COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 343

EXISTENCE OF A "LOCAL" COOPERATIVE EQUILIBRIUM IN A CLASS OF VOTING GAMES

Gerald H. Kramer and Alvin K. Klevorick

August 17, 1972

EXISTENCE OF A "LOCAL" COOPERATIVE EQUILIBRIUM IN A CLASS OF VOTING GAMES*

by

Gerald H. Kramer and Alvin K. Klevorick
Cowles Foundation, Yale University
August, 1972

I. Introduction

Many political and social choice processes, particularly those based on some form of voting rule, can be usefully represented as simple games. A simple game differs from the usual variety of n-person game in that certain combinations or coalitions of voters are "all-powerful," and are able to enforce their will irrespective of the desires or actions of the players who are not members of the coalition, while all of the remaining coalitions are essentially powerless to affect the outcome. A familiar example is a majority game, in which any coalition containing a majority of players can secure any outcome its members are able to agree upon, while no minority coalition can influence the outcome if the opposing majority is agreed upon a course of action. Other examples of simple games, some very complicated, are cited in [7] and [8, Chapter 10].

While simple games clearly represent a rather special form of the more general n-person game, they may nevertheless be quite complex in structure and difficult to analyze. Moreover, the existence of a cooperative solution, such as the core, is by no means assured in games of this type.

*The research described in this paper was undertaken under grants from the National Science Foundation and from the Ford Foundation.

Many simple games are known for which no core exists; the famous "voters' paradox" is a particularly simple three-player example of such a game. Indeed, much of the social choice and voting theory literature, though not cast in an explicitly game-theoretic framework, can be interpreted as showing how difficult it is to ensure the existence of a cooperative equilibrium in multi-person situations based on any reasonable kind of voting process or social choice mechanism.

A particular equilibrium condition for majority rule known as "single peakedness" has received considerable attention in the voting theory literature, and it has been the basis (sometimes only implicitly) of various studies of voting on public goods or other public sector problems. The condition, which is due to Duncan Black, is sufficient for the transitivity of majority rule, and hence for the existence of a core in a majority game. (We are ignoring various technical qualifications here.) In fact, the condition is even more powerful than this, for it can also be shown to be sufficient for the existence of a core in any simple game (Theorem 2 of the present paper). Roughly speaking, single-peakedness is equivalent to requiring that the relevant set of alternatives or possible outcomes be one-dimensional, and that voter preferences over this set be representable by strictly quasi-concave utility functions. Thus the condition, while clearly restrictive, is nonetheless a useful one, for in many economic problems, it often arises as a natural consequence of the convexity of individual preferences. The usual examples of voting on resource-allocation or other public sector problems [3], [6, Chapter 6] are of this type.

There have, however, also arisen examples of voting problems which, though essentially one-dimensional in character, are nevertheless such that the single-peakedness condition cannot be satisfied. In one such example described by Musgrave [6, Chapter 6], the electorate is to vote on the size of the budget for some single public service, and the budget is to be financed by means of a tax formula according to which the relative shares paid by different voters may vary arbitrarily. Another appears in [4], in which representatives of firms and households in a water-management area must vote to determine the water-quality standard which is to prevail in the region, with the revenue for constructing and operating the treatment facilities required to reach the agreed-upon standard being raised by effluent charged imposed on both households and firms. Under various conditions, such as increasing returns to scale in the technology of pollution treatment, non-convexities are introduced into the problem which cause the single-peakedness condition to fail.

In examples such as these, the failure of the single-peakedness assumption can easily lead to the non-existence of a core. Consider, for example, the situation in Figure 1. Points on the horizontal axis are possible values of the underlying decision variable to be voted on, while those on the vertical axis represent (ordinal) utility levels. The utility functions of each of the three voters are graphed together in the figure. Under simple majority rule, the existence of a core is equivalent to the existence of a value x^o of the decision variable with the property that no different value, y, is strictly preferred by a majority to x^o. It is readily

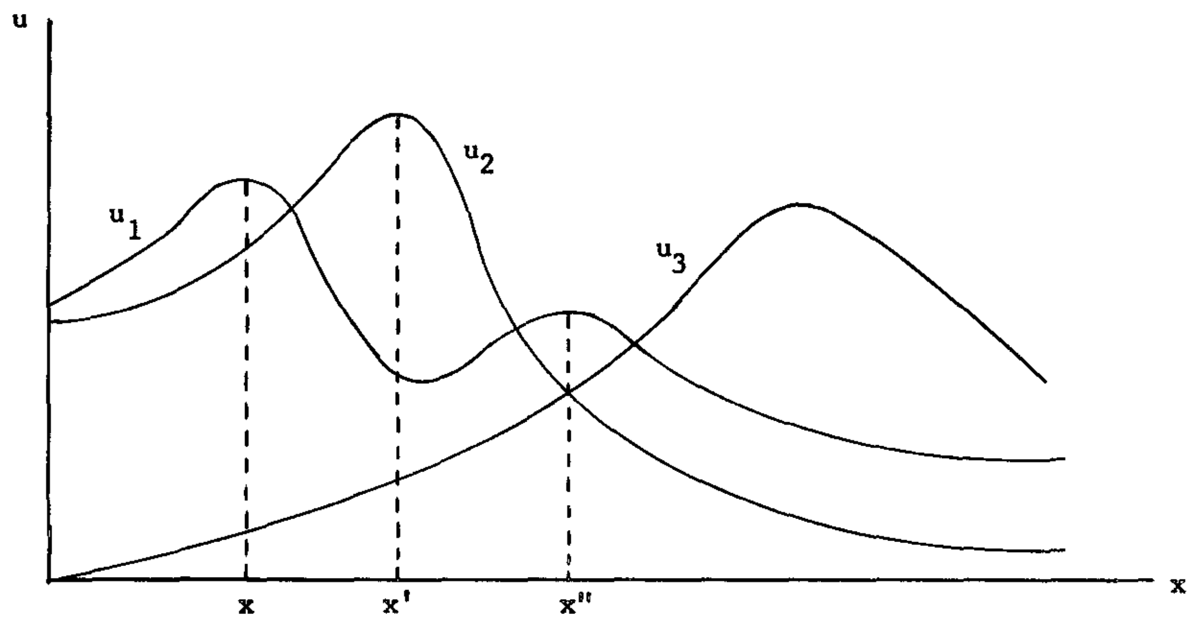

FIGURE 1

A Voting Game without Single-Peakedness and without a Core

verified that in this example no such value exists. For example, the value x' is preferred by voters 2 and 3 to any smaller value; x'' in turn defeats x', as well as any value in the interval $[x', x'')$, via the coalition $\{1,3\}$, and it also defeats any value larger than itself via the coalition $\{1,2\}$; yet x'' is itself defeated, in turn, by the point x, via the coalition $\{1,2\}$. Thus, any proposed resolution to the underlying choice problem will be blocked by some majority coalition, and no cooperative solution exists for this simple 3-person game.

In determining in this fashion whether a particular alternative constitutes a cooperative equilibrium in the game, it is assumed that the members of each winning coalition collectively examine the alternative to see whether they can improve their position relative to it. If they can, that is, if they can identify some other feasible alternative which every member of C prefers, then the alternative in question is vulnerable to collusive action by the coalition, and hence does not constitute an equilibrium. A possible equilibrium is thus pitted against every other feasible alternative, no matter how distant; the various coalitions are assumed to have the ability and foresight to examine and weigh every possible outcome, even those quite distant from the current state of affairs, with which the members of the coalition may have had no experience.

The equilibrium concept considered in this paper is based on a more limited view of decisionmakers' abilities to examine and evaluate alternative states. We shall suppose the voting process to be more "incremental" or "myopic" in character, in the sense that the various potential blocking coalitions consider only "nearby" alternatives in determining whether to block the current state. If some winning coalition prefers a neighboring alternative to the current state, then as before, the existing state of affairs is vulnerable to collusive action by the members of the coalition, and is not an equilibrium. Conversely, an alternative which is not blocked by any nearby outcome will be said to be a "local" cooperative equilibrium. In Figure 1, it is easily seen that the point x'' is a local equilibrium in this sense; moreover it is the only such equilibrium in that example. The principal result of this paper shows that in simple games of this kind, where the alternative set is essentially one-dimensional in character but the single-peakedness condition does not apply, a "local" cooperative equilibrium nevertheless exists quite generally.

II. Definitions and Assumptions

We denote by $N = \{1, 2, \ldots, n\}$ the set of voters or players, and by A the set of alternatives or possible outcomes of the game. A basic assumption of this paper is that the set of alternatives being considered is essentially one-dimensional. More precisely, A is assumed to be a point set homeomorphic to some closed interval. For simplicity, and without loss of generality, we can reformulate this premise as:

Assumption I. The set A of alternatives is the closed unit interval: $A = [0,1]$.

Each voter $i \in N$ has an ordinal utility function u_i , defined and continuous on the set A . Our principal result will depend on one further assumption about voter preferences. A real-valued function f defined on a point set will be said to have a proper relative maximum (p.r.m.) at a point $\bar{x}$ in the set if it has a relative maximum at $\bar{x}$, and does not have a relative minimum at $\bar{x}$. Our additional assumption on voter preferences is that each voter's utility function u_i satisfies the following:

Assumption II. Each u_i has only a finite number of proper relative maxima in A .

This "finite-peakedness" premise is a natural generalization of the well-known "single-peakedness" condition. In effect it constitutes a rather weak "smoothness" condition on individual utility functions: it excludes functions which "oscillate" infinitely many times in the interval, but permits functions with any finite number of oscillations, or with "flats" (regions over which u_i is constant). The assumption does not require differentiability, and it is clearly much weaker than quasi-concavity or "single-peakedness."

The outcome eventually adopted will be determined in part by the preferences of the players, and in part by the manner in which the rules or structure of the game distributes power among the various coalitions, that is sets $C \subseteq N$ of players. The distribution of power in a simple game is such that each of the coalitions is either all-powerful, in a sense to be made precise below, or is powerless. We denote by $\mathcal{W}$ the collection of "powerful" or winning coalitions. $\mathcal{W}$ is assumed to be:

1. Non-empty
2. Proper: If $C \in \mathcal{W}$ then $N-C \notin \mathcal{W}$.
3. Superadditive: If $C \in \mathcal{W}$ and $C \subseteq C' \subseteq N$, then $C' \in \mathcal{W}$ also.

An obvious consequence of these properties is that the set N of all players is winning and hence that any winning coalition is non-empty. Note also that it is not true, in general, that the complement (in N) of a non-winning coalition is necessarily winning.

The coalition of all players, acting together, can achieve any outcome. Whether a particular outcome $x \in A$ is a solution or equilibrium of the game depends on whether any player, or combination of players, could improve their position by acting differently. In particular, if there is some other alternative which is preferred to x by each member of a particular coalition $C \subseteq N$ of players, and if the coalition C is powerful enough to ensure or attain this outcome, then the original outcome x will be prevented, or "blocked," by this coalition. In a simple game only the winning coalitions are powerful enough to block in this sense. A winning coalition can block any outcome its members wish to prevent, while no outcome can be blocked by a coalition which is not winning. Thus we have:

Definition.* An alternative x is blocked by another, y, if and only if there exists some winning coalition $C \in \mathcal{W}$ for which $u_i(y) > u_i(x)$ for each player $i \in C$.

---

*Unfortunately, the same term is also used in the game theory literature to denote a quite different concept. A coalition which is not itself winning, and whose complement (in N) is also not winning, is often referred to as a "blocking" coalition. Such a coalition is unable to block (in our sense) any alternative itself, but is powerful enough to prevent (or "block") any disjoint coalition of players from blocking an alternative. (A simple game in which there are no such coalitions is strong.) None of our results depends on the existence or non-existence of such coalitions, and we shall use the term "block" in the sense of the above Definition throughout.

The usual cooperative solution, the core, can now be simply defined as the set of unblocked alternatives; that is, the core is $\{x \in A: x$ is not blocked by any $y \in A\}$. As discussed earlier the equilibrium concept with which we shall be concerned is based on a more restricted notion of blocking, wherein a particular alternative x can be blocked only by "nearby" alternatives. More precisely, we define a local cooperative equilibrium as follows:

Definition: A point $x^o \in A$ is a local cooperative equilibrium if there exists a neighborhood $(x^o - \epsilon, x^o + \epsilon)$ of x^o which contains no point $y \in A$ that blocks x^o.

III. Existence of a Local Cooperative Equilibrium

The principal result to be established can now be stated as follows:

Theorem 1. A simple game which satisfies Assumptions I and II has a local cooperative equilibrium.

Before proceeding to the proof of this proposition it will be useful to establish a preliminary result. At any point $x \in A$ there is a set of voters who "prefer small increases"--that is, a set for whom x is inferior to any slightly larger value--and a set who "prefer small decreases." Let us represent these sets by $I(x)$ and $D(x)$, respectively. More precisely, for any $x \in A$ we define $I(x) = \{i \in N : \text{there exists } \delta > 0 \text{ such that } u_i(z) > u_i(x) \text{ for all } z \in (x, x+\delta) \cap A\}$, and $D(x) = \{i \in N : \text{there exists } \delta' > 0 \text{ such that } u_i(z) > u_i(x) \text{ for all } z \in (x-\delta', x) \cap A\}$. (Clearly some voters may be contained in both of these sets, and some in neither.)

If at any $x \in A$ the set $I(x)$ constitutes a winning coalition, and if $x < 1$, then evidently every neighborhood of x will contain points $y \in A$ which block x. This will also be true if $D(x) \in \mathcal{W}$ and $x > 0$. If at some $x^o \in A$ neither of these conditions holds, however, then a neighborhood $(x^o-\delta, x^o+\delta)$ of x^o can be found which contains no points that block x^o, and hence x^o itself constitutes a local cooperative equilibrium. To show the existence of such a point we first establish the following property of the sets $I(x)$ and $D(x)$.

Lemma. Assumptions I and II imply that for any point $x^o \in A$ there exists a number $\delta > 0$ such that the utility function u_i of each voter $i \in I(x^o)$ is strictly increasing on the interval $[x^o, x^o+\delta] \cap A$, and a number $\delta' > 0$ such that for each $i \in D(x^o)$, u_i is strictly decreasing on $[x^o-\delta', x^o] \cap A$.

Proof. It suffices to prove the first part of the lemma, since the second part is argued in essentially the same fashion.

The first proposition is trivially true if $x^o = 1$. If $0 \leqq x^o < 1$, then for any $i \in I(x^o)$ we can define a point x_i^* as follows. If $u_i(x) > u_i(x^o)$ for all $x^o < x \leq 1$, let $\alpha = 1$; otherwise let α be the smallest point lying above x^o which is not preferred to x^o, that is, $\alpha = \min\{x \in (x^o,1] : u_i(x) \leq u_i(x^o)\}$. The continuity of u_i ensures the existence of this minimum. If the interval $[x^o,\alpha]$ contains a point which is a proper relative maximum on A of u_i, we define x_i^* as the smallest such point. (From Assumption II the number of such points in the interval must be finite and hence a smallest one exists.) If $[x_o,\alpha]$ contains no such point, we simply set $x_i^* = \alpha$. Since x^o is an absolute minimum of u_i on the non-degenerate interval $[x^o,\alpha]$, it cannot be a relative maximum of u_i on A. Hence x^o cannot be a p.r.m. of u_i on A, and thus $x_i^* > x^o$.

For any y such that $x^o < y < x_i^*$, the continuous function u_i must have a maximum on the interval $[x^o,y]$, and there must be a smallest point in the interval at which u_i takes on this maximum value. That is, there exists a point $\mu(y)$ such that $u_i(\mu(y)) \geq u_i(z)$ for all $z \in [x^o,y]$, and $u_i(\mu(y)) > u_i(z)$ for any $z \in [x^o, \mu(y))$.

The point $\mu(y)$ cannot lie in the interior of the interval $[x^o, y]$. To see why, suppose it did. Then since it is a maximum of u_i on this interval it would also be a maximum on some neighborhood of $\mu(y)$, $(\mu(y)-\delta, \mu(y)+\delta)$ with $\delta \leq \min\{|y-\mu(y)|, |y-x^o|\}$. Hence $\mu(y)$ would be a relative maximum of u_i on A. Moreover, by definition of $\mu(y)$, any neighborhood of $\mu(y)$ will contain points $z \in [x^o, \mu(y))$ for which $u_i(\mu(y)) > u_i(z)$ so that $\mu(y)$ could not be a relative minimum on A. Hence, $\mu(y)$ would be a p.r.m. of u_i on A, but yet it would lie in the interval (x^o, x_i^*). This would contradict the definition of x_i^*.

Thus, either $x^o = \mu(y)$ or $\mu(y) = y$ must hold. But $x^o = \mu(y)$ is impossible, since then we would have $u_i(x^o) = u_i(\mu(y)) \geqq u_i(z)$ for any $z \in [x^o, y]$ contradicting the original assumption that $i \in I(x^o)$. Therefore $\mu(y) = y$.

This implies that u_i is strictly monotonically increasing in the interval $[x^o, x_i^*]$, since if $x^o \leqq y < y' \leqq x_i^*$, then $\mu(y') = y'$, so that $x^o \leqq y < y'$ implies $u_i(y') = u_i(\mu(y')) > u_i(y)$. Hence, letting $\delta = \min_{i \in I(x^o)} \{|x^o - x_i^*|\}$, the proposition is established. Q.E.D.

Having obtained this result, we now turn to the proof of the main theorem.

<u>Proof of Theorem 1</u>: If the set $I(0) \subseteq N$ is not a winning coalition, then clearly a neighborhood $(0, \delta)$ can be found such that no point in the neighborhood blocks 0. Hence, $x = 0$ is a local cooperative equilibrium. Similarly, if $D(1) \notin \mathcal{W}$, the point $x = 1$ is an equilibrium.

The remaining possibility is that both $I(0) \in \mathcal{W}$ and $D(1) \in \mathcal{W}$. In this case let $x^o = \sup\{x \in A : I(x) \in \mathcal{W}\}$; clearly, $x^o > 0$. We shall show that x^o is an equilibrium, that is, $I(x^o) \notin \mathcal{W}$ and $D(x^o) \notin \mathcal{W}$.

Suppose, to the contrary, that $I(x^o) \in \mathcal{W}$. From the lemma, it follows that there exists an interval $[x^o, x']$ on which the utility function u_i of each voter $i \in I(x^o)$ is strictly increasing. Since $I(1) \notin \mathcal{W}$ by assumption, we have $x^o < 1$, and the interval $[x^o, x']$ must be non-degenerate; that is, $x^o < x'$. Hence, there exists a point y such that $x^o < y < x'$. But any u_i is strictly increasing on the interval $[y,x']$ if it is strictly increasing on $[x^o,x'] \supset [y,x']$. Since u_i strictly increasing on $[y,x']$ clearly implies $i \in I(y)$, we have $I(y) \supseteq I(x^o)$. If $I(x^o) \in \mathcal{W}$, then the superadditivity of $\mathcal{W}$ (property 3 of $\mathcal{W}$) implies that $I(y) \in \mathcal{W}$ also. Since $y > x^o$ and x^o is by definition an upper bound of the set $\{x \in A : I(x) \in \mathcal{W}\}$, this is a contradiction. Hence, $I(x^o)$ cannot be a winning coalition.

Now consider the remaining possibility, namely, that $D(x^o) \in \mathcal{W}$. The lemma then implies that there exists an interval $[x^*,x^o]$ on which u_i is strictly decreasing for each $i \in D(x^o)$. Since $x^o > 0$, this interval must be non-degenerate; that is, $x^o > x^*$. It must then be true that $D(x^o) \cap I(y) = \emptyset$ for every point y in the interior of $[x^*,x^o]$. Otherwise, using the other part of the lemma, there would exist a non-degenerate interval $[y,y'] \subseteq [x^*,x^o]$ on which some voter's utility function would be both strictly increasing and strictly decreasing, which is impossible. Hence, $I(y) \subseteq N-D(x^o)$ for all y such that $x^* < y < x^o$.

Since, by hypothesis, $D(x_o) \in \mathcal{W}$, property 2 of the collection $\mathcal{W}$ implies that $N-D(x^o) \notin \mathcal{W}$. But, then, superadditivity of $\mathcal{W}$ (property 3 of $\mathcal{W}$) and $I(y) \subseteq N-D(x^o)$ imply that for all $y \in (x^*, x^o)$, $I(y) \notin \mathcal{W}$. Since x^o is by definition a least upper bound of the set $\{x \in A : I(x) \in \mathcal{W}\}$, it is possible to have $I(y) \notin \mathcal{W}$ for all $y \in (x^*, x^o)$ only if $I(x^o)$ is itself a winning coalition. This is again a contradiction, since it has just been shown that $I(x^o) \notin \mathcal{W}$. Hence, $D(x^o) \notin \mathcal{W}$ also, and the theorem is established. Q.E.D.

IV. Discussion

Of the two assumptions on which Theorem 1 rests, Assumption II is a fairly weak "smoothness" condition on individual preferences. In contrast, Assumption I, requiring in effect that the set of alternatives be a single-dimensional point set, is quite strong. It is, however, essential to the result and cannot be substantially weakened.

Consider the following simple majority game, in which the alternative set is the unit square (rather than the unit interval as assumed heretofore): $A' = \{(x,y): x \in [0,1],\ y \in [0,1]\}$, and there are three voters, with utility functions $u_1(x,y) = -(x^2 + y^2)$, $u_2(x,y) = -[x^2 + (y-1)^2]$, $u_3 = -[(x-1)^2 + y^2]$. These strictly quasi-concave utility functions clearly satisfy Assumption II, since each has only a single proper relative maximum in A'. In Figure 2 the indifference curve of each voter through a representative point $a = (\bar{x},\bar{y})$ is shown. It is apparent from the diagram that the point $a = (\bar{x},\bar{y})$ is blocked by a neighboring point $a' = (\bar{x}-\delta,\ \bar{y})$ (for any $0 < \delta < \bar{x}$) if $\bar{x} > 0$, and by $a'' = (\bar{x},\ \bar{y}-\delta)$ (for any $0 < \delta < \bar{y}$)

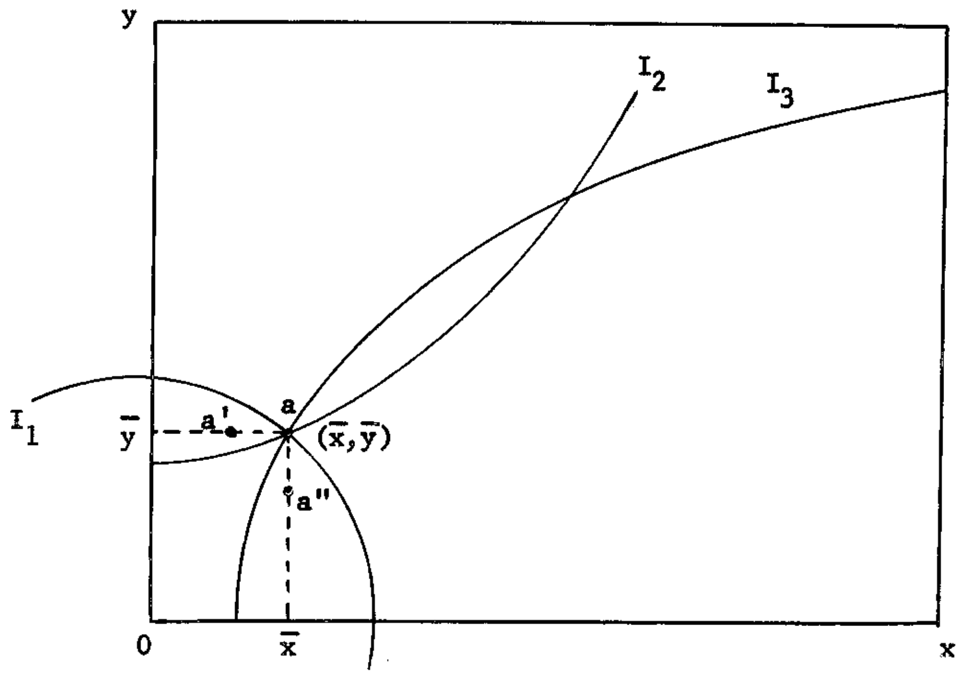

FIGURE 2

A Simple Majority Game with a Two-Dimensional Alternative Set and No Local Equilibrium

if $\bar{y} > 0$, while clearly the point (0,0) is blocked by (δ,δ) for any $0 < \delta < 1$. Hence, no local equilibrium exists even in this simple example. In general, no such equilibrium will exist when the set A is of dimensionality greater than one.

Theorem 1 establishes conditions for the existence of a local equilibrium, but it says nothing about uniqueness, or about the topological structure of the set of equilibria. It is easily verified by examples that in general there is not a unique equilibrium, and the set of equilibria need not be convex, or even closed. However this set does have maximal and minimal elements. In particular, if we denote by E the set of local cooperative equilibria, and let $x^u = \sup(E)$ and $x^L = \inf(E)$, then we can establish the following:

<u>Comment</u>: x^u and x^L are themselves equilibria. Moreover, if $x^u < 1$ (resp. $x^L > 0$), then $D(z) \in \mathcal{W}$ for all $z \in (x^u, 1]$ (resp. $I(z) \in \mathcal{W}$ for all $z \in [0, x^L)$), and x^u (resp. x^L) is a proper relative maximum of u_i for some $i \in N$.

<u>Proof</u>. It suffices to consider x^u, since the corresponding properties of x^L are argued similarly. We must first show that x^u is an equilibrium. If $D(1) \notin \mathcal{W}$, then clearly $x = 1$ is an equilibrium and $x^u = 1$. If $D(1) \in \mathcal{W}$, let $x^o = \sup\{x \in A : I(x) \in \mathcal{W}\}$. It was shown in the proof of Theorem 1 that x^o is an equilibrium, so $x^u \geq x^o$ by definition of x^u.

Suppose x^u is not an equilibrium. Then it follows from the definition of x^o that $I(x^u) \notin \mathcal{W}$. Hence, we must have $D(x^u) \in \mathcal{W}$. But the lemma states that there exists a non-degenerate interval $[x^*, x^u]$ on which the utility function of each $i \in D(x^u)$ is strictly decreasing. Hence, for all $z \in (x^*, x^u)$, $D(z) \supseteq D(x^u)$, and it follows from the superadditivity of $\mathcal{W}$ that $D(z) \in \mathcal{W}$. Thus, no $z \in (x^*, x^u)$ can be an equilibrium. Since x^u is by definition a least upper bound of the set of equilibria, the only remaining possibility is that x^u is an equilibrium.

Now suppose $x^u < 1$, and hence $D(1) \in \mathcal{W}$. Since $x^u \geqq x^o = \sup\{x \in A : I(x) \in \mathcal{W}\}$, it follows that if $z \in (x^u, 1]$, then $I(z) \notin \mathcal{W}$. It must therefore be true that $D(z) \in \mathcal{W}$ for otherwise z would be a local equilibrium, and this is a contradiction since $z > x^u = \sup(E)$.

Finally, suppose $x^u < 1$ but x^u is not a proper relative maximum of any voter's utility function. Then since the number of p.r.m. is finite, there exists a neighborhood $(x^u-\delta, x^u+\delta)$ of x^u, lying in A, which contains no p.r.m. of any voter. Let $x^* = x^u + \frac{1}{2}\delta$. For any $z \in (x^u-\epsilon, x^*)$ each voter's continuous utility function u_i has a maximum on the closed interval $[z,x^*]$. Let $\mu_i(z)$ be the largest point in the interval at which u_i takes on this maximum value. For any voter $i \in D(x^*)$, evidently $\mu_i(z) < x^*$ for all $z < x^*$.

If there were any voter $i \in D(x^*)$ such that $\mu_i(z) > z$ for some $z \in (x^u-\delta, x^*)$, then u_i would have a p.r.m. at $\mu_i(z)$, since $u_i(\mu_i(z))$ is a relative maximum on A, as can be seen by examining u_i in the neighborhood $(\mu_i(z)-\delta, \mu_i(z)+\delta)$ where $0 < \delta < \min\{\mu_i(z)-z, x^*-\mu_i(z)\}$, and, $u_i(\mu_i(z))$ is not a relative minimum since every neighborhood of $\mu_i(z)$ will contain a point w such that $\mu_i(z) < w \leqq x^*$ for which $u_i(\mu_i(z)) > u_i(w)$ by definition of $\mu_i(z)$. But $\mu_i(z)$ cannot be a p.r.m., since it lies in a neighborhood which by hypothesis contains no p.r.m. of any voter. Hence, it must be true that for all $i \in D(x^*)$, $\mu_i(z) = z$ for all $z \in (x^u-\delta, x^*]$, i.e. that u_i is strictly decreasing on $[x^u-\delta, x^*]$.

This implies, however, that $D(x^u) \supseteq D(x^*)$. Hence, from the superadditivity of $\mathcal{W}$ and the fact established previously that $x^* > x^u$ implies $D(x^*) \in \mathcal{W}$, it follows that $D(x^u) \in \mathcal{W}$. This is also a contradiction, however, since we have just shown x^u to be an equilibrium. Hence, x^u must be a proper relative maximum of some voter's utility function. Q.E.D.

In concluding, let us consider the relation between our assumptions and result, and the well-known "single-peakedness" condition. Though in principle the usual definition of single-peakedness is slightly more general, for our purposes we can define the concept as follows:

Definition. If the set A of alternatives satisfies Assumption I, and if for each $i \in N$, the utility function u_i is strictly quasi-concave over A, then the players have single-peaked preferences over A.

Single-peakedness is thus a special case of our ("finite-peakedness") Assumption II, since it is easily verified that any strictly quasi-concave function has a unique relative maximum, and hence only one p.r.m., on any closed interval. In the voting theory literature, one alternative y is said to be preferred by majority rule to another x, denoted $yP_{maj}x$, if and only if the number of voters who prefer y to x is strictly greater than the number who prefer x to y. Single-peakedness implies that the (antisymmetric) relation P_{maj} is transitive over A ([1, pp. 75-80], [2]), and hence that (for finite A at least) there exists a maximal element, or majority winner, in A. In a majority game (that is, a game in which the winning coalitions are those with more than $n/2$ members), $yP_{maj}x$ implies x is blocked by y, in the sense defined in Section II. Hence under single-peakedness a majority game will have a non-empty core (if A is finite). This result can be generalized to other simple games, however, as follows:

<u>Theorem 2</u>. A simple game whose players have single-peaked preferences possesses a core.*

---

*A related result of Wilson's [9, pp. 9-10] is that under single-peakedness (defined in slightly more general terms than we have defined it), the "blocking" relationship is transitive in any <u>strong</u> simple game, that is, in any simple game in which the complement (in N) of every non-winning coalition is necessarily winning.

<u>Proof</u>. The theorem is not difficult to establish directly. (Compare the proof of the first part of Lemma 3 in [5].) It can be more easily argued here as follows. A strictly quasi-concave function can have only a single relative maximum on A, and hence only one p.r.m. on A. Hence, Assumption II is satisfied, and from Theorem 1 the game must have a local cooperative equilibrium, say x^o. If x^o were not in the core, there would exist $y \in A$ which blocks x^o. That is, there would exist a $y \in A$ such that $C(y,x^o) \equiv \{i \in N : u_i(y) > u_i(x^o)\}$ is a winning coalition. Let $z(t) = ty + (1-t)x^o$, $t \in [0,1]$. Clearly, $u_i(y) > u_i(x^o)$ and quasi-concavity of u_i imply that for all $i \in C(y,x^o)$, $u_i(z(t)) > u_i(x^o)$ for all $t \in (0,1]$. Hence, $C(z(t), x^o) \supseteq C(y,x^o)$ and from superadditivity of $\mathcal{W}$ it follows that $C(z(t), x^o) \in \mathcal{W}$ for all such t. Thus, every neighborhood $(x^o-\delta, x^o+\delta)$ of x^o would contain a point $z(t')$ (where $0 < t' < \frac{\delta}{|y-x^o|}$) which blocks x^o. But this is a contradiction of the fact that x^o is a local cooperative equilibrium. Hence x^o must be unblocked and belong to the core. Q.E.D.

REFERENCES

[1] Arrow, K.J., Social Choice and Individual Values, John Wiley and Sons, Inc., New York, 1951.

[2] Black, D., The Theory of Committees and Elections, Cambridge University Press, 1958.

[3] Bowen, H.R., "The Interpretation of Voting in the Allocation of Economic Resources," Quarterly Journal of Economics, 58 (1943), pp. 27-48.

[4] Klevorick, A.K. and G.H. Kramer, "Social Choice on Pollution Management: The Case of the Genossenschaften," Unpublished manuscript, July 1972.

[5] Kramer, G.H., "Sophisticated Voting over Multidimensional Choice Spaces," Unpublished manuscript, Cowles Foundation, April 1971.

[6] Musgrave, R.A., The Theory of Public Finance; A Study in Public Economy, McGraw-Hill Book Company, Inc., New York, 1959.

[7] Shapley, L., "Simple Games: An Outline of the Descriptive Theory," Behavioral Science, 7 (1962), pp. 59-66.

[8] Von Neumann, J. and O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, New Jersey, 1944.

[9] Wilson, R., "The Game-Theoretic Structure of Arrow's General Possibility Theorem," Technical Report No. 53, Institute for Mathematical Studies in the Social Sciences, Stanford, California, February 1972.