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PRICE VARIATION DUOPOLY WITH DIFFERENTIATED PRODUCTS AND RANDOM DEMAND

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by

Richard Levitan and Martin Shubik*

1. Introduction

It is shown in this paper that under the appropriate conditions the introduction of a random component to demand in a duopolistic (or more generally oligopolistic) market has the competitive effect of increasing stability in the sense that the market without a random component may have no noncooperative equilibrium point (in pure strategies) whereas the market with a random component has a noncooperative equilibrium point.

This result is related to the previous results of Shubik,[1] Levitan,[2] and Shapley and Shubik[3] on duopolistic demand and competition. These are summarized below.

Suppose two firms with identical constant average costs are supplying identical goods to a market. As was noted by Bertrand[4] if the firms employ price as a strategic variable and if they have no capacity

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limits then after a process of price-cutting an equilibrium will be established with prices equal to cost. Edgeworth[5] analyzing the competition between two firms selling the identical product with rising average costs (or equivalently capacity constraints) observed that no (pure strategy) equilibrium need exist and that price will fluctuate in a range which we may call the Edgeworth Cycle.

It does not seem to be realistic or reasonable to expect prices to bounce up and down incessantly over a wide range in an otherwise static situation. It has been argued that at least Edgeworth's analysis was a step closer to reality to that of Cournot inasmuch as price rather than quantity offered for sale is a more natural independent variable for oligopolists. We feel that neither price nor quantity strategy models are satisfactory but that both price and quantity should be treated as simultaneous independent variables and that inventory shortages and stockouts need to be considered.

Chamberlin added a considerable amount of richness and relevance to the analysis of oligopolistic markets by introducing product differentiation.[6] He examined the equilibrium conditions for a group of firms whose products were symmetrically differentiated and suggested that by means of price policy they would achieve a non-cooperative equilibrium. Figure 1 illustrates this equilibrium for two firms, each with the same constant average costs of production, each selling a symmetrically differentiated product and each facing a linear demand when they charge the same price.

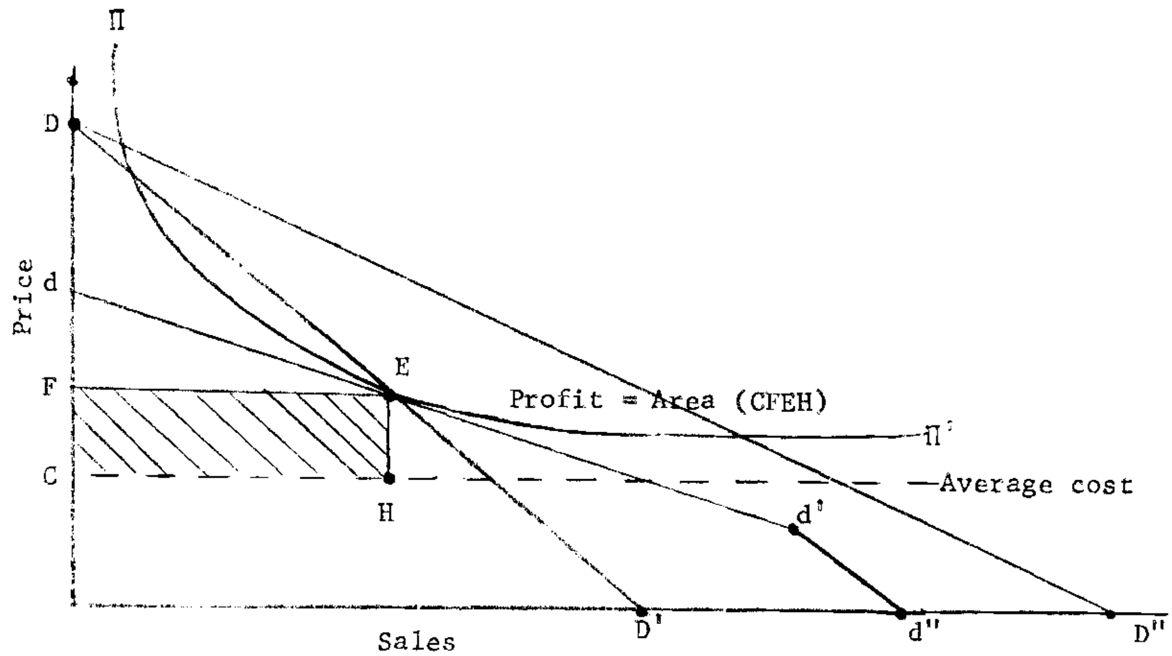

Figure 1

The line DD' is precisely the same as in Chamberlin's analysis[7] and indicates how much an individual will sell on the assumption that the other charges the same price. The line DD'' may be drawn and interpreted as the total amount sold in both markets when each firm charges the same price. Since they are selling differentiated products, we are adding "oranges and apples" when we draw this curve; quantity actually has two dimensions, not one. Nevertheless as the products are symmetric it is useful to draw DD'' provided it is correctly interpreted.

The line segments Dd , dED' and d'd'' describe the contingent demand for one firm as it varies its price from D to zero given that the other maintains its price at the level of E . These line segments are the complete description of the dd' curve of Chamberlin. Their properties and shapes have been investigated by Shubik,[8] Levitan[9] and Shapley and Shubik.[10] The point d is that point at which the higher priced firm has priced itself out of its market. The point d' is the point at which the lower priced firm will have priced its competitor out of his market. The point d'' is the maximum demand attainable for any firm in its market regardless of the price charged by the other. This is easy to see when the meaning of the segment $d''D''$ and the slope of dd' are considered. They both serve as measures of the lack of substitutability between the two commodities. Suppose that we smoothed away the product differentiation so that the firms sell closer and closer substitutes until in the limit they sell identical products. The line dd' will become more and more horizontal as the differentiation diminishes. Furthermore the distance $d''D''$ will shrink until in the limit d'' approaches D'' and the distance is zero. The interpretation is direct. If the two firms sell the identical product, a slight difference in price will give all of the demand to the lower priced firm. If their products are the same then all of the demand will be given by DD''. When there is product differentiation the amount available to any firm will be less than DD''.

The point E is the Chamberlinian noncooperative equilibrium point. Behavioralistically it is exactly the same as the Cournot, Bertrand, Edgeworth or Nash equilibria. The economic analysis may be regarded as more satisfactory than that in other models inasmuch as product differentiation is more realistic. The location of E is determined by examining the family of isoprofit curves and selecting a point on the intersection of DD' with an isoprofit curve such that the curve is tangent to dd' at that point. It is easy to see that the tangency condition guarantees that it is not profitable to undercut or to raise price if both firms are at E .

If both firms were being run solely for the good of the public they would set price equal to marginal cost which in this case is equal to average cost. The rectangle LFEH shows the "monopoly profit" attained by this noncooperative monopolistic competition. As the product differentiation is lessened the distance EH shrinks until (as was noted by Bertrand) with identical products price will equal costs and there will be no profit.

Shapley and Shubik[11] showed that Chamberlin's analysis is incomplete. The mere introduction of product differentiation does not destroy the instability encountered by Edgeworth. If average costs are increasing sufficiently; if there are low enough capacity constraints or if there are inventory carrying costs the Chamberlinian monopolistic competition equilibrium will not exist. Any of these conditions cause an extra "kink" or bend to appear in the segment of the contingent demand denoted by dE . This is shown in Figure 2. The presence of the added kink will destroy the equilibrium when it becomes sufficiently pronounced that a new tangency to the isoprofit curve is possible at G as shown in Figure 2.

The meaning of the additional kink and segment ss' is that although the firm would lose demand along dd' by raising price, this demand loss is based upon the assumption that the competitor can supply any increased demand going his way. If he is unable to do so (owing to capacity limits or inventory shortage) or unwilling to do so (owing to high costs) then the loss in demand will not be so severe to the higher priced firm. Instead of proceeding along s'd it becomes ss'. The existence of G destroys the equilibrium.

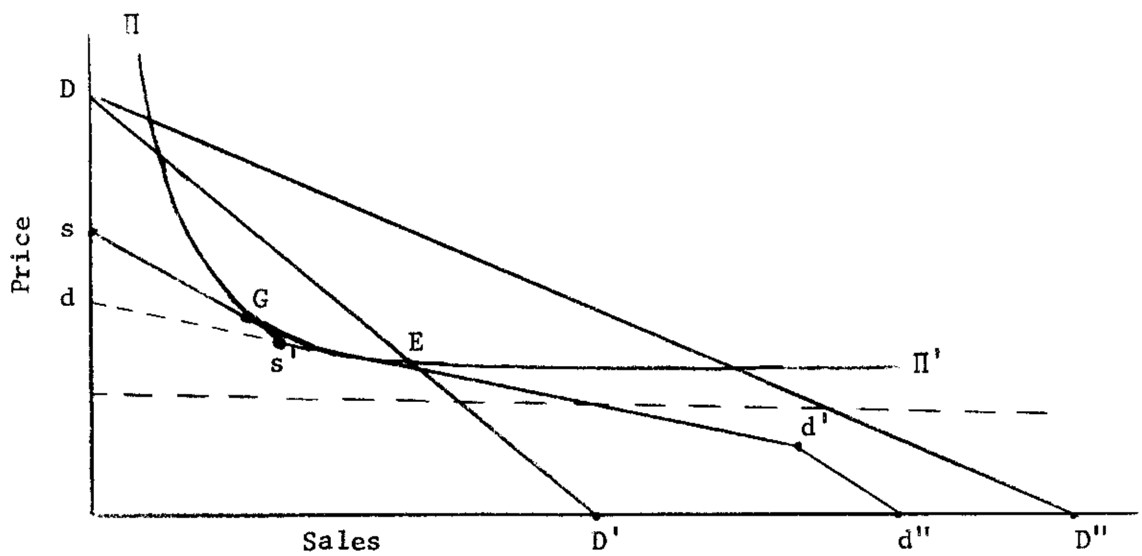

Figure 2

2. The Inventory Problem

It is easy to see that if we introduce inventory costs the noncooperative equilibrium must be destroyed. Let production costs be the same as in Figures 1 and 2, and let there be a positive cost of holding inventory excess to the sales of the current period; and assume further that the firm can supply consumers in the current period only from previously ordered production. The point E can no longer be an equilibrium point as is shown in Figure 3. If it were, then no firm would hold more inventory than exactly enough to satisfy demand at that point. However this is the equivalent of a very severe capacity limitation so that the contingent demand faced by a firm raising its price is Es rather than Ed . This means that the higher priced firm is virtually a monopolist in its price range as the lower priced firm has inventory sufficient only for E .

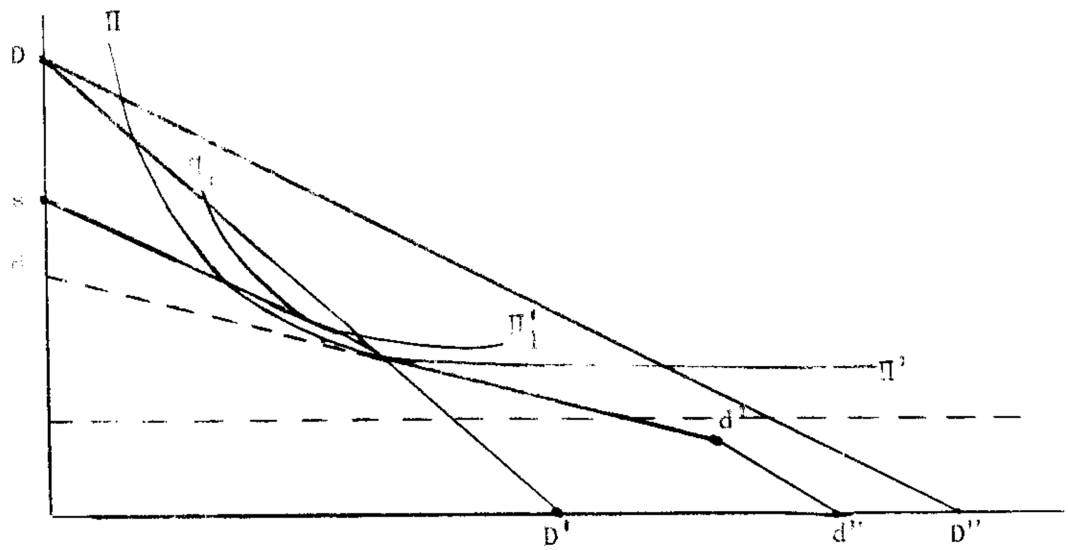

Figure 3

Following the desires expressed by many for more economically reasonable and realistic models in the study of oligopoly theory we maintain that the addition of inventory costs is a step closer to realism and relevance, yet it apparently helps to destroy equilibrium. We shall show that the addition of a further quite realistic complication helps to restore the existence of the equilibrium this complication is a degree of uncertainty in expected demand.

3. An Oligopolistic Market with Random Demand

The general condition needed for stability is that the minimum inventories that the firms are induced to keep by consideration of demand fluctuations at a potential equilibrium point provide sufficient extra stock for each to make it unprofitable for either to change his price. The test for the condition is straightforward: solve the model for a pure strategy equilibrium by using the first order conditions for a local maximum; then verify that no higher payoff can be achieved by having one firm change its price or quantity globally (i.e. over the whole range) given that the other firm is constrained to charging the price and supplying the quantity that is the candidate for the equilibrium.

In practice the stability will depend specifically upon the forms of the demand, production cost and inventory carrying cost functions. In this paper we are unable to offer a general characterization of the functional forms which satisfy the conditions but confine ourselves to the calculation of an example which established the validity of our assertion.

3.1. The Model

Π = profit for the first (or distinguished) firm

p = price

q = demand

x = supply

ρ = inventory carrying cost per unit

The same notation with a bar e.g. $\bar{p}$, stands for the variables of the other firm.

We assume production costs are zero.* Profits are given by:

$$\Pi = p \min(q,\ x) - \rho \max(0,\ x - q) \ . \tag{1}$$

To obtain the demand we follow the calculations for contingent demand illustrated in Figure 1 and 2 as shown in Section 1 and developed mathematically by Shubik,[12] Levitan[13] and Shapley and Shubik.[14] Here, however, we introduce a random component. The constant term is replaced by a random variable.

Let us give a derivation of the curve Dss'd'd" shown in Figure 2. We define the (normalized) ordinary linear demand curve,

$$q(p,\ \bar{p}) = \varepsilon - p - \gamma(p - \bar{p}) \ , \tag{2}$$

which describes the demand for the distinguished firm when there are no stockouts or "price-outs." This function is represented by segment s'd' . The segment d'd" describes the demand when the distinguished firm has priced its competitor out of the market. The demand on this segment is given by

$$\hat{g}(p) = \frac{(1 + 2\gamma)(\varepsilon - p)}{1 + \gamma} \ , \tag{3}$$

which is derived by imputing to the non-distinguished firm that price, $(\varepsilon - \gamma p)/(1 + \gamma)$ which causes its demand to be exactly zero. Finally, the segment ss' gives the demand of the distinguished firm when the non-distinguished firm's demand exceeds its supply, $\bar{x}$. The demand on ss' is given by

$$\hat{q}(p,\ \bar{x}) = \frac{(1 + 2\gamma)(\varepsilon - p) - \gamma\bar{x}}{1 + \gamma} \ . \tag{4}$$

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*This assumption is made for the sake of economy of notation in the sequel. All results are easily generalized to the case of constant costs by substituting p-c for p in the demand and profit functions.

This function is derived similarly by substituting for $\bar{p}$ the price, $(\epsilon - \bar{x} - \gamma p)/(1 + \gamma)$, which makes the second firm's demand equal to its supply.

It is easy to see that the demand function whose graph is Dss'd'd" is equal to

$$q_T(p, \bar{p}, \bar{x}) = \max(0, \hat{q}(p, \bar{x}), \min(q(p, \bar{p}), \hat{\hat{q}}(p))) ,$$

and the actual sales of the distinguished firm are given by

$$s = \min(q_T, x) = \min(x, \max(0, \hat{q}, \min(q, \hat{\hat{q}}))) .$$

In the sequel we introduce the random component into the demand by considering ϵ to be a random variable.

Expressing (1) as a function of actual sales, we can write the profit of the distinguished firm as

$$\Pi = ps - \rho(x - s) = (p + \rho)s - \rho x \tag{5}$$

and by linearity, we can write the expectation of Π as:

$$E(\Pi) = (p + \rho)E(s) - \rho x \;. \tag{6}$$

We may now derive expected sales. This involves deriving the conditions on ϵ for sales to have the values respectively of 0 , $\hat{\hat{q}}$, q , $\hat{q}$ and x .

$$s = 0 \iff \max(\hat{q},\ \min(q,\ \hat{\hat{q}})) \leq 0$$

this implies $\hat{q} \leq 0$ and $\min(q,\ \hat{\hat{q}}) \leq 0$

$$\iff \hat{q} \leq 0 \quad \text{and} \quad (q \leq 0 \quad \text{or} \quad \hat{\hat{q}} \leq 0) \;.$$

$$\iff \epsilon < \max(\epsilon_2,\ \min(\epsilon_1,\ p))$$

(7) where $\epsilon_1 = (1 + \gamma)p - \gamma\bar{p}$ and

$$\epsilon_2 = p + \frac{\gamma}{1 + 2\gamma}\,\bar{x} \;. \tag{8}$$

In a similar manner we may calculate the other four conditions:

$$s = \hat{q} \iff \max(\epsilon_2,\ \epsilon_5) \leq \epsilon \leq \epsilon_7 \;, \tag{9}$$

$$s = q \iff \max(\epsilon_1,\ \epsilon_4) \leq \epsilon \leq \min(\epsilon_5,\ \epsilon_6) \tag{10}$$

$$s = \hat{\hat{q}} \iff p \leq \epsilon \leq \min(\epsilon_3,\ \epsilon_4) \tag{11}$$

and

$$s = \chi \iff \epsilon \geq \max(\epsilon_3,\ \min(\epsilon_6,\ \epsilon_7)) \;. \tag{12}$$

where

$$\epsilon_3 = p + \frac{1 + \gamma}{1 + 2\gamma}\,x \quad , \tag{13}$$

$$(14) \qquad \epsilon_4 = (1 + \gamma)\bar{p} - \gamma p ,$$

$$(15) \qquad \epsilon_5 = (1 + \gamma)\bar{p} - \gamma p + \bar{x} ,$$

$$(16) \qquad \epsilon_6 = (1 + \gamma)p - \gamma\bar{p} + x ,$$

and

$$(17) \qquad \epsilon_7 = p + \frac{\gamma\bar{x} + (1 + \gamma)x}{1 + 2\gamma} .$$

There always is a range for $s = 0$ and $s = x$ to be satisfied, the other cases however may be degenerate. The conditions for non-degeneracy must be specified. Consider (9), for $s = \hat{q}$ to be a possibility we need

$$\epsilon_7 > \max(\epsilon_2, \epsilon_5)$$

$$\Longleftrightarrow \epsilon_7 > \epsilon_2 \text{ and } \epsilon_7 > \epsilon_5 ; \text{ since } \epsilon_7 > \epsilon_2 \text{ by definition}$$

$$\Longleftrightarrow \epsilon_7 > \epsilon_5 \Longleftrightarrow p + \frac{\gamma\bar{x} + (1 + \gamma)x}{1 + 2\gamma} > (1 + \gamma)\bar{p} - \gamma p + \bar{x}$$

or

$$(1 + 2\gamma)(p - \bar{p}) > \bar{x} - x$$

or

$$(17) \qquad p > \bar{p} + \frac{\bar{x} - x}{1 + 2\gamma} .$$

Similarly for the interval in which $s = q$ to be nondegenerate

$$(18) \qquad \bar{p} - \frac{x}{1 + 2\gamma} < p < \bar{p} + \frac{\bar{x}}{1 + 2\gamma} ;$$

and for $s = \hat{\hat{q}}$

(19) $$p < \bar{p}\ .$$

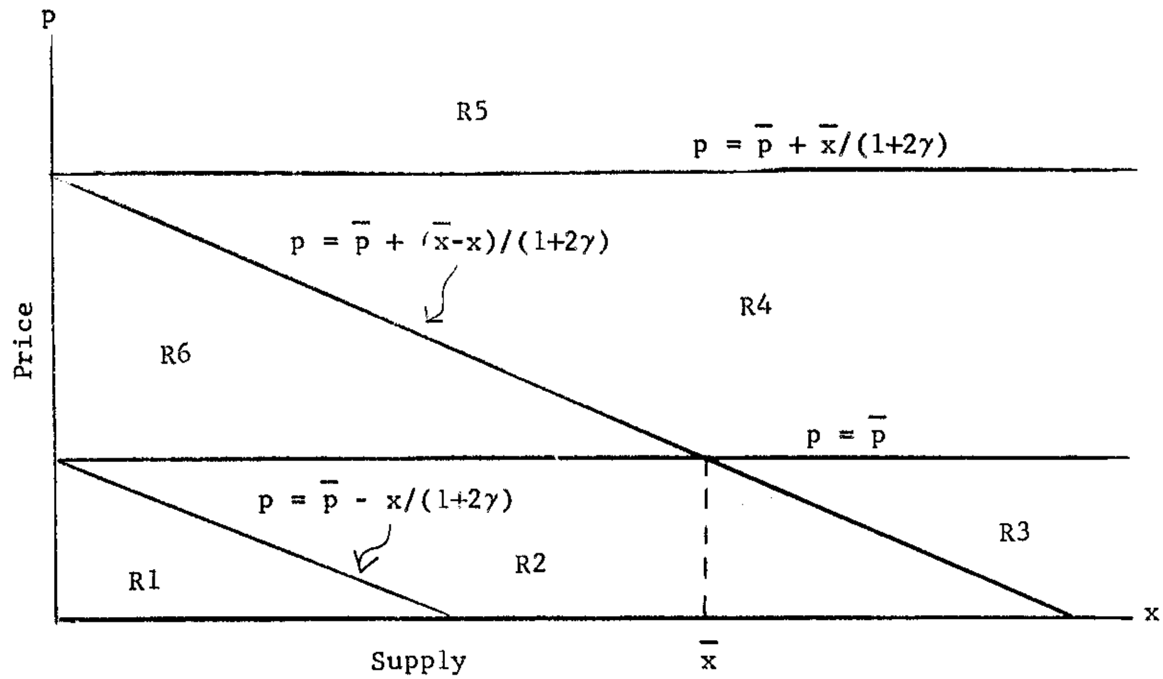

Figure 4

Figure 4 illustrates the six regions in the decision space of the first player in which sales exhibit different qualitative characteristics.

In R_1 $$p < \bar{p} - \frac{x}{1 + 2\gamma}$$

$$s = \begin{cases} x & \text{if} \quad \epsilon \geq \epsilon_3 \\ \hat{q} & \text{if} \quad p \leq \epsilon \leq \epsilon_3 \\ 0 & \text{if} \quad \epsilon \leq p \end{cases} .$$

In R_2 $$\min\left(\bar{p},\ \bar{p} + \frac{\bar{x} - x}{1 + 2\gamma}\right) > p > \bar{p} - \frac{x}{1 + 2\gamma}$$

$$s = \begin{cases} x & \text{if} \quad \epsilon \geq \epsilon_6 \\ q & \text{if} \quad \epsilon_4 \leq \epsilon \leq \epsilon_6 \\ \hat{q} & \text{if} \quad p \leq \epsilon \leq \epsilon_4 \\ 0 & \text{if} \quad \epsilon \leq p \end{cases}$$

In R_3 $$\bar{p} + \frac{\bar{x} - x}{1 + 2\gamma} < p < \bar{p}$$

$$s = \begin{cases} x & \text{if} \quad \epsilon \geq \epsilon_7 \\ \hat{q} & \text{if} \quad \epsilon_5 \leq \epsilon \leq \epsilon_7 \\ q & \text{if} \quad \epsilon_4 \leq \epsilon \leq \epsilon_5 \\ \hat{q} & \text{if} \quad p \leq \epsilon \leq \epsilon_4 \\ 0 & \text{if} \quad \epsilon \leq p \end{cases}$$

In R_4 $$\max\left(\bar{p} + \frac{\bar{x} - x}{1 + 2\gamma},\ \bar{p}\right) \leq p \leq \bar{p} + \frac{\bar{x}}{1 + 2\gamma}$$

$$s = \begin{cases} x & \text{if} \quad \epsilon \geq \epsilon_7 \\ \hat{q} & \text{if} \quad \epsilon_5 \leq \epsilon \leq \epsilon_7 \\ q & \text{if} \quad \epsilon_1 \leq \epsilon \leq \epsilon_5 \\ 0 & \text{if} \quad \epsilon \leq \epsilon_1 \end{cases}$$

In R_5: $$p \geq \bar{p} + \frac{\bar{x}}{1 + 2\gamma}$$

$$s = \begin{cases} x & \text{if} \quad \epsilon \geq \epsilon_7 \\ \hat{q} & \text{if} \quad \epsilon_2 \leq \epsilon \leq \epsilon_7 \\ 0 & \text{if} \quad \epsilon \leq \epsilon_2 \end{cases}$$

Finally in R_6: $$\bar{p} < p < \bar{p} + \frac{\bar{x} - x}{1 + 2\gamma}$$

$$s = \begin{cases} x & \text{if} \quad \epsilon > \epsilon_6 \\ q & \text{if} \quad \epsilon_1 \leq \epsilon \leq \epsilon_6 \\ 0 & \text{if} \quad \epsilon \leq \epsilon_1 \end{cases}$$

Expected profits have been denoted by $E(\Pi)$; for brevity Π will now be interpreted as $E(\Pi)$.

(20) $$\Pi = (p + \rho)E(s) - \rho x$$

where in, for example R_1

(21) $$E(s) = \int_p^{\epsilon_3} \frac{1 + 2\gamma}{1 + \gamma} (\epsilon - \gamma)dF + (1 - F(\epsilon_3))x$$

where F is the distribution function of the random variable ϵ .

In order to examine equilibrium conditions derivatives are needed. We shall compute them in detail in R_1 only.

(22) $$\frac{\partial \Pi}{\partial p} = E(s) + (p + \rho) \frac{\partial}{\partial p} E(s)$$

$$= E(s) - (p + \rho)\,\frac{(1 + 2\gamma)}{1 + \gamma}\,(F(\epsilon_3) - F(p))$$

(23) $$\frac{\partial \Pi}{\partial x} = (p + \rho)(1 - F(\epsilon_3)) - \rho$$

(24) $$\frac{\partial^2 \Pi}{\partial p^2} = - \frac{(1 + 2\gamma)}{1 + \gamma}\,(2(F(\epsilon_3) - F(p)) + (p + \rho)(f(\epsilon_3) - f(p))$$

where f is the density function of ϵ.

(25) $$\frac{\partial^2 \Pi}{\partial x^2} = - (p + \rho)\left(\frac{1 + \gamma}{1 + 2\gamma}\right)f(\epsilon_3) .$$

Finally

(26) $$\frac{\partial^2 \Pi}{\partial x \partial p} = 1 - F(\epsilon_3) - (p + \rho)f(\epsilon_3) .$$

In a like manner the expected profit formulae in the five other regions can be calculated. All are tabulated below.

Region 1: $$(p + \rho)\left[\int_p^{\epsilon_3} \hat{\hat{q}}dF + (1 - F(\epsilon_3)\right] - \rho x$$

Region 2: $$(p + \rho)\left[\int_p^{\epsilon_4} \hat{\hat{q}}dF + \int_{\epsilon_4}^{\epsilon_6} qdF + (1 - F(\epsilon_6)\right] - \rho x$$

Region 3: $$(p + \rho)\left[\int_p^{\epsilon_4} \hat{\hat{q}}dF + \int_{\epsilon_4}^{\epsilon_5} qdF + \int_{\epsilon_5}^{\epsilon_7} \hat{q}dF + x(1 - F(\epsilon_7))\right] - \rho x$$

Region 4: $$(p + \rho)\left[\int_{\epsilon_1}^{\epsilon_5} qdF + \int_{\epsilon_5}^{\epsilon_7} \hat{q}dF + x(1 - F(\epsilon_7))\right] - \rho x$$

Region 5: $$(p + \rho)\left[\int_{\epsilon_2}^{\epsilon_7} \hat{q}dF + x(1 - F(\epsilon_7))\right] - \rho x$$

Region 6: $$(p + \rho)\left[\int_{\epsilon_1}^{\epsilon_6} qdF + x(1 - F(\epsilon_6))\right] - \rho x$$

At a symmetric equilibrium, if any, $p = \bar{p}$ and $x = \bar{x}$ and (p, x) is at the intersection of R_2, R_3, R_4 and R_6.

At $(p, x) = (\bar{p}, \bar{x})$:

$$\epsilon_1 = p ,$$

$$\epsilon_2 = p + \left(\frac{\gamma}{1 + 2\gamma}\right)x ,$$

$$\epsilon_3 = p + \left(\frac{1 + \gamma}{1 + 2\gamma}\right)x ,$$

$$\epsilon_4 = p \quad \text{and}$$

$$\epsilon_5 = \epsilon_6 = \epsilon_7 = p + x .$$

At $(p, x) = (\bar{p}, \bar{x})$ in all the four regions R_2, R_3, R_4 and R_6 the values $\frac{\partial \Pi}{\partial p}$ and $\frac{\partial \Pi}{\partial x}$ are the same. Their values are:

$$\frac{\partial \Pi}{\partial p} = \int_p^{p+x} qdF + x(1 - F(p + x)) - (1 + \gamma)(p + \rho)(F(p + x) - F(p))$$

$$\frac{\partial \Pi}{\partial x} = (p + \rho)(1 - F(p + x)) - \rho .$$

We now limit our discussion to the special case of a rectangular distribution on the variability in demand.

$$\text{Let } F(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{\Delta} & \text{if } a \leq x \leq a+\Delta \\ 1 & \text{if } x \geq a+\Delta \end{cases}$$

$$\text{hence} \qquad f(x) = \frac{dF}{dx} = \begin{cases} \frac{1}{\Delta} & \text{if } x \in [a, a+\Delta] \\ 0 & \text{if } x \notin [a, a+\Delta] \end{cases}$$

At equilibrium $\frac{\partial \Pi}{\partial p} = \frac{\partial \Pi}{\partial x} = 0$ or since at $(\bar{p}, \bar{x})$ $q = \epsilon - p$

then

$$(27) \quad \frac{\partial \Pi}{\partial p} = \int_p^{p+x} (\epsilon - p) f(\epsilon) d\epsilon + x(1 - F(p+x)) - (1+\gamma)(p+\rho)(F(p+x) - F(p)) = 0$$

$$(28) \qquad \frac{\partial \Pi}{\partial x} = (p + \rho)(1 - F(p + x)) - \rho = 0$$

Let us convene that by $\hat{x}$, $\hat{\rho}$ we mean the equilibrium value of ρ , $\bar{p}$, γ and $\bar{x}$. We may use the last condition (28) to solve for $\hat{x}$ as a function of $\hat{p}$. We have

$$\frac{\rho}{p + \rho} = 1 - F(\hat{p} + \hat{x}) = 1 - \frac{\hat{p} + \hat{x} - a}{\Delta}$$

or
$$\hat{p} + \hat{x} = \frac{\Delta\hat{p}}{\hat{p} + \rho} + a$$

giving

(29)
$$\hat{x} = \frac{\Delta\hat{p}}{\hat{p} + \rho} + a - \hat{p} \ .$$

Returning to (27) and substituting in, we note $F(\hat{p} + \hat{x}) = \dfrac{\hat{p}}{\hat{p} + \rho}$

and

(30)
$$\frac{\partial\Pi}{\partial p} = \Phi(p) = \int_p^{\frac{\Delta p}{p+\rho}+a} (\epsilon-p)f(\epsilon) + \left(\frac{\Delta\rho}{p+\rho} + a-p\right)\frac{\rho}{p+\rho} - (\rho-p)(1+\gamma)\left(\frac{p}{\rho+p} - F(p)\right) = 0 \ .$$

There are two cases to consider namely $\hat{p} < a$ and $\hat{p} \geq a$.

In the first $F(\hat{p}) = 0$ and the integral above is

(31)
$$\Phi(p) = \Phi_1(p) = \frac{1}{\Delta}\int_a^{\frac{\Delta p}{p+\rho}+a} (\epsilon-p)d\epsilon + \frac{\Delta\rho p}{(p+\rho)^2} + \frac{(a-p)}{p+\rho} - (1+\gamma)p$$
$$= a + \frac{\Delta}{2} - \left(\frac{\Delta\rho^2}{2(p+\rho)^2} + (2+\gamma)p\right) \ .$$

In the second case, where $\hat{p} \geq a$

(32)
$$\Phi(p) = \Phi_2(p) = \frac{1}{\Delta}\int_p^{p+x} (\epsilon-p) + \frac{xp}{p+\rho} - (1+\gamma)(p+\rho)(F(p+x) - F(p))$$

$$= \frac{1}{2\Delta}\ x^2 + \frac{x\rho}{p+\rho} - (1+\gamma)(p+\rho)\frac{x}{\Delta} \ .$$

(33)
$$= \frac{x}{2\Delta(p+\rho)}\left[\Delta\rho + (a+\Delta+\rho)(p+\rho) - (3 + 2\gamma)(p+\rho)^2\right]$$

We may observe by application of the law of signs to the numerator that there is precisely one root greater than $-\rho$, if $\frac{x}{p+\rho} > 0$. Further:

$$\Phi(a) = \Phi_1(a) = \Phi_2(a) = \frac{\Delta}{2} - \left(\frac{\Delta\rho^2}{2(a+\rho)^2} + (1+\gamma)a\right) \quad (34)$$

Also Φ_1 may be written as:

$$\Phi_1(p) = \frac{2\rho^2 a + 2\rho(\Delta + 2a - \rho(2+\gamma))p + (\Delta + 2a - 4\rho(2+\gamma))p^2 - 2(2+\gamma)p^3}{2(p+\rho)^2}$$

Since all parameters are non-negative there is precisely one sign alteration in the above cubic numerator, hence $\phi_1(p)$ has precisely one positive root.

If $\Phi(a) > 0$, then Φ_2 has a zero greater than a and Φ_1 does not have a zero less than a. Hence, Φ has a unique zero above a. If $\Phi(a)$ is 0, then a is the unique zero of Φ, and finally, if $\Phi(a) < 0$, then Φ_1 has a zero in $[0, a)$ and Φ_2 has no zero above a, and again, Φ has a unique zero below a.

Hence there exists a unique point at which the first order necessary conditions for a pure strategy equilibrium is satisfied. The reader may verify for himself that if the zero of ϕ is greater than a the zero of ϕ_2 is less than $a+\Delta$.

In the second case (where $p \geq a$) we can write down the unique root of ϕ immediately from $\phi_2 = 0$ where ϕ_2 is defined in

(32). This gives

$$\hat{p} = \frac{a + \Delta + \rho + \sqrt{(a + \Delta + \rho)^2 + 4(3 + 2\gamma)\Delta\rho}}{3 + 2\gamma} - \rho . \tag{35}$$

When $\rho = 0$ the roots of both ϕ_1 and ϕ_2 have simple expressions.

$$\phi_1 = \bar{\epsilon} - (2 + \gamma)p$$

Thus if $\hat{p} < a$

$$\hat{p} = \frac{\bar{\epsilon}}{2 + \gamma} \tag{37}$$

and

$$\hat{x} = \frac{1 + \gamma}{2 + \gamma}\bar{\epsilon} + \frac{\Delta}{2} . \tag{38}$$

for $\hat{p} \geq a$

$$\hat{p} = \frac{a + \Delta}{3 + 2\gamma} \tag{39}$$

and

$$\hat{x} = \frac{2(1 + \gamma)}{3 + 2\gamma}(a + \Delta) . \tag{40}$$

We now proceed to discuss necessary and sufficient conditions for a pair of pure strategies to be in equilibrium. That is, given a specific pair of strategies $(\hat{p}, \hat{x})$ and $(\hat{\bar{p}}, \hat{\bar{x}})$, the expected profit function $\Pi(p, x, \bar{p}, \bar{x})$ defined as in (20) has the property

$$\Pi(\hat{p}, \hat{x}, \hat{\bar{p}}, \hat{\bar{x}}) \geq \Pi(p, x, \hat{\bar{p}}, \hat{\bar{x}}) , \quad \text{and}$$

$$\Pi(\hat{\bar{p}}, \hat{\bar{x}}, \hat{p}, \hat{x}) \geq \Pi(\bar{p}, \bar{x}, \hat{p}, \hat{x}) ,$$

for all $p, x, \bar{p}, \bar{x} \geq 0$. In the case of symmetric equilibrium strategies, the above two inequalities are identical.

It will be useful in the sequel to exploit the fact that $\frac{\partial \Pi}{\partial x}$ is a piecewise linear, decreasing continuous function of x. Therefore, for each p, $\bar{p}$, and $\bar{x}$, there is an optimal value of x; called $\hat{x}(p, \bar{p}, \bar{x})$ or $\hat{x}(p)$ which gives a maximum of Π. It can easily be shown that $\hat{x}(p)$ can be expressed as

$$\hat{x}(p, \bar{p}, \bar{\gamma}) = \begin{cases} \max(0, \frac{1+2\gamma}{1+\gamma}(\frac{\Delta p}{p+\rho} + a - p)) & \text{if} \quad (1+\gamma)\bar{p}-\gamma p > \frac{\Delta p}{p+\rho} + a \\ \max(0, \frac{\Delta p}{p+\rho} + a-(1+\gamma)p+\gamma\bar{p}) & \text{if} \quad (1+\gamma)\bar{p}-\gamma p \leq \frac{\Delta p}{p+\rho} + a \leq (1+\gamma)\bar{p}-\gamma p+\bar{x} \\ \max(0, \frac{1+2\gamma}{1+\gamma}(\frac{\Delta p}{p+\rho} + a-p) - \frac{\gamma}{1+\gamma}\bar{x}) & \text{if} \quad (1+\gamma)\bar{p}-\gamma p+\bar{x} \leq \frac{\Delta p}{p+\rho} + a . \end{cases}$$

Since $\Pi(p, \hat{x}(p), \bar{p}, \bar{x})$ dominates $\Pi(p, x, \bar{p}, \bar{x})$ for all $x > 0$, the sufficient condition for a symmetric pure strategy equilibrium at $\hat{p}$, $\hat{x}$ is that $\Pi(p, \hat{x}(p), \hat{p}, \hat{x})$ have its maximum at $\hat{p}$ and that $\hat{x}(\hat{p}) = \hat{x}$.

We note that $\Pi(p, \hat{x}(p), \bar{p}, \bar{x})$ has a continuous first derivative and a discontinuous second derivative. Second order conditions necessary for equilibrium could be written based on a two-sided second derivative. However, since it turns out that such conditions are not sufficient for equilibrium, we shall content ourselves with a remark that a result is derivable there from that at $\hat{p}$ the limit of the upper second derivative of Π is plus infinity when Δ approaches zero and $\gamma > 0$. This gives a proof for the remark noted above that in the nonstochastic case there is no pure strategy equilibrium.

For deriving sufficient conditions analytic methods turned intractable and we resorted to numeric methods of calculation. Our method consisted of a numerical calculation, for any given set of values of the parameters, $\bar{\epsilon}$, Δ , γ , and ρ , the pair of values $\hat{p}$, $\hat{x}$ which satisfy the first order conditions for equilibrium and then a search of the function $\Pi(p)$ for the location of its maximum. If the maximum is at $\hat{p}$ then the solution is indeed an equilibrium. In Figure 5 we give sample curves, representing $\Pi(p)$ and $\hat{x}(p)$ in a case where $\Pi(p)$ has a secondary maximum exactly equal to $\Pi(\hat{p})$.

Noting that the model is homogeneous in the parameters $\bar{\epsilon}$, Δ , and ρ , we conducted a search for the maximal value of γ consistent with equilibrium over a range of the ratios $\rho/\bar{\epsilon}$ and $\Delta/\bar{\epsilon}$, and tabulated them in Table 1. It was our computational experience that values of γ above the tabulated value always resulted in no equilibrium and those below always give an equilibrium solution in pure strategies.

We shall conclude our analysis of the stochastic duopoly model by examining the sensitivity of equilibrium price to changes in the values of the parameters. Since the result is quite apparent, we shall state without proof that $\bar{p}$ is a decreasing function of γ and an increasing function of $\bar{\epsilon}$. We been left to deal with changes in the parameters ρ and Δ . The analysis differs depending on whether $\hat{p}$, the equilibrium price is greater than or less than or equal to a , the lower bound of the random variable ϵ .

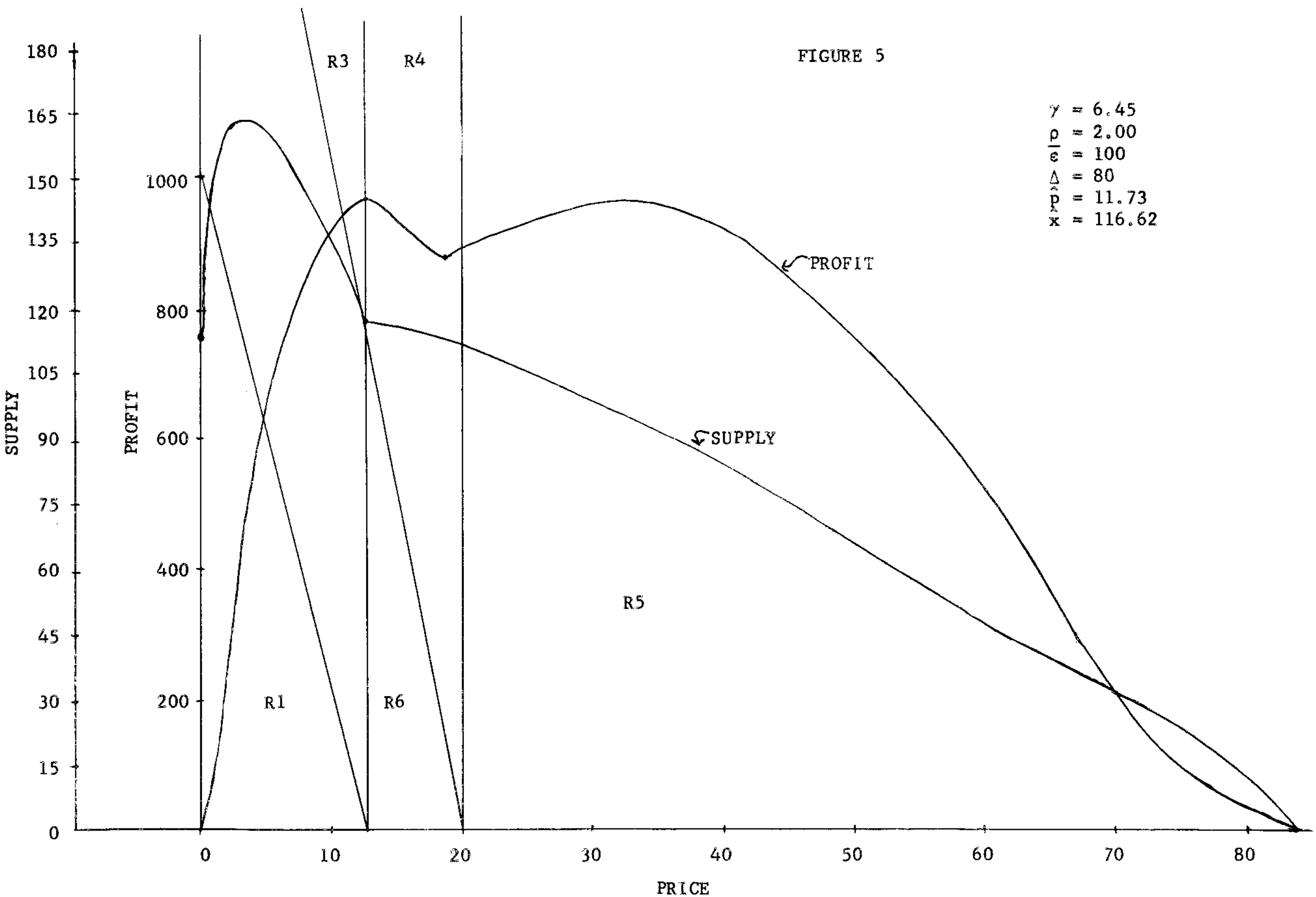
FIGURE 5
γ = 6.45
ρ = 2.00
$\bar{e}$ = 100
Δ = 80
$\hat{p}$ = 11.73
$\bar{x}$ = 116.62
R3
R4
R1
R6
R5
PROFIT
SUPPLY
SUPPLY
PROFIT
PRICE
180
165
150
135
120
105
90
75
60
45
30
15
0
1000
800
600
400
200
0
10
20
30
40
50
60
70
80

In the case where $\hat{p} \geq a$, we have from equation (35)

$$\hat{p} = \frac{\bar{\varepsilon} + \rho + \Delta/2 + \sqrt{(\bar{\varepsilon} + \rho + \Delta/2)^2 + 4(3 + 2\gamma)\Delta\rho}}{2(3 + 2\gamma)}$$

It is immediately apparent that $\hat{p}$ is an increasing function of Δ . The behavior of $\hat{p}$ as a function of ρ is more ambiguous. For large ρ , $\hat{p}$ behaves like

$$\frac{\bar{\varepsilon} + \rho + \Delta/2}{3 + 2\gamma} - \rho$$

which decreases linearly with ρ . However, at $\rho = 0$, we have

$$\frac{\partial p}{\partial \rho} = \frac{1}{3 + 2\gamma} + \frac{\Delta}{\bar{\varepsilon} + \Delta/2} - 1 .$$

Since $\hat{p} = (\bar{\varepsilon} + \Delta/2)/(3 + 2\gamma) \geq a = \bar{\varepsilon} - \Delta/2$, which implies that $1/(3 + 2\gamma) \geq (\bar{\varepsilon} + \Delta/2)/(\bar{\varepsilon} - \Delta/2)$, we have

$$\frac{\partial \hat{p}}{\partial \rho} \geq \frac{\bar{\varepsilon} - \Delta/2}{\bar{\varepsilon} + \Delta/2} + \frac{\Delta}{\bar{\varepsilon} + \Delta/2} - 1 = 0 .$$

Thus we have that $\hat{p}$ increases with ρ for small ρ and then decreases for large ρ until the regime changes and $\hat{p} < a$.

Finally we examine the sensitivities in the case where $\hat{p} < a$. Here we do not have a closed form solution for $\hat{p}$; but since we know that $\hat{p}$ is a zero of ϕ_1 at a point where $\partial\phi/\partial p < 0$, we can compute $\partial\hat{p}/\partial\Delta$ and $\partial\hat{p}/\partial\rho$ respectively as

TABLE 1

MAXIMAL GAMMA FOR PURE STRATEGY EQUILIBRIUM

	$\Delta/\bar{\epsilon}$										
$p/\bar{\epsilon}$	0.025	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.0	0.71	1.02	1.55	2.54	3.61	4.84	6.30	7.50	8.31	8.83	9.15
0.010	0.70	1.01	1.51	2.43	3.38	4.40	5.53	6.47	7.02	7.32	7.44
0.020	0.69	0.99	1.47	2.33	3.19	4.08	5.01	5.81	6.25	6.45	6.50
0.030	0.69	0.97	1.44	2.25	3.04	3.83	4.63	5.34	5.71	5.86	5.86
0.040	0.68	0.96	1.41	2.18	2.91	3.62	4.33	4.97	5.31	5.42	5.39
0.050	0.67	0.94	1.38	2.12	2.80	3.45	4.09	4.67	4.98	5.07	5.03
0.060	0.66	0.93	1.35	2.06	2.70	3.30	3.88	4.43	4.72	4.80	4.74
0.070	0.65	0.92	1.33	2.01	2.61	3.18	3.71	4.21	4.50	4.56	4.50
0.080	0.65	0.90	1.30	1.96	2.53	3.06	3.56	4.03	4.31	4.37	4.29
0.090	0.64	0.89	1.28	1.91	2.46	2.96	3.43	3.86	4.14	4.20	4.12
0.100	0.63	0.88	1.26	1.87	2.40	2.88	3.31	3.72	4.00	4.05	3.97
0.110	0.63	0.87	1.24	1.84	2.34	2.79	3.21	3.59	3.86	3.92	3.84
0.120	0.62	0.86	1.23	1.80	2.29	2.72	3.11	3.47	3.75	3.80	3.72
0.130	0.61	0.85	1.21	1.77	2.24	2.65	3.03	3.37	3.64	3.70	3.61
0.140	0.61	0.84	1.19	1.74	2.19	2.59	2.95	3.27	3.54	3.60	3.52
0.150	0.60	0.83	1.18	1.71	2.15	2.54	2.88	3.19	3.45	3.52	3.43
0.200	0.58	0.79	1.11	1.59	1.97	2.30	2.59	2.85	3.07	3.18	3.11
0.300	0.54	0.73	1.01	1.42	1.73	2.00	2.23	2.42	2.59	2.73	2.74
0.400	0.51	0.68	0.94	1.30	1.57	1.80	2.00	2.16	2.30	2.41	2.50
0.500	0.48	0.65	0.88	1.21	1.46	1.66	1.84	1.98	2.10	2.20	2.28
0.600	0.46	0.62	0.83	1.14	1.37	1.56	1.72	1.85	1.96	2.05	2.13
0.700	0.44	0.59	0.80	1.08	1.30	1.47	1.62	1.74	1.85	1.94	2.01
0.800	0.43	0.57	0.76	1.03	1.24	1.40	1.54	1.66	1.76	1.84	1.91
0.900	0.41	0.55	0.74	0.99	1.19	1.34	1.48	1.59	1.68	1.76	1.83
1.000	0.40	0.53	0.71	0.96	1.14	1.29	1.42	1.53	1.62	1.70	1.76
1.100	0.39	0.52	0.69	0.93	1.10	1.25	1.37	1.48	1.57	1.64	1.71
1.200	0.38	0.50	0.67	0.90	1.07	1.21	1.33	1.43	1.52	1.59	1.66
1.300	0.37	0.49	0.65	0.87	1.04	1.17	1.29	1.39	1.47	1.55	1.61
1.400	0.36	0.48	0.64	0.85	1.01	1.14	1.25	1.35	1.43	1.51	1.57
1.500	0.36	0.47	0.62	0.83	0.99	1.11	1.22	1.32	1.40	1.47	1.53

TABLE 1 (continued)

$\rho/\bar{\epsilon}$	$\Delta/\bar{\epsilon}$ 1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
0.0	9.30	9.33	9.27	9.13	8.95	8.72	8.46	8.17	7.88	7.88	7.88
0.010	7.44	7.35	7.20	7.01	6.78	6.54	6.28	6.01	5.88	5.88	5.88
0.020	6.43	6.30	6.12	5.91	5.68	5.43	5.18	4.92	4.88	4.88	4.87
0.030	5.77	5.61	5.42	5.20	4.96	4.71	4.46	4.24	4.24	4.23	4.23
0.040	5.28	5.11	4.91	4.68	4.44	4.20	3.95	3.78	3.78	3.77	3.77
0.050	4.90	4.72	4.51	4.28	4.04	3.80	3.55	3.43	3.42	3.41	3.41
0.060	4.60	4.41	4.20	3.96	3.72	3.48	3.23	3.15	3.14	3.13	3.13
0.070	4.35	4.16	3.94	3.70	3.46	3.21	2.97	2.91	2.90	2.90	2.89
0.080	4.14	3.94	3.72	3.48	3.24	2.99	2.74	2.72	2.71	2.70	2.69
0.090	3.96	3.76	3.53	3.29	3.94	2.79	2.56	2.55	2.54	2.53	2.52
0.100	3.81	3.60	3.37	3.13	2.88	2.63	2.41	2.40	2.39	2.38	2.38
0.110	3.67	3.46	3.23	2.98	2.73	2.48	2.28	2.27	2.26	2.25	2.24
0.120	3.55	3.34	3.10	2.85	2.60	2.34	2.16	2.15	2.14	2.13	2.13
0.130	3.45	3.23	2.99	2.74	2.48	2.23	2.06	2.05	2.04	2.03	2.02
0.140	3.35	3.13	2.89	2.64	2.38	2.12	1.97	1.95	1.94	1.93	1.93
0.150	3.26	3.05	2.80	2.54	2.28	2.02	1.88	1.87	1.86	1.85	1.84
0.200	2.94	2.72	2.46	2.19	1.92	1.64	1.54	1.53	1.52	1.51	1.50
0.300	2.60	2.37	2.10	1.81	1.50	1.18	1.13	1.11	1.10	1.08	1.07
0.400	2.43	2.23	1.96	1.64	1.30	0.94	0.88	0.86	0.84	0.82	0.81
0.500	2.32	2.18	1.93	1.60	1.24	0.83	0.70	0.68	0.66	0.65	0.63
0.600	2.12	2.16	1.96	1.64	1.26	0.82	0.58	0.56	0.53	0.52	0.50
0.700	2.06	2.09	2.00	1.73	1.35	0.89	0.49	0.46	0.44	0.42	0.40
0.800	1.96	2.00	2.01	1.83	1.49	1.03	0.42	0.39	0.36	0.34	0.32
0.900	1.88	1.92	1.94	1.90	1.64	1.21	0.60	0.33	0.30	0.27	0.25
1.000	1.82	1.86	1.88	1.89	1.78	1.42	0.85	0.28	0.25	0.22	0.20
1.100	1.76	1.80	1.83	1.85	1.84	1.62	1.12	0.24	0.21	0.18	0.15
1.200	1.71	1.75	1.78	1.80	1.81	1.76	1.40	0.68	0.18	0.15	0.11
1.300	1.66	1.71	1.74	1.76	1.78	1.78	1.63	1.07	0.15	0.12	0.08
1.400	1.62	1.67	1.70	1.73	1.74	1.75	1.74	1.42	0.13	0.10	0.06
1.500	1.58	1.63	1.67	1.70	1.72	1.72	1.72	1.66	1.02	0.08	0.04

$$- \frac{\partial \phi_1/\partial \Delta}{\partial \phi_1/\partial p} \quad \text{and} \quad - \frac{\partial \phi_1/\partial \rho}{\partial \phi_1/\partial p} \, .$$

We have, from equation (31),

$$\phi_1 = \bar{\epsilon} - (2 + \gamma)p - \frac{\Delta\rho^2}{2(p + \rho)^2} \, ,$$

which immediately implies that $\partial \phi_1/\partial \Delta \leq 0$ and hence $\partial \hat{p}/\partial \Delta \leq 0$. Finally,

$$\frac{\partial \phi_1}{\partial \rho} = - \frac{\Delta p \rho}{(p + \rho)^3} \leq 0 \; ;$$

and in this regime $\hat{p}$ is a decreasing function of both Δ and ρ .

Conclusions

We have shown that if one improves the realism of the Chamberlinian model by adding inventory carrying casts and making the assumptions that production takes time that the pure strategy noncooperative equilibrium postulated by Chamberlin never exists. Instability of the type suggested by Edgeworth is all that remains.

By introducing a further complication into the model the equilibrium may be restored. This too is a step towards realism. It is the introduction of a random element to overall market size.

We leave as open problems the generalizations and the extension of our work to the n-person symmetric and nonsymmetric cases. Our experience with models of this variety indicates that it is a safe conjecture that our results go through for the nonsymmetric market model we have developed elsewhere. Although our results are mathematically inelegant, our conjecture is of substantive and theoretical interest. While our proof of the <u>nonexistence</u> of the Chamberlinian equilibrium is perfectly general, our proof of the existence of equilibrium under uncertainty uses a specially simple example. In order to obtain a more general result, a more powerful and different type of mathematical approach is undoubtedly needed.

FOOTNOTES

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