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DUOPOLY WITH PRICE AND QUANTITY AS STRATEGIC VARIABLES

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1. Introduction

In this paper we provide an explicit mixed strategy equilibrium solution for an oligopoly game. In the specification of the model, we assume that each firm has to make a decision on the production level while it names its prices, and we introduce a fixed unit cost for unsold inventory. Hence, both price and quantity appear as strategic variables.

This paper can be considered as a continuation of our effort initiated in [1] to find explicit mixed strategy solutions to duopoly games. In [1], for example, price was treated as the only decision while production was considered to be under a capacity constraint. In this paper both price and quantity are decision variables, but there is no capacity limitation. In both papers, demand is assumed to be linear and nonstochastic, and products are assumed to be identical.

It turns out that while we consider duopoly with a linear demand function, and identical inventory carrying cost, our solution can be easily generalized to the case of oligopoly with nonlinear demand, and asymmetric inventory costs. Further, the solution is not sensitive to the model of demand redistribution when one firm's product is in short supply.

In Section 2, we review our assumptions concerning interrelated demand and specify the model. In Section 3, we state the mixed strategy solution to

the price quantity game, leaving the detailed proof for the Appendix. In section 4, we examine some alternative specifications of the duopoly game model, including a dynamic sequential game and a minorant version where one firm must make its choices first. In Section 5, we extend our solution to the n firm situation where penalty costs may be asymmetric. In the concluding section, we discuss the realism of the model and suggest extensions.

2. Specification of Duopoly Model

In Figure 1 the line AD represents the overall market demand for the product. If one firm charges a lower price than the other we expect that it will obtain all of the demand as they are selling identical products. If it can satisfy this demand then there is nothing left for its competitor.
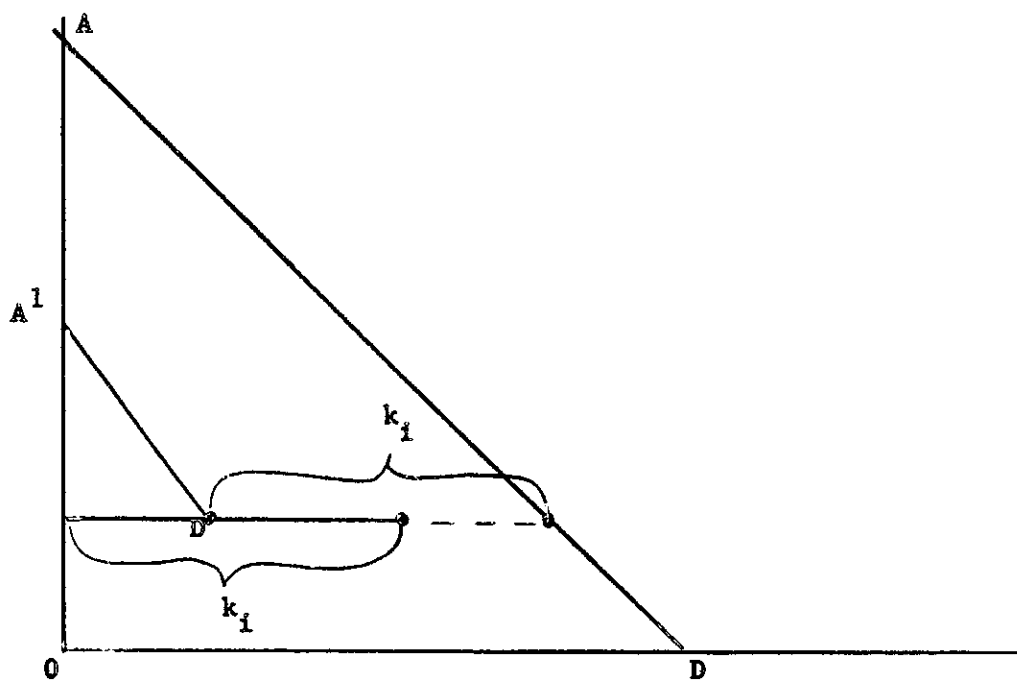

Figure 1

suppose, however that it runs out of inventory before it has satisfied all demand. What is left for its competitor? Looking at a relatively pessimistic possibility (from the viewpoint of the second firm) the demand to him is given by the line $A'D'$ which is obtained by shifting the original line AD in towards the left by a distance k_i where k_i is the level of inventories held by the lower priced player. Obviously if k_i were sufficient to supply all of the demand there would be nothing left for the individual with the higher price.

Mathematically we may describe demand under all circumstances as given

by:

$$
d_i = \begin{cases}
a - p_i & \text{when } p_i < p_j \quad \text{for } 0 \le p \le a \\
(a - p_i)/2 & \text{when } p_i = p_j \\
\max(a - k_j - p_i, 0) & \text{when } p_i > p_j .
\end{cases}
$$

Suppose that the firms had no capacity limitations and there were no costs to carrying inventories. For ease of argument we have selected an overall demand function of the form $d = a - p$ and furthermore we assume that there are no costs of production, i.e. $c_i = 0$. Neither of these restrictions is critical to the argument and they will be relaxed later.

If both firms had no capacity limitations and no inventory carrying costs then they each could always carry enough inventory to supply the whole market at any price. In particular if both were to carry an inventory of size $q_1 \ge a$ and $q_2 \ge a$ then, as with the unlimited capacity model of Bertrand [2] there would be a pure strategy equilibrium in the market at $p = 0$ with the firms splitting the market. In this case this is precisely the same price as the competitive equilibrium.

The profits made by each firm at the competitive equilibrium are zero. What happens to the equilibrium and the profits if we introduce an inventory carrying cost? Before we can investigate this we must completely specify our model.

Two firms compete in a market in which they are selling identical products. They each simultaneously select a price and an inventory level for

the period ahead. Production takes time, hence the inventory level selected is tantamount to a short term capacity constraint. All of the inventories are available for sale at the start of the period. There is an inventory cost of λ charged on each unit of inventory left in stock at the end of the period. There are several different ways of attaching an inventory cost which reflect the financing, physical handling and excess stock aspects of inventories.

Given inventory costs which vary as the size of the inventory, then even when there are no capacity limitations on the firms there can be no pure strategy equilibrium. This is true even if the products they are selling are differentiated. In particular this means that neither the Bertrand case nor the Chamberlinian large group equilibrium can exist if there are inventory carrying costs. The reason is quite simple. Assume the contrary. In which case with undifferentiated products both firms would be charging $p = 0$ and each would supply half of the market. If this were truly an equilibrium then each would cut down his inventories so that he had just enough to supply his half of the market. But if one firm knows that the other is charging $p = 0$ and has a supply of only $a/2$ it will face a contingent demand of $d = (a/2) - p$ if it raises its price. Hence it will pay it to raise its price to $p = a/4$ and there will be no equilibrium.

We have shown that the pure strategy equilibrium is destroyed by the introduction of inventory costs. Furthermore the effect of the inventory costs is to shade prices in an upward direction. What does it do to the profits of the firms? Is there a mixed strategy equilibrium to this game with price and quantity as simultaneous independent variables?

J. <u>Mixed-Strategy Solution to the Price-Quantity Game</u>

Two firms simultaneously name prices p_1 and p_2 for their products and select their supplies for the period at q_1 and q_2. The payoff (shown for Firm 1) is given by:

$$\Pi_1 = \begin{cases} p_1 \min[q_1, \ a - p_1] & \text{if } p_1 < p_2 \\ p_1 \max[0, \ \min(q_1, \ a - q_2 - p_1)] - \lambda \max[0, \ q_1 - (a - q_2 - p_1)] & \text{if } p_1 > p_2 \end{cases}$$

The case in which $p_1 = p_2$ is not formally written down. It plays no role in the determination of the mixed strategy equilibrium. For other purposes one might wish to define it. An <u>ad</u> <u>hoc</u> convention must be given. A reasonable one is that if they charge the same price they split the market. If one cannot supply his share then the other obtains the excess demand.

There is no pure strategy equilibrium. We assume that there is a mixed strategy equilibrium.

Let $F_i(p, q)$ = probability that $p_i \leq p$ and $q_i \leq q$.

In the sequel for ease of notation we shall refer to the price and inventory of the distinguished Firm i as p and q instead of p_i and q_i except where confusion might arise.

The profit of Firm i is:

$$\Pi_i = (p + \lambda)\{\min(a - p, \ q)[1 - F_j(p, \ a) + F_j(p, \ a - p - q)]$$

$$+ \int_{\substack{0 \leq x \leq p \\ a-p-q \leq y \leq a-p}} (a - p - y)dF_j(x, \ y)\} - \lambda q$$

Assuming that $0 \leq q \leq a - p$

$$\frac{\partial \Pi_i}{\partial q} = (p + \lambda)\{1 - F_j(p, a) + F_j(p, a - p - q) - \lambda\}$$

At $p = p_e$, the lowest price

$$F_j(p, a) = F_j(p, a - p - q) = 0$$

Hence $\dfrac{\partial \Pi_i}{\partial q} = (p_e + \lambda) - \lambda = p_e \geq 0$.

Let us assume that $F_j(p, q)$ assigns no positive probability to any price, we wish to examine two possibilities:

<u>Case 1</u> $p_e = 0$ and $\Pi_i = 0$

<u>Case 2</u> $p_e > 0$ and $\Pi_i > 0$.

We assert that if Case 1 were to hold $q = q(p) = a - p$. This would give:

$$\Pi_i = 0 = (a - p)\{(p + \lambda)(1 + F_j(p, a)) - \lambda\}$$

$$\text{or } F_j(p, a) = \frac{p}{p + \lambda} , \quad \text{for } 0 \leq p < a .$$

Thus we have an equilibrium point of the form:

$$F_j(p, q) = \Phi_j(p) = \frac{p}{p + \lambda} \text{ and } q = a - p , \text{ for } 0 \leq p < a ,$$

and as follows from $p = 0$ being an active strategy $\Pi_i = 0$.

It is shown in the Appendix that this is the only equilibrium point for Case 1 with $p_e = 0$. In the Appendix, Case 2 is examined and it is shown that there can be no solution if $p_e > 0$ thus the equilibrium point is unique. We now turn to examining its economic properties and meaning.

The solution is unique for all serious "moves" i.e., those moves which involve a positive supply. In our linear case, at equilibrium the players each make a non-serious move with probability $\lambda/(a+\lambda)$. When making a non-serious move, in our model the price has no effect and can have any value. Making a non-serious move corresponds to staying out of the market. This is reasonably consistent with economic sense. As the inventory carrying cost λ grows so does the probability that a firm stays away from the market. In this case its method of nonparticipation is to have nothing for sale.

Figure 2 shows the shape of $\beta_j(p)$ for several values. Suppose that $a = 10$ and $\lambda = .1$, 1 or 10. In the neighborhood of $\lambda = 0$, both firms are almost always ready to supply the whole market at a price close to zero. As λ increases, the chance that the market is not served increases until at around $\lambda = 40$ there is over a 50% chance that the market will not even be supplied. An inventory carrying cost this high is, in general, unreasonable hence one would expect to see risk being cut down. The range of λ that would cover most goods is close to zero.

It is easy to observe that the introduction of an inventory carrying cost in this instance, while it does not lower the payoffs to the firms, considerably lowers consumer welfare. This follows immediately from observing that for any finite λ there is a finite possibility that the
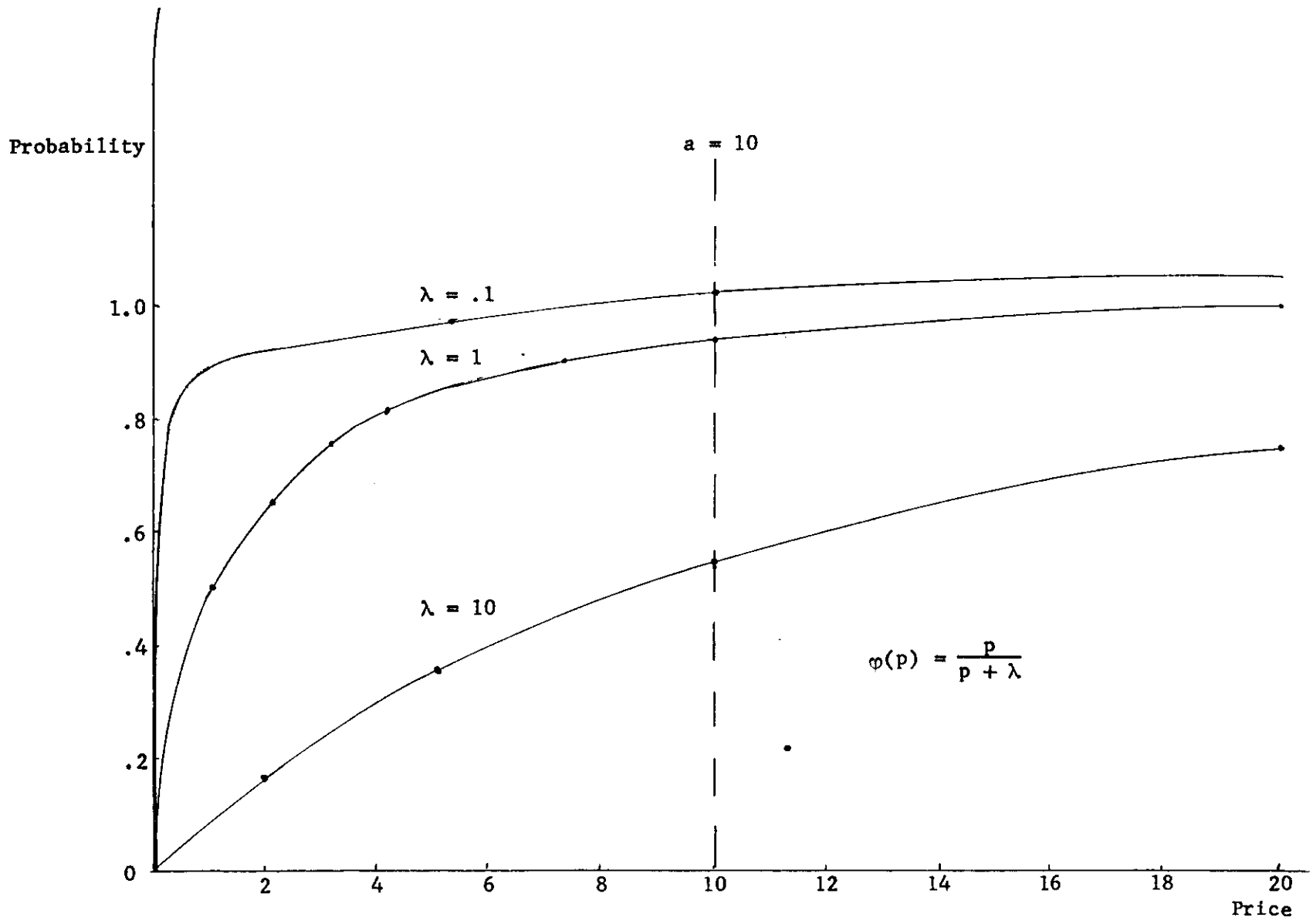
FIGURE 2

customers will not even be served. Furthermore as can be seen from Figure 2
as the inventory carrying cost is increased not only do the customers have
a smaller chance of being served, but if they are served the chances that
it will be for a small amonnt at a high price are increasing. This is con-
sistent with the observation that the firms have to earn enough to cover
expected inventory carrying costs.

identical constant

One could introduce/costs of production into the model by merely making
a transformation on price, replacing p by (p+c) where c is the average
cost of a unit of production. We have not worked out the more difficult
case of unequal production costs.

The equilibrium point described above is virtually independent of the
method used for calculating contingent demand. This is because if a firm
decides to produce anything it will produce enough to saturate the market
if it charges the lower price. The higher firm is not going to sell any-
thing anyhow hence the type of reconstitution is irrelevant. Furthermore
the equilibrium holds for a monotonically decreasing demand function. The
proof of its uniqueness becomes more difficult and has not been done.

4. The Modified Edgeworth Cycle, Maxmin and the Minorant Game

Suppose that the firms were in some sort of dynamic market where they
could undercut each other sequentially. The maxmin point for the firm
moving first is to announce a price of $p = 0$ and a production of $q = a$.
The best reply for the other is to produce nothing. This does appear to
be dynamically stable as long as the second firm is around to undercut the
first if it departs from its price policy. However modeling difficulties

now obviously appear in specifying the details of production and entry into competition. It is unlikely that there is going to be a nonproducing firm-in-being always present and always in a position to move immediately against any action of the production firm.

The minorant game has the pure strategy maxmin point at $p = 0$. This is an equilibrium point in this game; but it is not the only equilibrium point. There is another at which both firms make profits.

Suppose that Firm 1 must commit itself first. It selects a price and level of production such that when Firm 2 is informed it is slightly more profitable for Firm 2 to charge a higher price, taking advantage of the contingent demand that has been left for it, than to undercut Firm 1. This calculation obviously depends upon the specific form of contingent demand. We use that as described in Section 2.

If Firm 1 charges p_1 and produces q_1 then the calculation made by Firm 2 to decide if it should undercut or name a higher price is given by:

$$p(a - p) = \left(\frac{a - q_1}{2}\right)^2$$

hence $q_1 = a - 2\sqrt{p(a - p)}$.

Firm 1 may regard its problem as an attempt to maximize $p_1 q_1$ subject to $q_1 = a - 2\sqrt{p_1(a - p_1)}$. We obtain a maximum at $p_1 = .16a$ and $q_1 = .266a$ [*]

Hence $p_2 = .367a$ and $q_2 = .367a$.

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[*]More accurately the maximization condition gives $16p^3 - 24ap^2 + 10a^2p - a^3 = 0$.

It is of interest to note that this equilibrium is independent of the parameter λ . This we should expect, as no inventories are left over. Profits at this equilibrium are: $\Pi_1 = .0426a^2$ and $\Pi_2 = .1342a^2$.

The possibility of a cycle and the existence of an equilibrium with a positive value, both appear to depend delicately upon the formulation of the model. Furthermore they stress its limitations, in the sense that although the λ is a parameter connected with inventories, it is more of a discount parameter that one might apply to having to mark down goods left at the end of the season than a straight per unit charge. It is still necessary to investigate other forms of charges on inventory. An inventory charge of say $c/2$ on all units (implying an average time in stock of $1/2$ a period) can be treated as an addition to cost of production.

5. An n-Firm Asymmetric Model

Suppose that there are n firms in the market. Each firm has an inventory cost of λ_i . We shall assume that an equilibrium point with all firms obtaining zero profit exists and then attempt to calculate it. Instead of using the function $F_i(p_i, q_i)$ we assume that firms supply the whole market at any price they charge and let

$$\Pi_i = d_i(p)((p + \lambda_i) \prod_{j \neq i} \psi_j(p) - \lambda_i) = 0$$

Where $d(p)$ is the market demand at price p .

From which it follows that

[A]
$$\prod_{\substack{j\neq i}}^{n} \psi_j(p) = \frac{\lambda_i}{p + \lambda_i} \; ;$$

multiplying by ψ_i we obtain:

$$\prod_{j=1}^{n} \psi_j(p) = \frac{\lambda_i \psi_i}{p + \lambda_i} \quad \text{for} \quad i = 1, \ldots, n \; .$$

Multiply these n equations together, and we obtain

$$\left(\prod_{j=1}^{n} \psi_j(p) \right)^n = \prod_{i=1}^{n} \left(\frac{\lambda_i}{p + \lambda_i} \right) \prod_{i=1}^{n} \psi_i(p) \; ;$$

or
$$\prod_{j=1}^{n} \psi_j(p) = \prod_{i=1}^{n} \left(\frac{\lambda_i}{p + \lambda_i} \right)^{\frac{1}{n-1}} .$$

From Equation [A] this becomes

$$\frac{\lambda_i \psi_i}{p + \lambda_i} = \left[\prod_{i=1}^{n} \frac{\lambda_i}{p + \lambda_i} \right]^{\frac{1}{n-1}} , \quad \text{or}$$

[B]
$$\psi_i(p) = \frac{p + \lambda_i}{\lambda_i} \left[\prod_{i=1}^{n} \left(\frac{\lambda_i}{p + \lambda_i} \right) \right]^{\frac{1}{n-1}} .$$

Suppose that all $\lambda_i = \lambda$. This becomes:

$$\psi_i(p) = \frac{p + \lambda}{\lambda} \left(\frac{\lambda}{p + \lambda} \right)^{\frac{n}{n-1}} = \left(\frac{\lambda}{p + \lambda} \right)^{\frac{1}{n-1}} .$$

for $n = 2$ $\psi_i(p) = \dfrac{\lambda}{p + \lambda}$ hence

$$F_i(p, a) = 1 - \psi_i(p) = \frac{p + \lambda - \lambda}{p + \lambda} = \frac{p}{p + \lambda}$$

and this checks with the results in Section 2.

As n becomes large when all $\lambda_i = \lambda$ we observe that

$$\left(\frac{\lambda}{p + \lambda}\right)^{\frac{1}{n-1}} \to 1 \quad \text{for a bounded } p \ .$$

Hence, $F(p, a) \to 0$ for any positive price p. We obtain a result that at first might appear to be paradoxical. If there are many firms in the market the possibilities for being undercut are enormous. As they increase, a protective strategy is to almost always price yourself out of the market. When a firm does not price itself out, the tendency will be to offer very little at a high price in order to be able to recoup expected losses from being caught with unsold stock.

6. Concluding Remarks

Pure duopoly theory has always smacked somewhat of esoterica. Yet if we are willing to note explicitly the limitations of our results it is probably well worth pursuing. Sometimes in an extremely simple model it is possible to demonstrate that effects which have been regarded as personal or social idiosyncracies are manifested without needing recourse to socio-psychological or other explanations.

In parts of the fashion industry many competitors produce few high priced

items which are not marked down during the season. After the season is over a firm may take its losses by destroying or remaindering its inventory. This type of behavior is manifested in this model.

The model is obviously far too simple for any satisfactory application. We need to know how to handle dynamics and above all information conditions. Especially when firms are few in number a simple single period model is inadequate because it cannot reflect threat conditions. This is illustrated strikingly here when we observe the intuitively unsatisfactory equilibrium point in the nonsymmetric two firm market. From [B] we obtain

$$\psi_1(p) = \frac{\lambda_2}{p + \lambda_2} \quad \text{and} \quad \psi_2(p) = \frac{\lambda_1}{p + \lambda_1}$$

Suppose $\lambda_1 = 0$ and $\lambda_2 > 1$ this gives $\psi_2(p) = 0$ or $F_2(p, a) = 1$ immediately. The equilibrium suggested is such that the firm with inventory costs supply the whole market at the competitive equilibrium price while the other sells nothing. This is not particularly reasonable. Furthermore we note that if we changed the information conditions somewhat the solution would change drastically in favor of the more efficient firm.

Duopoly models in which both price and quantity are treated as independent variables simultaneously are difficult to analyze. Among the open problems are firms with different costs and the treatment of differentiated products.

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APPENDIX

In this Appendix we shall prove the principal results in this paper, the existence and uniqueness of the given mixed strategy equilibrium. We shall assume without proof that a mixed strategy equilibrium must have no price with a lumped probability except perhaps at the maximum of the range of active prices. Also we shall assume that the set of prices for which the probability density is positive is an interval. The proof of these propositions is quite tedious; those interested in such an exercise may consult Reference [3]. We shall make a further assumption which is quite innocuous but which makes it possible to define the equilibrium of serious offer uniquely. This assumption is that if a firm names a price for which there is a positive market demand then it must offer a positive quantity for sale.

We define $F_i(p, q)$ as the probability that Firm i charges a price less than or equal to p and supplies a quantity less than or equal to q. Since a is an upper bound on sales, optimality gives $F(p, \infty) = F(p, a)$ for any choice, p, such that $0 \leq p \leq a$.

Suppose Player i supplies $q \leq a - p$. He will undercut his competitor and sell q with probability $1 - F_j(p, a)$; and if he himself is undercut he might still sell q due to j's shortage with probability $F_j(p, a - p - q)$. If his competitor undercuts him and supplies y in the interval $[a - p - q, a - p]$ his sales will be $a - p - y$ with probability density $dF(x, y)$ given that $x \leq p$. Hence, the expected profit of firm i is given by

$$\Pi_i = \Pi_i(p, q) = (p + \lambda)q(1 - F_j(p, a) + F_j(p, a - p - q))$$

$$+ \int (a - p - y)dF_j(x, y) - \lambda q ,$$

$$q - p - q \leq y \leq a - p$$

$$x \leq p$$

subject to $0 \leq q \leq a - p$.

Suppose that $\Pi_i > 0$. Consider p_ℓ the minimum price for which the firms have a positive probability density. Now $\Pi_i(p_\ell, q) = (p_\ell + \lambda)q - \lambda q = p_\ell q$ has its maximum at $(a - p_\ell)$ since $p_\ell > 0$ is implied by $\Pi_i > 0$. Thus $\Pi_i = p_\ell(a - p_\ell)$. Now, for all (p, q) , defining $\Pi_i^!(p, a)$ as

$$\frac{\partial}{\partial q} \Pi_i = (p + \lambda)(1 - F_j(p, a) + F_j(p, a - p - q) - \lambda$$

is a non-increasing function of q , and $\frac{\partial}{\partial q} \Pi_i(p, a - p)$ is continuous and $\frac{\partial}{\partial q} \Pi_i(p_\ell, a - p_\ell) = p_\ell > 0$. Hence, there exists a maximal interval $(p_\ell, p*)$ in which $a - p$ is the unique optimal supply. Thus for $p \in (p_\ell, p*)$

$$\Pi_i = (a - p)[(p + \lambda)(1 - F_j(p, a)) - \lambda] , \quad \text{and}$$

$$F_j(p, a) = \frac{p - \Pi_i/(a - p))}{p + \lambda} , \quad p_\ell \leq p < p* .$$

Since the above function is decreasing near a, $p* < a$, and also in $(p_\ell, p*)$, $F_j(p*, a) < 1$. Firm j cannot lump his remaining probability at $p*$. If this were the case i would always undercut $p*$ and $p*$ would

give j a nonpositive profit. Thus $F_j(p, a)$ is continuous at p* and

for p > p* , a - p is no longer uniquely optimal; for p > p* ,

$\frac{\partial}{\partial q} \Pi_i(p, a - p) \leq 0$, while for p < p* , $\frac{\partial}{\partial q} \Pi_i(p, a - p) \geq 0$. By

continuity $\frac{\partial}{\partial q} \Pi_i(p*, a - p*) = 0$. However, $(a - p*) \frac{\partial}{\partial q} \Pi_i(p*, a - p*)$

is the limit of $\Pi_i(p, a - p)$ as p approaches p* from below. This

implies $\Pi_i = 0$, a contradiction and we have $\Pi_i \leq 0$.

The case $\Pi_i < 0$ is ruled out by the fact that q = 0 is an available

strategy which gives $\Pi_i = 0$. Hence $\Pi_i = 0$ and $p_\ell = 0$ is the remaining

case.

We assert that $F_i(p, q) = 0$ for all p > 0 and 0 < q < a - p ,

i = 1, 2 . Suppose it is not true; then for some (p_0, q_0), $F_1(p_0, q_0) > 0$

with $p_0 > 0$ and $0 < q_0 < a - p_0$.

For any optimal q and j = 1, 2 :

$$\Pi_1'(p_0, q) = \frac{\partial}{\partial q} \Pi_j(p_0, q) = (p_0 + \lambda)(1 - F_j(p_0, a) + F_i(p_0, a - p_0 - q)) - \lambda = 0 .$$

But $\Pi_j(p_0, q) = q\Pi_1'(p_0, q) + (p + \lambda)\int(a - p - y)dF(x, y) = 0$. Hence the

integral must equal zero and $\Pi_j'(p_0, q)$ must be non-positive for all positive

q or else a positive profit is attainable.

Now by our assumption about serious offers there exists a (p_1, q_1)

such that $p_1 > 0$, $0 < q_1 < a - p_1$ and $F_1(p, q)$ is strictly increasing

with both p and q in every neighborhood of (p_1, q_1) . Hence

$F_1(p_1, q_1 + \Delta) > F_1(p_1, q_1 - \Delta)$ for all $\Delta > 0$ and $\Pi_1'(p_1, q_1) = 0$.

Thus for $p_1 \leq p < a - q_1$, letting $0 \geq \Pi_2'(p, a - q_1 - p - \Delta) =$

$(p + \lambda)(1 - F_1(p, a) + F_1(p_1, q_1 + \Delta) - \lambda > (p + \lambda)(1 - F_1(p, a) + F_1(q_1 - \Delta))$

$. = \Pi_2'(p, a - q_1 - p + \Delta)$.

Consequently, for prices in $[p_1, a - q_1)$, $a - q_1 - p$ is an upper

bound for the supply of firm 2. Specifically, for $p_1 \leq p \leq a_1 - q_1$ and

$q \geq a - q_1 - p_1$

$$F_2(p, q) = F_2(p_1, q) .$$

Finally, for $0 < \Delta < a - q_1 - p_1$

$\Pi_1'(p_1 + \Delta, q_1 - \Delta) = (p + \lambda + \Delta)[1 - F_2(p_1 + \Delta, a) + F_2(p_1 + \Delta, a - p_1 - q_{11})) - \lambda$

$= (p + \lambda + \Delta)[1 - F_2(p_1, a) + F_2(p_1, a - p_1 - q_1)] - \lambda$

$= \dfrac{\Delta \lambda}{p + \lambda} + \Pi_1'(p_1, q_1) > 0$.

This contradicts $\Pi_1' \leq 0$ and the assertion is proved. The case remains that $F_j(p, a - p - q) = 0$ for all $a - p > q > 0$, and we have

$$\frac{\partial}{\partial q} \Pi_i = (p + \lambda(1 - F_j(p, a) - \lambda = 0 ,$$

or

$$F_j(p, a) = \frac{p}{p + \lambda} .$$

This gives

$$\Pi_i(p, q) = q[(p + \lambda)1 - \frac{p}{p + \lambda} - \lambda] = 0$$

for all $0 \leq q \leq a - p$ which completes the proof.