COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 193

Random Measure Preserving Transformations*

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September 2, 1965

* Research undertaken by the Cowles Commission for Research in Economics under Task NR 047-006 with the Office of Naval Research.

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June, 1965

Introduction.

Let $\underset{\sim}{I}$ be the measure algebra of Lebesgue measurable subsets of the unit interval I, modulo the sets of measure 0 (i.e. the algebra in which sets differing by sets of measure 0 are not distinguished). Let $\mathfrak{A}$ be the group of Lebesgue measure preserving automorphisms of $\underset{\sim}{I}$; the members of this group may be thought of as invertible measure-preserving transformations from I onto itself, where two transformations are identified if they differ on a set of measure 0 only. The group $\mathfrak{A}$ has been topologized in at least two different ways (see Halmos [2]); in one of those topologies (the "weak" topology) it has been proved that the set of ergodic transformations (and in fact, the set of weakly mixing transformations) is of the second category, and the set of strongly mixing transformations is of the first category (see [2], p. 77 ff.).

The question now arises as to whether it is possible to impose a probability structure on $\mathfrak{A}$, i.e. whether there is some natural way to define $\mathfrak{A}$ as a finite measure space. Then, corresponding to the information about the "topological size" of various subsets of $\mathfrak{A}$ that the category theorems yield, one could seek information about the probability - or measure - of those sets.

The purpose of this note is to show that there is no natural way to define $\mathfrak{A}$ as a probability space -- at least if "natural" is given the meaning that we give it here. This is additional evidence of the comparative intractability of function spaces when viewed from the measure-theoretic, rather than from the topological viewpoint (for some previous evidence, see Aumann [1]).

It seems likely that this theorem is not new; any contribution that this note may make is probably in the short and elementary proof.

2. Statement of the Theorem.

Throughout this note we will treat members of $\underset{\sim}{I}$ as if they were subsets of I, speaking of unions, intersections, inclusions, and so on. No confusion will result.

λ will denote Lebesgue measure throughout.

Theorem. *There is no pair* (Γ,μ), *where* Γ *is a* σ-*field of subsets of* $\mathfrak{T}$, *and* μ *is a probability measure on* Γ, *for which*

(2.1) *For each* $\mathscr{H} \in \Gamma$ *and* $T \in \mathfrak{T}$, *we have* $\mathscr{H}T \in \Gamma$, *and* $\mu(\mathscr{H}T) = \mu(\mathscr{H})$.

(2.2) *For all* E *and* F *in* $\underset{\sim}{I}$, *the function from* $\mathfrak{T}$ *to the reals defined by*

$$f(T) = \lambda(E \cap T\,F)$$

is Γ-*measurable*.

A few words of explanation are in order. "Probability measure", of course, means that $\mu(\mathfrak{T}) = 1$. Condition (2.1) is the right invariance condition; it says that if a Γ-measurable set of transformation is multiplied on the right by a single transformation, then it remains Γ-measurable, and its μ-measure (probability) remains unchanged. Without some such condition it would be trivially possible to construct a probability measure on $\mathfrak{T}$, for example by concentrating all the probability on one transformation T. Condition (2.2) is a measurability assumption which seems very reasonable.

The theorem remains true if $\mathscr{H}T$ is replaced by $T\mathscr{H}$ in Condition (2.1), i.e. if right-invariance is replaced by left-invariance. Condition (2.2) remains unchanged.

3. Proof of the Theorem.

It will be assumed throughout that there is a given pair (Γ,μ) obeying the specifications of the theorem, and this will lead eventually to a contradiction.

Often it will be convenient to use the language of probability, i.e. to replace μ by "Prob", $\int_{\mathfrak{T}} \mu(dT)$ by "Exp" (for "Expectation"), and so on. "Variance" will be abbreviated by "Var", and "covariance" by "Cov"; like "Exp", these two operators will be applied exclusively to random variables defined on the probability space $(\mathfrak{T},\Gamma,\mu)$.

Lemma 1. Let D, F_1, $F_2 \in \underset{\sim}{I}$, and $\lambda(F_1) = \lambda(F_2)$. Then

$$\operatorname{Exp} \lambda(D \cap T F_1) = \operatorname{Exp} \lambda(D \cap T F_2).$$

Proof. Let S be a member of $\mathcal{B}$ such that $S F_1 = F_2$. Define measures η_1 and η_2 on the closed unit interval $[0,1]$ by

$$\eta_i[0,\alpha] = \mu\{T: \lambda(D \cap T F_i) \leqq \alpha\}$$

for $i = 1, 2$. Then

$$\begin{aligned}\eta_2[0,\alpha] &= \mu\{ T: \lambda(D \cap T S F_1) \leqq \alpha\} \\ &= \mu\{TS: \lambda(D \cap T S F_1) \leqq \alpha\} \\ &= \mu\{ U: \lambda(D \cap U F_1) \leqq \alpha\} \\ &= \eta_1[0,\alpha],\end{aligned}$$

where the second equality follows from (2.1) and the third by setting $U = T S$ and noting that as T runs over $\mathcal{B}$, so does U. From this it follows that

$$\operatorname{Exp} \lambda(D \cap T F_2) = \int_0^1 \alpha \, \eta_2(d\alpha) = \int_0^1 \alpha \, \eta_1(d\alpha) = \operatorname{Exp} \lambda(D \cap T F_1),$$

which is the assertion of the lemma.

Lemma 2. For all $D, F, \in \underset{\sim}{I}$,

$$\operatorname{Exp} \lambda(D \cap T F) = \lambda(D) \lambda(F).$$

Proof. For an arbitrary but fixed positive integer m, let $F_1, \cdots, F_m$ be disjoint members of $\underset{\sim}{I}$ with equal measure, whose union is I. Then $\lambda(F_i) = 1/m$ for all i. From lemma 1 it follows that $\operatorname{Exp} \lambda(D \cap T F_i)$ does not depend on i; let us denote it by γ. Now

$$\begin{aligned}\lambda(D) &= \operatorname{Exp} \lambda(D) = \operatorname{Exp} \lambda(D \cap T I) \\ &= \operatorname{Exp} \lambda(D \cap T \textstyle\bigcup_{i=1}^m F_i) = \operatorname{Exp} \sum_{i=1}^m \lambda(D \cap T F_i) \\ &= \textstyle\sum_{i=1}^m \operatorname{Exp} \lambda(D \cap T F_i) = m\gamma.\end{aligned}$$

Hence $\gamma = \lambda(D)(1/m) = \lambda(D) \lambda(F_i)$, for $i = 1, \cdots, m$.

Now whenever $\lambda(F) = 1/m$ for some m, it is possible to set $F_1 = F$ and to find $m - 1$ sets $F_2, \cdots, F_m$ satisfying the above conditions; hence whenever

$\lambda(F)$ is the reciprocal of an integer, the assertion of the lemma is established. But each measurable set $F \in \underset{\sim}{I}$ is a countable union of sets whose measures are reciprocals of integers; and since Exp is countably additive for non-negative random variables, the assertion of the lemma follows in the general case as well.

Before stating the next lemma, we introduce the following notation: For $D, E, F \in \underset{\sim}{I}$ and $E \cap F = \emptyset$, we write

$$g(E,F) = g_D(E,F) = \text{Exp}\ [\lambda(D \cap T\ E)\ \lambda(D \cap T\ F)].$$

Lemma 3. Let $D, E, F_1, F_2 \in \underset{\sim}{I}$, and $F_1 \cap E = F_2 \cap E = \emptyset$, $\lambda(F_1) = \lambda(F_2)$. Then

$$g_D(E,F_1) = g_D(E,F_2).$$

Proof. The proof is similar to that of lemma 1. This time, let S be a member of $\mathcal{G}$ such that both $S\ F_1 = F_2$ and $S\ E = E$. Define measures η_1 and η_2 on $[0,1]$ by $\eta_i[0,\alpha] = \mu\ \{T: \lambda(D \cap T\ E)\ \lambda(D \cap T\ F_i) \leqq \alpha\}$; then because of (2.1),

$$\eta_2[0,\alpha] = \eta_1[0,\alpha]$$

for all x, and hence

$$\int_0^1 \alpha\ \eta_2(d\alpha) = \int_0^1 \alpha\ \eta_1(d\alpha);$$

but that is precisely what the lemma asserts.

Lemma 4. Let $D, E, F \in \underset{\sim}{I}$, and $E \cap F = \emptyset$. Then

$$g_D(E,F) \leqq \lambda^2(D)\ \lambda(E)\ \lambda(F).$$

Proof. If $\lambda(E)$ or $\lambda(F)$ vanish, there is nothing to prove; assume therefore that $\lambda(E) > 0$, $\lambda(F) > 0$, and so $\lambda(E) < 1$, $\lambda(F) < 1$. For an arbitrary but fixed positive inter m, let $F_1, \cdots, F_m$ be disjoint members of $\underset{\sim}{I}$ with equal measure, whose union is $I - E$. Then

$$\lambda(F_i) = (1 - \lambda(E)/m$$

for all i. From lemma 3 it follows that $g(E,F_i)$ does not depend on i; denote it by γ. Now

$$g(E,I\setminus E) = g(E,\textstyle\bigcup_{i=1}^m F_i) = \sum_{i=1}^m g(E,F_i) = m\gamma;$$

Hence

$$g(E,F_i) = \gamma = g(E,I\setminus E)/m = \lambda(F_i)\ g(E,I\setminus E)/(1 - \lambda(E)).$$

Whenever $F \subset I\setminus E$ and $\lambda(F) = (1 - \lambda(E))/m$ for some m, it is possible to set $F_1 = F$ and to find $m - 1$ sets $F_2, \cdots, F_m$ satisfying the above conditions; hence, for such F, we have

$$g(E,F) = \lambda(F)\ \frac{g(E,I\setminus E)}{(1 - \lambda(E))}\ . \tag{3.1}$$

But each set $F \subset I\setminus E$ is a countable union of such F; and so (3.1) follows for all F with $E \cap F = \emptyset$. Now

$$\begin{aligned} g(E,I\setminus E) &= \mathrm{Exp}\ [\lambda(D \cap T\ E)(\lambda(D) - \lambda(D \cap T\ E))] \\ &\leqq \mathrm{Exp}\ [\max_{0 \leqq \beta \leqq \lambda(d)} \beta(\lambda(D) - \beta)] \\ &= \mathrm{Exp}\ [\lambda^2(D)/4] = \lambda^2(D)/4. \end{aligned}$$

Then choosing E_0 so that $\lambda(E_0) = 1/2$, and applying (3.1), we find

$$g(E_0,F) \leqq \lambda(F)\ \frac{\lambda^2(D)/4}{1/2} = \lambda(F)\ \lambda(E_0)\ \lambda^2(D) \tag{3.2}$$

whenever $E_0 \cap F = \emptyset$. Now by using (3.1) and the symmetry of g in its two variables, we obtain

$$g(E,F) = \lambda(E)\ \frac{g(F,I\setminus F)}{(1 - \lambda(F))} \tag{3.3}$$

whenever $E \cap F = \emptyset$. Setting $E = E_0$ in (3.3) and combining with (3.2), we deduce

$$g(F,I\setminus F)/(1 - \lambda(F)) \leqq \lambda(F)\ \lambda^2(D)$$

whenever $E_0 \cap F = \emptyset$. Combining this with (3.3), we obtain

$$g(E,F) \leqq \lambda(E)\ \lambda(F)\ \lambda^2(D)$$

whenever $E_0 \cap F = \emptyset$ and $E \cap F = \emptyset$. Now whenever $\lambda(F) \leqq 1/2$, it is possible to choose E_0 so that $E_0 \cap F = \emptyset$; since $E \cap F = \emptyset$ by the hypothesis of the lemma, the lemma is proved in those cases. When $\lambda(F) > 1/2$, we may express F as the union of two disjoint subsets each of measure $\leqq 1/2$; the lemma then follows from the additivity in F both of $g(E,F)$ and of $\lambda(F)$.

Lemma 5. For all D, $F \in \underset{\sim}{I}$,

$$\operatorname{Var} \lambda(D \cap T F) = 0.$$

Proof. If $D = I$ there is nothing to prove; assume, therefore, that $\lambda(D) < 1$. Let $F_1, \cdots, F_n$ be disjoint members of $\underset{\sim}{I}$, with equal measures, such that $\bigcup_{i=1}^{n} F_i = F$; then $\lambda(F_i) = \lambda(F)/n$ for all i. Assume $n > 1$. Define random variables $\underset{\sim}{x}_1, \cdots, \underset{\sim}{x}_n$ by $\underset{\sim}{x}_i = \lambda(D \cap T F_i)$. Then

$$\operatorname{Var} \lambda(D \cap T F) = \Sigma_{i=1}^{n} \operatorname{Var} \underset{\sim}{x}_i + 2\Sigma_{i>j} \operatorname{Cov}(\underset{\sim}{x}_i, \underset{\sim}{x}_j).$$

Now by lemmas 2 and 4,

$$\begin{aligned} &\operatorname{Cov}(\underset{\sim}{x}_i, \underset{\sim}{x}_j) \\ &= g(F_i, F_j) - (\operatorname{Exp} \lambda(D \cap T F_i))(\operatorname{Exp} \lambda(D \cap T F_j)) \\ &\leqq \lambda(F_i)\, \lambda(F_j)\, \lambda^2(D) - (\lambda(D)\, \lambda(F_i))(\lambda(D)\, \lambda(F_j)) \\ &= 0. \end{aligned}$$

On the other hand, $\underset{\sim}{x}_j$ is clearly bounded by $\lambda(F_i) = \lambda(F)/n \leqq 1/n$, so $\operatorname{Var} \underset{\sim}{x}_i \leqq 1/n^2$. Hence

$$\operatorname{Var} \lambda(D \cap T F) \leqq n/n^2 = 1/n.$$

Letting $n \to \infty$, we deduce the conclusion of the lemma.

Suppose now that $F = [0, \frac{1}{2}]$. Then it follows from lemma 5 that with probability 1, $T F$ intersects every rational interval D in a set of measure $\frac{1}{2}\lambda(D)$. But then with probability 1, $T F$ is a set of density $\frac{1}{2}$ at each point; whereas it is known that there are no such Lebesgue measurable sets. This is the contradiction that establishes our theorem.

The corresponding theorem when right invariance is replaced by left invariance can be proved in a similar manner. Alternatively, if (Δ, ν) is a pair satisfying (2.2) and the left-invariant analogue of (2.1), define

$$\Gamma = \{\mathcal{H} \subset \mathcal{G} : \mathcal{H}^{-1} \in \Delta\},$$

where

$$\mathcal{H}^{-1} = \{T \in \mathcal{G} : T^{-1} \in \mathcal{H}\};$$

and define μ on Γ by $\mu(\mathcal{H}) = \nu(\mathcal{H}^{-1})$. Then it may be verified that (μ,Γ) satisfies (2.1) and (2.2), and so contradicts the main theorem; this establishes the left-invariant version.

REFERENCES

[1] R.J. Aumann, "Borel structures in function spaces," Illinois Jour. Math. 5 (1961), pp. 614-630.

[2] P.R. Halmos, Lectures on Ergodic Theory, New York, Chelsea, 1956.