A MARKET VALUE APPROACH TO APPROXIMATE EQUILIBRIA

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Abstract: We consider the market value of excess demand as a measure of disequilibrium. We show that, in a fixed exchange economy, there exist approximate equilibria whose measures of disequilibrium depend only on the endowments and not on the preferences. A related bound on the norm of excess demand, depending on the endowments and the approximate equilibrium price, is also obtained. We show the existence of allocations which are nearly competitive, as measured by the largest proportion of demand given up at the allocation by any trader. We use these results to obtain, for very general sequences of exchange economies, allocations giving all traders bundles close to norm to their demands. This result includes a $O(1/n)$ rate of convergence in the case of uniformly bounded endowments.

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1. Introduction

A number of authors have treated the existence of approximate equilibria in finite economies with non-convex preferences. The papers most relevant to this study are the following:

Ross M. Starr [8] introduced the Shapley-Folkman Theorem to the study of approximate equilibria. He showed the existence of a price so that excess demand could be bounded by a constant depending on the degrees of non-convexity of the individuals' preferences. The results were refined in Arrow-Hahn [2].

Hildenbrand-Schmeidler-Zamir [6] considered families of economies, where the preference-endowment pairs were taken from a fixed compact set. They showed that, given such a family and \( \delta > 0 \), there exists \( n \in \mathbb{N} \) so that any economy in the family with at least \( n \) traders possessed a price and an allocation \( f \) so that, for all traders \( a \), \( f(a) \) was within \( \delta \) of \( a \)'s demand correspondence. Their result gives no computable relationship between \( n \) and \( \delta \), and hence no rate of convergence. Their proof makes use of Aumann's existence theorem [3] for markets with a measure-theoretic continuum of traders. Hildenbrand [5] weakened the compactness assumption to tightness of preferences and uniform integrability of endowments.


Our approach focuses on what we shall call the market value measure of disequilibrium. Given a price, we consider the cost (with respect
to the approximate equilibrium price) of the absolute value of the excess demand. This measure is closely related to Starr's notion of an $\epsilon$-value equilibrium. We show that there exists a price so that this measure of disequilibrium is bounded by a constant depending on the endowments but not on the preferences or the number of individuals. This says that, in a large economy, the per capita cash value of any shortages or surpluses is small. Heuristically, this says that any inventory adjustments required to make markets clear would be easily absorbed by the economy.

Moreover, we can obtain a bound on the norm of excess demand depending on the bound determined above and on the approximate equilibrium price itself. Since the price is not known \textit{a priori}, this is somewhat unsatisfactory; however, we shall argue that in most situations, the bound could be determined from \textit{a priori} considerations.

We also show that there exists an allocation which is on average nearly competitive in the two senses—market value and norm—noted above. However, this does not preclude the possibility that the allocation gives a few individuals commodity bundles far from their demand sets.

The above-mentioned results have analogues in Starr's paper. The principal improvement is that we substitute easily computable bounds based on endowments in place of Starr's bounds, which involves measuring the non-convexity of the preferences.

We also obtain a uniform bound on the proportion of any commodity demanded by any individual which must be given up in going to the allocation. Thus, while a few individuals might be allocated bundles far from their demand sets, these individuals would necessarily have large endowments, and the amount of any commodity given up in going from the demand to the allocated bundle would be a small portion of the amount demanded.
of that commodity. Heuristically, this says that no one suffers a great injustice. There is no result analogous to this in any of the papers cited.

Neither Starr's results nor our theorems mentioned so far preclude the possibility that a few individuals may be allocated commodity bundles far from their demand sets. In order to obtain such results, we must consider sequential formulations. The first step is to show that, for very general sequences of economies, the approximate equilibrium prices can be chosen within a compact subset of the open simplex. These sequences generalize those considered by Hildenbrand-Schmeidler-Zamir [6], Hildenbrand [5], and Khan [7] through a weakening of the assumptions on the preference-endowment pairs. The proof is a translation of a nonstandard proof which appeared in the first version of this paper, and which in turn was adapted from arguments in Brown [4] and Khan [7].

It is an immediate corollary of the compactness of the sequence of approximate equilibrium prices and our earlier results that the norm of the per capita excess demand converges to 0. In case the endowments are uniformly bounded, the rate of convergence is \( O(1/n) \). No rate of convergence is specified by Hildenbrand-Schmeidler-Zamir [6], Hildenbrand [5], or Khan [7]. Subsequent to the writing of this paper, Weber [9] showed how the Hildenbrand-Schmeidler-Zamir argument could be modified to obtain a rate of convergence result. We also show that there exist allocations so that the maximum proportion of demand given up by any trader converges to 0, with the same rate.

In addition, we show that if the endowments are uniformly integrable, one can choose an allocation so that every individual is assigned a bundle close to his or her demand set. Uniform boundedness again yields a
$O(1/n)$ rate of convergence. Similar results (without the rate of convergence) are given under stronger assumptions on endowments and preferences in Hildenbrand-Schmeidler-Zamir [6] and Khan [7], and under stronger assumptions on preferences in Hildenbrand [5].

This paper is closely related to Anderson [1], which shows that core allocations are approximate equilibria according to a measure of disequilibrium rather similar to the market value criterion used here.

2. Results

We begin with some notation and definitions. Suppose $x, y \in \mathbb{R}^k$, $A \subseteq \mathbb{R}^k$, and $\delta > 0$.

$x^i$ denotes the $i^{th}$ component of $x$

$$||x||_\infty = \max_{1 \leq i \leq k} |x^i|, \quad ||x||_1 = \sum_{i=1}^k |x^i|$$

$x \preceq y$ if $x^i \leq y^i$ ($1 \leq i \leq k$)

$x \ll y$ if $x^i < y^i$ ($1 \leq i \leq k$)

$\text{conv } A$ is the convex hull of $A$

$\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x \geq 0\}$

$B(x, \delta) = \{y \in \mathbb{R}^k : ||y-x||_\infty < \delta\}$

$|x|$ is the vector with $|x|^i = |x^i|$

$e_i$ is the vector $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the $1$ appears in the $i^{th}$ place.

Let $P$ denote the set of preferences (i.e., binary relations on $\mathbb{R}^k_+$) satisfying the following conditions:

(i) transitivity: $x \succ y, \ y \succ z \implies x \succ z$

(ii) irreflexivity: $x \not\succ x$

(iii) continuity: $\{(x,y) : x \succ y\}$ is relatively open in $\mathbb{R}^k_+$

(iv) monotonicity: $x \succeq y, \ x \not\succ y \implies x \succ y$
Thus, $P$ is the family denoted by $P^{\text{eq}}$ in Hildenbrand [5].

An exchange economy is a map $\epsilon : A \rightarrow P \times R^k_+$, where $A$ is a finite set. For $a \in A$, let $\lambda_a$ be the projection of $\epsilon(a)$ onto $P$, and $e(a)$ the projection of $\epsilon(a)$ onto $R^k_+$. $\lambda_a$ is interpreted as the preference of trader $a$, and $e(a)$ his initial endowment. An allocation is a map $f : A \rightarrow R^k_+$ such that $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$. Let

$$M_\epsilon = \max\{\|e(a_1) + \ldots + e(a_k)\|_\infty : a_1, \ldots, a_k \in A, a_i \text{ distinct}\}.$$  

A price $p$ is a vector in $R^k_+$ such that $\|p\|_1 = 1$. Let $S$ be the set of all prices, $S^0$ the relative interior of $S$. If $p \in S^0$, let

$$C_p = 1/\min\{p^1, \ldots, p^k\}.$$  

If $p \in S^0$ and $a \in A$, let $D(p,a)$ be the demand set of $a$, i.e.,

$$D(p,a) = \{x \in R^k_+ : p \cdot x \leq p \cdot e(a), p \cdot y \leq p \cdot e(a) \implies y \not\prec x\}.$$  

Let $D(p) = \sum_{a \in A} D(p,a)/|A|$.

The following theorem asserts the existence of approximate equilibria.

Conclusion 1 says that the market value of the absolute per capita excess demand is small; in other words, the market value of any inventory adjustments required to make markets clear is small. Conclusion 2 gives a bound on the norm of excess demand. It is not entirely satisfactory since the constant $C_p$ depends on the approximate equilibrium price, which is not known a priori. However, it should be possible in practice to obtain an upper bound on $C_p$ for a given economy. For example, it should be possible in a particular case to determine $a_i$ so that lowering $p^i$ below $a_i$ would result in a shortage of some commodity $i$. Then the approximate equilibrium price $p$ determined by Theorem 1 would have $p^i \geq a_i$, so that

$$C_p \leq 1/\min\{a_1, \ldots, a_k\}.$$  

Conclusion 2 also shows that there is an allocation which is close to the demand in the aggregate; however, the deviations may be relatively large for a few individuals. Conclusion 3 says that there exists an allocation so that no individual gives up more than a
small fraction of his or her demand for each commodity. Heuristically, this says that no individual is significantly inconvenienced in accepting the allocation instead of taking his or her demand.

Theorem 1. Let $c : A \rightarrow P \times R^k_+$ be a finite exchange economy, with $|A| = n$, and $\sum_{a \in A} e(a) >> 0$. Then there exists $p \in S^o$ such that

$\sum_{a \in A} e(a)/n \in \text{con } D(p)$. For any such $p$, there is an allocation $h : A \rightarrow R^k_+$ such that $p \cdot h(a) = p \cdot e(a)$ for all $a$, and a selection $g(a) \in D(p,a)$ such that

1) $p \cdot \left| \sum_{a \in A} (g(a) - e(a)) \right| \leq p \cdot \sum_{a \in A} |g(a) - h(a)| \leq 2\gamma \varepsilon$.

2) $\left\| \sum_{a \in A} (g(a) - e(a)) \right\|_{\infty} \leq \left\| \sum_{a \in A} (g(a) - h(a)) \right\|_{\infty} \leq 2C \cdot M \varepsilon$.

Finally, there exists an allocation $f : A \rightarrow R^k_+$ such that $p \cdot f(a) = p \cdot e(a)$ for all $a$ and

3) $f(a)^j \geq g(a)^j (1 - (2C \cdot M \varepsilon / \sum_{a \in A} e(a)^j))$ for all $a \in A$, $1 \leq j \leq k$.

Proof. The familiar fixed point argument (see for example Hildenbrand [5, pp. 149-150, Proposition 3 and Lemma 1]) shows there exists $p \in S^o$ such that $\sum_{a \in A} c(a)/n \in \text{con } D(p)$. By the Shapley-Folkman Theorem (Starr [8]), there exist $h(a) \in \text{con } D(p,a)$ such that $\sum_{a \in A} e(a) = \sum h(a)$, and $h(a) \in D(p,a)$ for all but $k$ traders $\{a_1, \ldots, a_k\}$. Choose arbitrarily $g(a_i) \in D(p,a_i)$ for $1 \leq i \leq k$, and let $g(a) = h(a)$ for $a \in \{a_1, \ldots, a_k\}$. Thus, $g(a) \in D(p,a)$ for all $a$. Since $\gamma_a$ is monotone and $g(a) \in D(p,a)$ for all $a$, $p \cdot g(a) = p \cdot e(a)$ for all $a \in A$. Since $h(a) \in \text{con } D(p,a)$, $p \cdot h(a) = p \cdot e(a)$ for all $a$. 
\[ p \cdot \left| \sum_{a \in A} (g(a) - e(a)) \right| = p \cdot \left| \sum_{a \in A} (g(a) - h(a)) \right| \]

\[ \leq p \cdot \sum_{a \in A} |g(a) - h(a)| \]

\[ = p \cdot \sum_{i=1}^{k} |g(a_i) - h(a_i)| \]

\[ \leq (\sum_{i=1}^{k} p \cdot g(a_i) + p \cdot h(a_i)) \]

\[ = 2 \sum_{i=1}^{k} p \cdot e(a_i) \]

\[ = 2p \cdot \sum_{i=1}^{k} e(a_i) \]

\[ \leq 2\| \sum_{i=1}^{k} e(a_i) \| \]

\[ \leq 2M_c \cdot \]

\[ \left\| \sum_{a \in A} (g(a) - e(a)) \right\|_\infty \leq \sum_{a \in A} \left( \left\| g(a) - h(a) \right\|_\infty \right) \]

\[ \leq p \cdot \sum_{a \in A} |g(a) - h(a)| / \min\{p^1, \ldots, p^k\} \]

\[ \leq 2C \cdot M \cdot p^{-c} \cdot \]

To prove conclusion 3), let \( J = \{ j : \sum_{b \in A} e(a)^j < \sum_{b \in A} g(a)^j \} \) and

\[ r(a)^j = \begin{cases} 
  g(a)^j (\sum_{b \in A} e(b)^j / \sum_{b \in A} g(b)^j) & \text{if } j \in J \\
  g(a)^j & \text{if } j \notin J.
\end{cases} \]
Note \( r(a) \geq 0 \), \( \sum r(a) \leq \sum e(a) \), \( p \cdot r(a) \leq p \cdot g(a) = p \cdot e(a) \). Now let
\[
f(a) = r(a) + \sum_{b \in A} \left( e(b) - r(b) \right) \cdot \frac{p \cdot e(a) - r(a))}{p \cdot \sum_{b \in A} (e(b) - r(b))}.
\]
Thus \( f(a) \geq 0 \), \( \sum_{a \in A} f(a) = \sum r(a) + \sum (e(a) - r(a)) = \sum e(a) \). Thus, \( f \)
is an allocation. Also, \( p \cdot f(a) = p \cdot e(a) \).

If \( j \notin J \), \( f(a)^j \geq r(a)^j = g(a)^j \geq g(a)^j (1 - \frac{C_{M_c}}{p \cdot \sum_{b \in A} e(b)^j}) \).

If \( j \in J \), \( f(a)^j = r(a)^j \)
\[
= g(a)^j \frac{\sum_{b \in A} e(b)^j}{\sum_{b \in A} g(b)^j}
\]
\[
= g(a)^j \left[ 1 - \frac{\sum_{b \in A} g(b)^j - e(b)^j}{\sum_{b \in A} g(b)^j} \right]
\]
\[
\geq g(a)^j \left[ 1 - \frac{2C_{M_c}}{\sum_{b \in A} g(b)^j} \right]
\]
\[
\geq g(a)^j (1 - \frac{2C_{M_c}}{p \cdot \sum_{b \in A} e(b)^j})
\]

This completes the proof of Theorem 1.

We now turn to a limit theorem for sequences of economies. Let \( p_n \) be the price determined by Theorem 1 for the \( n \)th economy. We shall give sufficient conditions to ensure that the \( p_n \) stay within a compact subset of \( \mathbb{S}^D \). The argument is adapted from Brown [4] and Khan [7]. Then \( p_n \) will stay bounded, so Theorem 1 will give us the desired result.

**Definition.** A set of preferences \( P \subset P \) is said to be equi-monotone if, for all \( x \in \mathbb{R}^k_+ \), there exists \( \delta > 0 \) such that \( B(x + e_i, \delta) \succ B(x, \delta) \) for each \( i \) \( (1 \leq i \leq k) \) and all \( x \in P \).
Those familiar with the topology of closed convergence on $P$ (see Hildenbrand [5]) will recognize that any compact set $P \subset P$ is equi-
monotone; thus, the sequences we consider are more general than those
considered by Hildenbrand-Schmeidler-Zamir [6], Hildenbrand [5], or Khan
[7]. Our result also improves the previous results in obtaining the rate of
convergence.

**Theorem 2.** Let $\varepsilon_n : A_n \rightarrow P \times R^k_+$ be a sequence of exchange economies
such that

(i) $\sup_n \| \sum_{a \in A_n} e_n(a)/|A_n| \|_\infty < \infty$

(ii) there exists $P$ equi-monotone and $\delta > 0$ such that

$\left| \{ a \in A_n : x_a \in P, e(a)^i > \delta \} \right| / |A_n| > \delta$ for each $i$

$l \leq i \leq k$ and all $n \in \mathbb{N}$.

Then there exist allocations $h_n$, prices $p_n \in S^0$, $g_n(a) \in D_n(p_n, a)$
and a constant $L$ such that $p_n \cdot h_n(a) = p_n \cdot e_n(a)$ for all $a \in A_n$ and

$$
\left\| \sum_{a \in A_n} (g_n(a) - e_n(a)) \right\|_\infty \leq \sum_{a \in A_n} \left\| g_n(a) - h_n(a) \right\|_\infty \leq LM_{\varepsilon_n}.
$$

Moreover, there exist allocations $f_n$ such that $p_n \cdot f_n(a) = p_n \cdot e_n(a)$
for all $a \in A_n$ and such that

$$
2) \quad f_n(a) = g_n(a) (1 - LM_{\varepsilon_n} / |A_n|).
$$

Finally, if $|A_n| \rightarrow \infty$ and the endowments are uniformly integrable, i.e.,

(iii) $|E_n| / |A_n| \rightarrow 0 \implies \left\| \sum_{a \in A_n} e_n(a) \right\|_\infty / |A_n| \rightarrow 0$,

then the $f_n$ can be chosen so that they also satisfy

$$
3) \quad \max_{a \in A_n} \| f_n(a) - g_n(a) \|_\infty \leq LM_{\varepsilon_n} / |A_n| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
$$
Remark. If the endowments are uniformly bounded by a constant $M$, conclusion 3) establishes the rate of convergence

$$\max_{a \in A_n} \| f_n(a) - g_n(a) \|_{\infty} = O(1/|A_n|).$$

Before we prove Theorem 2, we need the following lemma.

**Lemma 3.** Under the hypotheses of the theorem, there exist prices $p_n$ and $L < \infty$ such that $\sum_{a \in A_n} e_n(a)/|A_n| \in \text{con} D_n(p_n)$ and $C_n p_n \leq L/2$, for all $n$.

**Proof.** The Lemma is analogous to Hildenbrand-Schmeidler-Zamir [7, Lemma 2], but their proof does not quite cover the case we are considering. Our original proof used Nonstandard Analysis. At the suggestion of a referee, we give a standard proof which is a translation of the nonstandard proof.

By Theorem 1, we can find prices $p_n$ such that $\sum_{a \in A_n} e_n(a)/|A_n| \in \text{con} D_n(p_n)$.

If the Lemma is false, we may pass to a subsequence and assume without loss of generality that $C_n p_n \to \infty$. Since $S$ is compact, we may pass to a further subsequence and assume that $p_n \to p \in S$. Since $C_n \to \infty$, we have $p^1 = 0$ for some $1$. Assume $p^1 = 0$, $p^2 > 0$. Choose $P$ equi-monotone and $\delta > 0$ such that $|E_n|/|A_n| > \delta$, where $E_n = \{ a \in A_n : a_p < P, e_n(a)^2 > 2\delta \}$.

Let $K = \sup\| e_n(a)/|A_n| \|_{\infty} k/\delta$. For each $x \in R^k_+$ and $\gamma \in P$, there exists $\gamma$ so that $B(x+e_1, \gamma) \supset B(x, \gamma)$. Hence, there exists $\gamma \in P$ so that $B(x+e_1, \gamma) \supset B(x, \gamma)$ for all $\| x \|_{\infty} \leq K$ and $\gamma \in P$. Choose $n$ sufficiently large that $(\delta^2/2k)\min(\delta^2/k, \gamma/2) > p^1_n$.

If $a \in E_n$ and $x \in D_n(p_n)$, $p_n \cdot x = p_n \cdot e_n(a) + p_n \cdot e_n(a)^2 > 2\delta p_n^2 + 2\delta p^2$. Hence, for $n$ sufficiently large, $a \in E_n$ and $x \in D_n(p_n)$ imply
\( p_n \cdot x > \delta p^2 \). Thus, there exists \( i \) such that \( p_n^i \cdot x > \delta p^2/k \). If \( \|x\|_\infty < K \), \( p_n^i > \delta p^2/kk \); since \( p_n^i < 1 \), \( \delta p^2/k < x^i \leq K \).

Consider \( y = x + e_1 - \min(\delta p^2/k, \gamma/2)e_1 \).

\[
p_n \cdot y = p_n \cdot x + p_n \cdot \left[ \frac{\delta p^2}{k} \cdot \frac{\gamma}{2} \right] \leq p_n \cdot x + p_n \cdot \left[ \frac{\delta p^2}{k} \cdot \frac{\gamma}{2} \right] \leq p_n \cdot x.
\]

\[
y_1 \geq x_1 - \min \left( \frac{\delta p^2}{k}, \frac{\gamma}{2} \right) > 0, \text{ so } y \in \mathbb{R}^k_+.
\]

Thus, \( y \in B(x + e_1, \gamma) \). Hence, \( y \succeq_a x \), contradicting \( x \in D_a(p_n) \).

Thus, we conclude that given \( K \), there exists \( n_0 \) such that \( n \geq n_0 \) implies \( \|x\|_\infty > K \) for all \( x \in D_a(p_n) \), \( a \in E_n \). Hence, if \( z \in \text{con}(D_n(p_n)) \), \( \|z\|_\infty > \delta K/k \) for \( n \geq n_0 \). This contradicts \( \sum_{a \in A_n} e_n(a)/|A_n| \in \text{con}(D_n(p_n)) \), and proves the lemma.

**Proof of Theorem 2.** By Lemma 3, \( \sup_{n} C_n < \infty \), so choosing \( L = 2 \sup_{n} C_n \) establishes conclusions 1) and 2).

In order to prove conclusion 3), we need to modify the definition of the \( f_n \) slightly. Note that since \( \|g_n(a)\|_\infty \leq C_n \|e_n(a)\|_\infty \), the sequence \( \{g_n\} \) is uniformly integrable. Hence there exists \( m \in M \) so that, if \( E_n = (a \in A_n : \|g_n(a)\|_\infty \leq m) \),

\[
\sum_{a \in E_n} g_n(a) > \frac{2}{3} \sum_{a \in A_n} g_n(a) .
\]

Since the endowments are uniformly integrable and \( |A_n| \to \infty \),
\[ M_{\mathcal{E}} / |A_n| \to 0. \] Hence, by conclusion 1), there exists \( n_1 \) such that

\[ \sum_{a \in \mathcal{E}} g_n(a) > \sum_{a \in A_n} e_n(a)/2 \quad \text{for } n > n_1. \]

We now define \( f_n \) (\( n > n_1 \)) as in the proof of Theorem 1, except that all the changes from \( g_n \) are absorbed on \( E_n \). For convenience, we drop the subscript \( n \) for the moment. Let

\[
 r(a)^j = \begin{cases} 
 g(a)^j \left( 1 - \frac{\sum (g(b) - e(b))^j}{\sum g(b)^j} \right) & \text{if } j \in J \text{ and } a \in E \\
 g(a)^j & \text{otherwise.}
\end{cases}
\]

\( f \) is now defined from \( r \) using the same formula as in the proof of Theorem 1. \( f \) will be an allocation provided it is positive.

\[
f(a)^j \geq g(a)^j \left( 1 - 2C_{\mathcal{E}} M_{\mathcal{E}} / \sum_{b \in E} g(b)^j \right)
\]

\[
\geq g(a)^j \left( 1 - 4C_{\mathcal{E}} M_{\mathcal{E}} / \sum_{b \in A} e(b)^j \right)
\]

\[
\geq g(a)^j \left( 1 - 4C_{\mathcal{E}} M_{\mathcal{E}} / \delta^2 |A| \right), \text{ by (ii).}
\]

If \( a \notin E \), \( f(a) = g(a) \). If \( a \in E \),

\[
\|f(a) - g(a)\|_\infty \leq 4C_{\mathcal{E}}^2 M_{\mathcal{E}} \|g(a)\|_\infty / \delta^2 |A| \leq 4C_{\mathcal{E}}^2 M_{\mathcal{E}} / \delta^2 |A| .
\]

Let \( L = \max\{4C_{\mathcal{E}} / \delta^2, 4MC_{\mathcal{E}}^2 / \delta^2\} \). We have shown that for each \( n > n_1 \), there exist \( f_n \) and \( g_n \) satisfying...
\[ f_n(a) \geq g_n(a)(1 - \frac{LM_{\varepsilon_n}}{|A_n|}) \text{ for all } a \in A_n \]

and

\[ \max_{a \in A_n} \left\| f_n(a) - g_n(a) \right\|_{\infty} \leq \frac{LM_{\varepsilon_n}}{|A_n|} . \]

\[ f_n \] is positive (and hence an allocation) for \( n \) such that

\[ \frac{LM_{\varepsilon_n}}{|A_n|} \leq 1 . \]

Since the endowments are uniformly integrable,

\[ \frac{M_{\varepsilon_n}}{|A_n|} \to 0 , \text{ so there exists } n_0 \text{ such that } f_n \text{ is an allocation for all } n > n_0 . \]

Redefine \( f_n \) to be an arbitrary allocation (say \( f_n = h_n \)) for \( n \leq n_0 \). Then (increasing \( L \) if necessary to bring the finite set of allocations \( f_1, \ldots, f_{n_0} \) within the desired bounds) the result follows.
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