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COMPETITIVE BIDDING AND PROPRIETARY INFORMATION

Richard Engelbrecht-Wiggans

Robert J. Weber

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by

Richard Engelbrecht-Wiggans[1]

Robert J. Weber[2]

Abstract

The auction of an object is considered, for the case in which one bidder is better-informed than the others concerning the actual value of the object. An equilibrium point solution of the competitive bidding game is determined; at this equilibrium, the expected revenue of the less-well-informed bidders is zero. The case of an object which can take only values from a discrete set is dealt with as the limit of auctions of continuously-valued objects.

The Basic Model

An object of uncertain value is to be sold at auction. Before the sale one of the participants obtains some information concerning the value of the object; the other participants are aware of this, but do not have access to private information of their own. An example of this situation is the auction of mineral rights for a tract of off-shore territory. All

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[1]Cowles Foundation for Research in Economics at Yale University, Box 2125, Yale Station, New Haven, Connecticut 06520.

[2]Graduate School of Management, Northwestern University, Evanston, Illinois 60201.

of the bidders have access to publicly-available geological data. In some instances one of the bidders also has (and is known to have) additional proprietary information as a result of work performed in an adjacent tract, or through a privately-commissioned line survey.

To be specific, assume that there are $m+1$ potential bidders, all of whom are aware of the joint distribution of a pair (X,Z) of random variables for which all conditional expectations exist. However, only one bidder is informed of the specific value assumed by X. Each bidder submits a sealed bid for an object of unknown value Z. The high bidder is awarded the object and pays the amount of his bid.

An Equilibrium Point Solution

A (pure) bidding strategy $\beta(\cdot)$ for the informed bidder indicates his bid as a function of his information. A (mixed) strategy for any uninformed bidder is a probability distribution $G(\cdot)$ from which his bid will be drawn. We seek a symmetric equilibrium point of the bidding game. That is, we seek a strategy β for the informed bidder, and a strategy G to be played independently by each of the uninformed bidders, such that each bidder's strategy is an optimal response to the indicated strategies of the others.

Theorem 1 Assume that X is absolutely continuous with marginal distribution F_X and density function F_X', and that $E(Z|X=x)$ is continuous and monotone increasing in x. Then equilibrium strategies for the bidders are given by

$$\beta(x) = E(Z|X \leq x) \text{ , for all } x \text{ in the support of } X \text{ ,}$$

and

$$G(b) = F_X(\beta^{-1}(b))^{\frac{1}{m}} \text{ , for all } b \text{ in the range of } \beta \text{ .}$$

(If β is constant on some interval, then by convention,

$$\beta^{-1}(b) = \max \{x : \beta(x) = b\}.)$$

<u>Proof</u> Let x_{min} be the smallest value in the support of X . Then the range of β is $[E(Z|X = x_{min}), E(Z)]$. Assume that the uninformed bidders choose their bids according to G , and that the informed bidder learns that $X = x$. Consider a bid $\hat{\beta}$ by the informed bidder. If $\hat{\beta} < E(Z|X = x_{min})$, then the bidder's expected return is zero (that is, he is certain to not be the high bidder); if $\hat{\beta} > E(Z)$, then a bid of precisely $E(Z)$ is strictly preferable (either bid will win with certainty). Hence, assume that for some t in the support of X , $\hat{\beta} = E(Z|X \leq t)$. Then the expected revenue of the informed bidder is

$$E(Z - \hat{\beta}|X = x \text{ , all other bids are } \leq \hat{\beta}) \cdot G(\hat{\beta})^m$$

$$= [E(Z|X = x) - E(Z|X \leq t)] \cdot F_X(t) \text{ .}$$

Since $E(Z|X \leq t) \cdot F_X(t) = \int_{-\infty}^{t} E(Z|X = w) \, dF_X(w)$, it follows that the derivative, with respect to t , of the informed bidder's expected revenue is $[E(Z|X = x) - E(Z|X = t)] \cdot F_X'(t)$. This expression is nonnegative for $t \leq x$, and nonpositive for $t \geq x$; consequently, an optimal bid when $X = x$ is $\hat{\beta} = E(Z|X \leq x)$.

For any uninformed bidder, a bid of less than $E(Z|X = x_{min})$ will always lose, yielding an expected revenue of zero; a bid greater than $E(Z)$ will always win, yielding a nonpositive expectation. Consider a bid $b = \beta(t)$.

Then the expected revenue of this bidder, when the remaining bidders follow the asserted equilibrium strategies, is

$$E(Z-b|X \le t \text{ , all other bids are } \le b)\cdot F_X(t)\cdot G(b)^{m-1} .$$

But $E(Z-b|X \le t) = E(Z|X \le t) - \beta(t) = 0$. Hence the distribution G has only optimal bids in its support.

It may be observed that the equilibrium strategy and the expected revenue of the informed bidder are independent of the number of uninformed bidders. Furthermore, the maximum of the m bids of the uninformed bidders (which is also independent of m) has the same marginal distribution as the bid $\beta(X)$ of the informed bidder.

At equilibrium, the expected revenue of an uninformed bidder is zero. This is related to a much more general result for auctions with reservation prices (that is, auctions for which there is a lower bound on acceptable bids). Consider the sale by auction of an object of uncertain value, where the high bidder obtains the object for the amount of his bid. Assume that among the bidders there are two, one of whom has access to all information available to the other. Then the latter, less-well-informed bidder has an expected revenue of zero at any equilibrium point of the bidding game.

In order to see this, note first that if the less-well-informed bidder's expectation were negative, he would prefer to abstain from the auction. On the other hand, assume that there is some situation in which he receives information, conditional on which his expectation is positive. Let b be the lowest bid in the support of his (possibly random) equilibrium bid in this situation; his expectation from all bids in the support must be equal, and hence positive. There must therefore be some case in which the better-informed bidder receives additional information conditional on which the expected revenue from a bid of b is still positive, yet in which case he enters a bid no greater than b. An alternative bid, slightly greater than b, would then improve the expected revenue of the better-informed bidder; hence the original situation could not have been in equilibrium. (This argument is due to Paul Milgrom of Northwestern University.)

Weverbergh [6] has considered the case in which there are only two bidders, one of whom knows the value of the object with certainty. He obtained a differential equation for the equilibrium strategy of the uninformed bidder. It can be verified that, for the case he considers, the closed-form strategies in the above theorem satisfy his equation. (In passing, we note that the strategy given at the bottom of page 293 of [6] should be $G(p) = F_V((\theta p + 1)/\theta) = e^{\theta p + 1}$, for all $-\infty < p \leq -1/\theta$). In an earlier paper, Wilson [8] considered the same situation and obtained an equilibrium point for the game in which the informed bidder publicly announces his mixed strategy prior to the informed player's choice of bid.

The preceding characterization of equilibrium strategies applies only when the informational random variable is absolutely continuous (that is, has an associated density function.) Similar restrictions appear in much of the theoretical literature on auctions and competitive bidding. We sketch below an approach to more general distributions.

<u>Mixed Strategies as Limits of Pure Strategies.</u> Consider an example of two bidders competing for a prize. The (subjective) value of the prize is v_1 to the first bidder, and v_2 to the second, where v_1 and v_2 are independent observations of a nonnegative random variable V with distribution $F(\cdot)$. Each bidder submits a sealed bid, and the prize is awarded to the

high bidder (or is assigned at random, in the case of a tie); both bidders are required to pay the amount of the lower bid.

Assume that V has density $f = F'$. Hirshleifer and Riley [2] have shown that an equilibrium point arises when each bidder ($i = 1, 2$) adopts the pure strategy of bidding

$$\beta(v_i) = \int_0^{v_i} \frac{t\,f(t)}{1-F(t)}\,dt \ .$$

This result does not cover the case in which V is concentrated at a single point $\bar{v}$, taking that value with certainty. However, consider the distribution of $\beta(V)$, if V has density f and is supported in a small neighborhood of $\bar{v}$. Then

$$\begin{aligned}\Pr(\beta(V) \le b) &= \Pr\left(\int_0^V \frac{t\,f(t)}{1-F(t)}\,dt \le b\right) \\ &\approx \Pr\left(\bar{v}\int_0^V \frac{f(t)}{1-F(t)}\,dt \le b\right) \\ &= \Pr\left(F(V) \le 1 - \exp\left(-\frac{b}{\bar{v}}\right)\right) \\ &= 1 - \exp\left(-\frac{b}{\bar{v}}\right) \ .\end{aligned}$$

Hence the observed bid of a bidder following his (pure) equilibrium strategy is approximately exponentially distributed with mean $\bar{v}$.

It is straightforward to verify that in the limiting case, when V takes the value $\bar{v}$ with certainty, an equilibrium point results if both bidders follow mixed strategies, choosing their bids at random according to exponential distributions with mean $\bar{v}$. This was first noted by Smith and Price [5] who used this bidding game to model ritual competition between individuals of the same species. It is of interest to note that this same game was also discussed by Shubik 4 as a model of escalating military commitment.

The limiting approach which links the Hirshleifer-Riley and Smith-Price results is closely related to work of Harsanyi 1 . A similar approach can be taken in other situations. Consider a game in which a player is informed of the value of a particular random variable. A pure strategy for the player will be a function of this random variable, and hence will correspond to an "apparent" mixed strategy. As the distribution of the underlying random variable is changed to approach a limiting distribution, the apparent mixed strategies will, under appropriate conditions, converge to an actual behavioral strategy in the limit game; that is, a probability distribution (over the player's strategy space) will be associated with each value in the support of the limit random variable. Furthermore, equilibrium properties will be retained in the limit. Details of this phenomenon will be presented elsewhere; the limiting idea is applied below to a discrete version of the proprietary information model.

The Discrete Case. An object is to be sold at auction, and will be awarded to the highest bidder for the amount of his bid. The value of the object is Z , a discrete random variable taking the values $a_1 \leq \cdots \leq a_\ell$ with probabilities $p_1 , \cdots , p_\ell$. One bidder knows the value taken by Z , while m other bidders know only the distribution from which Z is drawn.

This is, of course, a special case of the model of the first section, in which $X = Z$. Hughart [3] and Wilson [7] have considered this situation for the case $\ell = 2$; their results are corollary to those given below. More generally, the theorem below provides an equilibrium point solution to a competitive bidding case [10] based on a study written by Woods [9].

Let H_1 be the distribution concentrated at a_1, and for $2 \leq k \leq \ell$ define

$$H_k(t) = \frac{(t-a_1)\, p_1 + \ldots + (t-a_{k-1})\, p_{k-1}}{(a_k-t)p_k}$$

for $E(Z|Z \leq a_{k-1}) \leq t \leq E(Z|Z \leq a_k)$. Also, let G be a distribution with mass p_1 at a_1, and with

$$G(t) = p_1 + \ldots + p_{k-1} + p_k H_k(t)$$

for $E(Z|Z \leq a_{k-1}) < t \leq E(Z|Z \leq a_k)$ and $2 \leq k \leq \ell$.

<u>Theorem 2</u> Let the informed bidder bid a_1 if $Z = a_1$, and have him choose his bid according to the distribution H_k if $Z = a_k$ for some $2 \leq k \leq \ell$. Let each uninformed bidder choose his bid according to the distribution $G^{\frac{1}{m}}$. These strategies constitute an equilibrium point of the game under consideration.

This result can be obtained from Theorem 1 through the limiting procedure discussed in the previous section. Assume the value of the object being auctioned is an absolutely continuous random variable W , supported on a small disjoint neighborhoods $N_1, \ldots, N_\ell$ of $a_1, \ldots, a_\ell$, with $Pr(W \in N_k) = p_k$, let F be the distribution of W . Also, let

$\beta(w) = E(W|W \leq w)$ be the equilibrium strategy defined for the informed bidder in Theorem 1. Then, for $2 \leq k \leq \ell$, and all

$$\max\left(E(W|W \in \bigcup_{1}^{k-1} N_k),\ E(Z|Z \leq a_{k-1})\right) \leq t \leq \min\left(E(W|W \in \bigcup_{1}^{k} N_k),\ E(Z|Z \leq a_k)\right),$$

$$\Pr(\beta(W) \leq t|W \in N_k) = \Pr(E(W|W \leq w)|_{w=W} \leq t|W \in N_k)$$

$$\simeq \Pr\left(\frac{a_1 p_1 + \ldots + a_{k-1} p_{k-1} + a_k(F(W) - p_1 - \ldots - p_{k-1})}{F(W)} \leq t|W \in N_k\right)$$

$$= \frac{1}{p_k} \cdot \Pr\left(p_1 + \ldots + p_{k-1} \leq F(W) \leq \frac{(a_k - a_1)\, p_1 + \ldots + (a_k - a_{k-1})\, p_{k-1}}{a_k - t}\right)$$

$$= H_k(t) .$$

Hence, the informed bidder's apparent mixed strategy approaches the behavioral strategy of Theorem 2, as the distribution of W approaches that of Z. The equilibrium strategies of the uninformed bidders are chosen so that G, the distribution of the maximum of their bids, is also the (marginal) distribution of the informed bidder's bid.

These computations indicate how the equilibrium strategies of Theorem 2 were derived. A formal proof of the theorem can be obtained by direct verification that each bidder's strategy is a best response to the strategies of the other bidders.

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