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# EXISTENCE OF A COMPETITIVE EQUILIBRIUM IN A NONSTANDARD EXCHANGE ECONOMY

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# EXISTENCE OF A COMPETITIVE EQUILIBRIUM IN A NONSTANDARD EXCHANGE ECONOMY*

by

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## I. Introduction

Nonstandard exchange economies and the associated nonstantard concepts of the core and competitive equilibrium were first defined in Brown-Robinson [3]. In that paper they proved the equivalence between the nonstandard core and the set of nonstandard competitive equilibria. In a second paper [4], using the equivalence theorem for nonstandard exchange economies, they showed that core allocations in large standard exchange economies were approximately competitive equilibria.

In this paper, we shall modify slightly their definition of a competitive equilibrium and then prove the existence of this nonstandard competitive equilibrium. In a future paper, we hope to show that this existence theorem implies the existence of approximate or "$\varepsilon$"-equilibria in large standard exchange economies.

Our proof is based, in part, on Schmeidler's elegant proof of the existence of a competitive equilibrium for a continuous economy [8].

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As in the continuous case we do not assume convexity of preferences, since we use a nonstandard analogue of the Lyapunov theorem, on the convexity of vector-valued measures, due to P. Loeb [ 6 ].

Also we have benefitted a great deal from Auman's seminal papers [ 1 ], [ 2 ] and Debreu's paper on the integration of correspondences [ 5 ].

For an introduction to nonstandard analysis, as it will be used in this paper, we suggest the introduction of Brown-Robinson [ 3 ] or [ 4 ] and the references listed there.

## II. Mathematical Preliminaries

Let $T$ be an internal set in a nonstandard model of the reals where $|T| = \omega$, $\omega$ an infinite natural number. Define $\mathcal{J}(T)$ as the family of internal subsets of $T$ and note that $\mathcal{J}(T)$ is a Boolean algebra. Let $\{\epsilon_t\}_{t=1}^{\omega}$ be an internal set of positive infinitesimals such that $\sum_{t=1}^{\omega} \epsilon_t$ is a finite number. For any non-empty internal subset of $T$, say $S$, define $\mu_t(S) = \sum_{t \in S} \epsilon_t$. Let $\mu_t(\emptyset) = 0$, then $<\mathcal{J}(T), \mu_t>$ will be called an infinitesimal scalar measure space. Any n-tuple of infinitesimal scalar measures, $<\mu_{t_1}, \ldots, \mu_{t_n}>$ defines an infinitesimal vector measure denoted by $\bar{\mu}_t$. An infinitesimal vector measure $\bar{\mu}_t$ is said to be S-convex iff for all internal subsets of $T$, $S$, and all $\lambda \in (0,1)$, there exists an internal subset of $S$, $R$, such that $\bar{\mu}_t(R) \simeq \lambda \bar{\mu}_t(S)$. The following fundamental theorem is due to P. Loeb [6] and is the nonstandard analogue of Lyaponov's theorem.

Theorem 1. Every infinitesimal vector measure is S-convex.

A set $B$ is said to be S-convex if for all $\bar{x}, \bar{y} \in B$ and $\lambda \in (0,1)$, there exists $\bar{z} \in B$ such that $\bar{z} \simeq \lambda\bar{x} + (1-\lambda)\bar{y}$. Note that this definition differs from that given in [3]; they are equivalent for S-open sets.

Let $\varphi \in \mathcal{P}({}^*\Omega_n)^T$, i.e. $\varphi$ is a correspondence from $T$ to ${}^*\Omega_n$. Let $\eta_\varphi = \{f \in {}^*\Omega_n^T | f$ is internal and for all $t \in T$, $f(t) \in \varphi(t)$ and $f(t)$ is finite$\}$. We define $\frac{1}{\omega}\sum_{t \in T} \varphi = \{\bar{x} \in {}^*\Omega_n | \bar{x} \simeq \frac{1}{\omega} \sum_{t \in T} g(t)$ for some $g \in \eta_\varphi\}$.

<u>Theorem 2</u>. $\frac{1}{\omega}\sum_{t\in T}\varphi$ is S-convex.

<u>Proof</u>: Suppose $\bar{x}, \bar{y} \in \frac{1}{\omega}\sum_{t\in T}\varphi$, then $\exists$ infinitesimal vector measures $\bar{\mu}_v$, $\bar{\mu}_n$ such that $\sum_{t\in T}\bar{\mu}_v(t) \simeq \bar{x}$ and $\sum_{t\in T}\bar{\mu}_n \simeq \bar{y}$, where $\mu_v = \frac{1}{\omega} g$ and $\mu_n = \frac{1}{\omega} f$ for some $g, f \in \eta_\varphi$. Consider the infinitesimal measure $\bar{\mu} = \langle \bar{\mu}_v, \bar{\mu}_n \rangle$ on $\mathcal{J}(T)$. By Theorem 1 it is S-convex and $\bar{\mu}(\emptyset) \simeq \langle \bar{0}, \bar{0}\rangle$, $\bar{\mu}(T) \simeq \langle \bar{x}, \bar{y}\rangle$. Hence for any $\lambda$, $0 \leq \lambda \leq 1$, $(\exists S \in \mathcal{J}(T))$ such that $\bar{\mu}(S) \simeq \lambda\bar{\mu}(T) = \langle \lambda\bar{x}, \lambda\bar{y}\rangle$. Therefore $\bar{\mu}(T-S) \simeq (1-\lambda)\bar{\mu}(T) = \langle (1-\lambda)\bar{x}, (1-\lambda)\bar{y}\rangle$. Define the infinitesimal vector measure

$$\bar{\mu}' = \begin{cases} \bar{\mu}_v & \text{for } t \in S \\ \bar{\mu}_n & \text{for } t \in T-S \end{cases} \quad \text{then } \bar{\mu}'(T) = \bar{\mu}_v(S) + \bar{\mu}_n(T-S) \simeq \lambda\bar{x} + (1-\lambda)\bar{y}$$

which is a finite vector. Hence $\bar{\mu}' = \frac{h}{\omega}$, where $h = \begin{cases} g & \text{for } t \in S \\ f & \text{for } t \in T-S, \end{cases}$ which implies that $\lambda\bar{x} + (1-\lambda)\bar{y} \in \frac{1}{\omega}\sum_{t\in T}\varphi$.

$\varphi$ is said to be internal if the graph of $\varphi$, $\{\langle t, \bar{x}\rangle \mid t \in T, \bar{x} \in \varphi(t)\}$ is internal, $\varphi$ is said to be internally bounded if there exists an internal function $g \in {}^*\Omega_n^T$ such that for all $t \in T$ and for every $\bar{x} \in \varphi(t)$, $\bar{x} \leq g(t)$, and $g(t)$ finite for all $t \in T$.

<u>Theorem 3</u>. If $\varphi$ is internal, internally bounded, and for all $t \in T$, $\varphi(t) \neq \emptyset$, then $\frac{1}{\omega}\sum_{t\in T}\varphi(t) \neq \emptyset$.

Proof: By transfer there exists an internal selection, $f$, where $f(t) \in \varphi(t)$ for all $t \in T$. Since $\varphi$ is internally bounded, $f(t)$ is finite for all $t \in T$.

We will need several notions of convergence and sequential compactness. Let $\varphi$ be a sequence in ${}^*U$ taking values in ${}^*R_n$. That is $\varphi : {}^*N \to {}^*R_n$. If $\varphi$ is internal, then $\bar{x}$ is said to be a Q-Lim of $\varphi$ or $\varphi$ Q-converges to $\bar{x}$ iff $(\forall \delta \in {}^*R, \delta > 0)(\exists n \in {}^*N)(\forall m \in {}^*N) m \geq n \implies |\varphi(m) - \bar{x}| < \delta$, $\bar{x}$ is said to be the F-Lim of $\varphi$ or $\varphi$ F-converges to $\bar{x}$ iff $(\forall \delta \in R, \delta > 0)(\exists n \in N)(\forall m \in N) m \geq n \implies |\varphi(m) - \bar{x}| < \delta$. Note that the definition of F-convergence does not require that $\varphi$ be an internal sequence.

A subset $B$ of ${}^*R_n$ is said to be Q-sequentially compact if every internal sequence of elements in $B$ has an internal subsequence which Q-converges to a point in $B$. A subset $B$ of ${}^*R_n$ is said to be a Q-compact set if every internal cover of Q-open subsets has an internal star finite subcover. The Q and S topologies are defined in [7].

A subset $B$ of ${}^*R_n$ is said to be F-sequentially compact if every sequence of elements in $B$ has a subsequence which F-converges to a point in $B$. $B$ is S-bounded if $(\exists r \in R)(\forall \bar{x} \in B)|\bar{x}| < r$.

We note that a set $B$ is S-closed iff it is closed under F-Limits.

Theorem 4. If $B$ is a S-convex, S-closed, and S-bounded subset of ${}^*R_n$; and $\varphi \in \mathcal{P}(B)^B$ is S-convex, S-closed, and nonvoid. Then there exists a $b \in B$ such that $b \in \varphi(b)$.

Proof: Let ${}^{\circ}\varphi : {}^{\circ}B \to \mathcal{P}({}^{\circ}B)$, where ${}^{\circ}\varphi(b) = {}^{\circ}(\varphi(b))$. Then ${}^{\circ}\varphi$ has a fixed point by the Kakutani fixed point theorem, call it $b$. Therefore $b + \delta \in \varphi(b)$, where $\delta \simeq 0$. But $\varphi(b)$ is S-closed, hence $b \in (b)$.

Theorem 5. (Robinson) Let $\{A_t\}_{t \in T}$ be an internal family of nonempty subsets of ${}^*R_n$ and $B = \underset{t \in T}{X_{\mathfrak{J}}} A_t$, the internal set of internal selections from the $A_t$. Suppose $\{\bar{y}_t\}_{t \in T}$ an internal function such that $(\forall t \in T)(\exists \bar{z}_t \in A_t)$ $\bar{z}_t \simeq \bar{y}_t$, then there exists $g \in B$ such that $\forall t \in T$, $g(t) \simeq \bar{y}_t$.

Proof: The following sentence is true in our standard universe, $U$, for every positive $\delta \in R$: $(\forall T \subset N)(\forall \{A_t\}_{t \in T})[(f \in R_n^T) \wedge (\forall t \in T)(\exists b_t \in A_t)(|f(t) - b_t| < \delta) \Longrightarrow (\exists g \in \underset{t \in T}{X} A_t)(\forall t \in T)(|g(t) - f(t)| < \delta)]$. Hence this sentence is true when translated into ${}^*U$, our nonstandard universe. This implies that for every $n \in N$, $\exists g_n \in B$ such that $\forall t \in T |g_n(t) - \bar{y}_t| < 1/n$. Hence we have a sequence $\varphi : N \to B$ s.t. $\varphi(n) = g$. Extend to $\rho : {}^*N \to B$. Hence $\exists \nu \in {}^*N - N$ s.t. $g_\nu \in B$ and $|g_\nu(t) - \bar{y}_t| < 1/\nu$.

## III. Definitions and Assumptions

The nonstandard exchange economy, $\mathcal{E}_{\omega}$, we will consider is assumed to have the following properties:

(i) The function indexing the initial endowments, $I(t)$, is internal.

(ii) $I(t)$ is standardly bounded, i.e. there exists a standard vector $\bar{r}_0$ such that for all $t$, $I(t) \leq \bar{r}_0$. $I(t) \not\gtrsim \bar{0}$, i.e. $I(t)$ has at least one non-infinitesimal component.

(iii) $\frac{1}{\omega} \sum_{t \in T} I(t) \not\gg \bar{0}$, i.e., each component of $\frac{1}{\omega} \sum_{t \in T} I(t)$ is noninfinitesimal.

(iv) The relation, $Q$, where $Q = \{\langle t, >_t \rangle | t \in T, >_t \subseteq {}^*\Omega_n \times {}^*\Omega_n\}$ is internal. For all $t$:

(α) $>_t$ is a partial order

(β) If $\bar{x} \geq \bar{y}$ then $\bar{x} >_t \bar{y}$

(γ) $>_t$ is Q-continuous, i.e. for all $\bar{x}, \bar{y} \in \Omega_n$ $\{\bar{z} \in {}^*\Omega_n | \bar{z} >_t \bar{y}\}$ and $\{\bar{z} \in {}^*\Omega_n | \bar{x} >_t \bar{z}\}$ are Q-open subsets

(δ) For all $\bar{x}, \bar{y} \in {}^*\Omega_n$, if $\bar{x} \not\simeq \bar{y}$ and $\bar{x} > \bar{y}$ then for all $\bar{w} \in \mu(\bar{x})$ and $\bar{v} \in \mu(\bar{y})$, $\bar{w} >_t \bar{v}$.

$\bar{x} \gg_t \bar{y}$ iff $\bar{w} \in \mu(\bar{x})$, $\bar{v} \in \mu(\bar{y})$ implies that $\bar{w} >_t \bar{v}$. Note that this definition differs from [3].

An <u>assignment</u> is an internal function from $T$, the set of traders, into ${}^*\Omega_n$.

An <u>allocation</u> is a standardly bounded assignment $Y(t)$ from the set of traders $\{1, 2, \ldots, \omega\}$ into ${}^*\Omega_n$ such that $\frac{1}{\omega} \sum_{t=1}^{\omega} Y(t) \simeq \frac{1}{\omega} \sum_{t=1}^{\omega} I(t)$.

A <u>price vector</u>, $\bar{p}$, is a finite nonstandard vector such that $\bar{p} \not\gtrsim \bar{0}$.

The $t^{th}$ traders budget set, $B_{\overline{p}}(t)$, is $\{\overline{x} \in {}^*\Omega_n \mid \overline{p}\cdot\overline{x} \lesssim \overline{p}\cdot I(t)\}$.

$\overline{y}$ is said to be maximal in $B_{\overline{p}}(t)$ if $\overline{y} \in B_{\overline{p}}(t)$ and there does not exist an $\overline{x} \in B_{\overline{p}}(t)$ such that $\overline{x} \gg_t \overline{y}$.

A <u>competitive equilibrium</u> is defined as a pair $\langle\overline{p}, X\rangle$, where $\overline{p}$ is a price vector and $X$ an allocation such that there exists an internal set of traders $K$ where $|K|/\omega \simeq 1$; $X(t)$ is maximal in $B_{\overline{p}}(t)$ for all $t \in K$.

That the above assumptions are consistent follows from the consistency lemma proved in [3].

## IV. Theorems

$$\tilde{P} = \{\bar{p} \in {}^*\Omega_n \mid \sum_{i=1}^{n} p_i \simeq 1\}, \quad P = \{\bar{p} \in {}^*\Omega_n \mid \sum_{i=1}^{n} p_i = 1\}$$

$$\tilde{M} = \{\bar{x} \in {}^*\Omega_n \mid \bar{x} \lesssim K(\sum_{j=1}^{n} \frac{1}{\omega} \sum_{t \in T} I^j(t))\bar{e}\}, \quad \text{where } \bar{e} = (1, 1, \ldots, 1) \text{ and } K \text{ a standard integer}$$

$$M = \{\bar{x} \in {}^*\Omega_n \mid \bar{x} \leq K(\sum_{j=1}^{n} \frac{1}{\omega} \sum_{t \in T} I^j(t))\bar{e}\}$$

$$\tilde{B}_{\bar{p}}(t) = \{\bar{x} \in {}^*\Omega_n \mid \bar{p}\cdot\bar{x} \lesssim \bar{p}\cdot I(t)\}, \quad B_{\bar{p}}(t) = \{\bar{x} \in {}^*\Omega_n \mid \bar{p}\cdot\bar{x} \leq \bar{p}\cdot I(t)\}$$

$$\tilde{C}_{\bar{p}}(t) = \tilde{B}_{\bar{p}}(t) \cap \tilde{M}, \quad C_{\bar{p}}(t) = B_{\bar{p}}(t) \cap M$$

A K-bounded partial competitive equilibrium is a pair $\langle \bar{p}, X \rangle$ where $\bar{p} \gtrsim 0$, $X$ is an allocation, and for each $t$ such that $\bar{p}\cdot I(t) \gtrsim 0$ the point $X(t)$ is maximal with respect to $t$ in the "K-bounded budget set" $\tilde{C}_{\bar{p}}(t)$. By maximal in $\tilde{C}_{\bar{p}}(t)$ we mean maximal with respect to $\gg_t$.

Principal Lemma. For all $K \in N$, if $K > 1$ then under the assumptions of Section III, there is a K-bounded partial competitive equilibrium.

Proof: Let

$$\tilde{D}_{\bar{p}}(t) = \begin{cases} \{\bar{x} \in \tilde{C}_{\bar{p}}(t) \mid \bar{x} \text{ is maximal in } \tilde{C}_{\bar{p}}(t)\} & \text{if } \bar{p}\cdot I(t) \gnsim 0 \\ \tilde{C}_{\bar{p}}(t) & \text{if } \bar{p}\cdot I(t) \simeq 0 \end{cases}$$

$$D_{\bar{p}}(t) = \begin{cases} \{\bar{x} \in C_{\bar{p}}(t) \mid \forall \bar{y} \in C_{\bar{p}}(t), \text{ not } \bar{y} >_t \bar{x}\} & \text{if } \bar{p}\cdot I(t) > 0 \\ C_{\bar{p}}(t) & \text{if } \bar{p}\cdot I(t) = 0 . \end{cases}$$

Next we shall define the set-valued functions

$$\varpi : M \to P , \quad \tilde{\varpi} : \tilde{M} \to \tilde{P} , \quad \tilde{\psi} : \tilde{P} \to \tilde{M} , \quad \psi : P \to M$$

$$\tilde{\varpi}(\bar{x}) = \{\bar{p} \in \tilde{P} \mid \forall \bar{q} \in \tilde{P},\ \bar{p}\cdot(\bar{x} - \frac{1}{\omega} \sum_{t\in T} I(t)) \gtrsim \bar{q}\cdot(\bar{x} - \frac{1}{\omega} \sum_{t\in T} I(t))\}$$

$$\varpi(\bar{x}) = \{\bar{p} \in P \mid \forall \bar{q} \in P,\ \bar{p}\cdot(\bar{x} - \frac{1}{\omega} \sum_{t\in T} I(t)) \geq \bar{q}\cdot(\bar{x} - \frac{1}{\omega} \sum_{t\in T} I(t))\}$$

$$\tilde{\psi}(\bar{p}) = \{\bar{x} \in {}^*\Omega_n \mid \exists \text{ an assignment } X(t),\ \bar{x} \simeq \frac{1}{\omega} \sum_{t\in T} X(t) \text{ and } \forall t \in T,\ X(t) \in \tilde{D}_{\bar{p}}(t)\}$$

$$\psi(\bar{p}) = \{\bar{x} \in {}^*\Omega_n \mid \exists \text{ an assignment } X(t),\ \bar{x} = \frac{1}{\omega} \sum_{t\in T} X(t) \text{ and } \forall t \in T,\ X(t) \in D_{\bar{p}}(t)\}$$

Finally we define $\tilde{\theta} : \tilde{P} \times \tilde{M} \to \tilde{P} \times \tilde{M}$ as $\tilde{\theta}(\bar{p},\bar{x}) = \tilde{\varpi}(\bar{x}) \times \tilde{\psi}(\bar{p})$. We shall show that $\tilde{\theta}$ fulfills the conditions of Theorem II.4 and consequently has a fixed point. Suppose $\langle \bar{p},\bar{x}\rangle$ is a fixed point of $\tilde{\theta}$, then there exists an assignment $X(t)$ such that $\bar{x} \simeq \frac{1}{\omega} \sum_{t\in T} X(t)$ and $X(t) \in \tilde{D}_{\bar{p}}(t)$. We need only show that $\frac{1}{\omega} \sum_{t\in T} X(t) \simeq \frac{1}{\omega} \sum_{t\in T} I(t)$ to complete the proof.

$X(t) \in \tilde{D}_{\bar{p}}(t) \subset \tilde{C}_{\bar{p}}(t) \subset \tilde{B}_{\bar{p}}(t)$ implies that $\bar{p}\cdot X(t) \lesssim \bar{p}\cdot I(t)$ . We shall show that $\bar{p}\cdot X(t) \simeq \bar{p}\cdot I(t)$ . Suppose $\bar{p}\cdot X(t) \lneq \bar{p}\cdot I(t)$ and $\bar{p}\cdot I(t) \gneq 0$ . Then $X(t)$ maximal with respect to $\gg_t$ in $\tilde{C}_{\bar{p}}(t)$ . $K > 1$ implies that $K(\sum_{j=1}^{n} I^j(t))\bar{e} \ggneq I(t)$ since $\forall t \in T$ , $I(t) \gneq \bar{0}$ . $\exists i$ such that $p^i \gneq 0$ and $K(\sum_{j=1}^{n} I^j(t))\bar{e} \gneq X^i(t)$ . Because if not $\bar{p}\cdot K(\sum_{j=1}^{n} I^j(t))\bar{e} \simeq \bar{p}\cdot X(t)$ which contradicts $K(\sum_{j=1}^{n} I^j(t))\bar{e} \ggneq I(t)$ . Therefore $\exists \epsilon \gneq 0$ such that $p^i\epsilon \lesssim \bar{p}\cdot I(t) - \bar{p}\cdot X(t)$ and $\epsilon \lesssim K(\sum_{j=1}^{n} I^j(t)) - X^i(t)$ . Consider $X(t) + \epsilon\bar{e}_i$ , where $\bar{e}_i$ has a 1 in the $i^{th}$ place and 0 elsewhere, then $\bar{p}\cdot(X(t) + \epsilon\bar{e}_i) \lesssim \bar{p}\cdot I(t)$ and $X(t) + \epsilon\bar{e}_i \lesssim K(\sum_{j=1} I^j(t))\bar{e}$ . But $X(t) + \epsilon\bar{e}_i \in \tilde{C}_{\bar{p}}(t)$ , and $X(t) + \epsilon\bar{e}_i \gg_t X(t)$ which contradicts the maximality of $X(t)$ . Therefore we have shown that $\bar{p}\cdot X(t) \simeq \bar{p}\cdot I(t)$ .

Let $b = \frac{1}{\omega} \sum_{t\in T} X(t) - \frac{1}{\omega} \sum_{t\in T} I(t)$ , then $\bar{p}\cdot\bar{b} \simeq 0$ . Now by the definition of $\tilde{\omega}$ , $(\forall q \in \tilde{P}) 0 \simeq \bar{p}\cdot\bar{b} \gtrsim \bar{q}\cdot\bar{b}$ . Let $\bar{q} = \bar{e}_i$ , then $0 \gtrsim \bar{e}_i\cdot\bar{b} = b^i$ , i.e. $(\forall i) b^i \lesssim 0$ . If $b^i \lneq 0$ , then $p^i \simeq 0$ ; since if $(\exists i)$ such that $b_i \lneq 0$ and $p^i \gneq 0$ this would imply that $\bar{p}\cdot\bar{b} \lneq 0$ , a contradiction. For each $i$ such that $b^i \lneq 0$ , let $S_i^n = \{t \mid I^i(t) - X^i(t) \geq \frac{1}{n}\}$ and $S_i = \bigcup_{n\in N} S_i^n$ .

Suppose $(\forall n \in N)|S_i^n|/\omega \simeq 0$ , then $(\exists n_0 \in N)$ such that except for at most a negligible set of $t \in T$ , $I^i(t) - X^i(t) < \frac{1}{n_0} \lneq -b^i$ . Hence

$\frac{1}{\omega} \sum_{t \in T} (X^i(t) - I^i(t)) \gtrsim -\frac{1}{n_0} \not\gtrsim b^i$ , but $b^i = \frac{1}{\omega} \sum_{t \in T} X^i(t) - I^i(t)$ , i.e. $b^i \not\gtrsim b^i$ , a contradiction. Since $S_i^n \subseteq S_i^{n+1}$ for all $n \in N$ and $|S_i^m|/\omega \not\simeq 0$ for some $m \in N$ , it follows that there exists $\nu_i \in {}^*N - N$ such that $S_i^{\nu_i} = \{t | I^t(t) - X^i(t) \geq \frac{1}{\nu_i}\}$ , where $S_i^{\nu_i}$ internal, $|S_i^{\nu_i}|/\omega \not\simeq 0$ , $\bar{p} \cdot I(t) \simeq 0$ for all $t \in S_i^{\nu_i}$ , and $S_i \subseteq S_i^{\nu_i}$ . Let $c^i = \frac{1}{\omega} \sum_{t \in S_i^{\nu_i}} (I^i(t) - X^i(t))$

now $b^i = \frac{1}{\omega} \sum_{t \in T} (X^i(t) - I^i(t)) \not\lesssim 0$ implies that $|b^i| = -b^i$ . But

$$-b^i = \frac{1}{\omega} \sum_{t \in S_i^{\nu_i}} (I^i(t) - X^i(t)) + \frac{1}{\omega} \sum_{t \in T/S_i^{\nu_i}} (I^i(t) - X^i(t)) \simeq \frac{1}{\omega} \sum_{t \in S_i^{\nu_i}} I^i(t) - X^i(t) = c^i .$$

Therefore $|b^i| \simeq c^i$ .

$$\text{Let } Y^i(t) = \begin{cases} X^i(t) + \frac{|b^i|}{c^i} (I^i(t) - X^i(t)) , & t \in S_i^{\nu_i} \\ X^i(t) , & t \notin S_i^{\nu_i} \end{cases} ; \quad Y^j(t) = X^j(t) \text{ for } j \neq i$$

Since $p^i \simeq 0$ , it follows that $\bar{p} \cdot Y(t) \simeq \bar{p} \cdot X(t) \simeq \bar{p} \cdot I(t)$ . Hence $Y(t) \in \tilde{B}_{\bar{p}}(t)$ . $|b^i| \simeq c^i$ implies that $|b^i|/c^i \simeq 1$ and therefore $Y^i(t) \simeq I^i(t) \not\lesssim K(\sum_{j=1} I^j(t))$ . Hence $Y(t) \in \tilde{C}_{\bar{p}}(t)$ . Since $\frac{1}{\omega} \sum_{t \in T} Y(t) \simeq \frac{1}{\omega} \sum_{t \in T} I(t)$ , we see that $Y(t)$ is a competitive allocation.

In order to apply our fixed-point theorem, Theorem II.4, we must show that $\tilde{\mathcal{P}} \times \tilde{M}$ is S-convex, S-closed, and S-bounded; and that $\tilde{\mathcal{G}}$ is S-closed, S-convex, and nonvoid. That $\tilde{\mathcal{P}}$, $\tilde{M}$ are S-convex, S-closed, and S-bounded is immediate, hence their product has the required properties. It is sufficient to show that $\tilde{\varphi}$ and $\tilde{\psi}$ separately satisfy the conditions of Theorem II.4.

From transfer it follows that $\varphi(\bar{x})$ is nonvoid not only for all $\bar{x} \in M$ but also for all $\bar{x}$ in $\tilde{M}$. Since $\varphi(\bar{x}) \subseteq \tilde{\varphi}(\bar{x})$, we see that $\tilde{\varphi}(\bar{x})$ is nonvoid. The S-convexity of $\tilde{\varphi}(\bar{x})$ is obvious. We now show that $G_{\tilde{\varphi}}$ is S-closed. Suppose $\langle \bar{x}_n, \bar{p}_n \rangle \in G_{\tilde{\varphi}}$, and $\langle \bar{x}_n, \bar{p}_n \rangle$ $\xrightarrow{\mathcal{F}\text{-Lim}} \langle \bar{x}, \bar{p} \rangle$.

Suppose $\bar{p} \notin \tilde{\varphi}(\bar{x})$, then $\exists \bar{q} \in \tilde{\mathcal{P}}$ such that $\bar{p} \cdot (\bar{x} - \frac{1}{\omega} \sum_{t \in T} I(t)) \not\gtrsim \bar{q} \cdot (\bar{x} - \frac{1}{\omega} \sum_{t \in T} I(t))$. Hence $\exists m \in N$ such that $\bar{p}_m \cdot (\bar{x}_m - \frac{1}{\omega} \sum_{t \in T} I(t)) \not\gtrsim \bar{q} \cdot (\bar{x}_m - \frac{1}{\omega} \sum_{t \in T} I(t))$ which contradicts the definition of the sequence $\langle \bar{x}_n, \bar{p}_n \rangle$.

<u>Lemma 1</u>. $(\forall t \in T)\bar{p} \cdot I(t) \not\gtrsim 0 \Longrightarrow \tilde{D}_{\bar{p}}(t) \neq \emptyset$.

<u>Proof</u>: If $\bar{p} \cdot I(t) \not\gtrsim 0$ then $D_{\bar{p}}(t) \neq \emptyset$ by transfer. But $D_{\bar{p}}(t) \subseteq \tilde{D}_{\bar{p}}(t)$. Consequently $\tilde{D}_{\bar{p}}(t) \neq \emptyset$.

<u>Lemma 2</u>. $\tilde{D}_{\bar{p}}(t)$ is S-closed.

Proof: Let $\langle \bar{p}_n, \bar{x}_n \rangle \xrightarrow{F\text{-Lim}} \langle \bar{p}, \bar{x} \rangle$, $\bar{x}_n \in \tilde{D}_{\bar{p}_n}(t)$. Therefore $\bar{x}_n \in \tilde{C}_{\bar{p}_n}(t)$ which implies that $\bar{x} \in \tilde{C}_{\bar{p}}(t)$. If $\bar{p}\cdot I(t) \simeq 0$, then $\bar{x} \in \tilde{D}_{\bar{p}}(t)$. So suppose $\bar{p}\cdot I(t) \not\gtrsim 0$. Hence $(\exists n_0 \in N)(\forall n \in N) n \geq n_0$ implies that $\bar{p}_n \cdot I(t) \not\gtrsim 0$ which implies that $\bar{x}_n$ is maximal in $\tilde{C}_{\bar{p}}(t)$. Suppose $(\exists \bar{y} \in \tilde{C}_{\bar{p}}(t))\bar{y} \gg_t \bar{x}$, then $(\exists \bar{z} \in {}^*\Omega_n)\bar{z} \gg_t \bar{x} \wedge \bar{p}\cdot\bar{z} \not\lesssim \bar{p}\cdot I(t)$. So there is an $n_1 \in N$ such that $n \geq n_1$ implies that $\bar{p}_n \cdot \bar{z} \lesssim \bar{p}_n \cdot I(t)$ and there is an $n_2 \in N$ such that $n \geq n_2$ implies $\bar{z} \gg_t \bar{x}_n$. Let $m = \max\{n_0, n_1, n_2\}$, then $\bar{x}_m$ cannot be maximal in $\tilde{C}_{\bar{p}_m}(t)$, a contradiction.

Lemma 3. $\mathfrak{F}(\bar{p}) \neq \emptyset$ and $\mathfrak{F}(\bar{p})$ is S-convex.

Proof. $\mathfrak{F}(\bar{p}) = \frac{1}{\omega} \sum_{t \in T} \tilde{D}_{\bar{p}}(t)$, hence $\mathfrak{F}(\bar{p})$ is S-convex by Theorem II.2. $\phi(\bar{p})$ is defined for $\bar{p} \in \tilde{P}$ and $\phi(\bar{p}) \subseteq \mathfrak{F}(\bar{p})$ for all $\bar{p} \in \tilde{P}$. Since $\phi(\bar{p})$ is nonvoid by Theorem II.3, hence $\mathfrak{F}(\bar{p}) \neq \emptyset$.

Lemma 4. $\mathfrak{F}(\bar{p})$ is S-closed.

Proof: Given $\{\langle \bar{p}_n, \bar{x}_n \rangle\}_{n \in N}$, where $\bar{x}_n \in \phi(\bar{p}_n) = \frac{1}{\omega} \sum_{t \in T} \tilde{D}_{\bar{p}_n}(t)$, i.e., $(\forall n \in N)\bar{x}_n \simeq \frac{1}{\omega} \sum_{t \in T} X_n(t)$. By Theorem II.5 without loss of generality we may assume that $\forall n \in N$, $X_n(t) \in \mathop{X}_{t \in T} C_{\bar{p}_n}(t)$, the internal set of internal selections from the $C_{\bar{p}_n}(t)$. Extending the sequence $\{\langle \bar{p}_n, X_n(t) \rangle\}_{n \in N}$ to $\{\langle \bar{p}_n, X_n(t) \rangle\}_{n \in {}^*N}$, there exists a $\nu_1 \in {}^*N - N$ such that $(\forall \omega \in {}^*N)\omega \leq \nu_1 \Longrightarrow X(t) \in \mathop{X}_{t \in T} C_{\bar{p}}(t)$. Now suppose that $\langle \bar{p}_n, \bar{x}_n \rangle \xrightarrow{F\text{-Lim}} \langle \bar{p}, \bar{x} \rangle$, then $(\exists \nu_2 \in {}^*N - N)(\forall \rho \in {}^*N - N)\rho \leq \nu_2 \Longrightarrow \bar{P}_\rho \simeq \bar{P} \quad \bar{x} \simeq \frac{1}{\omega} \sum_{t \in T} X_\rho(t)$ .

Let $\nu_3 = \min\{\nu_1, \nu_2\}$ and $\varphi : {}^*N \to ({}^*N)$, where $\varphi(n) = \{\rho \in {}^*N | \rho \leq \nu_3,$ $(\exists t \in T)[(\exists \bar{y} \in C_{\bar{p}_\rho}(t))(\forall \bar{w} \in S_{1/n}(\bar{y}) \wedge \forall \bar{z} \in S_{1/n}(X_\rho(t)))\bar{w} >_t \bar{z} \wedge \bar{p}_\rho \cdot I(t)$ $\geq 1/n]$ . This is an internal mapping, hence $\theta : {}^*N \to {}^*N$ where $\theta(n) = \min\{\nu | \nu \in \varphi(n)\}$ is an internal mapping. Finally consider $g : {}^*N \to {}^*N$, where $g(n) = 1/\theta(n)$ . Since $(\forall n \in N)g(n) \simeq 0$, there exists a $\xi \in {}^*N - N$ such that $(\forall \nu \in {}^*N)\nu \leq \xi \Longrightarrow g(\nu) \simeq 0$ . That is $\theta(n) \in {}^*N - N$ for all $n \leq \xi$ . Hence $(\exists \xi, \rho \in {}^*N - N)(\forall t \in T)[(\forall \bar{y} \in \tilde{C}_{\bar{p}_\rho}(t))$ $(\exists \bar{w} \in S_{1/\xi}(\bar{y}) \wedge \exists \bar{z} \in S_{1/\xi}(X_\rho(t)))\bar{w} \not>_t \bar{z} \vee \bar{p}_\rho \cdot I(t) < 1/\xi]$ . In addition $\bar{p}_\rho \simeq \bar{p}$, $\bar{x} \simeq \frac{1}{\omega} \sum_{t \in T} X_\rho(t)$, and $X_\rho(t) \in \underset{t \in T}{X} C_{\bar{p}_\rho}(t)$ . Clearly $X_\rho(t) \in \underset{t \in T}{X} \tilde{D}_{\bar{p}_\rho}(t)$ . But by Lemma 2, $(\forall t \in T)\tilde{D}_{\bar{p}_\rho}(t)$ is S-closed. Therefore $\bar{p}_\rho \simeq \bar{p}$ implies that $X_\rho(t) \in \underset{t \in T}{X} \tilde{D}_{\bar{p}}(t)$, i.e. $\bar{x} \in \frac{1}{\omega} \sum_{t \in T} \tilde{D}_{\bar{p}}(t)$ .

<u>Theorem</u>. Under the assumptions of Section III, $\mathcal{E}_\omega$ has a competitive equilibrium.

<u>Proof</u>: $\tilde{P} = \{\bar{p} \in {}^*\Omega_n | \sum_{i=1}^{n} p_i \simeq 1\}$ is S-closed and S-bounded, hence every sequence $\{\bar{p}_k\}_{k \in N}$ in $\tilde{P}$ has a subsequence which F-converges to some point in $\tilde{P}$ . In particular consider $\{\langle \bar{p}_k, Y_k \rangle\}_{k \in N, k > 1}$, where $\langle \bar{p}_k, Y_k \rangle$ is a k-bounded partial competitive equilibrium. Since $\bar{p}_k \in \tilde{P}$ for each $k$, we can without loss of generality assume that $\bar{p}_k \xrightarrow{\text{F-Lim}} \bar{p}$, where $\bar{p} \in \tilde{P}$ .

Let $A_k^n = \{t \in T | \exists \bar{z} \in C_{\bar{p}_k}(t))(\forall \bar{w} \in S_{1/n}(\bar{z}) \wedge \forall \bar{z} \in S_{1/n}(Y_k(t)))\bar{w} >_t \bar{z}$ $\wedge \bar{p}_k \cdot I(t) > 1/n\}$ . Then $(\forall k, n \in N)$ $A_n^k$ is internal, $A_k^n \subseteq A_k^{n+1}$, and

$A_k^n$ negligible. Therefore $(\forall k \in N)(\exists \nu_k \in {}^*N - N)A_k^{\nu_k}$ is negligible.

Let $B_n = \bigcup_{k=2}^{n} A_k^{\nu_k}$, then $(\forall n \in N)B_n$ is internal, $B^n \subseteq B^{n+1}$, and $B^n$ negligible. Hence there exists an internal $B$ such that $\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_k^n \subseteq B$ and $|B|/\omega \simeq 0$. Let $V = T/B$.

Lemma. $\bar{p} \not\gg \bar{0}$.

Proof: Let $S_n = \{t \in V | \bar{p} \cdot I(t) \geq 1/n\}$ for $n \in N$. Suppose $S_n$ is negligible for all $n \in N$, then $\exists \nu \in {}^*N - N$, $\bigcup_{n \in N} S_n \subseteq S_\nu$ and $|S_\nu|/\omega \simeq 0$. Therefore $\frac{1}{\omega} \sum_{t \in T} \bar{p} \cdot I(t) = \frac{1}{\omega} \sum_{t \in S_\nu} \bar{p} \cdot I(t) + \frac{1}{\omega} \sum_{t \in T/S_\nu} \bar{p} \cdot I \simeq 0$, which contradicts $\frac{1}{\omega} \sum_{t \in T} I(t) \not\gg 0$. Consequently $(\exists n_0 \in N)(\forall n \in N) n \geq n_0 \Longrightarrow |S_n|/\omega \not\gtrsim 0$.

Let $\alpha = \frac{2}{|S_{n_0}|} \times \sum_{t \in T} (\sum_{j=1}^{n} I^j(t))$, $A = \{\bar{x} \in {}^*\Omega_n | \bar{x} \leq \alpha \bar{e}\}$. Let $A_k = \{t \in S_{n_0} | Y_k(t) \notin A\}$, then $A_k$ is internal, since $A$ and $Y_k(t)$ are internal entities. Suppose $\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \bar{A}_k = \emptyset$. This implies that $(\forall t \in S_{n_0})(\exists k_t \in N)(\forall k \in N) k \geq k_t \Longrightarrow Y_k(t) \notin A$. Consider the map $\varphi : T \times N \to {}^*\Omega_n$, where $\varphi(t,k) = Y_k(t)$. Extend $\varphi$ to an internal map $\theta$ from $T \times {}^*N \to {}^*\Omega_n$. Since $(\forall n \in N) \frac{1}{\omega} \sum_{t \in T} \theta(n,t) \simeq \frac{1}{\omega} \sum_{t \in T} I(t)$ there exists a $\nu \in {}^*N - N$ such that for all $\xi \in {}^*N$ if $\xi \leq \nu$, then $\frac{1}{\omega} \sum_{t \in T} \theta(\xi, t) \simeq \frac{1}{\omega} \sum I(t)$.

Let $\psi(n,t) = \begin{cases} \theta(n,t) & \text{if } n < \nu \\ \theta(\nu,t) & \text{if } n \geq \nu \end{cases}$. For each $t \in S_{n_0}$, let $j_t = \min\{j \in {}^*N \mid \psi(j,t) \in A,\ j \geq K_t\}$. Then $\{j_t\}_{t \in S_{n_0}}$ is a star finite internal set and hence has a smallest element, call if $\nu_0 + 1$. Note that $\nu_0 + 1 \in {}^*N - N$. Let $X_n(t) = \begin{cases} \psi(n,t) & \text{for } n < \nu_0 \\ \psi(\nu_0, t) & \text{for } n \geq \nu_0 \end{cases}$. $X_n(t) \notin A$ implies that $\sum_{i=1}^{m} X_n^i(t) > \alpha$. Since for each $t \in S_{n_0}$, there exists $k_t \in N$ such that for all $k \in {}^*N$, if $k \geq k_t$ then $X_k(t) \notin A$; let $\ell = \max\{k_t \mid t \in S_{n_0}\}$ then we see that $\min\limits_{\ell \leq n \leq \nu_0} \sum_{i=1}^{m} X_n^i(t) \geq \alpha$ for all $t \in S_{n_0}$. Therefore

$$\frac{1}{\omega} \sum_{t \in S_{n_0}} \min_{\ell \leq n \leq \nu_0} \left( \sum_{i=1} X_n^i(t) \right) \geq \frac{|S_{n_0}|}{\omega} \alpha .$$

But

$$\frac{1}{\omega} \sum_{t \in S_{n_0}} \min_{\ell \leq n \leq \nu_0} \sum_{i=1} X_n^i(t) =$$

$$= \min_{\ell \leq n \leq \nu_0} \frac{1}{\omega} \sum_{t \in S_{n_0}} \left( \sum_{i=1} X_n^i(t) \right) \lesssim \min_{\ell \leq n \leq \nu_0} \frac{1}{\omega} \sum_{t \in T} \left( \sum_{i=1} X_n^i(t) \right) \simeq \min_{\ell \leq n \leq \nu_0} \frac{1}{\omega} \sum_{t \in T} \left( \sum_{i=1} I^i(t) \right)$$

$$= \frac{1}{\omega} \sum_{t \in T} \left( \sum_{i=1} I^i(t) \right) = \frac{\alpha |S_{n_0}|}{2\omega} .$$

Hence $\frac{1}{2} \gtrsim 1$ which contradicts our assumption that $\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \bar{A}_k = \emptyset$. So suppose $t_0 \in \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \bar{A}_k$, then $\{Y_k(t_0)\}$ has a F-limit point in $\tilde{A}$, where $\tilde{A} = \{\bar{x} \in {}^*\Omega \mid \bar{x} \lesssim \alpha\bar{e}\}$, since $\tilde{A}$ is F-sequentially compact. Call the limit point $\bar{y}_0$.

By Theorem III.5, without loss of generality, we may assume that $\forall n \in N$, $Y_n(t) \in X\mathcal{S}C_{\bar{p}_n}(t)$, the internal set of internal selections from the $C_{\bar{p}_n}(t)$. Extending the sequence $\{\langle \bar{p}_n, Y_n(t) \rangle\}_{n \in N}$ to $\{\langle \bar{p}_n, Y_n(t) \}_{n \in {}^*N}$ the following conditions hold:

(1) $(\exists \nu_1 \in {}^*N - N)(\forall \xi \in {}^*N)\xi \leq \nu_1 \Longrightarrow Y_\xi(t) \in X\, C_{\overline{p}_\xi}(t)$

(2) $(\exists \nu_2 \in {}^*N - N)(\forall \xi \in {}^*N)\xi \leq \nu_2 \Longrightarrow \frac{1}{\omega} \sum_{t\in T} Y_\xi(t) \simeq \frac{1}{\omega} \sum_{t\in T} I(t)$

(3) $(\exists \nu_3 \in {}^*N - N)(\forall \xi \in {}^*N - N)\xi \leq \nu_3 \Longrightarrow \overline{p}_\xi \simeq \overline{p}$

(4) $(\exists \nu_4, \rho \in {}^*N - N)(\forall \xi \in {}^*N)\xi \leq \nu_4 \Longrightarrow (\forall t \in V)[(\forall \overline{y} \in \tilde{C}_{\overline{p}_\xi}(t))$

$(\exists \overline{w} \in S_{1/\xi}(\overline{y}) \wedge \exists \overline{z} \in S_{1/\xi}(\overline{y}))\overline{w} \not\succ_t \overline{z} \wedge \overline{p}_\xi \cdot I(t) < 1/\rho$

(5) $(\forall t \in S_{n_0})(\exists \nu_{t_0} \in {}^*N - N)(\forall \xi \in {}^*N)\xi \leq \nu_4 \Longrightarrow \overline{p}_\xi \cdot Y_\xi(t) \simeq \overline{p}_\xi \cdot I(t)$

Let $\nu+1 = \min\{\nu_1, \nu_2, \nu_3, \nu_4, \nu_{t_0}\}$ and consider $\langle \overline{p}_\nu, Y_\nu \rangle$. Suppose for some $j$ that $p^j \simeq 0$, then $(\forall \omega \in {}^*N - N)\omega \leq \nu+1 \Longrightarrow p^j_\omega \simeq 0$. Also $(\forall \xi \in {}^*N - N)\xi \leq \nu+1 \Longrightarrow \overline{p}_\xi \cdot Y_\xi(t_0) \simeq \overline{p}_\xi \cdot I(t_0) \not\simeq 0$. $\overline{p}_\nu \simeq \overline{p}_{\nu+1}$ implies that $Y_\nu(t_0) \in \tilde{C}_{\overline{p}_{\nu+1}}(t_0)$, since $\forall t \in T$, $\tilde{C}_{\overline{p}_{\nu+1}}(t)$ is S-closed. Let $Z(t_0) = Y_\nu(t_0) + \beta\overline{e}_j$, then $Z(t_0) \in \tilde{C}_{\overline{p}_{\nu+1}}(t_0)$, where $\beta = \sum_{j=1}^{m} I^j(t_0)$. $Z(t_0) \gg_{t_0} Y_\nu(t_0)$, and $\overline{p}_{\nu+1} \cdot I(t_0) \simeq \overline{p} \cdot I(t_0) \not\simeq 0$, imply that $\exists \overline{w} \in \tilde{C}_{\overline{p}_{\nu+1}}(t_0)$ such that $\overline{w} \gg_{t_0} Y_\nu(t_0)$. But $Y_\nu(t_0) \simeq \overline{y}_0 \simeq Y_{\nu+1}(t_0)$ which contradicts condition (4) above. Hence $\overline{p} \gg\!\!\!\not\simeq \overline{0}$.

Therefore there is a $\delta \not\simeq 0$ such that for $K$ sufficiently large, say $K > K_0$, we have $p^i_K \gtrsim \delta$, $i = 1, 2, \ldots, n$. For such a $K$, for each $\overline{x} \in \tilde{B}_{\overline{p}_K}(t)$ we have $\delta x^i \lesssim p^i_K \cdot x^i \lesssim \overline{p}_K \cdot \overline{x} \lesssim \overline{p}_K \cdot I(t) \lesssim \sum_{i=1}^{n} I^i(t) \Longrightarrow x^i \lesssim \frac{1}{\delta} \sum_{i=1}^{n} I^i(t)$. Choose $K > 1/\delta$ and $K > K_0$; then $x^i \lesssim K \sum_{i=1}^{n} I^i(t) \Longrightarrow$

$\bar{x} \underset{\sim}{<} K(\sum_{i=1}^{n} I^i(t))\bar{e} \Longrightarrow \tilde{B}_{\bar{p}_K}(t) \subset \{\bar{x} \in \Omega_n | \bar{x} \underset{\sim}{<} K(\sum_{i=1}^{n} I^i(t))\bar{e}\}$ . We claim that $\langle\bar{p}_K, Y_K\rangle$ is a competitive equilibrium. Suppose $\bar{p}_K \cdot I(t) \not\gtrsim 0$ , then $Y_K(t)$ is maximal with respect to $\gg_t$ in $\tilde{C}_{\bar{p}_K}(t)$ , according to the definition of K-bounded partial competitive equilibrium. But we showed above that $\tilde{B}_{\bar{p}_K}(t) = \tilde{C}_{\bar{p}_K}(t)$ . If $\bar{p}_K \cdot I(t) \simeq 0$ and $\bar{p}_K \not\gtrsim \bar{0}$ then $I(t) \simeq 0$ and so $\tilde{B}_{\bar{p}_K}(t)$ is $\mu(\bar{0})$ . Hence $D_{\bar{p}_K}(t) = \mu(\bar{0})$ and $Y_K(t) = \bar{0}$ . Since $\bar{0}$ is maximal with respect to $\gg_t$ in $\{0\}$ , the proof of the Theorem is complete.

## REFERENCES

[1] Aumann, R.J., "Existence of Competitive Equilibria in Markets with a Continuum of Traders," Econometrica, 34 (1966), 1-17.

[2] ________, "Integrals of Set-Valued Functions," Journal of Mathematical Analysis and Applications, 12 (1965), 1-12.

[3] Brown, D.J. and A. Robinson, "Nonstandard Exchange Economies," Econometrica (to appear).

[4] ________, "A Limit Theorem on the Cores of Large Standard Exchange Economies," Proc. Nat. Acad. Sci. U.S.A., Vol. 69, No. 5, pp. 1258-1260.

[5] Debreu, G., "Integration of Correspondences," Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, University Press, Berkeley (1968), pp. 351-372.

[6] Loeb, P., "A Combinatorial Analog of Lyapunov's Theorem for Infinitesimally Generated Atomic Vector Measures" (to appear).

[7] Robinson, A., Non-Standard Analysis, North Holland, 1966.

[8] Schmeidler, D., "Competitive Equilibria in Markets with a Continuum of Traders and Incomplete Preferences," Econometrica, 37 (1969), 578-585.