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by

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Abstract

In an earlier paper we presented a variable dimension algorithm for solving the linear complementarity problem (LCP). We now extend the class of LCP's that can be solved by this algorithm to include LCP's with copositive plus coefficient matrices. The extension, inspired by Lemke [1965], is obtained by introducing an artificial dimension and by applying the variable dimension algorithm to the enlarged LCP.

Key words: linear complementarity problem,
pivoting method,
variable dimension algorithm.

1. Introduction

The linear complementarity problem with matrix of coefficients $M \in R^{n\times n}$ and right hand side $q \in R^n$, LCP(M,q), consists in finding vectors $s = [s_1, \ldots, s_n] \in R^n$ and $z = [z_1, \ldots, z_n] \in R^n$ satisfying

(1.1) $\quad s - Mz = q$

(1.2) $\quad s,z \geq 0$

(1.3) $\quad s_i z_i = 0 \quad$ for $i = 1, \ldots, n$.

Two vectors s and z verifying (1.3) are called complementary. This problem arises in mathematical programming, economics, engineering, and game theory, and is well discussed, e.g., by Cottle and Dantzig [1968].

The standard way to solve LCP's is a pivoting method due to Lemke [1965]. Applied to LCP's with copositive plus coefficient matrices the algorithm either finds a solution or shows that none exists by demonstrating the infeasibility of linear system (1.1)-(1.2).

1.4. Definition. The matrix $M \in R^{n\times n}$ is copositive if $z^t Mz \geq 0$ for all $z \in R^n_+$. M is copositive plus if M is copositive and if $(M+M^t)z = 0$ for any $z \in R^n_+$ verifying $z^t Mz = 0$.

In an earlier paper [Van der Heyden, 1980] we presented a variable dimension algorithm for a fairly restrictive class of LCP's, characterized by Cottle [1980] as those with completely-Q coefficient matrices. Jones, Saigal, and Schneider [1983] have identified specially structured LCP's that can be more efficiently solved by a variable dimension algorithm. We now show that this variable dimension algorithm can also be applied to solve LCP's with copositive plus coefficient matrices.

To solve LCP's with copositive plus coefficient matrices we proceed as in Lemke [1965]. We introduce an artificial variable and increase the dimension of the LCP by one. The extension is obtained by applying our variable dimension algorithm to the larger dimensional LCP.

2. A variable dimension algorithm

Before reviewing the algorithm we recall the usual nondegeneracy assumption for pivoting algorithms [Dantzig, 1963] and introduce notation.

2.1. Assumption. Equation (1.1) is nondegenerate, i.e., every solution has at least n nonzero variables.

Given a positive integer $k \le n$ and given a vector $x \in R^n$, $x^{(k)}$ denotes the vector $[x_1, \ldots, x_k] \in R^k$. Similarly, the kth leading principal submatrix of M is denoted $M^{(k)}$. Given LCP(M,q) the k-problem is the subproblem LCP($M^{(k)}$, $q^{(k)}$).

Our variable dimension algorithm [Van der Heyden, 1980] starts at the point $[s,z] = [q,0]$ and follows particular lines of solutions for (1.1), called variable dimension lines.

2.2. Definition. A variable dimension line comsists of a line of solutions $[s,z]$ for (1.1) verifying the following statements:

a. there exists an index k with $s_k < 0$ and $z_k > 0$;

b. $z_j = 0$ for $j > k$;

c. if $k > 1$ then $s^{(k-1)}$ and $z^{(k-1)}$ are nonnegative and complementary.

The above line is a line of the k-problem in that a point on the line would solve the k-problem if not for $s_k < 0$. The intent of the algorithm in following the line is to reach an endpoint where s_k is zero. An endpoint is reached whenever a variable $w_h = s_h$ or z_h, $h \le k$, becomes nonbasic, that is, equal to zero. The nondegeneracy assumption (2.1) insures that only one variable becomes zero at an endpoint. Distinguishing three cases, we now show that an endpoint either solves the LCP or uniquely leads to another variable dimension line, which the algorithm follows next. We characterize the new variable dimension line by identifying the variable which is zero at the endpoint and which becomes nonzero along the line.

i. $h = k$, $w_k = s_k$ [Dimension increase].

This is the desired situation where the endpoint $[s,z]$ solves the k-problem, and also solves the LCP if $s \geq 0$. Otherwise let $g = \min (j \mid s_j < 0)$, $j > k$. The other variable dimension line incident to the endpoint is obtained by increasing z_g $(z_g > 0)$ and is a line of the g-problem.

ii. $h = k$, $w_k = z_k$ [Dimension decrease].

The fact that the algorithm never returns to its starting point $[s,z] = [q,0]$ implies that $k > 1$ at this endpoint. The endpoint is a new solution for the (k-1)-problem. Let $g = \max (j \mid z_j > 0)$, $g < k$. The other variable dimension line incident to the endpoint is a line of the g-problem obtained by decreasing s_g $(s_g < 0)$.

iii. $h < k$.

The endpoint is incident to another line of the k-problem obtained by increasing the variable complementary to w_h, namely z_h if $w_h = s_h$ and s_h if $w_h = z_h$.

Starting at $[s,z] = [q,0]$ the algorithm follows variable dimension lines and generates endpoints until it finds a solution for the LCP or generates an unbounded variable dimension line. Assuming nondegeneracy, the algorithm never visits an endpoint twice as the initial point is incident to precisely one variable dimension line, and any other endpoint is incident to at most two such lines. The finiteness of the number of variable dimension lines, and hence of endpoints, then proves that the algorithm must terminate with a solution if all variable dimension lines are bounded.

3. The enlarged problem

Following Lemke [1965] we extend the class of LCP's solvable by our variable dimension algorithm by embedding the LCP into a larger dimensional one and by then applying the variable dimension algorithm to the larger LCP.

The embedding consists for a given LCP(M,q) to consider LCP(M^*,q^*) with

$$M^* = \begin{bmatrix} 0 & -u^t \\ u & M \end{bmatrix} \quad \text{and} \quad q^* = \begin{bmatrix} q_0 \\ q \end{bmatrix} ,$$

where $u^t = [1, \ldots, 1] \in R^n$. The variables of LCP(M^*,q^*) are denoted $s^* = [s_0, s] \in R^{n+1}$ and $z^* = [z_0, z] \in R^{n+1}$. The right hand side constant q_0 is chosen positive and such that the nondegeneracy of LCP(M,q) is transmitted to LCP(M^*,q^*) .

To verify that the variable dimension algorithm when applied to LCP(M^*,q^*) always computes a solution, it suffices to show that the algorithm does not generate any unbounded variable dimension line. Such a line can be characterized by a nonzero directional vector $[\bar{s}^*, \bar{z}^*] \in R^{2(n+1)}$ verifying for a given integer k , $0 \le k \le n$,

$$\text{(3.1)} \qquad \begin{aligned} \bar{s}_0 \qquad\quad + u^t\bar{z} &= 0 \quad , \\ \bar{s} - u\bar{z}_0 + M\bar{z} &= 0 \end{aligned}$$

$$\text{(3.2)} \qquad \begin{aligned} \bar{s}_j &\ge 0 \quad \text{for} \quad j = 0, \ldots, k-1 \\ &\le 0 \qquad\qquad\; = k \quad , \end{aligned}$$

and

$$\text{(3.3)} \qquad \begin{aligned} \bar{z}_j &\ge 0 \quad \text{for} \quad j = 0, \ldots, n \\ &= 0 \quad \text{for} \quad j > k \quad . \end{aligned}$$

We now argue that no such line is generated by the algorithm when solving LCP(M^*,q^*). If $k \geq 1$ then the nonnegativity of $\bar{s}_0$ and $\bar{z}$, along with (3.1), yield $\bar{s}_0 = 0$ and $\bar{z} = 0$. Hence $\bar{z}_0 > 0$ (for otherwise $\bar{s}^* = \bar{z}^* = 0$) and $\bar{s} = u\bar{z}_0 > 0$. The latter inequality contradicts the nonpositivity of $\bar{s}_k$ and leaves us with having to consider the case $k = 0$. A point $[s^*,z^*]$ $\epsilon\ R^{2(n+1)}$ on the latter line verifies $s^* = [s_0,\ q + uz_0]$ and $z^* = [z_0,\ 0]$ with $z_0 > 0$ and $s_0 < 0$. However $s_0 = q_0 - u^t z = q_0 > 0$ then yields a new contradiction. Hence all variable dimension lines are bounded and the variable dimension algorithm always finds a solution for LCP(M^*,q^*). We examine this solution in the next section.

4. The copositive plus case

We now assume the coefficient matrix M to be copositive plus. For the variable dimension algorithm to solve LCP(M,q) the constant q_0 needs to be sufficiently large.

4.1. Assumption. q_0 is such that every extreme point of $\{[z_0, z] \in R^{n+1} \mid z_0 \geq 0 , z \geq 0 , \text{ and } uz_0 + Mz + q \geq 0\}$ verifies $u^t z < q_0$.

A lower bound for q_0 is available from Khachiyan [1979] who bounds the extreme points of a system of linear inequalities [see also Bland et al., 1981]. Alternatively, a lexicographic implementation [Dantzig, 1963] of the algorithm avoids the explicit determination of q_0 . In this implementation q_0 is assumed to be a very large parameter and the right hand side q^* is rewritten parametically as

$$q^* = \begin{bmatrix} 0 \\ q \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_0 \quad .$$

The pivoting operations are then applied to the tableau $[I^*, - M^*, r^*]$ with $I^* \in R^{(n+1)\times(n+1)}$ denoting a unit matrix and $r^* = [{0 \atop q}] \in R^{n+1}$. Throughout these operations the updated right hand side remains a linear function of q_0 with constant term being given by the update of column r^* and with the vector of coefficients of q_0 given by the update of the first column of I^* . For large q_0 the ratio tests determining the pivots consider these two columns in lexicographic fashion, the update of r^* being considered only if these are ties among the ratios based on the update of the first column of I^* . Should degeneracy occur and should ties remain after both updated columns have been considered, then the lexicographic method for meeting nondegeneracy assumption (2.1) requires that these ties be resolved by considering the updates of the other columns of I^* [Dantzig, 1963].

The previous section argues that the variable dimension algorithm applied to LCP(M^*,q^*) terminates finitely at a solution $[\overset{o}{s}{}^*, \overset{o}{z}{}^*]$. If $\overset{o}{z}_0 = 0$ then the coordinates $[\overset{o}{s}, \overset{o}{z}]$ of this solution clearly solve the original problem, LCP(M,q) . We conclude the paper by arguing that the alternative case, $\overset{o}{z}_0 > 0$, signals that LCP(M,q) admits no solution because of the infeasibility of linear system (1.1)-(1.2). The argument is essentially due to Lemke [1965].

The solution $[\overset{o}{s}{}^*, \overset{o}{z}{}^*]$ verifies

(4.2) $\quad s_0 = q_0 - u^t z = 0$,

(4.3) $\quad s - uz_0 - Mz = q$,

(4.4) $\quad z_0 > 0,\ s \geq 0,\ z \geq 0$,

(4.5) $\quad s_i z_i = 0 \quad \text{for} \quad i = 1, \ldots, n$.

Assumption (4.1) then implies that $[\overset{o}{s}, \overset{o}{z}]$ lies on an unbounded line of solutions for (4.3)-(4.5). The directional vector of this line, $[\bar{s}, \bar{z}_0, \bar{z}] \in R_+^{2n+3}$, verifies

(4.6) $\quad \bar{s} - u\bar{z}_0 - M\bar{z} = 0$

(4.7) $\quad \bar{s}_i\bar{z}_i = \bar{s}_i\overset{o}{z}_i = \overset{o}{s}_i\bar{z}_i = 0 \quad \text{for} \quad i = 1, \ldots, n$.

The complementarity between $\bar{s}$ and $\bar{z}$ and equation (4.6) yield $\bar{z}^t\bar{s} = (\bar{z}^t u)\bar{z}_0 + \bar{z}^t M\bar{z} = 0$. The copositivity of M then implies that $(\bar{z}^t u)\bar{z}_0 = 0$ and

$\bar{z}^t M\bar{z} = 0$. Should $\bar{z} = 0$ then $\bar{s} = u\bar{z}_0$ with $\bar{z}_0 > 0$ so that $\overset{o}{s}{}^* = [0, q + u\overset{o}{z}_0]$ and $\overset{o}{z}{}^* = [\overset{o}{z}_0, 0]$. However this cannot be as $\overset{o}{z} = 0$ implies $\overset{o}{s}_0 = q_0 > 0$. Hence $\bar{z} \neq 0$, $\bar{z}_0 = 0$, $\bar{z}^t M\bar{z} = 0$, and $(M + u^t)\bar{z} = 0$ by the copositive plus property of M .

We now show that $\bar{z}$ verifies

$$(4.8) \qquad \bar{z}^t M \leq 0 \quad \text{and} \quad \bar{z}^t q < 0 .$$

The first inequality follows from the substitution of $\bar{z}_0 = 0$ into (4.6) yielding $0 \leq \bar{s} = M\bar{z} = -M^t\bar{z}$ since $(M+M^t)\bar{z} = 0$. The second inequality uses the complementarity relations (4.7). The complementarity of $\overset{o}{z}$ and $\bar{s}$ along with (4.6) yields $0 = \overset{o}{z}{}^t\bar{s} = \overset{o}{z}{}^t M\bar{z}$. Since $(M+M^t)\bar{z} = 0$ we also have $0 = \overset{o}{z}{}^t M^t\bar{z} = \bar{z}^t M\overset{o}{z}$. From the complementarity of $\bar{z}$ and $\overset{o}{s}$ and equation (4.3) we then deduce $0 = \bar{z}^t\overset{o}{s} = (\bar{z}^t u)\overset{o}{z}_0 + \bar{z}^t M\overset{o}{z} + \bar{z}^t q = (\bar{z}^t u)\overset{o}{z}_0 + \bar{z}^t q$. Hence $\bar{z}^t q < 0$ as $(\bar{z}^t u)\overset{o}{z}_0 > 0$.

The infeasibility of linear system (1.1)-(1.2) is now an easy conclusion as the existence of a vector $\bar{z}$ verifying inequalities (4.8) is easily seen to be inconsistent with (1.1)-(1.2) having a solution.

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