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A MARKET VALUE APPROACH TO APPROXIMATE EQUILIBRIA

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by

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Abstract: We consider the market value of excess demand as a measure of disequilibrium. We show that, in a fixed exchange economy, there exist approximate equilibria whose measures of disequilibrium depend only on the endowments and not on the preferences. A related bound on the norm of excess demand, depending on the endowments and the approximate equilibrium price, is also obtained. We show the existence of allocations which are nearly competitive, as measured by the largest proportion of demand given up at the allocation by any trader. We use these results to obtain, for very general sequences of exchange economies, allocations giving all traders bundles close to norm to their demands. This result includes a $O(1/n)$ rate of convergence in the case of uniformly bounded endowments.

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1. Introduction

A number of authors have treated the existence of approximate equilibria in finite economies with non-convex preferences. The papers most relevant to this study are the following:

Ross M. Starr [8] introduced the Shapley-Folkman Theorem to the study of approximate equilibria. He showed the existence of a price so that excess demand could be bounded by a constant depending on the degrees of non-convexity of the individuals' preferences. The results were refined in Arrow-Hahn [2].

Hildenbrand-Schmeidler-Zamir [6] considered families of economies, where the preference-endowment pairs were taken from a fixed compact set. They showed that, given such a family and $\delta > 0$, there exists $n \in N$ so that any economy in the family with at least n traders possessed a price and an allocation f so that, for all traders a, $f(a)$ was within δ of a's demand correspondence. Their result gives no computable relationship between n and δ, and hence no rate of convergence. Their proof makes use of Aumann's existence theorem [3] for markets with a measure-theoretic continuum of traders. Hildenbrand [5] weakened the compactness assumption to tightness of preferences and uniform integrability of endowments.

M. Ali Khan [7] established a systematic collection of existence results in sequences of finite economies. His theorems were similar in form to those of Hildenbrand-Schmeidler-Zamir. He used Brown's theorem [4] on the existence of competitive equilibria in nonstandard exchange economies, and nonstandard analysis.

Our approach focuses on what we shall call the market value measure of disequilibrium. Given a price, we consider the cost (with respect to the approximate equilibrium price) of the absolute value of the excess demand. This measure is closely related to Starr's notion of an ε-value equilibrium. We show that there exists a price so that this measure of disequilibrium is bounded by a constant depending on the endowments but not on the preferences or the number of individuals. This says that, in a large economy, the per capita cash value of any shortages or surpluses is small. Heuristically, this says that any inventory adjustments required to make markets clear would be easily absorbed by the economy.

Moreover, we can obtain a bound on the norm of excess demand depending on the bound determined above and on the approximate equilibrium price itself. Since the price is not known *a priori*, this is somewhat unsatisfactory; however, we shall argue that in most situations, the bound could be determined from *a priori* considerations.

We also show that there exists an allocation which is on average nearly competitive in the two senses--market value and norm--noted above. However, this does not preclude the possibility that the allocation gives a few individuals commodity bundles far from their demand sets.

The above-mentioned results have analogues in Starr's paper. The principal improvement is that we substitute easily computable bounds based on endowments in place of Starr's bounds, which involves measuring the non-convexity of the preferences.

We also obtain a uniform bound on the proportion of any commodity demanded by any individual which must be given up in going to the allocation. Thus, while a few individuals might be allocated bundles far from their demand sets, these individuals would necessarily have large endowments, and the amount of any commodity given up in going from the demand to the allocated bundle would be a small portion of the amount demanded of that commodity. Heuristically, this says that no one suffers a great injustice. There is no result analogous to this in any of the papers cited.

Neither Starr's results nor our theorems mentioned so far preclude the possibility that a few individuals may be allocated commodity bundles far from their demand sets. In order to obtain such results, we must consider sequential formulations. The first step is to show that, for very general sequences of economies, the approximate equilibrium prices can be chosen within a compact subset of the open simplex. These sequences generalize those considered by Hildenbrand-Schmeidler-Zamir [6], Hildenbrand [5], and Khan [7] through a weakening of the assumptions on the preference-endowment pairs. The proof is a translation of a nonstandard proof which appeared in the first version of this paper, and which in turn was adapted from arguments in Brown [4] and Khan [7].

It is an immediate corollary of the compactness of the sequence of approximate equilibrium prices and our earlier results that the norm of the per capita excess demand converges to 0. In case the endowments are uniformly bounded, the rate of convergence is $O(1/n)$. No rate of convergence is specified by Hildenbrand-Schmeidler-Zamir [6], Hildenbrand [5], or Khan [7]. Subsequent to the writing of this paper, Weber [9] showed how the Hildenbrand-Schmeidler-Zamir argument could be modified to obtain a rate of convergence result. We also show that there exist allocations so that the maximum proportion of demand given up by any trader converges to 0, with the same rate.

In addition, we show that if the endowments are uniformly integrable, one can choose an allocation so that every individual is assigned a bundle close to his or her demand set. Uniform boundedness again yields a

$O(1/n)$ rate of convergence. Similar results (without the rate of convergence) are given under stronger assumptions on endowments and preferences in Hildenbrand-Schmeidler-Zamir [6] and Khan [7], and under stronger assumptions on preferences in Hildenbrand [5].

This paper is closely related to Anderson [1], which shows that core allocations are approximate equilibria according to a measure of disequilibrium rather similar to the market value criterion used here.

2. Results

We begin with some notation and definitions. Suppose $x, y \in R^k$, $A \subset R^k$, and $\delta > 0$.

x^i denotes the i^{th} component of x

$\|x\|_\infty = \max_{1 \le i \le k} |x^i|$, $\|x\|_1 = \sum_{i=1}^{k} |x^i|$

$x \le y$ if $x^i \le y^i$ $(1 \le i \le k)$

$x \ll y$ if $x^i < y^i$ $(1 \le i \le k)$

con A is the convex hull of A

$R^k_+ = \{x \in R^k : x \ge 0\}$

$B(x,\delta) = \{y \in R^k_+ : \|y-x\|_\infty < \delta\}$

$|x|$ is the vector with $|x|^i = |x^i|$

e_i is the vector $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 appears in the i^{th} place.

Let P denote the set of preferences (i.e., binary relations on R^k_+) satisfying the following conditions:

(i) transitivity: $x \succ y$, $y \succ z \Longrightarrow x \succ z$

(ii) irreflexivity: $x \not\succ x$

(iii) continuity: $\{(x,y) : x \succ y\}$ is relatively open in R^k_+

(iv) monotonicity: $x \ge y$, $x \ne y \Longrightarrow x \succ y$

Thus, $\mathcal{P}$ is the family denoted by $\mathcal{P}_{mo}$ in Hildenbrand [5].

An exchange economy is a map $\varepsilon : A \to \mathcal{P} \times R^k_+$, where A is a finite set. For $a \in A$, let $\succ_a$ be the projection of $\varepsilon(a)$ onto $\mathcal{P}$, and $e(a)$ the projection of $\varepsilon(a)$ onto R^k_+ . $\succ_a$ is interpreted as the preference of trader a , and $e(a)$ his initial endowment. An allocation is a map $f : A \to R^k_+$ such that $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$. Let $M_\varepsilon = \max\{\|e(a_1) + \dots + e(a_k)\|_\infty : a_1, \dots, a_k \in A, a_i \text{ distinct}\}$. A price p is a vector in R^k_+ such that $\|p\|_1 = 1$. Let S be the set of all prices, S^o the relative interior of S . If $p \in S^o$, let $C_p = 1/\min\{p^1, \dots, p^k\}$. If $p \in S^o$ and $a \in A$, let $D(p,a)$ be the demand set of a , i.e., $D(p,a) = \{x \in R^k_+ : p \cdot x \leq p \cdot e(a), p \cdot y \leq p \cdot e(a) \implies y \not\succ x\}$. Let $D(p) = \sum_{a \in A} D(p,a)/|A|$.

The following theorem asserts the existence of approximate equilibria. Conclusion 1 says that the market value of the absolute per capita excess demand is small; in other words, the market value of any inventory adjustments required to make markets clear is small. Conclusion 2 gives a bound on the norm of excess demand. It is not entirely satisfactory since the constant C_p depends on the approximate equilibrium price, which is not known a priori However, it should be possible in practice to obtain an upper bound on C_p for a given economy. For example, it should be possible in a particular case to determine α_i so that lowering p^i below α_i would result in a shortage of some commodity i . Then the approximate equilibrium price p determined by Theorem 1 would have $p^i \geq \alpha_i$, so that $C_p \leq 1/\min\{\alpha_1, \dots, \alpha_k\}$. Conclusion 2 also shows that there is an allocation which is close to the demand in the aggregate; however, the deviations may be relatively large for a few individuals. Conclusion 3 says that there exists an allocation so that no individual gives up more than a small fraction of his or her demand for each commodity. Heuristically, this says that no individual is significantly inconvenienced in accepting the allocation instead of taking his or her demand.

Theorem 1. Let $\varepsilon : A \to P \times R^k_+$ be a finite exchange economy, with $|A| = n$, and $\sum_{a\in A} e(a) >> 0$. Then there exists $p \in S^o$ such that $\sum_{a\in A} e(a)/n \in \text{con } D(p)$. For any such p, there is an allocation $h : A \to R^k_+$ such that $p\cdot h(a) = p\cdot e(a)$ for all a, and a selection $g(a) \in D(p,a)$ such that

$$1)\quad p\cdot\left|\sum_{a\in A}(g(a)-e(a))\right| \leq p\cdot\sum_{a\in A}|g(a)-h(a)| \leq 2M_\varepsilon$$

$$2)\quad \left\|\sum_{a\in A}(g(a)-e(a))\right\|_\infty \leq \sum_{a\in A}\|g(a)-h(a)\|_\infty \leq 2C_pM_\varepsilon .$$

Finally, there exists an allocation $f : A \to R^k_+$ such that $p\cdot f(a) = p\cdot e(a)$ for all a and

$$3)\quad f(a)^j \geq g(a)^j\left(1-\left(2C_pM_\varepsilon/\sum_{a\in A} e(a)^j\right)\right) \quad \text{for all } a \in A,\ 1 \leq j \leq k .$$

Proof. The familiar fixed point argument (see for example Hildenbrand [5, pp. 149-150, Proposition 3 and Lemma 1]) shows there exists $p \in S^o$ such that $\sum e(a)/n \in \text{con } D(p)$. By the Shapley-Folkman Theorem (Starr [8]), there exist $h(a) \in \text{con } D(p,a)$ such that $\sum e(a) = \sum h(a)$, and $h(a) \in D(p,a)$ for all but k traders $\{a_1, \ldots, a_k\}$. Choose arbitrarily $g(a_i) \in D(p, a_i)$ for $1 \leq i \leq k$, and let $g(a) = h(a)$ for $a \notin \{a_1, \ldots, a_k\}$. Thus, $g(a) \in D(p,a)$ for all a. Since $\succ_a$ is monotone and $g(a) \in D(p,a)$ for all a, $p\cdot g(a) = p\cdot e(a)$ for all $a \in A$. Since $h(a) \in \text{con } D(p,a)$, $p\cdot h(a) = p\cdot e(a)$ for all a.

$$p\cdot\Big|\sum_{a\in A}(g(a)-e(a))\Big| = p\cdot\Big|\sum_{a\in A}(g(a)-h(a))\Big|$$

$$\leq p\cdot\sum_{a\in A}|g(a)-h(a)|$$

$$= p\cdot\sum_{i=1}^{k}|g(a_i)-h(a_i)|$$

$$\leq (\sum_{i=1}^{k}p\cdot g(a_i)+p\cdot h(a_i))$$

$$= 2\sum_{i=1}^{k}p\cdot e(a_i)$$

$$= 2p\cdot\sum_{i=1}^{k}e(a_i)$$

$$\leq 2\|\sum_{i=1}^{k}e(a_i)\|$$

$$\leq 2M_\varepsilon .$$

$$\|\sum_{a\in A}(g(a)-e(a))\|_\infty \leq \sum_{a\in A}\|g(a)-h(a))\|_\infty$$

$$\leq p\cdot\sum_{a\in A}|g(a)-h(a))|/\min\{p^1, \ldots, p^k\}$$

$$\leq 2C_pM_\varepsilon .$$

To prove conclusion 3), let $J = \{j : \sum e(a)^j < \sum g(a)^j\}$ and

$$r(a)^j = \begin{cases} g(a)^j(\sum_{b\in A}e(b)^j/\sum_{b\in A}g(b)^j) & \text{if } j\in J \\ g(a)^j & \text{if } j\notin J . \end{cases}$$

Note $r(a) \geq 0$, $\sum r(a) \leq \sum e(a)$, $p \cdot r(a) \leq p \cdot g(a) = p \cdot e(a)$. Now let

$$f(a) = r(a) + \sum_{b \in A} (e(b) - r(b)) \frac{p \cdot (e(a) - r(a))}{p \cdot \sum_{b \in A} (e(b) - r(b))} .$$

Thus $f(a) \geq 0$, $\sum_{a \in A} f(a) = \sum r(a) + \sum (e(a) - r(a)) = \sum e(a)$. Thus, f is an allocation. Also, $p \cdot f(a) = p \cdot e(a)$.

If $j \notin J$, $f(a)^j \geq r(a)^j = g(a)^j \geq g(a)^j (1 - (C_p M_\varepsilon / \sum_{b \in A} e(b)^j))$.

If $j \in J$, $f(a)^j = r(a)^j$

$$= g(a)^j \frac{\sum e(b)^j}{\sum g(b)^j}$$

$$= g(a)^j \left(1 - \frac{\sum g(b)^j - e(b)^j}{\sum g(b)^j}\right)$$

$$\geq g(a)^j \left(1 - \frac{2C_p M_\varepsilon}{\sum g(b)^j}\right)$$

$$\geq g(a)^j (1 - 2C_p M_\varepsilon / \sum_{b \in A} e(b)^j) .$$

This completes the proof of Theorem 1.

We now turn to a limit theorem for sequences of economies. Let p_n be the price determined by Theorem 1 for the n^{th} economy. We shall give sufficient conditions to ensure that the p_n stay within a compact subset in S^o . The argument is adapted from Brown [4] and Khan [7]. Then C_{p_n} will stay bounded, so Theorem 1 will give us the desired result.

<u>Definition</u>. A set of preferences $P \subset \mathcal{P}$ is said to be equi-monotone if, for all $x \in R^k_+$, there exists $\delta > 0$ such that $B(x + e_i, \delta) \succ B(x, \delta)$ for each i $(1 \leq i \leq k)$ and all $\succ \in P$.

Those familiar with the topology of closed convergence on $\mathcal{P}$ (see Hildenbrand [5]) will recognize that any compact set $P \subset \mathcal{P}$ is equi-monotone; thus, the sequences we consider are more general than those considered by Hildenbrand-Schmeidler-Zamir [6], Hildenbrand [5], or Khan [7]. Our result also improves the previous results in obtaining the rate of convergence.

Theorem 2. Let $\varepsilon_n : A_n \to \mathcal{P} \times R^k_+$ be a sequence of exchange economies such that

(i) $\sup_n \| \sum_{a \varepsilon A_n} e_n(a)/|A_n| \|_\infty < \infty$

(ii) there exists P equi-monotone and $\delta > 0$ such that $|\{a \varepsilon A_n : \succ_a \varepsilon P, e(a)^i > \delta\}|/|A_n| > \delta$ for each i $(1 \leq i \leq k)$ and all $n \varepsilon N$.

Then there exist allocations h_n, prices $p_n \varepsilon S^o$, $g_n(a) \varepsilon D_n(p_n, a)$ and a constant L such that $p_n \cdot h_n(a) = p_n \cdot e_n(a)$ for all $a \varepsilon A_n$ and

$$1) \quad \| \sum_{a \varepsilon A_n} (g_n(a) - e_n(a)) \|_\infty \leq \sum_{a \varepsilon A_n} \| g_n(a) - h_n(a) \|_\infty \leq LM_{\varepsilon_n} .$$

Moreover, there exist allocations f_n such that $p_n \cdot f_n(a) = p_n \cdot e_n(a)$ for all $a \varepsilon A_n$ and such that

$$2) \quad f_n(a) \geq g_n(a)(1 - LM_{\varepsilon_n}/|A_n|) .$$

Finally, if $|A_n| \to \infty$ and the endowments are uniformly integrable, i.e.,

$$\text{(iii)} \quad |E_n|/|A_n| \to 0 \Longrightarrow \| \sum_{a \varepsilon A_n} e_n(a) \|_\infty / |A_n| \to 0 ,$$

then the f_n can be chosen so that they also satisfy

$$3) \quad \max_{a \varepsilon A_n} \| f_n(a) - g_n(a) \|_\infty \leq LM_{\varepsilon_n}/|A_n| \to 0 \quad \text{as} \quad n \to \infty .$$

Remark. If the endowments are uniformly bounded by a constant M, conclusion 3) establishes the rate of convergence

$$\max_{a \varepsilon A_n} \|f_n(a) - g_n(a)\|_\infty = O(1/|A_n|) \ .$$

Before we prove Theorem 2, we need the following lemma.

Lemma 3. Under the hypotheses of the theorem, there exist prices p_n and $L < \infty$ such that $\sum_{a \varepsilon A_n} e_n(a)/|A_n| \ \varepsilon$ con $D_n(p_n)$ and $C_{p_n} \leq L/2$, for all n.

Proof. The Lemma is analogous to Hildenbrand-Schmeidler-Zamir [7, Lemma 2], but their proof does not quite cover the case we are considering. Our original proof used Nonstandard Analysis. At the suggestion of a referee, we give a standard proof which is a translation of the nonstandard proof.

By Theorem 1, we can find prices p_n such that $\sum_{a \varepsilon A_n} e_n(a)/|A_n| \ \varepsilon$ con $D_n(p_n)$. If the Lemma is false, we may pass to a subsequence and assume without loss of generality that $C_{p_n} \to \infty$. Since S is compact, we may pass to a further subsequence and assume that $p_n \to p \ \varepsilon \ S$. Since $C_{p_n} \to \infty$, we have $p^i = 0$ for some i. Assume $p^1 = 0$, $p^2 > 0$. Choose P equi-monotone and $\delta > 0$ such that $|E_n|/|A_n| > \delta$, where $E_n = \{a \ \varepsilon \ A_n : \succ_a \varepsilon \ P, \ e_n(a)^2 > 2\delta\}$.

Let $K = \sup_n \|\sum e_n(a)/|A_n|\|_\infty k/\delta$. For each $x \ \varepsilon \ R^k_+$ and $\succ \varepsilon \ P$, there exists γ so that $B(x+e_1, \gamma) \succ B(x,\gamma)$. Hence, there exists $\gamma > 0$ so that $B(x+e_1, \gamma) \succ B(x,\gamma)$ for all $\|x\|_\infty \leq K$ and $\succ \varepsilon \ P$. Choose n sufficiently large that $(\delta p^2/Kk)\min\{\delta p^2/k, \gamma/2\} > p^1_n$.

If $a \ \varepsilon \ E_n$ and $x \ \varepsilon \ D_a(p_n)$, $p_n \cdot x = p_n \cdot e_n(a) \geq p^2_n e_n(a)^2 > 2\delta p^2_n \to 2\delta p^2$. Hence, for n sufficiently large, $a \ \varepsilon \ E_n$ and $x \ \varepsilon \ D_a(p_n)$ imply

$p_n \cdot x > \delta p^2$. Thus, there exists i such that $p_n^i x^i > \delta p^2/k$. If $\|x\|_\infty \le K$, $p_n^i > \delta p^2/Kk$; since $p_n^i \le 1$, $\delta p^2/k < x^i \le K$.

Consider $y = x + e_1 - \min\{\delta p^2/k, \gamma/2\} e_i$.

$$\begin{aligned} p_n \cdot y &= p_n \cdot x + p_n^1 - p_n^i \min\left\{\frac{\delta p^2}{k}, \frac{\gamma}{2}\right\} \\ &\le p_n \cdot x + p_n^1 - \frac{\delta p^2}{Kk} \min\left\{\frac{\delta p^2}{k}, \frac{\gamma}{2}\right\} \\ &< p_n \cdot x . \end{aligned}$$

$$y_i \ge x_i - \min\left\{\frac{\delta p^2}{k}, \frac{\gamma}{2}\right\} > 0 , \quad \text{so} \quad y \in R_+^k .$$

Thus, $y \in B(x + e_1, \gamma)$. Hence, $y \succ_a x$, contradicting $x \in D_a(p_n)$.

Thus, we conclude that given K , there exists n_o such that $n \ge n_o$ implies $\|x\|_\infty > K$ for all $x \in D_a(p_n)$, $a \in E_n$. Hence, if $z \in \text{con}(D_n(p_n))$, $\|z\|_\infty > \delta K/k$ for $n \ge n_o$. This contradicts $\sum_{a \in A_n} e_n(a)/|A_n| \in \text{con}(D_n(p_n))$, and proves the lemma.

<u>Proof of Theorem 2</u>. By Lemma 3, $\sup_n C_{p_n} < \infty$, so choosing $L = 2 \sup_n C_{p_n}$ establishes conclusions 1) and 2).

In order to prove conclusion 3), we need to modify the definition of the f_n slightly. Note that since $\|g_n(a)\|_\infty \le C_{p_n} \|e_n(a)\|_\infty$, the sequence $\{g_n\}$ is uniformly integrable. Hence there exists $m \in M$ so that, if $E_n = \{a \in A_n : \|g_n(a)\|_\infty \le m\}$,

$$\sum_{a \in E_n} g_n(a) \ge \frac{2}{3} \sum_{a \in A_n} g_n(a) .$$

Since the endowments are uniformly integrable and $|A_n| \to \infty$,

$M_{\varepsilon_n}/|A_n| \to 0$. Hence, by conclusion 1), there exists n_1 such that

$$\sum_{a\varepsilon E_n} g_n(a) > \sum_{a\varepsilon A_n} e_n(a)/2 \quad \text{for} \quad n > n_1 .$$

We now define f_n $(n > n_1)$ as in the proof of Theorem 1, except that all the changes from g_n are absorbed on E_n . For convenience, we drop the subscript n for the moment. Let

$$r(a)^j = \begin{cases} g(a)^j\left(1 - \left(\dfrac{\sum_{b\varepsilon A} (g(b) - e(b))^j}{\sum_{b\varepsilon E} g(b)^j}\right)\right) & \text{if} \quad j \varepsilon J \quad \text{and} \quad a \varepsilon E \\ g(a)^j & \text{otherwise.} \end{cases}$$

f is now defined from r using the same formula as in the proof of Theorem 1. f will be an allocation provided it is positive.

$$f(a)^j \geq g(a)^j(1 - 2C_pM_\varepsilon/\sum_{b\varepsilon E} g(b)^j)$$

$$\geq g(a)^j(1 - 4C_pM_\varepsilon/\sum_{b\varepsilon A} e(b)^j)$$

$$\geq g(a)^j(1 - 4C_pM_\varepsilon/\delta^2|A|) \text{ , by (ii).}$$

If $a \notin E$, $f(a) = g(a)$. If $a \varepsilon E$,

$$\|f(a) - g(a)\|_\infty \leq 4C_p^2M_\varepsilon\|g(a)\|_\infty/\delta^2|A|$$

$$\leq 4C_p^2M_\varepsilon m/\delta^2|A| .$$

Let $L = \max\{4C_p/\delta^2, 4mC_p^2/\delta^2\}$. We have shown that for each $n > n_1$, there exist f_n and g_n satisfying

$$f_n(a) \geq g_n(a)(1 - LM_{\varepsilon_n}/|A_n|) \quad \text{for all} \quad a \in A_n$$

and

$$\max_{a \in A_n} \|f_n(a) - g_n(a)\|_\infty \leq LM_{\varepsilon_n}/|A_n| \ .$$

f_n is positive (and hence an allocation) for n such that $LM_{\varepsilon_n}/|A_n| \leq 1$. Since the endowments are uniformly integrable, $M_{\varepsilon_n}/|A_n| \to 0$, so there exists n_0 such that f_n is an allocation for all $n > n_0$. Redefine f_n to be an arbitrary allocation (say $f_n = h_n$) for $n \leq n_0$. Then (increasing L if necessary to bring the finite set of allocations $f_1, \ldots, f_{n_0}$ within the desired bounds) the result follows.

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