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ALGORITHMS FOR THE LINEAR COMPLEMENTARITY PROBLEM WHICH ALLOW AN ARBITRARY STARTING POINT

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1. Introduction

The linear complementarity problem with data $q \in R^n$ and $M \in R^{n \times n}$ consists in finding two vectors s and z in R^n such that

(1.1) $s = Mz + q$,

(1.2) $s, z \geq 0$,

(1.3) $s_i z_i = 0$, $i = 1, 2, \ldots, n$.

We denote this problem LCP or LCP(q,M). Two vectors s and z satisfying (1.3) are said to be <u>complementary</u>.

The LCP is an important problem in mathematical programming [see, e.g., Garcia and Gould (1980) for references]. Lemke (1965) first presented a solution for this problem. His ideas were later exploited by Scarf (1967) in his work on fixed point algorithms. The relationship between the LCP and the fixed point problem is well described by Eaves and Scarf (1976) and by Eaves and Lemke (1979).

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Recently, van der Laan and Talman (1979, 1981) proposed a class of variable dimension restart algorithms for approximating fixed points. These methods allow a start at an arbitrary point in the domain of the fixed point problem. One among several directions is followed to leave the starting point. These directions at the starting point define a collection of cones of variable dimensions in which the search for an approximate fixed point takes place. Properties of the function govern the movement of the procedure between the conical regions. In each region movement occurs through simplicial pivoting, but continuous path-following could be applied too [see Allgower and Georg (1980)].

The intimate relation between the fixed point problem and the LCP raises the question of the significance of van der Laan and Talman's work for the LCP. This paper shows that the main features of the variable dimension fixed-point algorithms developed by van der Laan and Talman (1981) can be adapted to the linear complementarity problem. One distinguishing feature of the resulting algorithms is that they allow an arbitrary starting point $z \geq 0$ whereas most LCP algorithms start at the origin $z = 0$. The only other algorithms sharing this feature are the homotopy procedure of Garcia and Gould (1980) and two algorithms sketched by Reiser (1978) in the appendix to his dissertation. In fact, one of the Reiser algorithms appears as a special case in our framework, while another special case of our framework is very similar to Reiser's second algorithm. This relationship is not surprising because of the similarity between Reiser's and one of van der Laan and Talman's fixed point algorithms [see Reiser (1981)]. One of the merits of our framework is that it encompasses both of Reiser's algorithms. Another interesting feature of our class of algorithms is that they coincide with Lemke's original algorithm when they are started at the origin $z = 0$. Finally, our framework can be motivated by considering the artificial column of Lemke's algorithm as a measure of infeasibility when a solution has not yet been reached.

The paper is organized as follows. In Section 2 we define the line segments that are followed by the procedure and the typical positions of the algorithm. The procedure itself is explained in Section 3 which also deals with convergence issues. The implementation of the algorithm can be found in Section 4. Finally, Section 5 contains our concluding remarks.

2. Movement and Positions of the Algorithm

We only consider points (s,z) in R^{2n} satisfying

$$(2.1) \qquad s = Mz + q \;, \quad z \geq 0 \;.$$

A measure of infeasibility for such points is

$$(2.2) \qquad t_0 = \max\begin{cases} -s_i & \text{for } i \in N = \{1, 2, \ldots, n\}, \\ s_i & \text{for } i \in N^+ = \{j \in N : z_j > 0\}. \end{cases}$$

This measure checks for the nonnegativity of s and for its complementarity with z. It is clear that z is a solution if and only if $t_0 \leq 0$. The measure t_0 can be negative only at $z = 0$ when q has all its coordinates positive $(q > 0)$. A positive value for t_0 corresponds to the value of the leading infeasibility or infeasibilities at z. Starting at the arbitrary starting point z^0, our algorithm increases z_i if $-s_i = t_0$ at z^0, or decreases z_i if $s_i = t_0$ with $z_i^0 > 0$. In other words, each term in maximand (2.2) is associated with a direction parallel to a coordinate axis (either e^i or $-e^i$, $i \varepsilon N$, e^i being the i^{th} unit vector). The direction followed to leave z^0 is that associated with the leading infeasibility--assumed unique--at z^0. The algorithm maintains this property: it only moves into regions associated with leading infeasibilities. For example, when the algorithm moves into the region $z = z^0 + e^1y_1 + e^2y_2 - e^3y_3$ with $y_1, y_2, y_3 > 0$, then it will be true that $t_0 = -s_1 = -s_2 = s_3$. This complementarity between directions incident at z^0 and leading infeasibilities is central to our procedure, and will be shown to identify a path of line segments which starts at z^0 and under certain conditions converges to a solution for the LCP. Before formalizing this complementarity, we slightly generalize expression (2.2) and define directions corresponding to the terms in the new expression. Our exposition will also be simplified by assuming that z^0 has only positive coordinates $(z^0 > 0)$. The general case $(z^0 \geq 0)$ will be discussed in Section 5.

Given a partitition $\{I_j : j = 1, 2, \ldots, k\}$ of the index set N, and given that all coordinates of z^0 are positive, we generalize the definition of t_0 at $z = z^0$:

$$(2.3) \qquad t_0 = \max\begin{cases} -s_j & \text{for } j \varepsilon N, \\ \sum\limits_{i\varepsilon I_j} s_i & \text{for } j \varepsilon K = \{1, 2, \ldots, k\}. \end{cases}$$

Again, we can write that z^0 is a solution if and only if $t_0 \leq 0$ at $z = z^0$. The above measure of infeasibility is valid at every $z \geq 0$, but will be modified on the boundary of the nonnegative orthant. With each of the $n+k$ terms in (2.3) we associate a direction which can be used to leave the starting point z^0. The directions $(d^i, i \varepsilon N)$

associated with the first n terms are still the ones mentioned earlier:

$$d^i = e^i \quad \text{for} \quad i \in N .$$

Leaving z^0 along d^i, $i \in N$, amounts to increasing z_i. With the $(n+j)^{th}$ term, $\sum_{i \in I_j} s_i$, we associate direction d^{n+j} where:

$$\begin{aligned} d_i^{n+j} &= -z_i^0 \quad \text{for} \quad i \in I_j , \\ &= 0 \quad \text{for} \quad i \in N - I_j . \end{aligned}$$

If we leave z^0 along d^{n+j}, we simultaneously decrease all coordinates of z with indices in I_j. The specific choice for d^{n+j} derives from a requirement by the procedure that, leaving z^0 along direction d^{n+j}, all coordinates with index in I_j should simultaneously become equal to zero. The different directions are illustrated in Figure 1.

The figure illustrates well that the starting point z^0 and the directional matrix $D = (d^1, d^2, \ldots, d^{n+k})$ partition the nonnegative orthant of z-space into relatively open areas $\{z \in R_+^n : z = z^0 + Dy,$ $y \in R_+^{n+k}, y_j > 0$ for $j \in P\}$, where P is a feasible subset of $N_0 = \{1, 2, \ldots, n+k\}$. A subset P is said to be <u>feasible</u> if, for any $j \in K$, if does not contain both index $n+j$ and all indices in I_j. An infeasible P leads to multiple representations for a vector z in terms of the y-coordinates. If we consider only y-vectors with feasible subsets of positive coordinates, the correspondence between vectors $z \in R_+^n$ and $y \in R_+^{n+k}$, with $y_j \leq 1$ for $n+1 \leq j \leq n+k$, is one-to-one. We will equivalently refer to z or to its representation in terms of y.

Our algorithm maintains a generalized form of complementarity, called

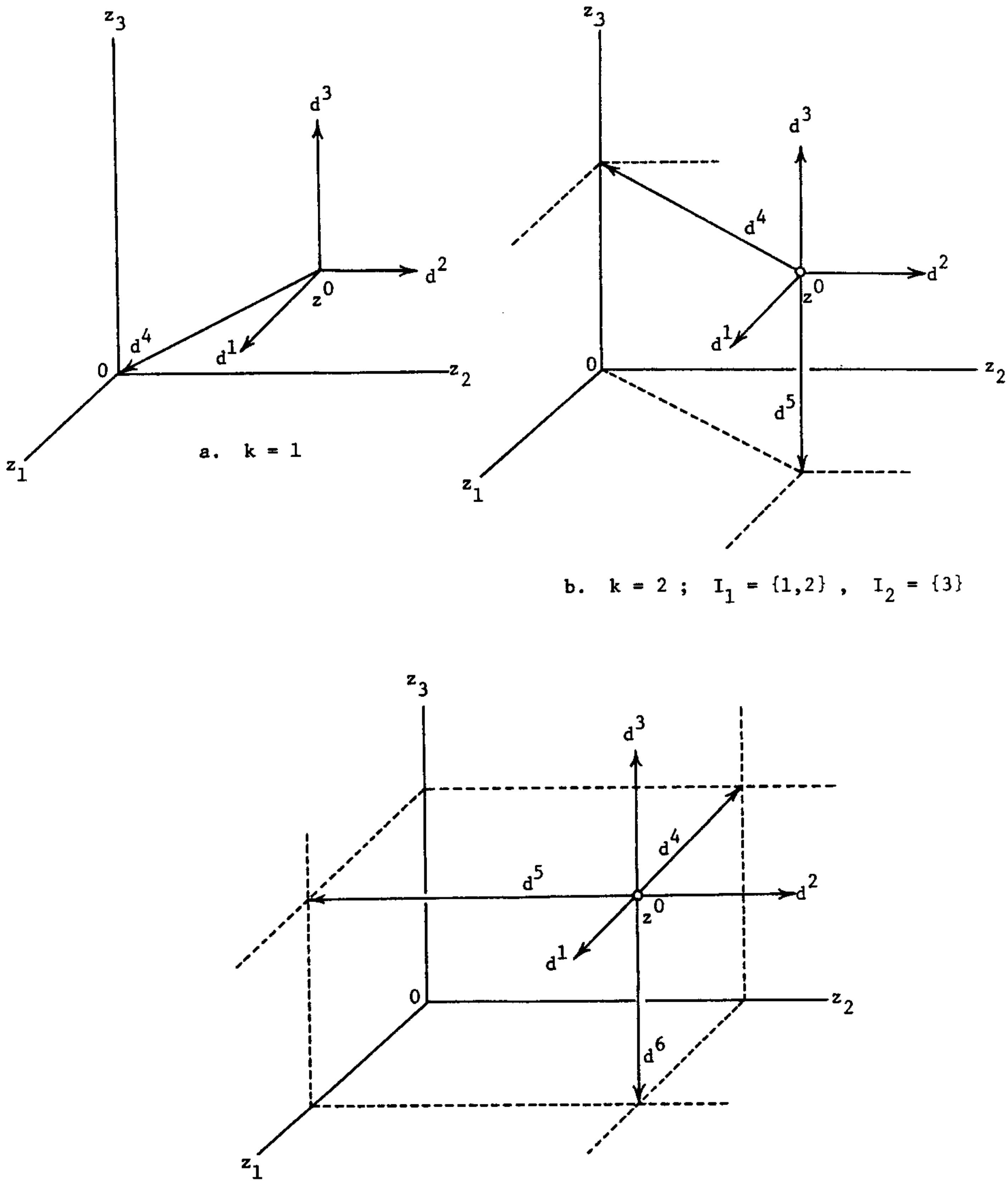


FIGURE 1. The directions d^j , $1 \leq j \leq n+k$, in three special cases of our algorithm ($n = 3$; $k = 1, 2$, and 3).

t_0-complementarity, between leading infeasibilities in maximand (2.3) and directions represented as columns of the matrix D . t_0-complementarity is more easily explained by introducing a vector whose components are the terms in maximand (2.3). Let $t = (t_1, t_2, \ldots, t_{n+k})$ be defined as:

(2.4) $$t_j = -s_j \quad \text{for} \quad j \in N ,$$

$$t_{n+j} = \sum_{i \in I_j} s_i \quad \text{for} \quad j \in K .$$

With this notation, (2.3) can be rewritten

(2.5) $$t_0 = \max(t_j : j \in N_0)$$

$$\text{with} \quad N_0 = \{1, 2, \ldots, n+k\} .$$

It will also be convenient to partition N_0 into sets $N_1 = N$ and $N_2 = \{n+1, n+2, \ldots, n+k\}$. Except for boundary issues, t_0-complementarity means that if $y_j > 0$, $j \in N_0$, then $t_0 = t_j$. We now motivate the changes to t_0-complementarity on the boundary of the nonnegative orthant in z-space.

Assume that initially $t_0 = t_j > 0$ for $j \in N_2$. Maintaining t_0-complementarity, the algorithm leaves z along direction d^j . The coordinates of z_i with $i \in I_h$ and $j = n+h$, are decreased along this line. In later stages, other directions are considered by the algorithm and larger dimensional regions explored. If at some point, one or more coordinates z_i , $i \in I_h$, become equal to zero--implying $y_{n+h} = 1$ --further movement along direction d^{n+h} is excluded for we don't want to leave the nonnegative orthant. Variable y_{n+h} is then fixed at 1, while t_{n+h} is free again to differ from t_0 although $y_{n+h} > 0$. This is achieved by completing the definition of t_0 on the boundary in the following way:

(2.6) $$t_0 = \max\{t_j : j \in N_0,\ y_j < 1 \text{ if } j \in N_2\} .$$

Hence, on the boundary we delete from maximand (2.5) any term t_{n+h} associated with a zero coordinate z_i , $i \in I_h$. Notice that $y_j < 1$ with $j \in N_2$ implies that $z_i > 0$ for <u>all</u> $i \in I_h$.

Definition (2.6) presents one complication in that t_0 can vary discontinuously when reaching the boundary. For example, assume that after leaving z^0 along direction $d^{n+h}(t_0 = t_{n+h})$, we reach the boundary where $y_{n+h} = 1$ and $z_i = 0$ for $i \in I_h$. Following (2.6) t_0 then decreases discontinuously--if all other t_j's are smaller than t_{n+h} --and becomes equal to the second leading infeasibility. At the boundary point $t_{n+h} > t_0$ and $t_j = t_0$ for some $j \in N_0 - \{n+h\}$. On the boundary t_0-complementarity thus takes a different form: the pair (y_{n+h}, t_{n+h}) is said to be t_0-complementary also when $y_{n+h} = 1$, although $t_{n+h} > t_0$ and $y_{n+h} > 0$. It is interesting to observe how the algorithm continues. With $y_{n+h} = 1$, the pair (y_{n+h}, t_{n+h}) is t_0-complementary, and so is the pair (y_j, t_j) , where $y_j = 0$ and where t_0 just became equal to t_j . Moving so as to maintain t_0-complementarity, the algorithm then makes y_j positive. Notice that in this movement $t_{n+h} > t_0$. It is important to observe that while $y_{n+h} = 1$ the latter inequality will be maintained. If t_{n+h} becomes equal to t_0 again, the algorithm continues by decreasing y_{n+h} away from 1, so that t_0-complementarity is maintained.

We now formalize t_0-complementarity by introducing basic and nonbasic variables.

Definition 2.1. A variable y_j , $j \in N_1$, is said to be nonbasic if $y_j = 0$. A variable y_j , $j \in N_2$, is said to be nonbasic if $y_j = 0$ or 1. With t_0 as defined in (2.5), t_j is said to be nonbasic if $t_j = t_0$. t_0 is nonbasic if $t_0 = 0$. When not nonbasic, a variable is said to be basic.

Notice that one among the variables t_j , $j \in N_0$, is nonbasic by definition. At first glance, the above definition may not appear similar to the conventional one of linear programming. The link with linear programming will be clarified later in this paper.

Definition 2.2. A pair (t,y) is t_0-complementary if for every $j \in N_0$ either or both y_j and t_j are nonbasic. We also call the point (s,z) t_0-complementary when the corresponding pair (t,y) is t_0-complementary.

A technical point has to be dealt with in order to have a well defined algorithm. The following assumption entails no loss of generality as a slight perturbation of the data will be shown to yield nondegeneracy.

Nondegeneracy Assumption 2.1. Among the $2(n+k)+1$ variables (t_0, t, y) at most $n+k+1$ are nonbasic at any given time.

The algorithm can now be described more precisely. The starting point z^0 is t_0-complementary as $y^0 = 0$. The nondegeneracy assumption ensures that there is exactly one nonbasic variable t_j , $j \in N_0$. To leave z^0 while maintaining t_0-complementarity means that we can move only by increasing y_j . We thus either increase z_j if $j \in N_1$ or decrease the z_i's with $i \in I_h$ if $j \in N_2$ with $j = n+h$. We pursue this movement until one—precisely one by the nondegeneracy assumption—basic variable becomes nonbasic. As long as $t_0 > 0$, a solution has

has not yet been reached and there is precisely one pair of nonbasic variables (t_j, y_j) . The algorithm makes one of these variables basic and continues its movement along another line of t_0-complementary points where precisely one variable in each pair (t_h, y_h) is nonbasic.

We formally introduce the lines followed by the algorithm. Recall that on the boundary of the nonnegative orthant $y_j = 1$ for some $j \varepsilon N_2$. For that j , the requirement along the line is $t_j > t_0$. Figure 2 identifies the nonbasic t-variables in different regions of z-space for the case $n = 2$.

<u>Definition 2.3</u>. A line of our algorithm consists of points on a line of solutions for (2.1) such that:

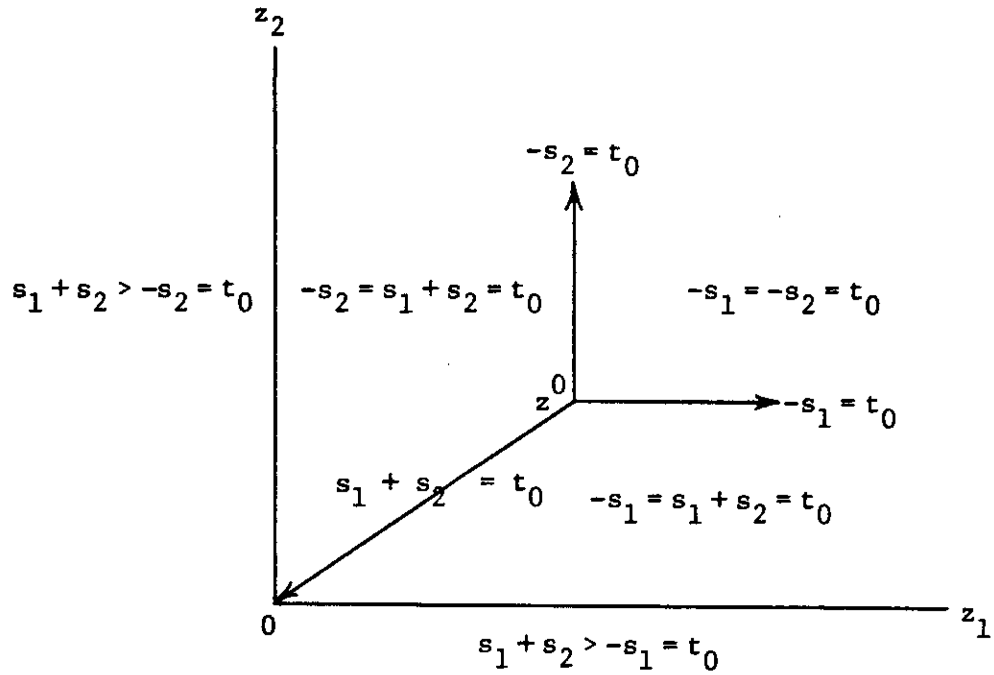
a. exactly one variable in each pair (t_j, y_j) is nonbasic (t_0-complementarity);

b. $t_j > t_0$ if $y_j = 1$ and $j \varepsilon N_2$;

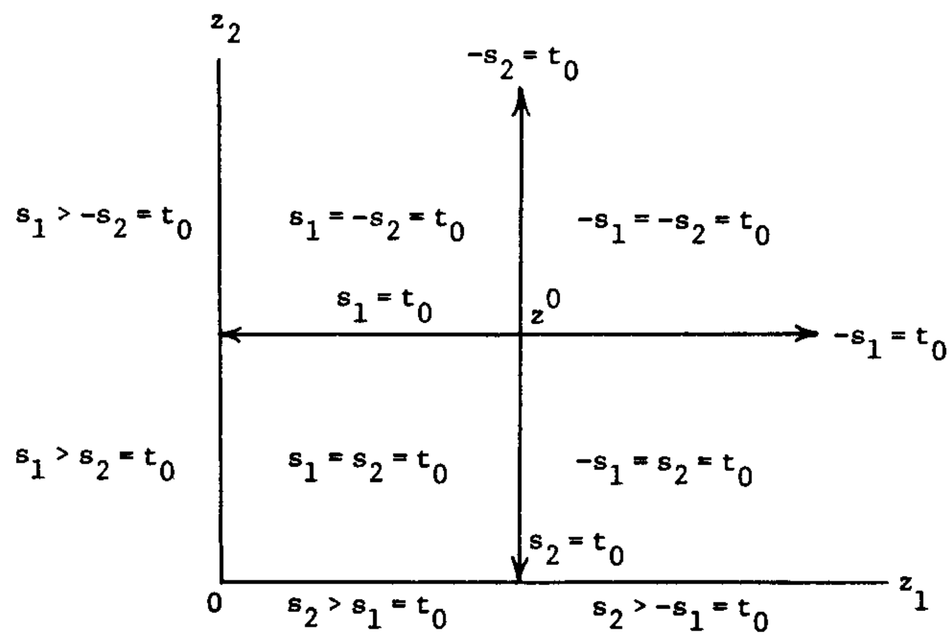
c. $t_0 > 0$.

The number of free (or basic) y-variables is equal to the number of constraints on the s-variables minus one (one t-variable is always nonbasic by definition; alternatively, eliminate t_0 and count the number of constraints--all independent--imposed on the s-variables). Hence, a set of t_0-complementary solutions for (2.1) is indeed a line segment if the set of nonbasic variables is fixed. The line segment is relatively open if exactly one variable in every pair (t_j, y_j) is nonbasic.

Let us examine the endpoints of the lines of our algorithm. An endpoint is reached when the set of nonbasic variables changes. If there is no discontinuity in the value of t_0 , and if t_0 is still basic, there is exactly one pair of t_0-complementary variables which are both



a. $n = 2$; $k = 1$.



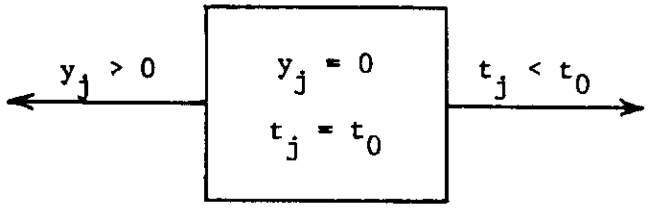
b. $n = 2$; $k = 2$.

FIGURE 2. The constraints imposed on the s-variables in different regions of z-space for a 2-dimensional example. Notice that we have omitted the inequalities that are implicit in the definition of t_0 .

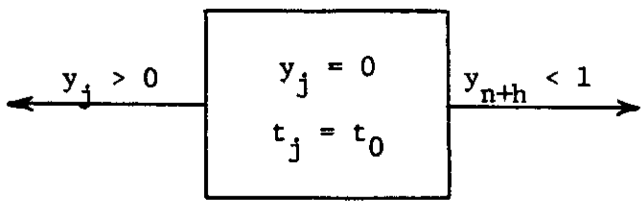
nonbasic. This gives rise to two types of positions for the algorithms. At a <u>position of type</u> a we have that, for some $j \varepsilon N_0$, $y_j = 0$ and $t_j = t_0 > 0$. At a <u>position of type</u> b we have that, for some $j \varepsilon N_2$, $y_j = 1$ and $t_j = t_0 > 0$. If an endpoint is reached where t_0 is nonbasic, then the endpoint will be shown to be a solution. The same is true if t_0 becomes nonpositive during a discontinuous decrease at the endpoint. A discontinuity in t_0 arises if y_j, $j \varepsilon N_2$, increases to 1 with y_j being the only nonbasic variable along the line. Upon reaching this endpoint, t_j is no longer considered in the determination of t_0. t_0 then decreases discontinuously since no other variable t_h is equal to t_0 along the line leading to the endpoint. If after the discontinuous decrease t_0 is still positive, there is one nonbasic pair (y_h, t_n) with $t_h = t_0 > 0$ and $y_h = 0$ for some $h \varepsilon N_0$. Such an endpoint is a position of type a. This completes our classification of endpoints into positions of type a or b and t_0-complementary points with $t_0 \leq 0$.

We now prove the important fact that if t_0 becomes nonpositive $(t_0 \leq 0)$ at an endpoint, then a solution has been found. Since $t_0 \geq \max(t_i = -s_i : i \varepsilon N)$, it is clear that $s \geq 0$ at such an endpoint. We still need to argue that $s_i = 0$ whenever $z_i > 0$. We distinguish two cases. If $y_{n+h} < 1$, with $i \varepsilon I_h$, then $s_i = 0$ easily follows from the fact that $0 \geq t_0 \geq t_{n+h} = \sum_{i \varepsilon I_h} s_i \geq 0$. If $y_{n+h} = 1$, then the positivity of z_i requires the positivity of y_i along the line leading to the endpoint. Hence, t_i is nonbasic along the line: $-s_i = t_i \geq 0$ (since $t_0 > 0$ along the line). This inequality is still valid at the endpoint and implies $s_i = 0$ at the endpoint.

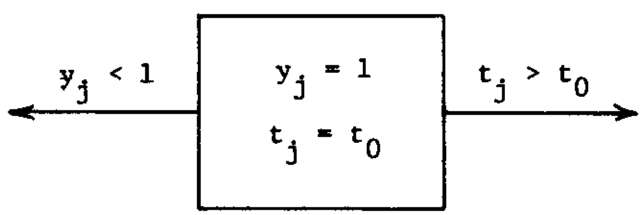
As a way of summary, we illustrate the incidence between positions



i. Position of type a: at least one y-variable is basic. No discontinuity in value of t_0 .



ii. Position of type a: all y-variables are nonbasic. Let $n+h = \text{argmin}(t_{n+j} : y_{n+j} = 1)$. Discontinuous increase in value of t_0 when leaving position along line drawn at the right of the position.



iii. Position of type b: $j \in N_2$. No discontinuity in value of t_0 for there is another $t_h = t_0$.

FIGURE 3. The incidence between positions (with $t_0 > 0$) and lines of our algorithm. Notice that in case ii, the line drawn at the right of the position is defined only if $\{n+i : y_{n+i} = 1\}$ is nonempty. If the set is empty, we are at the initial position. The initial position is the only position (with $t_0 > 0$) incident to only one line of our algorithm.

and lines of our algorithm in Figure 3. The algorithm leaves the initial position along the unique line incident to it. Every other position, which is not a solution, has two lines incident to it. If the position is reached along one line, then the algorithm leaves it along the other line. Solutions can be shown to be incident to only one line of our algorithm.

3. Convergence Issues

The previous section set the stage for an application of the well-known Lemke-Howson argument. The initial position is incident to one line of the algorithm. Every other position which is not a solution is incident to two lines of our algorithm. The Lemke-Howson argument proves that under these conditions no position will ever be visited twice. The number of lines is finite, hence, so is the number of positions. The algorithm thus either stops at a solution for the LCP or follows an unbounded line. Following Lemke (1965), we present a class of matrices --characterized by Garcia (1973)--for which the algorithm fi ds a solution for any right-hand side vector q. We then show that for copositive plus matrices [Lemke (1965)] the existence of an unbounded line implies that the LCP is not feasible. Of course, the point behind both results is that they hold for every initial starting point z^0 in R^n_+. [Garcia and Gould (1980) discuss the possibility of convergence for a particular set of starting points.]

<u>Theorem 3.1.</u> Let M satisfy the property that LCP(q,M) admits the unique solution $z = 0$ both when $q = 0$ and when $q = e$, where $e = (1, 1, \ldots, 1)^t$. Then no line of our algorithm is unbounded.

Proof. An unbounded line of our algorithm implies the existence of a (2n+1)-directional vector $(\bar{t}_0, \bar{s}, \bar{z})$ verifying the following conditions:

(3.1)
a. $\bar{s} = M\bar{z}$ with $\bar{z} \geq 0$;
b. if $\bar{z}_i > 0$ then $-\bar{s}_i = \bar{t}_0$;
c. if $\bar{z}_i = 0$ then $-\bar{s}_i \leq \bar{t}_0$;
d. $\bar{t}_0 \geq 0$.

(Notice that the directional vector $\bar{y}$ associated with $\bar{z}$ always has $\bar{y}_j = 0$ for $j \in N_2$, for we can't leave the nonnegative orthant in z-space. Hence, $\bar{y}_i = \bar{z}_i$ for $i \in N$.) It is clear from (3.1) that $\bar{z}$ is nonzero. If $\bar{t}_0 = 0$ then $\bar{s}$ is nonnegative and complementary with $\bar{z}$, which itself is nonnegative. $\bar{z}$ represents a nontrivial solution for LCP(0,M), which is impossible. If $\bar{t}_0 > 0$, we rescale $\bar{s}$ and $\bar{z}$ so that $\bar{t}_0 = 1$. $\bar{z}$ then satisfies the inequalities $M\bar{z} + e \geq 0$, where the i^{th} inequality is an equality if $\bar{z}_i > 0$. This shows that LCP(e,M) admits a nonzero solution, again contradicting our assumption.

Theorem 3.2. Let M be copositive plus: $u^tMu \geq 0$ when $u \geq 0$, with $u^tMu = 0$ implying that $(M+M^t)u = 0$. If the algorithm generates an unbounded line then the LCP is infeasible.

Proof. The LCP is infeasible if $s = Mz + q$, s and $z \geq 0$, is an infeasible linear system. Farkas's lemma states that this infeasibility is equivalent with the existence of a nonnegative vector u such that $u^tM \leq 0$ and $u^tq < 0$.

The arguments of Theorem 3.1 show that an unbounded line implies the existence of a vector $(\bar{t}_0, \bar{s}, \bar{z})$ verifying (3.1). If $\bar{t}_0 > 0$, then $\bar{z}^tM\bar{z} = \bar{z}^t\bar{s} = -(\bar{z}^te)\bar{t}_0 < 0$ since z is nonzero. This contradicts the copositive plus character of M . Hence $\bar{t}_0 = 0$.

A zero value for t_0 implies that $\bar{z}^t M\bar{z} = 0$ and, hence, that $M^t\bar{z} = -M\bar{z} \leq 0$, since $-M\bar{z} = -\bar{s} \leq \bar{t}_0 e = 0$. Obviously, $\bar{z}$ is our candidate for the Farkas direction. To conclude our proof, we only need to show that $\bar{z}^t q < 0$.

Consider the unique endpoint of the unbounded line, say (t_0^*, s^*, z^*) , where

$$(3.2) \qquad s^* = Mz^* + q,\ z^* \geq 0 \ , \quad \text{and} \quad t_0^* > 0 \ .$$

Premultiplying (3.2) with $\bar{z}^t$ yields

$$\begin{aligned} \bar{z}^t s^* &= \bar{z}^t M z^* + \bar{z}^t q \\ &= -\bar{s}^t z^* + \bar{z}^t q \ . \end{aligned}$$

Because of t_0-complementarity at (s^*, z^*) and along the lines of our algorithm, we have $-s_i^* = t_0^*$ whenever $\bar{z}_i > 0$, for even if y_i is not basic at z^* , it is made basic along the line. Hence $\bar{z}^t s^* = -(\bar{z}^t e)t_0^* < 0$ implying that

$$-\bar{s}^t z^* + \bar{z}^t q < 0 \ .$$

If we can argue that $\bar{s}^t z^* = 0$, then our result is obtained.

If $\bar{s}_i > 0 = -\bar{t}_0$, then it must be that y_i is nonbasic along the path $(\bar{y}_i = \bar{z}_i = 0)$. At the same time,

$$(3.3) \qquad \sum_{h \in I_j} \bar{s}_h \geq \bar{s}_i > 0 = \bar{t}_0 \ ,$$

where $i \in I_j$. The first inequality follows from the nonnegativity of

$\bar{s}$. Inequality $\sum_{h \in I_j} \bar{s}_h > \bar{t}_0$ implies that $y_{n+j} = 1$ along the unbounded line, and thus at its endpoint z^* . Since $y_{n+j} = 1$ and $y_i = 0$ along the line, we have $z_i = 0$ along the line, and hence $z_i^* = 0$ at the end-point. This concludes the argument establishing that $\bar{s}^t z^* = 0$.

4. Implementation

We first introduce a matrix $E = (E_{ij})$ to identify the partition $\{I_i : i \in K\}$ of N :

$$E_{ij} = 1 \quad \text{if} \quad j \in I_i \ , \quad i \in K \ ,$$
$$\quad = 0 \quad \text{otherwise.}$$

Vector t can be expressed in matrix form as

$$(4.1) \qquad t = \begin{bmatrix} -s \\ Es \end{bmatrix} = \begin{bmatrix} -M \\ EM \end{bmatrix} z + \begin{bmatrix} -q \\ Eq \end{bmatrix} .$$

It is also convenient to introduce a vector to represent the deviations of the components of t from t_0 :

$$t^+ = t_0 e - t \ .$$

By definition of t_0 , at least one component of t^+ must be zero and $t_i^+ \geq 0$ unless $i \in N_2$ and $y_i = 1$. Introducing t^+ and y , (4.1) becomes

$$(4.2) \qquad t_0 e - t^+ = \begin{bmatrix} -M \\ EM \end{bmatrix} Dy + \begin{bmatrix} -M \\ EM \end{bmatrix} z^0 + \begin{bmatrix} -q \\ Eq \end{bmatrix} ,$$

which can be written more simply as

(4.3) $\quad t_0 e - t^+ = Ay + a$

with $[A,a] = \begin{bmatrix} -I \\ E \end{bmatrix} [MD, q^0]$, where $q^0 = Mz^0 + q$. t_0-complementarity between t and y requires that in every pair (t_j, y_j) at least one variable be nonbasic:

$$y_j = 0 \text{ or } t_j^+ = 0 \text{ for } j \in N_1 ,$$

$$y_j = 0 \text{ or } 1 \text{ or } t_j^+ = 0 \text{ for } j \in N_2 .$$

The definition of nonbasic variables given in Definition 2.1 thus coincides with the conventional one of linear programming when upper bounds are imposed on some of the variables. Nondegeneracy Assumption 2.1 then entails no loss of generality as the classical perturbation technique of linear programming applied to linear system (4.3) yields nondegeneracy. A line of our algorithm consists in a partition N_{11}, N_{12} of N_1 and a partition N_{21}, N_{22}, N_{23} of N_2 such that:

$$y_j > 0 \text{ and } t_j^+ = 0 \text{ for } j \in N_{11} ,$$

$$y_j = 0 \text{ and } t_j^+ > 0 \text{ for } j \in N_{12} ,$$

$$1 > y_j > 0 \text{ and } t_j^+ = 0 \text{ for } j \in N_{21} ,$$

$$y_j = 0 \text{ and } t_j^+ > 0 \text{ for } j \in N_{22} ,$$

$$y_j = 0 \text{ and } t_j^+ < 0 \text{ for } j \in N_{23} .$$

The algorithm starts with $y = 0$, $t_0 = \max(a_i : i \in N_0)$, and $t^+ = t_0 e - a \geq 0$. By nondegeneracy, t^+ has only one zero coordinate, say $t_i^+ = 0$. The first step of the algorithm increases y_i until a new position is reached. The signal for that is when, for some $j \in N_0$, both y_j and t_j^+ become nonbasic. The algorithm leaves a position by making basic the variable which was nonbasic along the line that was followed to reach the position. All steps involve pivot steps of linear programming except that the pivot rules need to pay attention to basic variables of both positive and negative sign. Introducing $t^{+i} = (t_j^+ : j \in N_i)$, $i = 1, 2$, it is easy to see that the last k equations in (4.3) are equivalent to

$$(4.4) \qquad -(Ee^1 + e^2)t_0 + Et_1^+ + t_2^+ = 0 \ ,$$

where e^1 and e^2 are vectors of ones of dimensions n and k respectively. These equations are of the GVUB type [Schrage (1978)] since every variable with a positive coefficient appears only once in (4.4). At every position $(t_0 > 0)$ at least one among the variables t_{n+j}^+ and $(t_i^+ : i \in I_j)$ is basic. This implies that the basic matrix, after suitable permutation of its columns, contains an identity submatrix of order k . Schrage shows that this property allows an implicit treatment of the last k equations of (4.3). Every step then involves the updating of a basic submatrix of order n rather than $n+k$ in an explicit treatment of (4.4).

Between successive LP-like pivot steps, there may be intermediary steps of a different kind due to a discontinuity in the value of t_0 . First, a discontinuous decrease in t_0 may occur when reaching the boundary of R_+^n where y_j becomes equal to 1 for some $j \in N_2$. The term t_j , nonbasic along the line leading toward the boundary, drops from maximand (2.5). This causes a discontinuity in t_0 if all other coordinates of t are basic along the line. In that case, t_0 is reduced by an amount

$$\Delta t_0 = \min(t_i^+ : i \in N_0, y_i = 0) > 0$$

and becomes equal to $t_0' = t_0 - \Delta t_0$, while $(t^+)' = t^+ - \Delta t_0 e$. Notice that $(t_j^+)' = -\Delta t_0 < 0$ is now basic while $(t_i^+)' = t_i^+ - \Delta t_0$ is still negative if $t_i^+ < 0$, $i \in N_2$. Assuming $t_0' > 0$, a new nonbasic variable has been generated, say $(t_h^+)' = 0$, and the algorithm leaves the position by increasing y_h .

The reverse movement of that described in the preceding paragraph causes a discontinuous increase in the value of t_0 . This discontinuity occurs whenever y_j becomes zero and the algorithm calls for the increase of t_h^+ where t_h^+ is the only nonbasic t^+-variable at the position. Such increase violates the constraint that at least one t^+-variable be nonbasic. This situation is described in Figure 3 (case ii) when one reaches the position along the line drawn at the left. Leaving along the other line, the algorithm first increases t_0 by $\Delta t_0 = \min(-t_i^+ : y_i = 1) > 0$ and updates $(t^+)' = t^+ + \Delta t_0 e$. The algorithm then leaves the position by decreasing y_g from 1 where $\Delta t_0 = -t_g^+$ and $(t_g^+)' = 0$.

The most interesting cases in our class of algorithms appear to be $k = 1$ and $k = n$. In these two extreme cases, the algorithm treats all coordinates symmetrically, which is a desirable property unless the matrix M presents very special structure. For $k = n$, the appropriate linear system is

$$(4.5) \qquad t_0 e - \begin{bmatrix} t^{+1} \\ t^{+2} \end{bmatrix} = \begin{bmatrix} -M & M \\ M & -M \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} + \begin{bmatrix} -q^0 \\ q^0 \end{bmatrix}$$

where after rescaling the bounds on y^2 can be written $0 \leq y^2 \leq z^0$. Since $y_j^1 y_j^2 = 0$ we can omit y^2 by allowing y^1 to take on negative values. The bounds on y^1 then are $-z^0 \leq y^1$ and y_j^1 is nonbasic when equal to 0 or $-z_j^0$. The case $k = n$ is analogous to one of Reiser's algorithms [Reiser (1978)].

For $k = 1$, linear system (4.3) becomes

$$(4.6) \qquad t_0 e - \begin{bmatrix} t^{+1} \\ t^{+2} \end{bmatrix} = \begin{bmatrix} -M & -Mz^0 \\ e^t M & -e^t Mz^0 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} + \begin{bmatrix} -q^0 \\ eq^0 \end{bmatrix},$$

where t^{+2} and y^2 are two scalars, and $q^0 = Mz^0 + q$. The second Reiser algorithm, considers only the first n equations of (4.6). That algorithm corresponds to movements along t_0-complementary lines where $t_0 = \max(-s_1, -s_2, \ldots, -s_n, 0)$ as compared with $t_0 = \max(-s_1, -s_2, \ldots, -s_n, \sum_{i \in N} s_i)$ for our algorithm. The complementarity conditions along a line in Reiser's algorithm are

$$(4.7) \qquad (t^{+1})^t y^1 = 0 \quad \text{and} \quad t_0 y^2 = 0 .$$

In this setting $t_0 = 0$ no longer identifies a solution. The algorithm terminates either when y^2 reaches its upper bound of 1 or when an end-point is reached where the variables t_0 and t^{+1} are all nonbasic. In the first case, $t_0 = 0$ by complementarity along a line and the first n equations of (4.6) can be written $t^{+1} = My^1 + q$. This, along with the complementarity conditions (4.7), shows that (t^{+1}, y^1) is a solution for the LCP. In the second case, $(t_0, t^{+1}) = 0$ and it is easily seen that $(s,z) = (0, y^1 + (1 - y^2)z^0)$ is a solution for the LCP.

5. Concluding Comments

The ideas central to the van der Laan and Talman fixed point algorithms have been shown to yield a class of algorithms for the LCP. Similar ideas can be applied to modify other LCP algorithms, like the variable dimension algorithm of Van der Heyden (1980) [see also Yamamoto (1981)], to accept an arbitrary starting point. Flexibility in the choice of the starting point is desirable, e.g., in solving nonlinear complementarity problems via a succession of approximating LCP's [Josephy (1979)].

So far we assumed the starting point to have only positive coordinates. For an initial z^0 on the boundary, definition (2.4) is modified for an index $n+j \in N_2$ to

$$t_{n+j} = \sum_{i \in I_j^+} s_i$$

where $I_j^+ = \{i \in I_j : z_i^0 > 0\}$. The associated direction d^{n+j} is left unchanged. If I_j^+ is empty, the term t_{n+j} disappears from the maximand defining $t_0(d^{n+j} = 0)$. The number of rows in linear system (4.3) then also decreases by one. If $z^0 = 0$, then $D = I$ and $z = z^0 + Dy = y$ so that (4.3) can be rewritten

$$Mz + t_0 e + q = t^+ .$$

Since N_2 is empty, t^+ is always nonnegative and t_0-complementarity takes the form $(t^+)^t z = 0$. Our algorithms thus all generalize Lemke's original algorithm.

The relation of our algorithms with Lemke's algorithm reminds us that we can scale each coordinate of t before taking the largest one which defines t_0 . A vector of scaling factors $f = (f_1, f_2, \ldots, f_{n+k}) > 0$

leads us to rewrite (4.3) as

$$t_0 f - t^+ = Ay + a .$$

Due to the possibility of this scaling, the vector e in the statement of Theorem 3.1 need no longer be a vector of 1's, but instead can be a vector with positive but otherwise arbitrary coordinates.

Another way of generalizing our algorithms is to consider different directional matrices D. However, in order to have an algorithm which generalizes that of Lemke, one needs n positive unit directions at z^0. Other reasonable choices for the remaining k directions, all pointing towards the boundary, could not be found. Different directions, however, could lead to new convergence conditions.

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