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ON A CLASS OF EQUILIBRIUM CONDITIONS FOR MAJORITY RULE

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by

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1. Introduction

The possibility of intransitivity under majority rule has been of consequence in both political science and welfare economics. The indeterminateness or instability implied by the voting paradox has been a major challenge to the understanding of political processes based on majority rule. Political scientists have so far not developed analytical models capable of coping with this instability, and as a consequence our understanding of voting mechanisms is still rudimentary. In welfare economics, the paradox is in a sense fundamental to Arrow's impossibility theorem, since (by Arrow's Lemma 3) any social welfare function satisfying his conditions 2-5 is equivalent to majority rule, while condition 1 requires that the domain of the function be large enough to include sets of preference orderings of sufficient diversity to produce intransitivity.

It is evident that intransitivities cannot arise if there exists sufficient concensus or similarity of preferences among the members of

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society--e.g., in the extreme case, if all members have identical preferences. Thus one hope for developing useful models of voting mechanisms has been that less restrictive similarity conditions on individual preferences could be shown sufficient to avoid the paradox. The first specific such condition, single-peakedness, was discovered by Black [1948]. Arrow, though referring to single-peakedness as a "radical" restriction on individual preferences, also commented that Black's results at least "show that the condition of unanimity is mathematically unnecessary to the existence of a social welfare function, and we may well hope that there are still other conditions [than single-peakedness] under which the formation of social welfare functions...will be possible." (83)

A number of such conditions have in fact been discovered subsequently. The most important are those of Vickrey [1960], Dummet and Farquharson [1961], Inada [1964], Ward [1965], Sen [1966], Sen [1968], Pattanaik [1968], Inada [1969], and Sen and Pattanaik [1969].[1]

These various conditions, including single-peakedness, are the subject of the present paper. We refer to them as equilibrium conditions for majority rule, where "equilibrium" is used in the generic sense to mean, variously, a social welfare function or social ordering, a social decision function, or a stable outcome. (These concepts are defined in Section 2.)

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[1] The equilibrium conditions introduced in the important paper of Plott [1967] are different in nature from the above. They are considered separately in the last section of this paper.

Let $\mathcal{A}$ be a set of alternatives, and let Π be the set of all possible distinct preference orderings over $\mathcal{A}$. (There are three possible orderings of two alternatives, thirteen of three, etc.) A voting population is described by specifying the number $n(\pi)$ of voters who are assigned the preference ordering π, for all $\pi \in \Pi$. In general there exist assignments $n(\pi)$ which generate voting populations in which there is no equilibrium under majority rule. The equilibrium conditions mentioned above consist of restrictions on $n(\pi)$ sufficient to ensure that in any voting population generated by an assignment satisfying the restrictions, there will be an equilibrium. All of the conditions also have in common the following two characteristics:

1. The set $\mathcal{A}$ of alternatives is taken to be an arbitrary (in some cases finite) set, about which no topological assumptions are made. Hence the usual economic conditions on preferences, such as convexity or representability, have played no role in this literature.

2. The conditions are all <u>exclusion</u> restrictions. By this we mean they are of the following special form: there is a specified class $\mathcal{R}$ of subsets of Π, such that an assignment $n(\pi)$ satisfies the restriction if and only if every subset $R \in \mathcal{R}$ contains an ordering $\pi \in R$ such that $n(\pi) = 0$. Thus certain combinations of preference orderings are excluded, or prohibited, by the restriction, but there are no constraints at all upon the non-excluded orderings. In particular, nothing is said about the relative numbers of voters assigned each of the orderings in any non-excluded combination.

With the partial exception of single-peakedness, none of these conditions have so far had any serious application to even slightly realistic problems. Because of 1., it has not been possible to translate these conditions into the more familiar and intuitive terms of indifference loci and utility functions. For these reasons, it is not obvious how general or potentially useful these various conditions really are.

In the present study it is assumed that the alternatives in question are points in some multidimensional commodity or policy space, and that the preferences of voters are convex and representable by differentiable utility functions over this space. It is shown that under these circumstances, all of the above conditions are extraordinarily restrictive. If at some point x there are three voters whose marginal rates of substitution for any pair of commidities differ, all conditions but one fail; if they differ in a slightly stronger sense, all conditions fail. Hence the conditions are incompatible with even a very modest degree of heterogeneity of tastes, and for most purposes are not significantly less restrictive than the condition of complete unanimity.

The argument is easily extended to show that <u>no</u> exclusion condition on preferences can be less restrictive and still sufficient for equilibrium.

Certain non-exclusionary conditions have been shown by Plott [1967] to be both sufficient and necessary for equilibrium. These conditions are themselves so restrictive, however, that any attempt to base the existence of social welfare functions or other equilibria or restrictions on preferences appears futile.

2. Definitions and Assumptions

One alternative x is said to be weakly (strictly) preferred by majority rule to another, y, or $xR_{maj}y$ $(xP_{maj}y)$, if and only if the number of voters who weakly prefer x to y is not less than (is greater than) the number who weakly prefer y to x. R_{maj} is clearly a complete binary relation over $\mathcal{A}$. With respect to a given set $\mathcal{P}$ of preference orderings over a set $\mathcal{A}$ of alternatives there exists a stable outcome under majority rule if there is some $x \in \mathcal{A}$ such that $xR_{maj}y$ for all $y \in \mathcal{A}$. There exists a social decision function on $\mathcal{A}$ if every nonempty subset of alternatives has stable outcome, i.e. for all $A \subset \mathcal{A}$ there is some $x \in A$ such that $xR_{maj}y$ for all $y \in A$. A social ordering of $\mathcal{A}$, or a social welfare function over $\mathcal{A}$, exists if the binary relation R_{maj} is transitive. (Clearly a social decision function on $\mathcal{A}$ is sufficient for a stable outcome in $\mathcal{A}$. If $\mathcal{A}$ is compact a social ordering on $\mathcal{A}$ is sufficient for a stable outcome, and if $\mathcal{A}$ is finite it is also sufficient for a social decision function.)

There are $\ell \geq 3$ voters, indexed $i = 1, 2, \ldots, \ell$.

A social alternative is a vector $x \in R^m$. It can be thought of as a bundle of m collective goods; a state of income distribution (in which case x_i is the income of voter i, and $m = \ell$); or more generally a vector describing the final consumptions of both public and private goods of all voters. The set of feasible alternatives contains an interior relative to some affine subspace of dimensionality $n \geq 3$, which we take without loss of generality to be R^n. All subsequent discussion is relative to R^n. The set of alternatives hence contains an open set $S \subset R^n$.

Each voter has a preference ordering $\underset{i}{\succsim}$ of the alternatives, which is complete, transitive, and convex, i.e. for any $x, x' \in R^n$ and $0 < \lambda < 1$

(2.1)(a) $x \underset{i}{\succ} x'$ implies $\lambda x + (1-\lambda)x' \underset{i}{\succ} x'$, and

(b) $x \underset{i}{\sim} x'$ implies $\lambda x + (1-\lambda)x' \underset{i}{\succsim} x'$

Each such preference ordering is representable by a continuous ordinal utility function $u_i(x)$, each of whose partial derivatives $\partial u_i/\partial x_j$, $j = 1, \ldots, n$, exist and are continuous at all $x \in R^n$.

This last assumption implies that the gradient $\nabla_i(x)$ of $u_i(x)$ exists at all $x \in R^n$:

$$\nabla_i(x') = \begin{bmatrix} \left.\dfrac{\partial u_i(x)}{\partial x_1}\right|_{x=x'} \\ \left.\dfrac{\partial u_i(x)}{\partial x_2}\right|_{x=x'} \\ \left.\dfrac{\partial u_i(x)}{\partial x_h}\right|_{x=x'} \end{bmatrix} .$$

The directional derivative of u_i at x in the direction v is

$$\lim_{h\to 0} = \frac{u_i(x + hv) - u_i(x)}{h||v||} = \frac{[v, \nabla_i(x)]}{||v||} ,$$

where we denote by $[a, b]$ the inner product of two vectors a , b , and by $||a|| = \sqrt{[a, a]}$ the norm of a . We now establish for future reference the following straightforward

Lemma 1 For any $x, y \in R_n$,

(i) $[y, \nabla_i(x)] < [x, \nabla_i(x)]$ implies $x \succ_i y$

(ii) $[y, \nabla_i(x)] > [x, \nabla_i(x)]$ implies that there exists $\lambda^* > 0$ such that

$$\lambda x + (1-\lambda)y \succ_i y \quad \text{for} \quad 0 < \lambda < \lambda^* .$$

Proof (i) Suppose $y \succsim_i x$. From the convexity of $\succsim_i$, $x \precsim_i \lambda y + (1-\lambda)x = x + \lambda(y-x)$ for all $0 < \lambda < 1$, i.e. $u_i(x + \lambda(y-x)) \geq u_i(x)$. Hence

$$0 \leq \frac{u_i(x + \lambda(y-x)) - u_i(x)}{\lambda||y-x||} , \quad \text{and}$$

$$0 \leq \lim_{\lambda\to 0} \frac{u_i(x + \lambda(y-x)) - u_i(x)}{\lambda||y-x||} = \frac{[(y-x), \nabla_i(x)]}{||y-x||} ,$$

since the above limit is simply the directional derivative of u_i at x in the direction of $(y-x)$. But then $[y, \nabla_i(x)] \geq [x, \nabla_i(x)]$, contrary to the hypothesis of (i).

(ii) Suppose $y \underset{i}{\succsim} \lambda y + (1-\lambda)x = x + \lambda(y-x)$ for all $\lambda > 0$. Then, arguing as above,

$$0 \geq \lim_{\lambda \to 0} \frac{u_i(x + \lambda(y-x) - u_i(x))}{\lambda ||y-x||} = \frac{[(y-x), \nabla_i(x)]}{||y-x||},$$

whence $[x, \nabla_i(x)] \geq [y, \nabla_i(x)]$, contrary to hyposithis. Hence there exists $\lambda^* > 0$ such that $\lambda y + (1-\lambda)x \underset{i}{\succ} y$ for $\lambda = \lambda^*$; from the convexity of $\underset{i}{\succsim}$, this also holds for $0 < \lambda < \lambda^*$. Q.E.D.

A convex cone C is <u>pointed</u> provided it never contains both x and $-x$, for any $x \neq 0$. Also for future reference we recall the following property of pointed convex cones:

<u>Theorem 1</u> If C is a closed, pointed convex cone in R^n, there is some $p \epsilon R^n$ such that $[p, x] > 0$ for all $x \epsilon C$.

A proof can be found in Nikaido [1968], pp. 43-4.

3. Single-peakedness

The first "similarity" condition, and probably still the most useful, is Black's single-peakedness restriction. If a set of preference orderings is single-peaked (defined below) over a compact set $\mathcal{A}$ of alternatives, there exists a stable outcome in $\mathcal{A}$. If in addition the number of voters is odd (or if a chairman can break ties), majority rule will yield a social ordering of the alternatives--i.e., an ordinal social welfare function satisfying Arrow's Axioms 2-5.

Our definition of single-peakedness follows that of Arrow. A strong ordering Q of a set $\mathcal{A}$ is a binary relation over $\mathcal{A}$ satisfying:

(i) for all distinct x, $y \in \mathcal{A}$, xQy or yQx but not both

(ii) xQy and yQx implies xRz

Let $\mathcal{A}$ be an arbitrary set of alternatives, and $\mathcal{P} = \{\underset{1}{\succsim}, \underset{2}{\succsim}, \ldots, \underset{\ell}{\succsim}\}$ be a set of individual preference orderings (complete and transitive, but not necessarily, at this point, convex or representable). We then define single-peakedness as follows:

(3.1) A set $\mathcal{P}$ of preference orderings is <u>single-peaked</u> over a set $\mathcal{A}$ of alternatives provided there exists a strong ordering Q over $\mathcal{A}$ such that for any x, y, $z \in \mathcal{A}$, xQyQz or zQyQx implies that for each voter i, if $x \underset{i}{\succsim} z$ then $y \underset{i}{\succ} z$, while if $z \underset{i}{\succsim} x$ then $y \underset{i}{\succ} x$.

Less formally, there must exist an ordering Q of the alternatives (with no ties) such that whenever one alternative is "between" two others (according to the ordering Q), each voter must strictly prefer it to at least one of other two.

A more intuitive interpretation of this concept is as a kind of convexity requirement on individual preferences. In particular, if the set of alternatives is an interval in R^1, and if each individual preference ordering is strictly convex, then the single-peakedness condition is fulfilled (letting the ordering Q be given by $>$). One-dimensional social choice problems do occur in practice, and in such cases, single-peakedness often arises as a natural consequence of the convexity of individual preference orderings.

It is clearly of interest to inquire whether any comparable statement can be made when the set of alternatives is of higher dimensionality. The answer to this, as we shall now show, is unfortunately negative; indeed, under the assumptions of Section 2, the only obvious condition on individual preferences which will ensure single-peakedness is the condition of complete unanimity of individual preference orderings.

To show this informally, suppose that the choice set is two-dimensional (e.g., the amounts of each of two public goods must be chosen), and that there is some point x in the alternative space at which the indifference curves of any two voters cross. In Figure 3.1, I_1 and I_2 are indifference curves of voters 1 and 2 crossing at the point x. Consider the neighboring points w, y, and z (I_1' and I_2' are the indifference curves of 1 and 2 through y).

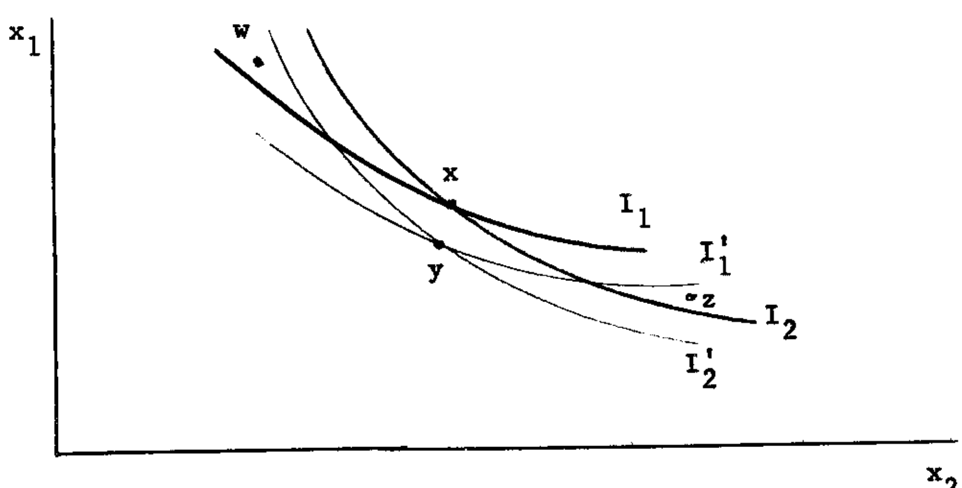

FIGURE 3.1

Evidently voter 1 orders the points $w \succ_1 x \succ_1 y \succ_1 z$, while voter 2 ranks them $z \succ_2 x \succ_2 y \succ_2 w$. It is easily verified that there is no ordering Q of these four points which will satisfy the single-peakedness condition. This can be done by examining each of the 24 possible orderings in turn, or more simply as follows: If w were between any two of x, y, z the condition would fail for $\succsim_2$, while if z were between any two of w, x, y, it would fail for $\succsim_1$. Hence when the points are ordered by Q the sequence must either begin with w and end with z, or conversely. But then y is between either x and w --yet is preferred to neither according to $\succsim_1$ --or else is between x and z, but is preferred to neither according to $\succsim_2$. Thus no ordering Q of the points can satisfy the definition, and the set of preferences are not single-peaked.

A more careful and slightly more general statement of this result, and a formal proof, are as follows:

<u>Proposition 1</u> If there exists a point $x \epsilon S$ at which the gradients $\nabla_i(x)$, $\nabla_j(x)$ of any two voters i, j are linearly independent, the set of preference orderings is not single-peaked over S.

<u>Proof</u> Let 1 and 2 be the voters in question, and let C_{12}, C_{-12}, etc. denote the convex polyhedral cones spanned by $\nabla_1(x)$, $\nabla_2(x)$, by $\nabla_1(x)$, $-\nabla_2(x)$, etc.

C_{1-2} is pointed; for otherwise there would be some $y \neq 0$ such that

$$y = \alpha_1 \nabla_1(x) - \alpha_2 \nabla_2(x) \in C_{1-2}\,, \quad \alpha_1,\ \alpha_2 \geq 0$$

$$-y = \beta_1 \nabla_1(x) - \beta_2 \nabla_2(x) \in C_{1-2}\,, \quad \beta_1,\ \beta_2 \geq 0$$

implying

$$0 = (\alpha_1 + \beta_1)\nabla_1(x) - (\alpha_2 + \beta_2)\nabla_2(x)\ .$$

Since $y \neq 0$ either $(\alpha_1 + \beta_1) > 0$ or $(\alpha_2 + \beta_2) > 0$, and hence $\nabla_1(x)$ and $\nabla_2(x)$ would be linearly dependent, contrary to hypothesis.

Since C_{1-2} is a closed, pointed convex cone, Theorem 1 applies and there exists $p \in R^n$ such that

$$[p,\ z] > 0 \quad \text{for all} \quad z \in C_{1-2}\ .$$

This implies $[p, \nabla_1(x)] > 0$, $[p, \nabla_2(x)] < 0$. Let $y(\lambda) = x + \lambda p$, $\lambda > 0$. Evidently

$$[y(\lambda), \nabla_2(x)] = [x, \nabla_2(x)] + \lambda[p, \nabla_2(x)] < [x, \nabla_2(x)]\ ,$$

so (i) of the Lemma 1 applies, and $y(\lambda) \underset{1}{\prec} x$ for all $\lambda > 0$. Similarly,

$$[y(\lambda), \nabla_1(x)] = [x, \nabla_1(x)] + \lambda[p, \nabla_1(x)] > [x, \nabla_1(x)]\ ,$$

so from (ii) of Lemma 1 there exists $\lambda^* > 0$ such that $y(\lambda) > x$ for

$0 < \lambda < \lambda^*$. Finally since x is an interior point of s , there exists some $\epsilon > 0$ such that

$$||y(\lambda) - x|| = \lambda||p|| < \epsilon \text{ implies } y(\lambda)\epsilon S .$$

Hence take $y_1 = y(\lambda')$ for some $0 < \lambda' < \min\left\{\frac{\epsilon}{||p||}, \lambda^*\right\}$. We have shown that $y_1 \epsilon S$, $y_1 \underset{1}{\succ} x$ and $y_1 \underset{2}{\not\prec} x$.

Arguing similarly from C_{-12} , there exists $y_2 \epsilon S$ satisfying $y_2 \underset{1}{\not\prec} x$ and $y_2 \underset{2}{\succ} x$.

Proceeding in the same fashion from C_{-1-2} , there exists $q \epsilon R^n$ such that $[q, \nabla_1(x)] < 0$ and $[q, \nabla_2(x)] < 0$. Letting $w(t) = tq + (1-t)x$, by previous arguments there exists $t' > 0$ such that

$$w(t) \epsilon S \text{ for } t < t' ,$$

$$w(t) \underset{1}{\prec} x \text{ for } t > 0 , \text{ and}$$

$$w(t) \underset{2}{\prec} x \text{ for } t > 0 .$$

Moreover there also exists $t^* > 0$ such that $w(t) \underset{2}{\succ} y_1$ for $0 < t < t^*$; for if $q \underset{2}{\not\prec} y_1$, note that $g(t) = u_2(w(t))$ is continuous on the interval [0, 1], and that

$$g(0) = u_2(x) > u_2(y_1) > u_2(q) = g(1) ,$$

so that for some $0 < t^* < 1$, $g(t^*) = u_2(y_1)$; otherwise take $t^* = 1$.

In either case, the convexity of $\underset{2}{\succsim}$ implies $x \underset{2}{\succ} w(t) \underset{2}{\succ} w(t^*) \underset{2}{\succsim} y_1$ for $0 < t < t^*$. By the same type of argument there exists $t^{**} > 0$ such that $w(t) \underset{1}{\succ} y_2$ for $0 < t < t^{**}$. Take $y_3 = w(t'')$ for some $0 < t'' < \min(t', t^*, t^{**})$

Thus there exist points y_1, y_2, y_3 which (using the transitivity of $\underset{1}{\succsim}$ and $\underset{2}{\succsim}$) satisfy

$$y_1 \underset{1}{\succ} x \underset{1}{\succ} y_3 \underset{1}{\succ} y_2 \quad \text{for voter 1, and}$$

$$y_2 \succ x \succ y_3 \succ y_1 \quad \text{for voter 2.}$$

We now show that no ordering Q over these points can satisfy the single-peakedness condition. Suppose aQy_1Qb or where $a, b \in \{x, y_2, y_3\}$. Then y would be between some two of $\{x, y_2, y_3\}$, but would be strictly preferred to neither of them by voter 2, thus violating the condition. Similarly, if aQy_2Qb, $a, b \in \{x, y_1, y_3\}$, the condition would be violated for voter 1. Hence the only possible orderings are:

$$y_1QxQy_3Qy_2$$

$$y_1Qy_3QxQy_2$$

$$y_2QxQy_3Qy_1$$

$$y_2Qy_3QxQy_1$$

In the first and last of these y_3 is between x and y_2, but is preferred to neither of them by voter 2. In the remaining two y_3 is between x and y_1, yet is preferred to neither by voter 1. Hence there is no strong ordering Q of the points satisfying the single-peakedness condition.

<u>Q.E.D.</u>

Thus if at any point in the alternative space any two voters have different marginal rates of substitution between any pair of commodities, their set of preference orderings is not single peaked. If we take $S = R^n$, we conjecture that unless all goods are perfect substitutes, unanimity is necessary, as well as sufficient, for a rate of preference orderings to be single-peaked over R^n, $n \geq 2$.

4. <u>Subsequent Conditions</u>

A number of alternative and in some cases more general conditions have been subsequently introduced, by Dummet and Farquharson [1961], Inada [1964], [1969], Pattanaik [1968], Sen [1966], Sen and Pattanaik [1968], Vickrey [1960], and Ward [1965]. Without attempting to do full justice to the literature we will briefly describe the major conditions and principle results achieved to date. In doing so we rely heavily upon the papers by Sen [1966] and Sen and Pattanaik [1969], to which the reader is referred for more detailed discussion.

All post-single-peakedness conditions require that every <u>triple</u> of alternatives satisfy certain restrictions. With respect to a set $\mathcal{P}$ of preference orderings over a set $\mathcal{A}$ of alternatives, let x, y, z be a triple of alternatives in $\mathcal{A}$. We then define the principle conditions as follows:

(WSP) The triple is <u>weakly single-peaked</u> if it contains an alternative a such that each voter either strictly prefers a to one or both of the other two, or is indifferent between all three members of the triple.

(SC) The triple is <u>single-caved</u> if it contains a member a such that each voter either strictly prefers at least one of the other two alternatives to a , or is indifferent to all three.

(Sep) The triple is <u>separable</u> if it contains a member a such that each voter strictly prefers a to both others, or strictly prefers both others to a , or is indifferent to all three.

(VR) The triple is <u>Value-Restricted</u> if it satisfies the WSP, or SC, or Sep conditions.

(ER) The triple satisfies the <u>Extremal Restriction</u> provided that whenever $a \underset{i}{\succ} b \underset{i}{\succ} c$ for some voter i , there is no voter j such that $c \underset{j}{\succ} a \underset{j}{\succ} b$, a , b , and c being members of the triple.

(LA) The triple satisfies the <u>Limited Agreement</u> restriction if it contains two members a , b such that each voter weakly prefers a to b .

A set $\mathcal{P}$ of preference orderings is said to be WSP, SC, etc., provided every triple in $\mathcal{Q}$ satisfies the WSP, SC, etc. condition on triples.

Clearly a triple (or set of preference orderings) which is WSP, SC, or Sep is also VR. In general the conditions VR, ER, and LA are logically independent; however, for triples (or larger sets of alternatives) such that no voter is indifferent between any two of their members, satisfaction of the ER or LA conditions implies satisfaction of the VR condition.

We now briefly summarize the principle results employing the above conditions as follows: If $\mathcal{P}$ is WSP, SC, Sep, VR, ER, or LA over set A of Pareto-optimal alternatives in $\mathcal{A}$ (i.e., $A = \{x \in \mathcal{A}:$ for no $y \in \mathcal{A}$ is $y \underset{i}{\succ} x$ for any $\underset{i}{\succsim} \in \mathcal{P}\}$), there exists a stable outcome in $\mathcal{A}$. If $\mathcal{P}$ is WSP, SC, Sep, VR, ER, or LA over $\mathcal{A}$, there exists a social decision function on $\mathcal{A}$ (and therefore also a stable outcome in $\mathcal{A}$). If $\mathcal{P}$ is ER over $\mathcal{A}$, or else if $\mathcal{P}$ is WSP, SC, Sep, or VR over $\mathcal{A}$ and for every triple in $\mathcal{A}$ the number of voters not indifferent to all members of the triple is odd, there exists a social ordering on $\mathcal{A}$. (These results assume $\mathcal{A}$ finite.)

In addition to these sufficiency results, certain of the conditions have been shown to be necessary, in the following special sense. It will be recalled that an exclusion restriction requires that for all $R \in \mathcal{R}$, $n(\pi) = 0$ for some $\pi \in R$, where $\mathcal{R}$ is a set of particular subsets of Π specified by the restriction. Suppose we weaken the restriction somewhat by replacing the class of specified sets $\mathcal{R}$ by some proper subset $\mathcal{R}' \subset \mathcal{R}$. If for every such way of relaxing the restriction there is an assignment $n(\pi)$, satisfying the weakened restriction but otherwise unconstrained, such that with the voting population so described majority rule does not yield an equilibrium of a particular type, then the original condition is said to be <u>necessary</u> for that type of equilibrium. (Note that there may also exist other assignments satisfying the relaxed constraint which do lead to an ordering or social decision function.) The following necessity results have been established.

For a social decision function on $\mathcal{A}$ it is necessary (in the above sense) that every triple of alternatives in $\mathcal{A}$ satisfy VR or ER or LA. For a social ordering it is necessary that every triple satisfy ER.

The necessity results suggest that substantial further generalizations of these equilibrium conditions are unlikely. While these various conditions and results are technically generalizations of Black's single-peakedness results, the practical import of the generalization achieved is not clear, for the reasons mentioned in the introduction.

Under the assumptions of Section 2, all of the above conditions are, in fact, exceedingly restrictive. If at any $x \epsilon S$ the gradient vectors of any three voters are linearly independent, the set of preference orderings containing the orderings of these three voters will fail to satisfy any of the above conditions. In fact, the three gradients need not even be linearly dependent, as seen in the following two-dimensional example. Let there be a point x in the alternative set at which the indifference loci I_1, I_2, I_3 of any three voters 1, 2, 3, cross in the manner indicated in Figure 4.1, and consider the neighboring points w, y, z.

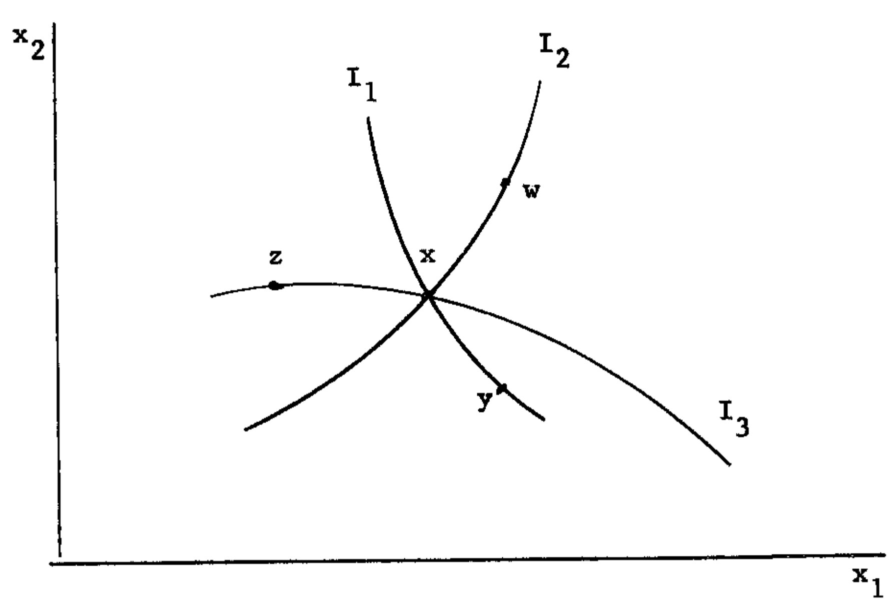

FIGURE 4.1

Evidently these three points are ordered:

$$w \underset{1}{\succ} y \underset{1}{\succ} z$$

$$z \underset{2}{\succ} w \underset{2}{\succ} y$$

$$y \underset{3}{\succ} z \underset{3}{\succ} w \,.$$

It is straightforward to verify that the triple $\{w, y, z\}$ fails to satisfy any of the previously-listed conditions. Rather than go through the exercise at this point, we do so in the course of proving the following:

<u>Proposition 2</u>. If there exists a point $x \epsilon S$ at which three gradients $\nabla_i(x)$, $\nabla_j(x)$, $\nabla_k(x)$ are linearly independent, the set of preference orderings does not satisfy any of the conditions WSP, SC, Sep, VR, ER, or LA over S .

Proof: As before, let 1, 2, 3, be the voters in question, and let C_{12-3} , etc. denote the cone spanned by $\nabla_1(x)$, $\nabla_2(x)$, $-\nabla_3(x)$, etc.

The cone C_{12-3} is pointed; for otherwise it contains some nonzero

$$y = \alpha_1\nabla_1(x) + \alpha_2\nabla_2(x) - \alpha_3\nabla_3(x) \,, \quad \alpha_i \geq 0 \,, \quad \text{and also}$$

$$-y = \beta_1\nabla_1(x) + \beta_2\nabla_2(x) - \beta_3\nabla_3(x) \,, \quad \beta_i \geq 0 \,,$$

implying $\nabla_1(x)$, $\nabla_2(x)$, $\nabla_3(x)$ linearly dependent (since at least one α and one β must be strictly positive). Hence there exists p such that

$$[p,\ z] > 0 \quad \text{for all} \quad z \in C_{12-3}\ , \quad \text{whence}$$

$$[p,\ \nabla_1(x)] > 0\ , \quad [p,\ \nabla_2(x)] > 0\ , \quad [p,\ \nabla_3(x)] < 0\ .$$

Let $y(\lambda) = x + \lambda p$, $0 < \lambda < 1$. Evidently

$$[y(\lambda),\ \nabla_3(x)] = [x,\ \nabla_3(x)] + \lambda[p,\ \nabla_3(x)] < [x,\ \nabla_3(x)]\ .$$

Since S is open and $x \in S$, there exists some $\epsilon > 0$ such that $||z - x|| < \epsilon$ implies $z \in S$. Hence $y(\lambda) \in S$ for $\lambda < \frac{\epsilon}{||p||}$.

From (ii) of Lemma 1 there exists $\lambda^* > 0$ such that

$$y(\lambda) \underset{1}{\succ} x \quad \text{for} \quad 0 < \lambda < \lambda^*\ , \quad \text{and}$$

there exists $\lambda^{**} > 0$ such that

$$y(\lambda) \underset{2}{\succ} x \quad \text{for} \quad 0 < \lambda < \lambda^*\ .$$

Let $y_1 = y(\lambda')$ for some $0 < \lambda' < \min\left\{\frac{\epsilon}{||p||},\ \lambda^*,\ \lambda^{**}\right\}$. Clearly $y_1 \in S$, $y_1 \underset{1}{\succ} x$, $y_1 \underset{2}{\succ} x$, and $[y_1,\ \nabla_3(x)] < [x,\ \nabla_3(x)]$.

By an entirely analagous argument from C_{1-2-3} there exists $y_2 \in S$ such that

$$y_2 \underset{1}{\succ} x$$

$$[y_2,\ \nabla_2(x)] < [x,\ \nabla_2(x)]$$

$$[y_2,\ \nabla_3(x)] < [x,\ \nabla_3(x)]\ .$$

The second statement implies, from (i) of Lemma 1, that

$$y_2 \prec_2 x \; .$$

Consider now $y(\alpha) = Ay_1 + (1 - \alpha)y_2$, $0 \leq \alpha \leq 1$. From the transitivity and convexity of $\succsim_1$, if $y_1 \succsim_1 y_2$, $y(\alpha) \succsim_1 y_2 \succ_1 x$, while if $y_2 \succsim_1 y_1$, $y(\alpha) \succsim_1 y_2 \succ_1 x$. Hence $y(\alpha) \succ_1 x$ for all $0 \leq \alpha \leq 1$. Moreover,

$$[y(\alpha), \nabla_3(x)] = \alpha[y_1, \nabla_3(x)] + (1 - \alpha)[y_2, \nabla_3(x)]$$

$$< [x, \nabla_3(x)] \; ,$$

so Lemma 1 implies $y(\alpha) \prec_3 x$ for all $0 \leq \alpha \leq 1$. Finally, consider $g(\alpha) = u_2(y(\alpha))$; this is evidently continuous on the closed interval $\alpha \epsilon [0, 1]$, and

$$g(0) = u_2(y_2) > u_2(x) > u_2(y_1) = g(1) \; ,$$

so that $g(\alpha^*) = u_2(x)$ for some $0 < \alpha^* < 1$. Let $a = y_2(\alpha^*)$; evidently the point a satisfies $a \succ_1 x$, $a \sim_2 x$, and $a \prec_3 x$.

Arguing similarly from C_{1-23} and C_{-1-23} , there exists a point $b \epsilon S$ satisfying $b \sim_1 x$, $b \prec_2 x$, $b \succ_3 x$. Arguing from C_{-123} and C_{-12-2} there is $c \epsilon S$ such that $c \prec_1 x$, $c \succ_2 x$, $c \sim_3 x$. Hence, applying transitivity, we have

$$a \underset{1}{\succ} b \underset{1}{\succ} c \quad \text{for voter 1,}$$

$$c \underset{2}{\succ} a \underset{2}{\succ} b \quad \text{for voter 2, and}$$

$$b \underset{3}{\succ} c \underset{3}{\succ} a \quad \text{for voter 3.}$$

We now show the triple $\{a, b, c\}$ fails to satisfy any of the previously listed conditions. None of the three voters is indifferent to all three members of the triple. Moreover, both $b \underset{3}{\succ} a$ or $c \underset{3}{\succ} a$ for voter 3, so a cannot be the alternative required to satisfy the WSP conditions; similarly $c \underset{2}{\succ} b$ and $a \underset{2}{\succ} b$, so b will not do; and finally $a \underset{1}{\succ} c$ and $b \underset{1}{\succ} c$, so c will not do either. Hence the required alternative does not exist, and the triple is not weakly single peaked.

Next, $a \underset{1}{\succ} b$ and $a \underset{1}{\succ} c$ so a will not do for SC; nor will b, since $b \underset{3}{\succ} a$ and $b \underset{3}{\succ} c$; nor c, since $c \underset{2}{\succ} a$ and $c \underset{2}{\succ} b$. Hence single-cavedness fails.

Next, note that $c \underset{2}{\succ} a \underset{2}{\succ} b$ so that a is neither strictly preferred to both others, nor are both others strictly preferred to a, by voter 2; similarly since $a \underset{1}{\succ} b \underset{1}{\succ} c$, and $b \underset{3}{\succ} c \underset{3}{\succ} a$, neither b nor c satisfies the requirement, so the triple is not separable.

Since the triple is not WSP, SC, or Sep, it is not value-restricted.

Since both $a \underset{1}{\succ} b \underset{1}{\succ} c$ and $c \underset{2}{\succ} a \underset{2}{\succ} b$, the triple does not satisfy the extremal restriction.

Since both $a \underset{1}{\succ} b$ and $b \underset{3}{\succ} a$, a, b cannot be the pair required for LA; similarly $b \underset{1}{\succ} c$ and $c \underset{2}{\succ} b$, so b, c will not do, and $a \underset{1}{\succ} c$ and $c \underset{3}{\succ} a$, so a, c also fails. Hence the triple does not satisfy the limited agreement condition. <u>Q.E.D.</u>

The condition of Proposition 2 is stronger than necessary. Progressively weaker alternative conditions are that at some point $x \varepsilon S$ there exist

a) three gradients, no two of which are linearly dependent and no one of which can be expressed as a positive linear combination of the other two.

b) three gradients, no two of which are linearly dependent

c) two linearly independent gradients.

Under (a), the proposition would still be true. Under (b), the WSP, SC, Sep, ER, and LA conditions would fail to hold. Under (c) (which is the condition of Proposition 1) the ER and LA conditions would fail. All of these conditions are so restrictive, however, as to make the differences between them and the original condition of Proposition 3 of little practical importance.

5. Conclusions

We have shown that all presently known exclusion conditions for equilibrium under majority rule are extremely restrictive, in the sense that if the preferences of individual voters over a multidimensional space of commodities or policy vectors are convex and representable by a differentable utility function, then a very modest degree of heterogeneity of tastes makes all known exclusion conditions inapplicable. (The differentiability assumption was made for convenience and is not essential to the argument.) It is difficult to believe that conditions of so a restrictive nature will be of much help in making social welfare judgements, or in understanding political processes based on majority rule.

It might still be hoped that there are other, still-undiscovered exclusion conditions which are not so severely restrictive. However it is straightforward to show that no such conditions can exist. Under any exclusion condition which permits as much heterogeneity of tastes as assumed by Proposition 2, there must exist preference orderings π_1, π_2, π_3 having the properties of the $\underset{1}{\succsim}$, $\underset{2}{\succsim}$ and $\underset{3}{\succsim}$ described in the proof of that proposition, such that any assignment of the following form is admissable:

$$n(\pi_1) = n(\pi_2) = n(\pi_3) > 0 , \quad n(\pi') = 0 \quad \text{for} \quad \pi' \notin \{\pi_1, \pi_2, \pi_3\}$$

With such an assignment majority rule will lead to a being strictly preferred to b , b to c , and c to a . Hence there can be no social ordering over any set $\mathcal{A}$ containing $\{a, b, c\}$ since transitivity fails. Continuing, since no member of $\{a, b, c\}$ is preferred by a majority to both others, there is no social decision function over any $\mathcal{A} \supset \{a, b, c\}$. Even if the definition of a social decision function were weakened by restricting its domain to the subsets of $\mathcal{A}$ which are closed, or compact, or compact and convex, such a function would still not exist, for it can be shown that there exist points a' , b' , c' near x (where $a' = \lambda a + (1-\lambda)x$ for some $0 < \lambda \leq 1$, etc.) whose convex hull contains no stable outcome.

There remains the final possibility that a stable outcome exists in some particular class of sets $\mathcal{A}$ containing $\{a, b, c\}$. Necessary and sufficient conditions for this case have been given by Plott [1967]. These conditions require that for every voter assigned a preference ordering of one type, another voter must be assigned some preference ordering of a complementary type. A condition of this form is not an exclusion restriction nor can it be restated as an exclusion restriction.

Plott's conditions are necessary, in the strong sense that any set of preference orderings not satisfying them will fail to have an equilibrium. Hence no exclusion condition can be sufficient for equilibrium.

Our concluding remarks are directed to the feasibility of other, non-exclusionary equilibrium conditions. No general conditions of this sort have been suggested for the existence of social orderings or social decision functions. However if we restrict the domain of definition to compact sets of alternatives, it is clear that the existence of either of these will imply the existence of a stable outcome, and that Plott's conditions must be satisfied. These conditions are extremely restrictive; they require the set P of preference orderings to satisfy a certain, severe symmetry requirement.[1] The likelihood of their being satisfied in any realistic problem is nil. Cyclical majorities, or the absence of stable outcomes under majority rule, are simply a fact of life.

Thus for welfare economists committed to Arrow's Axioms 2-5, similarity of preferences holds little promise as a possible basis for making social welfare judgements. For political scientists, the hope that useful models might be obtained by abstracting away from all the complexities of actual political processes and focussing on their majority-rule aspects also seems questionable, because of the inherent instability of pure majority rule. It seems likely that useful models will have to employ more flexible equilibrium concepts capable of dealing with this instability, and to take more account of the detailed structure of political mechanisms.

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[1] E.g., an interior point x is a stable outcome if and only if the voters can be paired off as follows there is a one-one correspondence $f : \mathcal{L} \to \mathcal{L}$ (where $\mathcal{L} = \{1, 2, \ldots, \ell\}$) such that

(a) if $\nabla_i(x) = 0$, $f(i) = i$

(b) if $\nabla_i(x) \neq 0$, then $\nabla_{f(i)}(x) = -\alpha\nabla_i(x)$ for some $\alpha > 0$.

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