Every admissible Malinvaud optimal allocation in a proper overlapping generations model is a valuation equilibrium that can be supported on* $\phi_{\mathbf{E}}$ *by a price of* $\mathbf{E}'$.

Proof. Let $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ be an admissible Malinvaud optimal allocation for our overlapping generations model. By Theorem 5.2 there exists a price $\mathbf{p} = (p_1, p_2, \ldots) \in \mathbf{A}^{\sim}$ supporting $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ on $\mathbf{A}$ and satisfying $\mathbf{p} \cdot \mathbf{x}_t > 0$ for $t = 0, 1, 2, \ldots$. We have $p_t \in \Theta_t'$ for each t.

Now by a Theorem of N. C. Yannelis and W. R. Zame [18] (see also [2, Theorem 9.2]) each linear functional $p_t\colon \Theta_t \longrightarrow \mathcal{R}$ is $\sigma(E_t, E_t')$-continuous. Since Θ_t is $\sigma(E_t, E_t')$-dense in E_t, it follows that p_t has a continuous extension p_t^* to all of E_t. Thus, $\mathbf{p}^* = (p_1^*, p_2^*, \ldots) \in \mathbf{E}'$, and we claim that $\mathbf{p}^*$ supports $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ on $\phi_{\mathbf{E}}$.

To see this, let $\mathbf{y} \succeq_t \mathbf{x}_t$ in $E_t^+ \times E_{t+1}^+$. Fix $\delta > 0$ and note that $\mathbf{y} + \delta\omega_t \succ_t \mathbf{x}_t$. Since each Θ_i is $\sigma(E_i, E_i')$-dense in E_i, it follows that Θ_i is also dense in E_i for the locally convex-solid topology $|\sigma|(E_i, E_i')$, and so for each i the ideal Θ_i is order dense in E_i; see [3, Exercise 6, p. 50]. In particular, the ideal $\Theta_t \times \Theta_{t+1}$ is order dense in $E_t \times E_{t+1}$. Thus, there exists a net $\{\mathbf{y}_\alpha\} \subseteq \Theta_t^+ \times \Theta_{t+1}^+$ with $\mathbf{y}_\alpha \uparrow \mathbf{y} + \delta\omega_t$, and so $\mathbf{y}_\alpha \xrightarrow{w} \mathbf{y} + \delta\omega_t$. In view of $\mathbf{y} + \delta\omega_t \succ_t \mathbf{x}_t$ and the order continuity of u_t, we can assume that $\mathbf{y}_\alpha \succ_t \mathbf{x}_t$ holds for all α. Thus, by the supportability of $\mathbf{p}$ on $\mathbf{A}$, we get $\mathbf{p} \cdot \mathbf{y}_\alpha \geq \mathbf{p} \cdot \mathbf{x}_t$ for all α, and by the weak continuity of $\mathbf{p}^*$ on $E_t \times E_{t+1}$, we see that $\mathbf{p}^* \cdot \mathbf{y} + \delta \mathbf{p}^* \cdot \omega_t \geq \mathbf{p}^* \cdot \mathbf{x}_t$ for all $\delta > 0$. Therefore, $\mathbf{y} \succeq_t \mathbf{x}_t$ in $E_t^+ \times E_{t+1}^+$ implies $\mathbf{p}^* \cdot \mathbf{y} \geq \mathbf{p}^* \cdot \mathbf{x}_t$, and the proof is finished. ■

Finally, we remark that the above discusssion (with some obvious modifications) shows that Theorems 5.3 and 5.4 are, in fact, true for the general overlapping generations model. That is, Theorems 5.3 and 5.4 are true for an overlapping generations model where

1) r persons are born in each time period, and
2) each person lives ℓ periods.

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