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TOWARD A THEORY OF DISCOUNTED REPEATED GAMES WITH IMPERFECT MONITORING*

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ABSTRACT

This paper investigates pure strategy sequential equilibria of repeated games with imperfect monitoring. The approach emphasizes the equilibrium value set and the static optimization problems embedded in extremal equilibria. We characterize these equilibria, and provide computational and comparative statics results. The "self-generation" and "bang-bang" propositions which were at the core of our analysis of optimal cartel equilibria [2], are generalized to asymmetric games and infinite action spaces. New results on optimal implicit reward functions include the <u>necessity</u> (as opposed to sufficiency) of bang-bang functions, and the nature of optimal punishment regions.

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1. Introduction

A recent paper of ours [2] demonstrates the existence of equilibria of the Green-Porter model [6],[15] that are optimal in terms of the degree of implicit collusion they sustain, and yet have an unexpectedly simple intertemporal structure. Here we exploit the same analytic approach to develop a general theory for a broad class of asymmetric discounted repeated games with imperfect monitoring. The results characterize efficient sequential equilibria, facilitate their computation, and establish a strong relationship between the equilibrium value set and the discount factor. More generally, they demonstrate the advantages of a perspective which views these repeated games in terms of a particular intertemporal decomposition.

Our analysis is in the spirit of dynamic programming, whose impact on game theory has, of course, been substantial (see, for example, Shapley [22], Abreu [1] and Radner, Myerson and Maskin [17]). It proceeds via a succession of propositions, central among which is "self-generation" (see Section 3), which reduce the study of the equilibria in question to the solution of a class of static problems. The latter are in turn greatly simplified, and the class to be confronted drastically reduced, by the following "bang-bang" proposition. It is unrestrictive to limit the reward function implicitly facing players after any history,[1] to taking only those values that are

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[1] The way in which an equilibrium induces an implicit reward function in each contingency is discussed in Section 3, and at greater length in [2].

extremal in the set of equilibrium payoffs of the supergame. We consider sequential equilibria in pure strategies only.

The result that it is sufficient to consider reward functions of the bang-bang form leaves lingering doubts about the appropriateness of the restriction. If the "natural" solution were a smooth function, which could be replaced by one with the bang-bang property at the cost of creating a complex pattern of rapid alternations among extremal values, one kind of simplicity would be traded against another. Reassurance is provided by a much stronger characterization in Section 7. Under certain conditions, the reward functions faced by players in Pareto efficient equilibria must be bang-bang: efficiency demands that non-extremal points of the payoff set are never used.

In most of the paper, players' choice sets are assumed discrete. Section 8 gives conditions under which our results hold when players instead have continuous choice variables. In symmetric games such as the Green-Porter model, if the first-order approach (see, for example, Mirrlees [14], Grossman and Hart [7], and Rogerson [20]) to the static problems mentioned earlier is valid, the implicit punishment regions occur where a certain "local likelihood ratio" is high. If, for example, a monotone likelihood ratio property holds (see Section 9), the reward function takes the form of a tail-test. Optimal supergame strategies are then trigger strategies of the simplest possible kind.

Ronald Howard's value-iteration [9] has an analogue in repeated games discussed in Section 5. It is an iterative procedure for computing the set of equilibrium values. The novelty here is the presence of sequential incentive constraints and the fact that the map that is iterated is set-valued. Apart from its importance for the numerical computation of equilibria of specific supergames, the algorithm is an alternative characterization of the equilibrium value set, and as such will have a variety of theoretical applications.

The ways in which this paper furthers the research reported in [2] may be summarized as follows. First, it relaxes the restriction of symmetry, showing the theory capable of embracing both asymmetric equilibria of symmetric games and arbitrary asymmetric games. Secondly, the sufficiency of using bang-bang reward functions in efficiently collusive equilibria is strengthened to a necessity theorem. The dual approach taken also proves helpful in relaxing the assumption that choice sets of players are discrete. Reward functions are characterized in detail for certain environments. Finally, we provide an algorithm useful in computing the sequential equilibrium value set.

2. The Model

The model outlined below features unobservable actions, stochastic outcomes and a publicly observable random variable correlated with players' private choices. It lends itself naturally to the study of a number of economic questions. Important examples are oligopoly [6],[15] and partnership problems [16] of various kinds.

The Single-Period Game

G denotes the N-person component game. Each player i has a compact strategy set S_i and a payoff function $\Pi_i : S \to \mathbb{R}$, where $S := S_1 \times \ldots \times S_N$. Π_i is an expected value. Payoffs actually received, π_i, are stochastic and depend on realizations of a random variable P. P has density function $g(p;q)$ that depends on the vector of actions $q \in S$. π_i depends on $q_{-i} := (q_1,\ldots,q_{i-1},q_{i+1},\ldots,q_N)$ only through the effect of the latter on the distribution of P, i.e., $\pi_i : \Omega \times S_i \to \mathbb{R}$. Finally, $\Pi_i(q) = \int_\Omega \pi_i(p,q_i)g(p;q)dp$.

The Repeated Game

$G^\infty(\delta)$ denotes the infinitely repeated game with component game G and discount factor $\delta \in (0,1)$. A strategy σ_i for player i in $G^\infty(\delta)$ is a sequence of functions $\{\sigma_i(t)\}_{t=1}^\infty$, where $\sigma_i(1) \in S_i$, and for $t > 1$, $\sigma_i(t) : \Omega^{t-1} \times S_i^{t-1} \to S_i$. The interpretation is that players can observe (and therefore condition upon) only their <u>own</u> past actions and past realizations of the random variable P. Let $p^t = (p(1),\ldots,p(t))$ and $q^t = (q(1),\ldots,q(t))$ denote t-period price and quantity histories, respectively. As is standard $\sigma|_{p^t,q^t}$ denotes the strategy profile induced by σ after the t-period history (p^t,q^t). P is assumed to be drawn independently in every period according to the density $g(p;q)$. Associated with any strategy profile σ of $G^\infty(\delta)$ is a stochastic stream of payoff-vectors. $v(\sigma) = (v_1(\sigma),\ldots,v_N(\sigma))$ denotes the expected present discounted value of this stream. Note for later use that period t payoffs are received at the end of period t and discounted to the beginning of period 1. We assume that:

(A1) S_i is compact $i = 1,\ldots,N$;

(A2) $\pi_i(p,q_i)$ is continuous in its arguments;

(A3) $\Omega = \{p|g(p,q) > 0\}$ is independent of $q \in S$;

(A4) G has a Nash equilibrium in pure strategies.

Additional assumptions are introduced as required.

(A1) and (A2) guarantee that $v(\sigma)$ is well defined. The solution concept used is sequential equilibrium (see Kreps and Wilson [11]). Hereafter, we use S.E. to denote a sequential equilibrium in pure strategies. $V := \{v(\sigma) | \sigma$ is an S.E.$\}$ is the set of S.E. payoffs. (A4) implies that V is non-empty; the strategy profile specifying that in every period independently of the history each player uses his one-period Nash equilibrium action, is an S.E. .

3. Factorization and Self-Generation

Consider the maximization problem faced by a player in the first period of an equilibrium σ. His choice of action q_i has two consequences: it affects payoffs in period 1, and also influences the distribution of the first period price $p(1)$. The player is in effect maximizing the sum of current payoffs and the expectation of the future reward (a function of $p(1)$) implicitly "promised" by σ. The reward function must be drawn from V: an S.E. can offer only S.E. rewards. Furthermore, $\sigma_i(1)$ must yield at least as high a value of the sum as any other action available to i. The same remarks apply to player i's choice after any t-period history.

We proceed rather abstractly by studying structures suggested by the above observations but no longer in the context of any particular equilibrium.

Definition: Let $L^\infty(\Omega;\mathbb{R}^N)$ denote the set of all (essentially) bounded measurable functions u from Ω into $\mathbb{R}^N$. For any pair $(q,u) \in S \times L^\infty(\Omega;\mathbb{R}^N)$, $E(q;u) := \delta\{\Pi(q) + \int_\Omega u(p)g(p;q)dp\}$.

Definition: For any set $W \subseteq \mathbb{R}^N$, a pair $(q,u) \in S \times L^\infty(\Omega;\mathbb{R}^N)$ is called admissible with respect to W if

(i) $u(\Omega) \subseteq W$, and

(ii) $E_i(\gamma_i,q_{-i};u) \leq E_i(q;u)$ for all $\gamma_i \in S_i$ and $i = 1,\ldots,N$.

These conditions mimic the two requirements noted above on pairs of the form (recommended action, reward function) arising in an S.E. .

Definition: For each set $W \subseteq \mathbb{R}^N$, $B(W) := \{E(q;u) \mid (q,u)$ is admissible w.r.t. $W\}$.

By definition there exist functions Q, U with domain $B(W)$ and ranges S and $L^\infty(\Omega;W)$ respectively such that for any $w \in B(W)$, the pair $(Q(w),U(w))$ is admissible w.r.t. W and $E(Q(w);U(w)) = w$. Such functions will be used in the proof of the next proposition.

That admissibility successfully captures the information essential for studying V is evidenced in Propositions 1 (Self-Generation) and 2 (Factorization). These combine to say that V is the largest bounded fixed point of the set-valued map B. This is remarkable insofar as the definition of B is quite simple and makes no reference to the complex strategic structure of an infinite-horizon game.

The proofs of the propositions below are very similar to those presented for the symmetric case in [2]. We have included them to provide a self-contained, although terse, treatment. The reader may refer to [2] for a more extended account.

Definition: $W \subseteq \mathbb{R}^N$ is said to be self-generating if $W \subseteq B(W)$.

Proposition 1 (Self-Generation): For any bounded set $W \subseteq \mathbb{R}^N$, if W is self-generating, then $B(W) \subseteq V$.

Proof For each $w \in B(W)$, let $U^1(w) := U(w)$ and $U^t(w)(p^t) := U(U^{t-1}(w)(p^{t-1}))(p(t))$ for all $p^t \in \Omega^t$ and $t = 2,3,\ldots$. Since $W \subseteq B(W)$, these functions are well defined. They are used now to construct sequential equilibria $\hat{\sigma}(w)$ which yield payoffs $v(\hat{\sigma}(w)) = w$, as required:

$$\hat{\sigma}(w)(1) = Q(w)$$

$$\hat{\sigma}(w)(t+1)(p^t;q^t) = Q(U^t(w)(p^t)) \quad t = 1,2,3,\ldots$$

Successively using $E(Q(x),U(x)) = x$ and noting that $\delta < 1$, one may check that this construction implies $v(\hat{\sigma}(w)) = w$. Again by construction, $\hat{\sigma}(x)|_p = \hat{\sigma}(U(x)(p))$, and

$$\delta\left[\Pi_i\left(\gamma_i,\sigma_{-i}(x)(1)\right) + \int_\Omega v_i\left(\hat{\sigma}(U(x)(p))\right)g\left(p;\gamma_i,\hat{\sigma}_{-i}(x)(1)\right)dp\right]$$

$$\leq v_i(\hat{\sigma}(x)) \quad \text{for all } x \in B(W).$$

For any $w \in B(W)$ and any history $p^t \in \Omega^t$, $t = 1,2,\ldots$, $\hat{\sigma}(w)|_{p^t} = \hat{\sigma}(x)$ for $x = U^t(w)(p^t) \in W \subseteq B(W)$. Hence, the inequalities above imply that given $\hat{\sigma}_{-i}(w)$, $\hat{\sigma}_i(w)$ is an "unimprovable", and hence optimal, strategy for player i (see, for example, Kreps [10]). Thus it is a best response to $\hat{\sigma}_{-i}(w)$, and $\hat{\sigma}(w)$ is a Nash equilibrium for all $w \in B(W)$.

The strategy profiles $\hat{\sigma}(w)$ are by construction independent of quantity histories $q^t \in S^t$; they are functions only of price histories $p^t \in \Omega^t$. Therefore, to check that $\hat{\sigma}(w)$ is an S.E., it is sufficient to verify that $\hat{\sigma}(w)|_{p^t}$ is a Nash equilibrium for all $p^t \in \Omega^t$, $t = 1,2,\ldots$. As noted above, $\hat{\sigma}(w)|_{p^t} = \hat{\sigma}(x)$ for some $x \in B(W)$.

Finally recall that we have just shown that $\hat{\sigma}(x)$ is a Nash equilibrium for all $x \in B(W)$. □

Remark (A3) implies that the set of Nash equilibrium payoffs is identical to the set of sequential equilibrium payoffs, though not every Nash equilibrium is a sequential equilibrium. The argument is straightforward and is left to the reader; it hinges on the fact that all possible price histories occur in equilibrium.

Proposition 2 (Factorization): $V = B(V)$.

Proof: By Proposition 1, we need only show that $V \subseteq B(V)$. Consider $w \in V$ and an S.E. σ such that $v(\sigma) = w$. Consider the pair (q,u) such that $q := \sigma(1)$ and $u(p) := v(\sigma|_{p,\sigma(1)})$ for all $p \in \Omega$. Clearly $w = \delta[\Pi(\sigma(1)) + \int_\Omega v(\sigma|_{p,\sigma(1)})g(p;\sigma(1))dp] = E(q;u)$. By (A3), the information sets $(p,\sigma_i(1))$ are reached in equilibrium for all $p \in \Omega$; hence $\sigma|_{p,\sigma(1)}$ is an S.E. and $u(p) \in V$ for all $p \in \Omega$.

Consider $\hat{\sigma}_i$ such that $\hat{\sigma}_i(1) = \gamma_i \in S_i$, and $\hat{\sigma}_i|_{p,\gamma_i} = \sigma_i|_{p,q_i}$ for all $p \in \Omega$. By definition, $(\hat{\sigma}_i,\sigma_{-i})|_{p,(\gamma_i,q_{-i})} = \sigma|_{p,q}$. Since σ is an S.E., for all i

$$\Pi_i(q) + \int_\Omega u_i(p)g(p;q)dp = \Pi_i(q) + \int_\Omega v_i(\sigma|_{p,q})g(p;q)dp$$

$$\geq \Pi_i(\gamma_i,q_{-i}) + \int_\Omega v_i(\sigma|_{p,(\gamma_i,q_{-i})})g(p;\gamma_i,q_{-i})dp$$

$$= \Pi_i(\gamma_i,q_{-i}) + \int_\Omega u_i(p)g(p;\gamma_i,q_{-i})dp \quad \text{for all } \gamma_i \in S_i.$$

Hence the pair (q,u) is admissible w.r.t. V and $E(q;u) = w$, as required. □

4. Bang-Bang Reward Functions and the Structure of Equilibria

This section proves that any reward function can be replaced by one yielding each player the same expected value (without affecting incentive compatibility) and taking on values only on the set of extreme points of V. Apart from the obvious practical advantages this offers in working with particular games, it has theoretical applications: examples are provided in the proofs of Propositions 5 and 6.

For $W \subseteq \mathbb{R}^N$, let $\text{co } W$ denote the convex hull of W and $\text{Ext } W$ the set of extreme points of $\text{co } W$.

Definition: $u \in L^\infty(\Omega;W)$ has the bang-bang property if $u(p) \in \text{Ext } W$ a.e. $p \in \Omega$.

Propositon 3 below implies that the function U of Section 3 can be chosen so that for each w, $U(w)$ has the bang-bang property. Now consider the nature of an equilibrium with value w, and summarized by (Q,U) with U chosen as above. For any price $p(1)$ arising in the first period, an extremal reward $U(w)(p(1))$ is "delivered" by the pair $(Q(U(w)(p(1))), U(U(w)(p(1))))$. When $p(2)$ is observed, a new reward function comes into effect, and so on. Since after any t-period history, players' future payoffs are in $\text{Ext } V$, a play of the game can be viewed as an alternation among extreme points of V, where the particular pattern of extreme points is determined by the sequence of realized prices. For the special case in which V is one-dimensional (as it is, for example, when attention is restricted to symmetric equilibria of symmetric games), this means that only two extreme points, and hence two action profiles, ever arise after the first period of the game.

We now replace (A1) by

(A1*) $\quad S_i$ is finite $i = 1,\ldots,N$.

This assumption is retained up to Section 7, but is used only to ensure the sufficiency of bang-bang reward functions.

Proposition 3: Let $W \subseteq \mathbb{R}^N$ be compact and $(q,\hat{u})$ be an admissible pair w.r.t. co W. Then there exists a function $\bar{u} : \Omega \to \text{Ext } W$ such that $(q,\bar{u})$ is admissible w.r.t W and $E(q,\bar{u}) = E(q,\hat{u})$.

Proof: Let

$$F := \{u \in L^{\infty}(\Omega,\text{co } W) \mid (q,u) \text{ is admissible w.r.t. co } W \text{ and } E(q,u) = E(q,\hat{u})\}.$$

By assumption F is nonempty ($\hat{u} \in F$), and it may easily be checked that F is convex. By Alaoglu's theorem [21], F is compact when $L^{\infty}(\Omega;\text{co } W)$ is endowed with the weak-* topology. Hence, by the Krein-Milman theorem, F has an extreme point.

By (A1*), the set of integral constraints defining F is finite and Proposition 6.2 of Aumann [3] applies directly. It implies that any extreme point $\bar{u}$ of F satisfies $\bar{u}(\Omega) \subseteq \text{Ext } W$. Since $\text{Ext } W \subseteq W$, $(q,\bar{u})$ is also admissible w.r.t. W, and the proof is complete. □

Corollary. Let $W \subseteq \mathbb{R}^N$ be compact. Then $B(W) = B(\text{co } W)$.

Lemma 1: If $W \subseteq \mathbb{R}^N$ is compact, $B(W)$ is compact.

Proof: Let $W \subseteq \mathbb{R}^N$ be compact. An easy argument shows that $B(W) \subseteq \delta[\Pi(S) + \text{co } W]$, so $B(W)$ is bounded. We need only show that $B(W)$ is closed. Let $\{w_n\}$ be a sequence in $B(W)$ such that $\lim w_n = w$. We will argue that $w \in B(\text{co } W) = B(W)$. For each n there exists (q^n,u^n), an admissible pair w.r.t. W, such that $E(q^n,u^n) = w_n$. Since S is compact, $W \subseteq \text{co } W$, and $L^\infty(\Omega;\text{co } W)$ is weak-* compact, w.l.o.g. we can assume that there exist $q \in S$ and $u \in L^\infty(\Omega;\text{co } W)$ such that $q^n \to q$ and $u^n \overset{*}{\to} u$. In the Appendix we show that $E : S \times L^\infty(\Omega;\text{co } W) \to \mathbb{R}^N$ is continuous when $L^\infty(\Omega;\text{co } W)$ is endowed with the weak-* topology. Therefore

$$w = \lim w_n = \lim \ E(q^n;u^n) = E(q;u) \ .$$

Similarly, one can show that $E(q;u) \geq E(\gamma_i,q_{-i};u)$ for all $\gamma_i \in S_i$ and $i = 1,\ldots,N$. Hence (q,u) is admissible with respect to W, and $w \in B(\text{co } W)$. □

Proposition 4: V is compact.

Proof: See proof of Corollary 2 of [2]. □

5. Computation

For many purposes it is important to have an algorithm capable of finding the set V in particular supergames, thereby making possible a study of the details of the equilibria themselves. To do so, it is necessary to find the largest bounded fixed point of the set-valued map B. It turns out that V may be computed by an elegant procedure analogous to Howard's "value-iteration" [9] for dynamic programs. The algorithm starts with a set $W_0 \subseteq \mathbb{R}^N$ such that $V \subseteq B(W_0) \subseteq W_0$. It then proceeds by computing the monotonically decreasing sequence of sets $W_n = B(W_{n-1})$ $n = 1,2,\ldots$. V is the limit of this process: $V = \lim_{n\to\infty} W_n := \cap_{n=1}^{\infty} W_n$.

Observe that the operator B is monotone, in the sense that for each $W_1 \subseteq W_2 \subseteq \mathbb{R}^N$, $B(W_1) \subseteq B(W_2)$. The next two lemmas follow directly from Factorization and the monotonicity of B; their proofs are left to the reader.

Lemma 2: Let $W := [\delta/(1-\delta)]$ co $\{\Pi(q) | q \in S\}$. Then $V \subseteq B(W) \subseteq W$.

Lemma 3: If $W \subseteq \mathbb{R}^N$ satisfies $V \subseteq B(W) \subseteq W$, then $V \subseteq B(B(W)) \subseteq B(W)$.

Lemma 4: Let $\{W_n\}$ be a decreasing sequence of compact sets in $\mathbb{R}^N$. Then co $\cap W_n = \cap$ co W_n.

Proof: See Appendix.

Proposition 5 (Algorithm): Let $W \subseteq \mathbb{R}^N$ be compact and satisfy $V \subseteq B(W) \subseteq W$. Define $W_0 := W$ and for $n = 1,2,\ldots$ let $W_n := B(W_{n-1})$. Then $\{W_n\}$ is a decreasing sequence and $V = \lim_{n\to\infty} W_n$.

Proof: By Lemmas 1 and 3, $\{W_n\}$ is a decreasing sequence of compact sets, so $W_\infty := \lim_{n\to\infty} W_n = \cap W_n$ and W_∞ is compact. Again by Lemma 3, $V \subseteq W_\infty$. To complete the proof we need to show that $W_\infty \subseteq V$. By Self-Generation and the corollary to Proposition 3, it is sufficient to show that $W_\infty \subseteq B(\text{co } W_\infty)$. Consider any $w \in W_\infty$. By definition, for each $n = 1,2,\ldots,$ there exists (q^n,u^n) admissible w.r.t. W_n such that $E(q^n,u^n) = w$. Since $q^n \in S$, where S is compact, and $L^\infty(\Omega;W_n) \subseteq L^\infty(\Omega;\text{co } W)$, where $L^\infty(\Omega;\text{co } W)$ is a weak-* compact set, we may w.l.o.g. assume $q^n \to q$ and $u^n \overset{*}{\to} u$ for some $q \in S$ and $u \in L^\infty(\Omega;\text{co } W)$. We argue that (q,u) is an admissible pair w.r.t. co W_∞, and $w = E(q,u)$. Since for all $n = 1,2,\ldots,$ $u^m(\Omega) \subseteq \text{co } W_m \subseteq \text{co } W_n$ for all $m \geq n$ (modulo sets of measure 0), we have $u(\Omega) \subseteq \text{co } W_n$ for all n. Hence, by Lemma 4, $u(\Omega) \subseteq \cap \text{ co } W_n = \text{co} \cap W_n = \text{co } W_\infty$. Since $E : S \times L^\infty(\Omega;\text{co } W) \to \mathbb{R}^N$ is continuous when $L^\infty(\Omega;\text{co } W)$ is endowed with the weak-* topology,

$$E(q,u) = \lim_{n\to\infty} E(q^n;u^n) = w.$$

Also $E_i(q^n;u^n) \geq E_i(\gamma_i,q^n_{-i};u^n)$ for all $\gamma_i \in S_i$ and each $n = 1,2,\ldots$ imply $E_i(q;u) \geq E_i(\gamma_i,q_{-i};u)$ for all $\gamma_i \in S_i$ and each $i = 1,\ldots,N$. □

6. Comparative Statics: Monotonicity in δ

Intuition suggests that the equilibrium set should in some sense increase with the discount factor. Plausibly "cooperation" becomes easier as players become more patient and thereby increasingly willing to forego immediate gain for a possible future reward. One is led to conjecture a monotonic relationship between equilibrium outcomes and the number δ, where outcomes are thought of as average discounted payoffs. Despite the complexity and generality of the model, this conjecture can be proved correct without invoking any assumptions beyond those of Proposition 3. When the discount factor increases from δ_1 to δ_2, and payoffs are appropriately normalized, the original set of equilibrium values is contained in the new set of values associated with

δ_2. The proof is short and simple and illustrates the power of self-generation as an analytical tool.

We now write $V(\delta)$, $B(W|\delta)$ and $E(q;u|\delta)$ to make explicit the dependence on the particular value of the discount factor.

<u>Proposition 6</u> (Monotonicity in Discount Factor): Let δ_1 and δ_2 be two discount factors such that $0 < \delta_1 < \delta_2 < 1$. Then $[(1-\delta_1)/\delta_1]\, V(\delta_1) \subseteq [(1-\delta_2)/\delta_2]\, V(\delta_2)$

<u>Proof</u>: As may be easily checked, we need to show $(1+k)V(\delta_1) \subseteq V(\delta_2)$, where $k := (\delta_2-\delta_1)/(\delta_1(1-\delta_2))$. For any $w \in V(\delta_1)$ let (q,u) be an admissible pair w.r.t. $V(\delta_1)$ such that $w = E(q;u|\delta_1)$. Define the function u^+ on Ω by $u^+(p) = u(p) + kw$. Then it may be verified that (q,u^+) is an admissible pair w.r.t. $\{kw\} + V(\delta_1)$, and $E(q;u^+|\delta_2) = (1+k)w$. Hence, $(1+k)w \in B(\{kw\} + V(\delta_1)|\delta_2)$ for all $w \in V(\delta_1)$. Since for any $z \in \mathbb{R}^n$, $z + kw = \lambda(1+k)z + (1-\lambda)(1+k)w$ for $\lambda := 1/(1+k) \in (0,1)$, $\{kw\} + V(\delta_1) \subseteq \text{co}\,(1+k)V(\delta_1)$. Therefore, $(1+k)V(\delta_1) \subseteq B(\text{co}\,(1+k)V(\delta_1)|\delta_2)$. Finally, by the corollary of Proposition 3 and Self-Generation, $(1+k)V(\delta_1) \subseteq V(\delta_2)$. □

7. <u>Optimization and the Necessity of Bang-Bang Reward Functions</u>

This section explores the idea that efficient incentive schemes must necessarily have a bang-bang structure. Consider $W \subseteq \mathbb{R}^N$ compact and some $q \in S$ which is the first element of an admissible pair yielding an extremal payoff in the set $B(W)$. An implication of Proposition 3 is that among the reward functions which support q and maximize a linear function of player payoffs, at least one has the bang-bang property. We show here that under certain conditions <u>all</u> optimal solutions must be bang-bang. The proof takes a dual approach to the optimization problem which highlights the way in which considerations of efficiency lead to the use of rewards that are extreme points of V (or, more generally, of the compact set W from which rewards are to be drawn).

Establishing the necessity of bang-bang solutions requires mild conditions not needed for the sufficiency result. A discussion follows the statement and proof.

<u>Definition</u>: Let $A \subseteq \mathbb{R}^N$. $\alpha \perp A$ denotes "$\langle\alpha, x-y\rangle = 0$ for all $x,y \in A$". $\alpha \not\perp A$ denotes "not $\alpha \perp A$".

<u>Definition</u>: $q \in S$ satisfies the <u>Slater constraint qualification</u> w.r.t. W if there exists $u \in L^\infty(\Omega;\text{co } W)$ such that

$$E_i(q;u) > E_i(\gamma_i,q_{-i};u) \quad \text{for all} \quad \gamma_i \in S_i,\ \gamma_i \neq q_i, \quad \text{and} \quad i = 1,\ldots,N .$$

For all $\beta \in \mathbb{R}^N$ and $W \subseteq \mathbb{R}^N$ compact, let $F(\beta,W) := \arg\min\{\langle\beta,w\rangle | w \in W\}$ and $\mathbb{F}(W) = \{F(\beta,W) | F(\beta,W) \not\subseteq \text{Ext } W\}$.

<u>Proposition 7</u>: Let $W \subseteq \mathbb{R}^N$ be compact, and consider $(\bar{q},\bar{u}) \in \arg\min\{\langle\alpha,E(q;u)\rangle | (q,u)$ is admissible w.r.t. $W\}$ for some $\alpha \in \mathbb{R}^N$, $\alpha \neq 0$. Suppose that (i) $g(p,\bar{q})$ is analytic in p, (ii) $\bar{q}$ satisfies the Slater constraint qualification w.r.t. W, (iii) $\mathbb{F}(W)$ is a countable collection of sets, and (iv) $\alpha \not\perp F$ for all $F \in \mathbb{F}(W)$. Then $\bar{u}$ satisfies the bang-bang property.

<u>Proof</u>: Let α, $\bar{q}$, and $\bar{u}$ be as above. By Proposition 3, $\bar{u}$ is a solution to:

(P1) $\min \langle \alpha, \int_\Omega u(p) g(p;\bar{q}) dp \rangle$

subject to $u \in L^\infty(\Omega; \text{co } W)$

$$E_i(\gamma, \bar{q}_{-i}; u) \leq E_i(\bar{q}; u) \quad \text{for each} \quad \gamma \in S_i \quad \text{and} \quad i = 1, \ldots, N.$$

We show that any solution to (P1) that has range W must have the bang-bang property. The Lagrangean associated with (P1) is

$$L(u,\lambda) = \begin{cases} +\infty & \text{if } u \notin L^\infty(\Omega; \text{co } W) \\ \int_\Omega \langle u(p), \xi(p,\lambda) \rangle dp + b(\lambda) & \text{if } u \in L^\infty(\Omega; \text{co } W) \text{ and } \lambda \geq 0 \\ -\infty & \text{if } u \in L^\infty(\Omega; \text{co } W) \text{ and } \lambda \ngeq 0, \end{cases}$$

where λ is the vector of Lagrange multipliers $\{\lambda_{i\gamma} | \gamma \in S_i, \; i = 1, \ldots, N\}$,

$$\xi_i(p,\lambda) := (\alpha_i - \textstyle\sum_{\gamma \in S_i} \lambda_{i\gamma}) g(p,\bar{q}) + \textstyle\sum_{\gamma \in S_i} \lambda_{i\gamma} g(p; \gamma, \bar{q}_{-i})$$

and

$$b(\lambda) := \textstyle\sum_{i=1}^N \sum_{\gamma \in S_i} \lambda_{i\gamma} [\pi_i(\gamma, \bar{q}_{-i}) - \pi_i(\bar{q})] .$$

Note that $\xi_i(p,\lambda)$ is analytic in p. By (ii), optimal Lagrange multipliers $\bar{\lambda} \geq 0$ exist and any solution to (P1) also solves

(P2) $\min L(u,\bar{\lambda})$ subject to $u \in L^\infty(\Omega; \text{ co } W)$.

It is clear that any optimal solution u of (P2) which has range W must be such that $u(p) \in \arg\min \{\langle \xi(p,\bar{\lambda}), w \rangle | w \in W\}$ a.e. $p \in \Omega$. Therefore, to complete the proof it is sufficient to show that there does not exist $\Omega_0' \subset \Omega$ such that $\mu(\Omega_0') > 0$ (μ denotes the Lebesgue measure) and $F(\xi(p,\bar{\lambda})) \in \mathbb{F}$ for all $p \in \Omega_0'$. (For simplicity we write $F(\beta,W)$, $\mathbb{F}(W)$ as $F(\beta)$, $\mathbb{F}$ respectively.) Suppose there does. Since $\mathbb{F}$ is a countable collection there exist $\Omega_0 \subseteq \Omega_0'$ and $\eta \in \mathbb{R}^N$ s.t. $\mu(\Omega_0) > 0$ and $F(\xi(p,\bar{\lambda})) = F(\eta) \in \mathbb{F}$ for all $p \in \Omega_0$. Let $h(p;x,y) := \langle\xi(p,\bar{\lambda}),x-y\rangle$ for all $p \in \Omega$ and any $x,y \in F(\eta)$. By assumption, $h(p;x,y) = 0$ for all $p \in \Omega_0$. Since $h(p;x,y)$ is analytic in p and $\mu(\Omega_0) > 0$, $h(p;x,y) = 0$ for all $p \in \Omega$. Therefore, $0 = \int_\Omega h(p;x,y)dp = \langle\int_\Omega \xi(p,\bar{\lambda})dp,x-y\rangle = \langle\alpha,x-y\rangle$ for each $x,y \in F(\eta)$. Hence, $\alpha \perp F(\eta) \in \mathbb{F}$, contradicting (iv). □

Conditions (i) and (ii) are technical assumptions that facilitate the dual line of proof we pursue. Assumption (iii) is relatively innocuous. It is worth noting that it is always satisfied when $V \subseteq \mathbb{R}^2$. (iv) serves to exclude exceptional cases such as the following. Suppose that for prices in some set Ω_0 of positive measure, a reward function u supporting q optimally w.r.t. α takes on values in some face F of V. Consider another function $\hat{u}$ supporting q, which on Ω_0 also takes on values in F, and coincides with u elsewhere. (Typically there are many ways to satisfy the incentive constraints, but some are more efficient than others.) If α happens to be perpendicular to F, $\hat{u}$ yields the same value of the objective function as does u, and hence is a distinct solution to the optimization problem. Any non-degenerate convex combination of u and $\hat{u}$ is also an optimal solution, and fails to have the bang-bang property.

(iv) is unrestrictive in the following sense: it must be satisfied for a dense subset of extreme points of $B(W)$.

Lemma 5: Let $W \subseteq \mathbb{R}^N$ be compact, and assume $\mathbb{F}(W)$ is a countable collection. Then for every $x \in \text{Ext } B(W)$ and any $\varepsilon > 0$, there exist $y \in \text{Ext } B(W)$ and $\alpha \in \mathbb{R}^N$ s.t. $\|y-x\| < \varepsilon$, $y \in \arg\min \{\langle\alpha,w\rangle | w \in B(W)\}$ and $\alpha \not\perp F$ for all $F \in \mathbb{F}(W)$.

Proof: See Appendix.

8. Continuous Choice Sets

When players' choice sets are not finite, the methods of Section 4 fail. Specifically, the presence of an infinity of incentive constraints rules out the possibility of a straightforward adaptation of the proof of Proposition 3. On the other hand, the approach of the previous section is particularly well-suited to this situation: duality allows us to replace the original problem with one having no incentive constraints. Proposition 8 below gives conditions under which an optimal reward function must be bang-bang. It is exactly analogous to Proposition 7, except that assumption (iii) is now much more demanding[2] (although for $V \subseteq \mathbb{R}^2$, it is still satisfied automatically).

Assume now that the S_i's are compact subsets of $\mathbb{R}$. We only sketch the proof since it is very similar to that of Proposition 7.

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[2] A useful example is the following: suppose $N = 3$ and $W = \{(x,y,z) | x^2 + y^2 \leq 1,\ z = 0\} \cup \{(0,0,1)\}$. Then $\mathbb{F}(W) = \emptyset$ and $\mathbb{F}(\text{co } W)$ is an uncountable collection.

Proposition 8: Let W, $\bar{q}$, $\bar{u}$, α and (P1) be as in Proposition 7. Suppose that $g(p;q)$ is analytic[3] in p and continuous in q, and that the dual problem to (P1) has a solution. If $\mathbb{F}(\text{co } W)$ is a countable collection of sets and $\alpha \notin F$ for all $F \in \mathbb{F}(\text{co } W)$, then $\bar{u}$ satisfies the bang-bang property.

Proof: Let $\hat{u}$ be any solution to (P1). The Lagrangean associated with (P1) is as before. Now, however,

$$\xi_i(p,\lambda) := \left(\alpha_i - \int_{S_i} \lambda_i(\gamma)d\gamma\right) g(p,\bar{q}) + \int_{S_i} \lambda_i(\gamma) g(p;\gamma,\bar{q}_{-i}) d\gamma$$

and

$$b(\lambda) := \sum_{i=1}^{N} \int_{S_i} \lambda_i(\gamma)[\pi_i(\gamma,\bar{q}_{-i}) - \pi_i(\bar{q})]d\gamma .$$

ξ is analytic in p (see Appendix). Proceeding to (P2) as in Proposition 7 we have

$$\hat{u}(p) \in \arg\min \{\langle \xi(p,\bar{\lambda}), w\rangle \mid w \in \text{co } W\} \quad \text{a.e.} \quad p \in \Omega .$$

Replacing $F(\beta,W)$, $\mathbb{F}(W)$ by $F(\beta,\text{co } W)$, $\mathbb{F}(\text{co } W)$ and continuing as in the earlier proof, we conclude that $\hat{u}(p) \in \text{Ext } W$ a.e. $p \in \Omega$. Hence, $\bar{u}$ solves (P1) and the proof is complete. □

An alternative proof of necessity of bang-bang reward functions (for continuous choice sets) is provided in the next section. It emerges from the investigation of the nature of punishment regions when the first-order approach is valid.

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[3] We require that there exists an open set $\Omega_0 \subseteq \mathbb{C}$ such that $\Omega \subseteq \Omega_0$ and $g : \Omega_0 \times S \to \mathbb{C}$ is analytic in its first argument. $\mathbb{C}$ denotes the complex numbers.

9. Punishment Regions: First Order Conditions and Generalized Tail Tests

A major determinant of the equilibria constructed earlier is the nature of the punishment and reward regions they implicitly use. This section characterizes these regions for symmetric equilibria of symmetric games[4,5] having continuous strategy spaces and satisfying a strong condition: solutions to the static optimization problems of Section 8 coincide with those of the relaxed problem in which the incentive constraints are replaced by one first-order condition for the players. (The scope of the "first-order approach" is explored in the context of the principal-agent problem by Mirrlees [14], Grossman and Hart [7], and Rogerson [20], among others.) We show that the punishment region used in supporting the symmetric action profile q, consists of all prices for which the ratio $r(p;q) := \dfrac{\partial g(p;q)/\partial q_1}{g(p;q)}$ lies to the appropriate side (depending on the sign of $\partial \Pi_1(q)/\partial q_1$) of some critical value. This is a close analogue of the Neyman -Pearson lemma (see, for example, De Groot [5]). It is a generalized tail test in the sense that one is re-ordering price space according to the value of the ratio specified above, and then applying a tail test. When this ratio is monotonic in p (that is, g has the monotone likelihood ratio property (M.L.R.P.)), no re-ordering is necessary, and a conventional

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[4] These were the focus of attention in [2].

[5] The one-shot game is symmetric if $S_i = S_1$ $i = 2,\ldots,N$ and for any permutation $(i_1,\ldots,i_N)$ of $(1,\ldots,N)$, $g(p;(q_{i_1},\ldots,q_{i_N})) = g(p;(q_1,\ldots,q_N))$ and $\Pi_1(q_{i_1},\ldots,q_{i_N}) = \Pi_{i_1}(q_1,\ldots,q_N)$. A strategy profile of the repeated game is symmetric if in all contingencies all players are required to take the same action.

tail test is optimal. In this case, the best symmetric sequential equilibrium is described completely by two quantities and two trigger prices (see Section 6 of [2]). This kind of application of M.L.R.P. is familiar from the principal-agent literature, especially Mirrlees [13], Holmstrom [8], Milgrom [12] and Grossman and Hart [7].

We assume in this section that $\Pi_1(q)$ and $g(p;q)$ are differentiable w.r.t. q_1. The statement of Proposition 9 requires some additional definitions. Let $e_N := (1,\ldots,1) \in \mathbb{R}^N$.

Definition: Let $q = xe_N \in S$ and $W \subseteq \mathbb{R}$. The pair (q,u) is locally admissible w.r.t W if $u \in L^\infty(\Omega;W)$ and

$$\frac{\partial \Pi_1(q)}{\partial q_1} + \int_\Omega u(p)\,\frac{\partial g(p;q)}{\partial q_1}\,dp = 0 .$$

Also define $H(q;W) := \{u \mid (q,u) \text{ is locally admissible w.r.t. } W\}$

Let $\mathbb{R}^* := \mathbb{R} \cup \{-\infty,+\infty\}$. For all $R \in \mathbb{R}^*$, let $\Gamma(R) := \{p \in \Omega \mid r(p;q) \geq R\}$ and $\Gamma_0(R) := \{p \in \Omega \mid r(p;q) = R\}$. For compact $W \subseteq \mathbb{R}$, let $\underline{w} = \min W$ and $\bar{w} = \max W$. For each $R \in \mathbb{R}^*$, define the generalized tail test functions:

$$u^e(R)(p) := \begin{cases} \underline{w} & \text{if } p \in \Gamma(R) \\ \bar{w} & \text{otherwise} \end{cases}$$

$$u^c(R)(p) := \begin{cases} \bar{w} & \text{if } p \in \Gamma(R) \\ \underline{w} & \text{otherwise} \end{cases}$$

$u^e(R)$ (respectively, $u^c(R)$) punishes prices which are relatively more likely when output is expanded (contracted).

Proposition 9 (Generalized Tail Tests): Fix $q = xe_N \in S$ and define $d := \partial\Pi_1(q)/\partial q_1$. Let $W \subseteq \mathbb{R}$ be compact and $H(q,W) \neq \emptyset$. Assume that $\mu(\Gamma_0(R)) = 0$ for all $R \in \mathbb{R}$. Then

(i) if $d \geq 0$ and $H(q;W) \neq \{u^e(0)\}$, there exist $R_1, R_2 \in \mathbb{R}^*$, $R_1 < 0 < R_2$ such that $u^e(R_i) \in H(q;W)$, $i = 1,2$ and $E_1(q;u^e(R_1)) < E_1(q;u) < E_1(q;u^e(R_2))$ for any other $u \in H(q;W)$.

(ii) if $d < 0$ and $H(q;W) \neq \{u^c(0)\}$ there exist $R_1, R_2 \in \mathbb{R}^*$, $R_2 < 0 < R_1$ such that $u^c(R_i) \in H(q;W)$, $i = 1,2$ and $E_1(q;u^c(R_1)) < E_1(q;u) < E_1(q;u^c(R_2))$ for any other $u \in H(q;W)$.

Proof: The result is proved for $d > 0$. The proof for $d < 0$ is analogous and for $d = 0$ trivial. Consider the program:

$$\begin{aligned} P1(\alpha) \qquad & \min \ \alpha \int_\Omega u(p)g(p;q)dp \\ & \text{s.t. } u \in L^\infty(\Omega;\text{co } W) \\ & \qquad \int_\Omega u(p)r(p;q)g(p;q)dp = -d \ . \end{aligned}$$

We need to show that for some $R_1 < 0 < R_2$, $u^e(R_1)$ and $u^e(R_2)$ are unique solutions to $P1(\alpha)$ for $\alpha = 1$ and $\alpha = -1$ respectively. For notational convenience we usually omit the argument q from the functions g and r.

Define $F^e(R) = \int_\Omega u^e(R)(p)r(p)g(p)dp$. Clearly F^e is continuous. It attains a unique minimum at $R = 0$ and increases monotonically to the right and left of $R = 0$. Since $\int_\Omega r(p)g(p)dp = 0$, $F^e(\infty) = F^e(-\infty) = 0$. The Lagrangean associated to $P1(\alpha)$ is

$$K_\alpha(u,\lambda) = \int_\Omega u(p)(\alpha + \lambda r(p))\, g(p)dp \quad \text{if} \quad u \in L^\infty(\Omega;\text{co } W).$$

By assumption there exists $u \in L^{\infty}(\Omega;W)$, $u \neq u^e(0)$ such that $\int_\Omega u(p)r(p)g(p)dp = -d$. Since $\int_\Omega u^e(0)(p)r(p)g(p)dp <$ $\int_\Omega u(p)r(p)g(p)dp$, it is clear from the earlier discussion that there exist $a,b \in \mathbb{R}$ such that $\int_\Omega u^e(a)(p)r(p)g(p)dp = F^e(a) < -d <$ $F^e(b) = \int_\Omega u^e(b)(p)r(p)g(p)dp$. Hence the Slater constraint qualification is satisfied (see [19]). Let $\lambda^*(\alpha)$ be an optimal Lagrange multiplier. If u solves $P1(\alpha)$, then it also solves the half saddlepoint problem

$$P2(\alpha) \qquad \min K_\alpha(u,\lambda^*(\alpha)) \qquad \text{s.t.} \quad u \in L^{\infty}(\Omega;\text{co } W).$$

We consider the cases $\alpha = +1$ and $\alpha = -1$. For either suppose $\lambda^*(\alpha) < 0$. Since by assumption $\mu\big(\Gamma_0(-\alpha/\lambda^*(\alpha))\big) = 0$, the solution to $P2(\alpha)$ is unique and must be of the form $u^c(R)$ for some $R \in \mathbb{R}^*$. If $\lambda^*(\alpha) = 0$, the solution to $P2(\alpha)$ is either $u^c(\infty)$ $(\alpha = +1)$ or $u^c(-\infty)$ $(\alpha = -1)$. Since for all $R \in \mathbb{R}^*$ $\int_\Omega u^c(R)(p)r(p)g(p)dp \geq 0$, for $\lambda^*(\alpha) \leq 0$ we must have $d \leq 0$, a contradiction. Hence, $\lambda^*(\alpha) > 0$, $\alpha = -1, +1$. From $P2(\alpha)$ we now conclude that the solution is unique and has the form $u^e(R(\alpha))$ for some $R(\alpha) \in \mathbb{R}$. Returning to $P1(\alpha)$, it is clear that $R(\alpha)$ must satisfy $F^e(R(\alpha)) = -d$. The properties of F^e discussed earlier imply that this equation has two distinct solutions, one of them negative and the other positive. Clearly $R(1) = R_1$ is the former and $R(-1) = R_2$ the latter. □

Remark: If g is analytic in p, then $\mu(\Gamma_0(R)) > 0$ if and only if $R = 0$ and $r(p;q)$ is identically zero. Thus, we may replace "Assume that $\mu(\Gamma_0(R)) = 0$ for all $R \in \mathbb{R}$" by "Assume g is analytic in p."

Assume now that $\Omega \subseteq \mathbb{R}$.

<u>Definition</u>: g has the <u>Monotone Likelihood Ratio Property</u> (M.L.R.P.) if for all $q = xe_N \in S$, $r(p;q)$ is strictly decreasing in p.

If g satisfies M.L.R.P., then for any $R \in \mathbb{R}^*$ there exists $t \in \mathbb{R}^*$ such that $\Gamma(R) = \{p \in \Omega | p \leq t\}$. Thus, a trigger value of the likelihood ratio may be equivalently expressed as a <u>trigger-price</u>. For each trigger-price $t \in \Omega$ define the tail-test functions:

$$u^{\ell}(t)(p) := \begin{cases} \underline{w} & \text{if } p \leq t \\ \bar{w} & \text{otherwise} \end{cases}$$

$$u^{h}(t)(p) := \begin{cases} \bar{w} & \text{if } p \leq t \\ \underline{w} & \text{otherwise .} \end{cases}$$

$u^{\ell}(t)$ punishes "low" prices while $u^{h}(t)$ punishes "high" prices.

<u>Corollary</u> (Tail Tests): Fix $q = xe_N \in S$ and $d = \partial\Pi_1(q)/\partial q_1$. Let $W \subseteq \mathbb{R}$ be compact and $H(q;W) \neq \emptyset$. Assume g satisfies M.L.R.P. and let p_0 solve $r(p_0;q) = 0$. Then

(i) if $d \geq 0$ and $H(q;W) \neq \{u^{\ell}(p_0)\}$, there exist $t_1, t_2 \in \Omega$, $t_2 < p_0 < t_1$ such that $u^{\ell}(t_i) \in H(q;W)$ $i = 1,2$ and $E_1(q;u^{\ell}(t_1)) < E_1(q;u) < E_1(q;u^{\ell}(t_2))$ for any other $u \in H(q;W)$.

(ii) if $d < 0$ and $H(q;W) \neq \{u^{h}(p_0)\}$, there exist $t_1, t_2 \in \Omega$, $t_1 < p_0 < t_2$ such that $u^{h}(t_i) \in H(q;W)$ $i = 1,2$ and $E_1(q;u^{h}(t_1)) < E_1(q;u) < E_1(q;u^{h}(t_2))$ for any other $u \in H(q;W)$.

10. Conclusion

Our purpose in this paper has been to lay foundations for a systematic theory of repeated discounted games with imperfect monitoring. The results suggest that ultimately a rather rich and satisfying theory will emerge. Already available for a broad class of these games are powerful characterizations of the equilibrium value set, a variety of results on the nature of implicit reward functions generated by extremal equilibria, and comparative static and computational theorems. While some of the propositions, notably the bang-bang principle, specifically address the problems caused by imperfect monitoring, those in Section 3 (and, with appropriate qualifications, Sections 5 and 6) apply also to games with perfect monitoring. Not yet covered are hybrid cases falling between models with perfect monitoring and those having a publicly observed random signal with constant support. Also awaiting study are mixed strategy equilibria of repeated discounted games. These problems deserve much attention.

APPENDIX

<u>Lemma</u> (Continuity of E): Assume that the density g and the expected payoff function Π_i are continuous in q, and let $W \subseteq \mathbb{R}^N$ be compact. Then $E_i : S \times L^\infty(\Omega;W) \to \mathbb{R}$ is continuous when $L^\infty(\Omega;W)$ is endowed with the weak-* topology.

<u>Proof</u>: It is easy to show that for each $q \in S$ and each $\varepsilon > 0$, there exists $\alpha > 0$ such that

$$\int_\Omega |g(p;\gamma) - g(p;q)|dp < \varepsilon \quad \text{for all} \quad \gamma \in S \quad \text{with} \quad \|\gamma - q\| < \alpha \ .$$

Let $\varepsilon > 0$ and $(q,u) \in S \times L^\infty(\Omega;W)$. Define the following neighborhood of u in the weak-* topology of $L^\infty(\Omega;W)$:

$$N(u) := \{v \in L^\infty(\Omega;W) | \qquad |\int_\Omega (u_i(p) - v_i(p))g(p;q)dp| < \varepsilon/3\} \ .$$

We now show that there exists $\alpha > 0$ such that $|E_i(\gamma;v) - E_i(q;u)| < \varepsilon$ for all $v \in N(u)$ and $\gamma \in S$ with $\|\gamma - q\| < \alpha$. Let $M := \max \{|w_i| \ |(w_1,\ldots,w_N) \in W\}$. There exists $\alpha_1 > 0$ such that

$$\int_\Omega |g(p;\gamma) - g(p;q)|dp < \varepsilon/(3M) \quad \text{for all} \quad \gamma \in S \quad \text{with} \quad \|\gamma - q\| < \alpha_1 .$$

By continuity of Π_i, there exists $\alpha_2 > 0$ such that $|\Pi_i(q) - \Pi_i(\gamma)| < \varepsilon/3$ for all $\gamma \in S$ with $\|\gamma - q\| < \alpha_2$. Let $\alpha := \min \{\alpha_1,\alpha_2\}$. Then for each $\gamma \in S$ with $\|\gamma - q\| < \alpha$, and any $v \in N(u)$,

$|E_i(\gamma;v) - E_i(q;u)|$

$\leq \delta|\Pi_i(\gamma) - \Pi_i(q)| + \delta|\int_\Omega v_i(p)g(p;\gamma)dp - \int_\Omega u_i(p)g(p;q)dp|$

$\leq \delta\varepsilon/3 + \delta|\int_\Omega(v_i(p) - u_i(p))g(p;q)dp| + \delta|\int_\Omega v_i(p)(g(p;\gamma) - g(p;q))dp|$

$\leq \delta\varepsilon/3 + \delta\varepsilon/3 + \delta M \int_\Omega |g(p;\gamma) - g(p;q)|dp \leq \delta\varepsilon < \varepsilon.$

Therefore, E_i is continuous. □

<u>Lemma 4</u>: Let $\{W_n\}$ be a decreasing sequence of compact sets in $\mathbb{R}^N$. Then $\text{co} \cap W_n = \cap \text{ co } W_n$.

<u>Proof</u>: Clearly $\cap W_n \subseteq \cap \text{ co } W_n$, and since $\cap \text{ co } W_n$ is convex, $\text{co} \cap W_n \subseteq \cap \text{ co } W_n$. Conversely, we argue that $\text{co} \cap W_n \supseteq \cap \text{ co } W_n$. Let $x \in \cap \text{ co } W_n$. Then for each n, there exist $\lambda^n \in \mathbb{R}^{N+1}$ and $(w_1^n,\ldots,w_{N+1}^n) \in W_n^{N+1}$ such that $\lambda^n \geq 0$, $\sum_{i=1}^{N+1} \lambda_i^n = 1$, and $x = \sum_{i=1}^{N+1} \lambda_i^n w_i^n$. Since $\{\lambda^n\}$ is bounded, $\{(w_1^n,\ldots,w_{N+1}^n)\} \subseteq W_1^{N+1}$, and W_1 is compact, we can assume w.l.o.g. that $\lambda^n \to \lambda$ and $(w_1^n,\ldots,w_{N+1}^n) \to (w_1,\ldots,w_{N+1})$, where $\lambda \geq 0$ and $\sum_{i=1}^{N+1} \lambda_i = 1$. Since $(w_1^m,\ldots,w_{N+1}^m) \in W_n^{N+1}$ for each $m \geq n$, and W_n is compact, $(w_1,\ldots,w_{N+1}) \in W_n^{N+1}$ for all n. Thus $(w_1,\ldots,w_{N+1}) \in [\cap W_n]^{N+1}$, and by continuity $\sum_{i=1}^{N+1} \lambda_i w_i = x$. Therefore $x \in \text{co} \cap W_n$. □

<u>Lemma 5</u>: Let $W \subseteq \mathbb{R}^N$ be compact, and assume $\mathbb{F}(W)$ is a countable collection. Then for every $x \in \text{Ext } B(W)$ and any $\varepsilon > 0$, there exist $y \in \text{Ext } B(W)$ and $\alpha \in \mathbb{R}^N$ s.t. $\|y-x\| < \varepsilon$, $y \in \arg\min \{\langle\alpha,w\rangle | w \in B(W)\}$ and $\alpha \not\perp F$ for all $F \in \mathbb{F}(W)$.

Proof:

Let $\mathbb{F}(W) = \{F_k\}$ and denote by $F_k^{\perp}$ the set of all vectors perpendicular to F_k, that is,

$$F_k^{\perp} := \{\beta \in \mathbb{R}^N \mid \langle \beta, f_1 - f_2 \rangle = 0 \quad \text{for each} \quad f_1, f_2 \in F_k\} .$$

Finally define $F^{\perp} := \cup F_k^{\perp}$. Since $F_k^{\perp}$ is a subspace of $\mathbb{R}^N$ of dimension at most $N - 1$, $F_k^{\perp}$ is closed and nowhere dense for each k. Hence, by Baire's Category Theorem, $F^{\perp}$ is nowhere dense as the countable union of nowhere dense closed sets.

A point z is an exposed point of $B(W)$ if there exists a hyperplane H such that $\{z\} = H \cap \text{co } B(W)$. Take $x \in \text{Ext } B(W)$ and $\varepsilon > 0$. By Straszewicz's Theorem (see Theorem 18.6 in Rockafellar [18]), there exists an exposed point z of $B(W)$ such that $\|x - z\| < \varepsilon/2$. Let $\beta \in \mathbb{R}^N$ be such that $\{z\} = F(\beta, B(W))$. For each $\alpha \in \mathbb{R}^N$ consider the problem

$$P(\alpha) \qquad \min \langle \alpha, w \rangle \quad \text{subject to} \quad w \in B(W).$$

Let $\mathbb{B}$ denote the unit ball. By the Maximum Theorem (see, for example, Debreu [4]), there exists $\eta > 0$ such that for each $\alpha \in \beta + \eta\mathbb{B}$, $F(\alpha, B(W)) \subseteq F(\beta, B(W)) + (\varepsilon/2)\mathbb{B} = \{z\} + (\varepsilon/2)\mathbb{B}$. Since $F^{\perp}$ is nowhere dense, we can choose $\alpha \in \{\beta\} + \eta\mathbb{B}$ such that $\alpha \notin F^{\perp}$. The set $F(\alpha, B(W))$ contains an extreme point $y \in B(W)$. Therefore, $y \in \text{Ext } B(W)$, $\|y - x\| \leq \|y - z\| + \|z - x\| < \varepsilon$, and y can be supported by a hyperplane whose normal α is not perpendicular to any $F_k \in \mathbb{F}(W)$. □

Lemma (Analyticity of the index function ξ): Assume that there exists an open set $\Omega_0 \subseteq \mathbb{C}$ such that $\Omega \subseteq \Omega_0$ and $g : \Omega_0 \times S \to \mathbb{C}$ is analytic in its first argument and continuous in its second. Let $q \in S$ and $\lambda_i \in L^1(S_i;\mathbb{R}_+)$. Then the function $\zeta : \Omega_0 \to \mathbb{C}$, defined by

$$\zeta(p) := \int_{S_i} \lambda_i(\gamma)g(p;\gamma,q_{-i})d\gamma ,$$

is analytic.

Proof: We first show that ζ is continuous. Let $p \in \Omega_0$ and $N(p)$ be a compact neighborhood of p in Ω_0, and define

$$m(\gamma) := \max \{g(z;\gamma,q_{-i}) \mid z \in N(p)\}, \quad \gamma \in S_i .$$

By the Maximum Theorem, $m : S_i \to \mathbb{R}_+$ is continuous. Since S_i is compact, $m \in L^\infty(S_i;\mathbb{R}_+)$. Therefore, the function $\bar{m} : S_i \to \mathbb{R}_+$ defined by $\bar{m}(\gamma) := \lambda_i(\gamma)m(\gamma)$, $\gamma \in S_i$, belongs to $L^1(S_i;\mathbb{R}_+)$. By Lebesgue's dominated convergence theorem, ζ is continuous at p.

Let $\Delta \subseteq \Omega_0$ be a triangle. Then

$$\int_{\partial\Delta} \zeta(p)dp = \int_{\partial\Delta} \int_{S_i} \lambda_i(\gamma)g(p;\gamma,q_{-i})d\gamma\, dp = \int_{S_i} \lambda_i(\gamma) \int_{\partial\Delta} g(p;\gamma,q_{-i})dp\, d\gamma = 0$$

by Cauchy's theorem. Then Morera's theorem [21] implies that ζ is analytic in Ω_0. □

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