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A DYNAMIC MODEL OF THE COMPETITIVE FIRM

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by

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I. Introduction

In the past few years a number of authors have attempted to develop a dynamic theory of monopoly, variously taking into account uncertainty of demand or lags in demand and allowing holdings of the final good, explicitly or implicitly, either by the monopolist or consumers [1, 2, 3, 4, 6]. Little attention, however, has been given to an explicit dynamic model of the purely competitive firm. The only recent, though fragmentary, attempt is contained in a study by Edwin Mills [Ch. 4, 4].

Apart from a number of possible uses, a dynamic theory of the competitive firm would seem to be particularly useful in the study of dynamic adjustment and stability in competitive markets. In this paper we offer a contribution in this direction by beginning an investigation of the competitive firm's behavior over a finite or infinite horizon. In the next section we specify the model, and in subsequent sections we consider the questions of existence and uniqueness and the properties of optimal behavior. We are able to show specific circumstances in which behavior in the static theory and the dynamic models coincide or differ and are able to obtain important features of optimal production and sales policies in a number of cases.

II. A Model of the Competitive Firm

We assume that time is discrete and that the horizon consists of a finite or infinite number of time periods of equal length. In each period the firm, which produces a single good with a given plant, has two decisions. It has to decide how much to produce prior to learning the market price and how much of its stock of the good to sell after the market price is revealed. In other words, we assume that the firm makes its production decision at the beginning of the period before the competitive price is decided and its sale decision at the end of the period when price is known.

To guide its decisions the firm supposes some probability distribution of prices to prevail at the start of the present and each subsequent period, and it takes as its objective the maximization of the present discounted value of expected profits. It obtains revenue by selling any amount of its stock of the good at prevailing market prices, and it incurs costs, above any fixed costs, by producing and by holding inventories of the good.

To present the model formally we use the following notation for non-negative variables:

x = the firm's initial inventory, showing the stock of the good on hand at the start of a period,

y = the starting stock of the good, showing the stock on hand in a period after production but prior to any sales, supposing no lag in production,

$q = y - x$ = production,

s = the amount of sales at the end of a period after price is known,

$y - s$ = the ending inventory, the stock of the good remaining at the end of a period after any sales. The ending inventory of a period is the initial inventory of the next period,

p = price of the good.

Unless a subscript is added, these variables refer to activities in the first period.

We assume that the firm holds inventories only of the final good, not of inputs, so that we can represent production cost in each period by the familiar variable production cost function, say, $c(y-x)$. We suppose that this function is twice differentiable and has the properties that marginal production cost is positive, $c' > 0$, and increasing, $c'' > 0$.[2] The cost of holding inventory in each period is assumed to depend on the ending inventory. Represent this cost by the function $h(y - s)$ which we suppose is twice differentiable with positive marginal holding cost, $h' > 0$. We shall examine cases in which the marginal holding cost is constant, $h'' = 0$, or increasing, $h'' > 0$. The subjective density of prices is given by $\varphi(p)$ which is differentiable with domain $0 \leq p \leq \infty$.[3] In this paper we only examine the case where $\varphi(p)$ is the same in each period. Finally, future revenues and costs are discounted by a discount factor α, bounded by zero and one.

The model we have outlined is similar to one proposed by Mills, though Mills had only limited success in obtaining properties of optimal behavior [Ch. 4, 4].[4] As a preliminary step in our investigation, we propose to consider first a pure sales model in which the firm does not produce but begins with some stock of the good and has to decide how to sell this stock over a finite or infinite horizon, supposing marginal holding cost is constant. We then introduce production into the model. Next we consider a pure sales model with increasing marginal holding cost, followed by the same model with production. Before turning to the sales model we end this section by introducing the functional equation representations of the sales model and the production and sales model which we use subsequently.

In the pure sales model the firm knows the market price in the first period at the time of sales but does not know future prices. For any initial stock and any current market price, and supposing an infinite horizon, it has to select a sales policy which maximizes the present discounted value of expected profits over the horizon. Let $r(x;p)$ represent this maximum discounted expected profits as a function of the initial stock x and the current price p. The profit in the first period is $p \cdot s - h(x-s)$, and the maximum discounted expected profits from the start of the second period is given by $r(x-s;p)$ where $(x-s)$ is the initial inventory and p is the market price of this period. Since only the density of p is known beforehand, the discounted expectation of $r(x-s;p)$ in the first period is $\alpha \int_0^\infty r(x-s;p)\varphi(p)dp$. Combining these terms the relevant functional equation for the sales model of infinite horizon is given by,

$$(1) \qquad r(x;p) = \max_{x \geq s \geq 0} \left\{ p \cdot s - h(x-s) + \alpha \int_0^\infty r(x-s;p)\varphi(p)dp \right\} .$$

One obvious property of r is $r(0;p) = 0$.

In the model with production and sales decisions each period, hereafter named the production model, and an infinite horizon, define $f(x)$ as the maximum discounted expected profits over the horizon as a function of the initial inventory x . The relevant functional equation then becomes,

$$(2) \qquad f(x) = \max_{y \geq x} \left\{ - c(y-x) + \int_0^\infty \left[\max_{y \geq s \geq 0} \left(p \cdot s - h(y-s) + \alpha f(y-s) \right) \right] \varphi(p)dp \right\}$$

In the following sections, as we develop the models in detail, we also introduce the appropriate functional equations in cases of finite horizon.

III. Constant Marginal Holding Cost

In establishing existence and uniqueness and properties of optimal behavior of equations (1) and (2), supposing constant marginal holding cost, we propose to use a mathematical induction followed by a limiting process. Turning now to the sales model, we consider first a one-period model, a two-period model, and a model of arbitrary finite horizon. In these models the variables now have subscripts to indicate particular periods in which activities occur and the expected profit functions have subscripts to specify the number of periods in the horizon.

The functional equation for the one-period model is simply,

$$(3) \qquad r_1 (x;p) = \max_{x \geq s_1 \geq 0} \left\{ p \cdot s_1 - h \cdot (x-s_1) \right\} ,$$

where h now denotes the constant marginal holding cost. Obviously, it is optimal to sell the entire stock no matter what the price so that the optimal sales policy is $s_1^* (x;p) = x$ and $r_1(x;p) = p \cdot x$. Taking $\overline{p}$ as the mean price, the expectation of r_1 is $E(r_1) = \int_0^\infty r_1(x;p)\varphi(p)dp = \overline{p} \cdot x$. These conclusions, clearly, coincide with the static theory of the competitive firm under similar circumstances. In the context of the static theory we have also shown that the "momentary" or "very short-run" supply curve for the firm is a vertical line intersecting the horizontal quantity (sales) axis at the amount of the available supply x . To facilitate comparisons with the dynamic models we shall hereafter identify the static theory with the one-period sales model and, later on, with the one-period production model.

If the horizon has two periods we obtain,

$$(4) \quad r_2(x;p) = \max_{x \geq s_2 \geq 0} \left\{ p \cdot s_2 - h \cdot (x-s_2) + \alpha \int_0^\infty r_1(x-s_2;p)\varphi(p)dp \right\}$$

$$= \max_{x \geq s_2 \geq 0} \left\{ p \cdot s_2 - h \cdot (x-s_2) + \alpha \, \bar{p} \cdot (x-s_2) \right\} .$$

By rearranging terms in (4), it is easy to see that if $\alpha \, \bar{p} > h$, the optimal sales policy is,

$$(5) \quad s_2^*(x;p) = \begin{cases} x & \text{if } p \geq \alpha \, \bar{p} - h \\ 0 & \text{otherwise} \end{cases} .$$

The firm now has a "reservation price" $(\alpha\bar{p}-h)$ so that the momentary supply curve runs along the vertical price axis to this price and then becomes a vertical line at the available supply x . We may also derive $E(r_2) = [\hat{p}_2 + (\alpha\bar{p}-h) \cdot \Phi_2] \cdot x = p_2^* \cdot x$ where

$$(6) \quad \hat{p}_2 = \int_{\alpha\bar{p}-h}^{\infty} p \, \varphi(p)dp \quad \text{and} \quad \Phi_2 = \int_0^{\alpha\bar{p}-h} \varphi(p)dp .$$[5]

Since $(\alpha\bar{p}-h) \cdot \Phi_2 > \bar{p} - \hat{p}_2$, it follows that $p_2^* > \bar{p}$. Thus, in the two-period model, if the holding cost is low enough, the expected profits exceed those in the static theory since the firm enjoys the prospect of sales in the second period if the first period price is too low.

If $\alpha\bar{p} \leq h$, then $s_2^*(x;p) = x$ and $E(r_2) = \bar{p} \cdot x$ as in the static theory. The holding cost now induces the firm to sell the entire stock in the first period, even if price is very low. As we proceed, we show that this result remains true in any horizon. If $\alpha\bar{p} \leq h$, the static theory and the dynamic models will yield the same sales behavior, independently of the length of the horizon.

To obtain an induction let an horizon be specified by any integer $n \geq 3$ so that,

$$(7) \qquad r_n(x;p) = \max_{x \geq s_n \geq 0} \left\{ p \cdot s_n - h \cdot (x - s_n) + \alpha \int_0^\infty r_{n-1}(x - s_n;p)\varphi(p)dp \right\} .$$

Suppose,

$$(8) \quad \begin{aligned} &(a) \qquad \alpha p_{n-2}^* - h > 0 \\ &(b) \qquad s_{n-1}^*(x;p) = \begin{cases} x & \text{if } p \geq \alpha p_{n-2}^* - h \\ 0 & \text{otherwise} \end{cases} \\ &(c) \qquad E(r_{n-1}) = [\hat{p}_{n-1} + (\alpha p_{n-2}^* - h)\Phi_{n-1}] \cdot x = p_{n-1}^* \cdot x , \\ &(d) \qquad p_{n-1}^* > p_{n-2}^* , \end{aligned}$$

where $\hat{p}_{n-1}$ and Φ_{n-1} are interpreted according to equation (6) and footnote 5 with $(\alpha p_{n-2}^* - h)$ as the appropriate $(\alpha p^* - h)$. Then,

$$(9) \qquad r_n(x;p) = \max_{x \geq s_n \geq 0} \left\{ p \cdot s_n - h \cdot (x-s_n) + \alpha p^*_{n-1} \cdot (x-s_n) \right\} .$$

It now follows that,

$$\text{(a)} \qquad s^*_n(x;p) = \begin{cases} x & \text{if } \; p \geq \alpha p^*_{n-1} - h \\ 0 & \text{otherwise} \end{cases}$$

$$(10) \qquad \text{(b)} \qquad E(r_n) = [\hat{p}_n + (\alpha p^*_{n-1} - h)\, \Phi_n] \cdot x = p^*_n \cdot x$$

and since,

$$(\alpha p^*_{n-1} - h) \cdot (\Phi_n - \Phi_{n-1}) > \int_{\alpha p^*_{n-2} - h}^{\alpha p^*_{n-1} - h} p \cdot \varphi(p)dp ,$$

$$\text{(c)} \qquad p^*_n = \hat{p}_n + (\alpha p^*_{n-1} - h) \cdot \Phi_n > \hat{p}_{n-1} + (\alpha p^*_{n-1} - h) \cdot \Phi_{n-1} > p^*_{n-1} .$$

The induction is complete if we let $p^*_1 = \bar{p}$ and suppose $\alpha\bar{p} > h$. We have now shown that the reservation price and expected profits increase as the finite horizon increases in length.

We may also use equation (10)(b) iteratively to compute p^*_n. Moreover, by a similar process of induction, we may show that if $\alpha\bar{p} \leq h$, then $s^*_n(x;p) = x$, $p^*_n = \bar{p}$, and $E(r_n) = \bar{p} \cdot x$ for any n. The inequality $\alpha\bar{p} \leq h$ thus gives a measure of holding costs which are too high to allow a positive expected profit by carrying inventory into subsequent periods, independently of the current price and length of the finite horizon.

In turning to the infinite horizon we now have available the fact that if $\alpha\bar{p} > h$ the sequence $\{p^*_n\}$ is monotonic increasing, and we can show, as follows, that this sequence is bounded above by $(\bar{p}-h)/(1-\alpha)$. Expanding any element of the sequence, $p_n^* = \hat{p}_n + (\alpha\hat{p}_{n-1}-h) \cdot \Phi_n + \alpha\,(\alpha\hat{p}_{n-2}-h) \cdot \Phi_{n-1}\,\Phi_n + \cdots + \alpha^{n-2}\,(\alpha\bar{p}-h) \cdot \Phi_2 \cdots \Phi_n$. From $p_n^* > p_{n-1}^*$ it follows that $\bar{p} > \hat{p}_{n-1} > \hat{p}_n$ and $1 > \Phi_n > \Phi_{n-1}$. It is now easily seen that $(\bar{p}-h)/(1-\alpha) > \bar{p} + (\alpha\bar{p}-h) + \cdots + \alpha^{n-2}\,(\alpha\bar{p}-h) > p_n^*$. Consequently, the sequence converges to some $p^* > p_n^*$ and $E(r_n) = p_n^* \cdot x$ converges uniformly to $E(r) = p^* \cdot x$ for any interval for x which assures the existence of an optimal sales policy in the infinite period case. The functional equation in this case may now be written,

$$(11) \qquad r(x;p) = \max_{x \geq s \geq 0} \left\{ p \cdot s - h \cdot (x-s) + \alpha p^* \cdot (x-s) \right\},$$

and the optimal sales policy is,

$$(12) \qquad s^*(x;p) = \begin{cases} x & \text{if } \alpha p^* - h \\ 0 & \text{otherwise}, \end{cases}$$

where (αp^*-h) gives the reservation price. Substituting the optimal sales policy into equation (11) and taking the expectation gives,

$$(13) \qquad E(r) = [\hat{p} + (\alpha p^*-h) \cdot \Phi] \cdot x = p^* \cdot x \; .$$

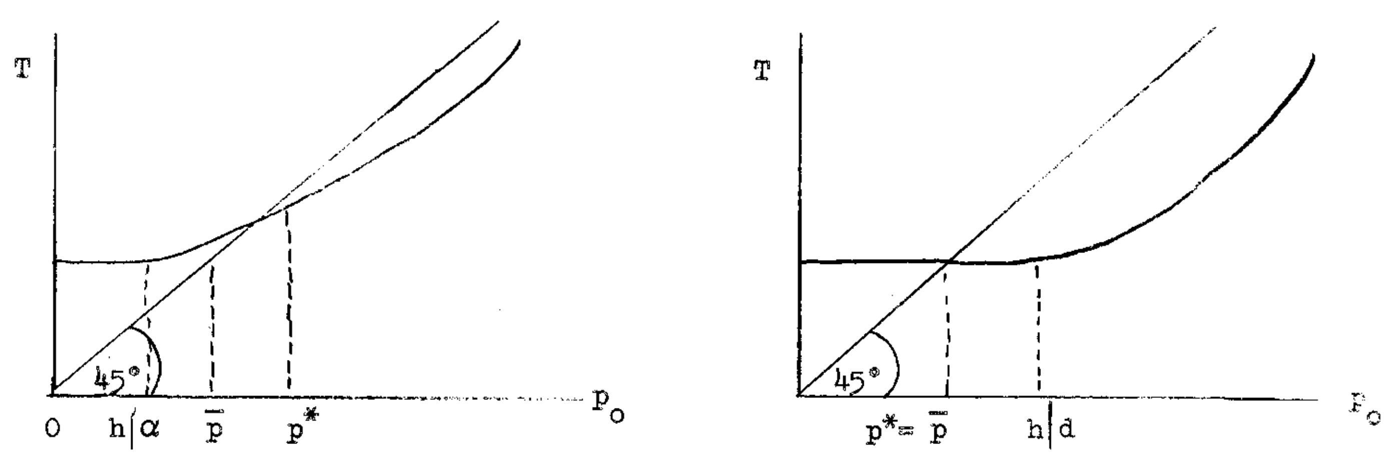
If $\alpha\bar{p} \leq h$, once again we may show that the static theory and the dynamic model yield identical results.

Since equation (13) is a single equation in a single unknown, we may use this equation to compute p^*. To be certain of obtaining p^*, however, we need some way of distinguishing possible multiple roots of (13). In this respect let $T(p_o) = \hat{p}_o + (\alpha p_o - h) \cdot \Phi_o$ be a function of a given price p_o where $\hat{p}_o$ and Φ_o are interpreted as before. The problem is then to find an appropriate $p_o = T$. By differentiation,

$$
(14) \quad
\begin{aligned}
&\frac{dT}{dp_o} = \alpha \Phi_o > 0 \quad \text{and} \quad \frac{d^2T}{dp_o^2} = \alpha^2 \varphi > 0 \quad \text{if} \quad \alpha p_o \geq h \,, \\
&\frac{dT}{dp_o} = \frac{d^2T}{dp_o^2} = 0 \quad \text{and} \quad T = \bar{p} \quad \text{if} \quad \alpha p_o < h \,.
\end{aligned}
$$

The graph of T is now given in Figure 1 in the cases where $\alpha\bar{p} \geq h$ and $\alpha\bar{p} < h$.

Figure 1



(a) $\alpha\bar{p} > h$ (b) $\alpha\bar{p} \leq h$

In each case two solutions for p^* are possible. However, only solutions with smaller values qualify because, in the second case, it is already known that $p^* = \bar{p}$, and in the first case because of the monotonic convergence of the sequence $\{p_n^*\}$. That is, if we let $p_o = p_{n-1}^*$ then $p_n^* = T(p_{n-1}^*) > p_{n-1}^*$ so that for values of p_o close to p^*, T must lie above the 45° line.

We could also, of course, approximate p^* by some p_n^*. To illustrate the computation of p^*, in an example where price has a finite upper bound, let prices be specified by the uniform distribution. Suppose $0 \leq p \leq 2$ so that $\varphi(p) = 1/2$ and $\bar{p} = 1$. If $\alpha = .6$ and $h = .5$ then $(\alpha\bar{p}-h) = .1$. If we let $p_o = \bar{p}$ then $p_2^* = T(\bar{p}) = 1.0025$. One more iteration gives $p_3^* = T(p_2^*) = 1.002574$. Taking 1.0026 as an estimate of p^*, we obtain $T(1.0026) = 1.002579$ which shows that $1.002574 < p^* < 1.002579$. It is easy enough to improve the estimate if desired.

In the production model we first consider the one-period or static theory case where

$$
\begin{aligned}
f_1(x) &= \max_{y_1 \geq x} \left\{ -c(y_1-x) + \int_0^\infty \left[\max_{y_1 \geq s_1 \geq 0} \left(p \cdot s_1 - h \cdot (y_1 - s_1) \right) \right] \varphi(p)dp \right\}, \\
(15) \qquad &= \max_{y_1 \geq x} \left\{ -c(y_1-x) + \bar{p} \cdot y_1 \right\},
\end{aligned}
$$

since $s_1^*(y_1;p) = y_1$. If $c'(0) < \bar{p}$, the optimal starting stock is derived by finding the $y_1^*(x)$ satisfying $c'(y_1^* - x) = \bar{p}$. Since the expected marginal revenue $\bar{p}$ is constant, it follows that optimal output $\bar{q} = y_1^* - x$

is independent of x. (The proof is obtained by differentiating the equation $c'(y_1^*-x) = \bar{p}$ with respect to x). If $c'(0) \geq \bar{p}$, then $y_1^* = x$ and $\bar{q} = 0$. Geometrically, in a diagram with expected price $\bar{p}$ on the vertical axis and starting stock y on the horizontal axis, the short-run supply curve is given by the marginal cost curve shifted to the right by the amount of the initial inventory x if $c'(0) < \bar{p}$ and by a vertical line at the initial inventory if $c'(0) \geq \bar{p}$. Also, we obtain $f_1(x) = -c(\bar{q}) + \bar{p} \cdot (x + \bar{q})$.

In the two-period horizon,

$$(16) \quad f_2(x) = \max_{y_2 \geq x} \left\{ -c(y_2-x) + \int_0^\infty r_2(y_2;p)\varphi(p)dp \right\},$$

where

$$(17) \quad r_2(y_2;p) = \max_{y_2 \geq s_2 \geq 0} \left\{ p \cdot s_2 - h \cdot (y_2-s_2) + \alpha f_1(y_2-s_2) \right\}.$$

Since $f_1(y_2-s_2) = -c(\bar{q}) + \bar{p} \cdot (y_2-s_2 + \bar{q})$ and $f_1'(y_2-s_2) = \bar{p}$, the optimal sales policy is the same as in the two-period sales model and the expectation $E(r_2) = [\hat{p}_2 + (\alpha\bar{p}-h) \cdot \Phi_2] \cdot y_2 + \alpha[\bar{p} \cdot \bar{q} - c(\bar{q})] = p_2^* \cdot y_2 + D_2$ where the constant $D_2 = \alpha[\bar{p} \cdot \bar{q} - c(\bar{q})]$. Substituting into (16),

$$(18) \qquad f_2(x) = \max_{y_2 \geq x} \left\{ -c(y_2-x) + p_2^* \cdot y_2 + D_2 \right\}.$$

If $c'(0) < p_2^*$, the optimal starting stock $y_2^*(x)$ satisfies $c'(y_2^*-x) = p_2^*$ and if $c'(0) \geq p_2^*$, $y_2^*(x) = x$ and output is zero. Again, since expected marginal revenue p_2^* is constant, optimal output $q_2^* = y_2^* - x$ is independent of x, and $q_2^* > \bar{q}$ if $\alpha\bar{p} > h$ and $c'(0) < p_2^*$. Under the combination $\alpha\bar{p} > h$

and $\bar{p} \leq c'(0) < p_2^*$, the firm would produce a positive amount in the first period but not in the second period of the two-period horizon. To specify the short-run supply curve as a function of $\bar{p}$ now requires determining $dp_2^*/d\bar{p}$. Since this derivative depends on the nature of the shift in $\varphi(p)$ with changes in $\bar{p}$, we consider this question at some length in the next section.

An induction and a limiting process yield similar outcomes in cases of arbitrary finite or infinite horizons. In particular, optimal sales policies are the same as those for sales models of corresponding horizons; optimal starting stocks satisfy $c'(y_n^*-x) = p_n^*$ or $c'(y^*-x) = p^*$ if $c'(0) < p_n^*$ or $c'(0) < p^*$; if $c'(0) < \bar{p}$ and $\alpha\bar{p} > h$, optimal outputs satisfy $q^* > q_n^* > \text{-----} > q_2^* > \bar{q}$, or if $p_n^* \leq c'(0) < p_{n+1}^*$ for some n, then $q_n^* = \text{-----} = \bar{q} = 0$; and if $\alpha\bar{p} \leq h$, optimal sales and output behavior in the dynamic models and in the static theory coincide. One additional obvious result is that production in the current period is independent of production costs in subsequent periods so that we could easily omit the assumption of identical production costs in each period. For example, if the horizon is infinite and $c_1(y-x)$ gives production cost in the first period, with possibly different production costs in later periods, optimal output is obtained simply by solving $c_1'(y^*-x) = p^*$ if $c_1'(0) < p^*$, or if $c_1'(0) \geq p^*$, optimal output is zero.

IV. Changes in Parameters

In cases with constant marginal holding cost the important outcome is the existence of critical prices p_n^* or p^* which are essential in finding reservation prices and optimal outputs. In understanding optimal behavior more completely it is useful to consider the shifts in these prices consequent upon changes in parameters such as the holding cost, the discount factor, and the mean and variance of the density of prices. In the static theory and in dynamic models with $\alpha\bar{p} \leq h$ since the critical prices and the mean price are the same, it is obvious enough that only changes in the mean price affect the critical price. In dynamic models with $\alpha\bar{p} > h$ it is also apparent that while a change in production cost in any particular period may cause output in that period to shift, the critical prices are invariant to such a change.

In evaluating changes in critical prices if $\alpha\bar{p} - h > 0$, we successively compute the derivatives of p_2^*, p_n^*, and p^* with respect to the particular parameter. If the discount factor changes,

$$\text{(19)} \qquad \frac{dp_2^*}{d\alpha} = \frac{d[\hat{p}_2 + (\alpha\bar{p}-h)\Phi_2]}{d\alpha} = \bar{p}\,\Phi_2 > 0$$

Suppose $dp_{n-1}^*/d\alpha > 0$ for $n \geq 3$; then

$$\text{(20)} \qquad \frac{dp_n^*}{d\alpha} = \frac{d[\hat{p}_n + (\alpha p_{n-1}^* - h)\Phi_n]}{dp} = p_{n-1}^*\,\Phi_n + \alpha\,\Phi_n\,\frac{dp_{n-1}^*}{d\alpha} > 0 .$$

Finally,

$$\text{(21)} \qquad \frac{dp^*}{d\alpha} = \frac{d[\hat{p} + (\alpha p^* - h)\Phi]}{dp} = \frac{p^*\Phi}{1-\alpha\,\Phi} > 0 .$$

These results are in accord with intuition; an increase in the present value of future returns causes the critical prices, the discounted expected marginal profits, to increase. If the holding cost changes, a similar process of differentiation and induction yields,

$$\frac{dp_2^*}{dh} = -\Phi_2 < 0 \; ; \quad \frac{dp_n^*}{dh} = -\Phi_n + \alpha \frac{dp_{n-1}^*}{dh} < 0 \, ,$$

(22)

$$\frac{dp^*}{dh} = -\frac{\Phi}{1-\alpha\,\Phi} < 0 \, .$$

Thus, as the holding cost increases, or, in other words, as h approaches $\alpha\bar{p}$, the critical prices approach the mean price.

If we assume changes in the mean and the variance of the density of prices, the results are not clearcut unless we are specific about the manner in which the density shifts. To avoid a lengthy exploration we shall assume only the simplest of shifts in the density. First, it is obvious enough that as the variance approaches zero in any manner, the critical prices approach the mean price. Also, if the following result holds with a change in variance, for some p_n^*,

$$(23) \qquad \int_0^t [\varphi(p) - \varphi_1(p)]dp > 0 \, , \qquad t \leq \alpha p_n^* - h \, , \quad t \neq 0 \, ,$$

where $\varphi_1(p)$ is the new density, then the critical price decreases. For example, if the horizon is two periods, the expected gain from future sales, $(\alpha\bar{p}-h)\Phi_2$, decreases more than the expected gain from current sales, $\hat{p}_2$, increases.

In considering a change in the mean price, we suppose that as the mean increases the density shifts to the right by the same amount. We represent this type of change, formally, by supposing that price expectations are additive. That is, let $p = \bar{p} + \gamma$ where $\tau(\gamma)$ is the density of γ with mean zero and $-b \leq \gamma \leq \infty$ with b as some constant such that $\bar{p} \geq b$. We then have that

$$(24) \qquad \int_{-b}^{\infty} \tau(\gamma)d\gamma = \int_{\bar{p}-b}^{\infty} \varphi(p)dp = \int_{0}^{\infty} \varphi(p)dp = 1 .$$

The last integral equals one since $\varphi(p) = 0$ for $0 \leq p \leq \bar{p} - b$. Also, for some p_n^*,

$$(25) \qquad \Phi_n = \int_{\bar{p}-b}^{\alpha p_{n}^*-h} \varphi(p)dp = \int_{-b}^{\alpha p_n^*-(h+\bar{p})} \tau(\gamma)d\gamma ,$$

and

$$(26) \qquad \hat{p}_n = \int_{\alpha p_n^*-h}^{\infty} p\varphi(p)dp = \bar{p}(1-\Phi_n) + \int_{\alpha p_n^*-(h+\bar{p})}^{\infty} \gamma\tau(\gamma)d\gamma .$$

By differentiation and induction,

$$(27) \qquad \begin{aligned} \frac{dp_2^*}{d\bar{p}} &= 1 - \Phi_2 + \alpha\,\Phi_2 < 1 , \\ \frac{dp_n^*}{d\bar{p}} &= 1 - \Phi_n + \alpha\,\Phi_n \frac{dp_{n-1}^*}{d\bar{p}} < 1 , \\ \frac{dp^*}{d\bar{p}} &= \frac{1 - \Phi}{1 - \alpha\,\Phi} < 1 ; \end{aligned}$$

so long as the probabilities Φ_2, Φ_n, and Φ are positive. Eventually, however, at some finite mean prices, these probabilities become zero since $(\alpha\bar{p}-h)$, (αp_{n-1}^*-h), and (αp^*-h) increase at a slower rate than $(\bar{p}-b)$, the highest price at which the density equals zero. The derivatives then equal one and the critical prices equal the mean prices.

Returning to the question of the short-run supply curves, it is now apparent that these curves lie to the right of the static supply curve over some range of mean prices, approaching the static curve as the mean price increases and eventually coinciding with it, and as the mean price decreases enough, coinciding with the vertical line at the initial stock.[6] The fact that the supply curves eventually coincide with the static curve and the difficulty noted in footnote 6 are consequences of the assumption of additive price expectations. The reader might wish to examine the outcomes under somewhat different assumptions.

V. Increasing Marginal Holding Cost and Finite Horizons

Suppose now that the marginal holding cost is increasing, $h'' > 0$. Using the same notation as before, the functional equation for the one-period sales model is obtained from equation (3) by substituting $h(x-s)$ for $h\cdot(x-s)$ to indicate that marginal holding costs are no longer constant. Again, independently of price, it is optimal to sell the entire stock so that $s_1^*(x;p) = x$ and $E(r_1) = \bar{p} \cdot x$. In the two-period sales model we then obtain,

$$(28) \qquad r_2(x;p) = \max_{x \geq s_2 \geq 0} \left\{ p \cdot s_2 - h(x-s_2) + \alpha\bar{p} \cdot (x-s_2) \right\} .$$

A measure of holding costs which are too large to allow a positive expected profit by future sales is now given by the inequality $\alpha\bar{p} \leq h'(0)$. If $\alpha\bar{p} > h'(x)$, the optimal sales policy has three phases,

$$(29) \qquad s_2^*(x;p) = \begin{cases} x & \text{if } p > \alpha\bar{p} - h'(0) \\ \pi_2(x;p) & \text{if } \alpha\bar{p} - h'(0) \geq p \geq \alpha\bar{p} - h'(x) \\ 0 & \text{otherwise} \end{cases}$$

where π_2 satisfies: $p + h'(x-\pi_2) - \alpha\bar{p} = 0$. Properties of π_2 are obtained by differentiating this equation with respect to x and p,

$$(30) \qquad \begin{gathered} h''(x-\pi_2) \cdot [1 - \frac{\partial\pi_2}{\partial x}] = 0 \text{ , and } \frac{\partial\pi_2}{\partial x} = 1 \text{ ,} \\ 1 - h''(x-\pi_2) \cdot \frac{\partial\pi_2}{\partial p} = 0 \text{ , and } \frac{\partial\pi_2}{\partial p} = \frac{1}{h''(x-\pi_2)} > 0 \text{ .} \end{gathered}$$

If $\alpha\bar{p} > h'(0)$ and $\alpha\bar{p} \leq h'(x)$, the sales policy is modified so that sales, or "disposal," occur even at a zero price. If $p = 0$, sales are the amount, $x-\hat{x}$, which ensures $\alpha p = h'(\hat{x})$. These characteristics of the optimal sales policy, or the momentary supply curve (MS), are shown in Figure 2 for the initial stocks x_1 and x_2 which satisfy $\alpha\bar{p} > h'(x_1)$ and $\alpha\bar{p} < h'(x_2)$.[7]

Now let $\bar{r}_2(x)$ represent the expectation $E(r_2)$ and suppose $\alpha\bar{p} > h'(x)$. Substituting the optimal sales policy into equation (28), taking the expectation, and differentiating, we obtain,

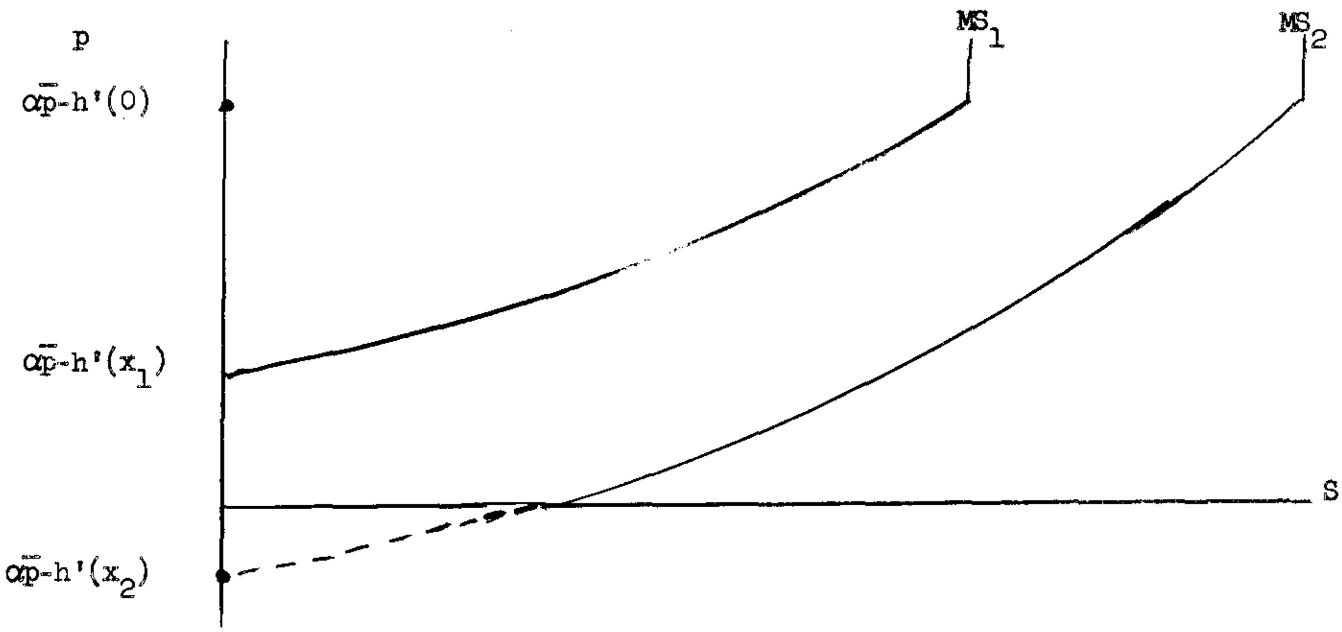
$$(31) \qquad \bar{r}_2'(x) = \int_{\alpha\bar{p}-h'(x)}^{\infty} p\varphi(p)dp + [\alpha\bar{p} - h'(x)]\, \Phi_2 \text{ .}$$

From this equation, and the findings of the previous section, $\bar{r}_2'(0) = p_2^*$, where p_2^* is the appropriate critical price if the constant marginal holding

Figure 2

Momentary Supply Curves

Two-Period Horizon



cost $h = h'(0)$, and $\bar{r}_2'(x) \le p_2^*$. If $\alpha\bar{p} \le h'(x)$, we also easily derive $\bar{r}_2'(x) = \bar{p}$. The expected marginal profit thus satisfies $\bar{p} \le \bar{r}_2'(x) \le p_2^*$. Taking the second derivative of the expectation yields $\bar{r}_2''(x) = - h''(x) \cdot \Phi_2$, which is negative if $\alpha\bar{p} > h'(x)$ and zero if $\alpha\bar{p} \le h'(x)$. The reader could now easily derive the optimal sales policy if the horizon is three periods, and we also have the essentials needed for an induction in an n-period horizon.

Since the induction is straightforward, we only present the results. If $n \ge 3$ and $\alpha\bar{p} \le h'(0)$, $s_n^*(x;p) = x$ and $\bar{r}_{n-1}'(x) = \bar{p}$. So, again, if holding costs are high enough, the results of the static theory and the dynamic models coincide. If $\alpha\bar{p} > h'(0)$ and $\alpha\bar{r}_{n-1}'(x) > h'(x)$,

$$(32) \qquad s_n^*(x;p) = \begin{cases} x & \text{if } p \geq \alpha p_{n-1}^* - h'(0) \\ \pi_n(x;p) & \text{if } \alpha p_{n-1}^* - h'(0) \geq p \geq \alpha \bar{r}_{n-1}'(x) - h'(x) \\ 0 & \text{otherwise} \end{cases}$$

where π_n satisfies: $p + h'(x-\pi_n) - \alpha\bar{r}_{n-1}'(x-\pi_n) = 0$. Properties of π_n are consequences of the following characteristics of $\bar{r}_{n-1}(x)$, in the range $\alpha\bar{r}_{n-1}'(x)-h'(x) > 0$: $\bar{r}_{n-1}'(0) = p_{n-1}^*$; $p_{n-1}^* \geq \bar{r}_{n-1}'(x) \geq \bar{r}_{n-2}'(x) > \bar{p}$; and $\bar{r}_{n-1}''(x) \leq \bar{r}_{n-2}''(x) \leq 0$. From these characteristics,

$$(33) \qquad \frac{\partial\pi_n}{\partial x} = 1 , \quad \text{and} \quad \frac{\partial\pi_n}{\partial p} = \frac{1}{h''(x-\pi_n)-\bar{r}_{n-1}''(x-\pi_n)} < 0 .$$

We also obtain that $\alpha\bar{r}_{n-1}' > h'(x)$ if and only if $\alpha\bar{p} > h'(x)$. If $\alpha\bar{p} \leq h'(x)$, the sales policy, again, is modified to allow sales even at zero price, and $\bar{r}_{n-1}'(x) = \bar{p}$. Geometrically, if $\alpha\bar{p} > h'(0)$, the momentary supply curves are much like those described in Figure 2. In general for the same initial stocks, they have a higher intercept on the price axis, or a lower intercept on the sales axis, and coincide with the vertical lines at higher prices.

In the production model, since $\bar{r}_1'(x) = \bar{p}$, the outcomes in the one-period horizon are that the optimal sales and production policies coincide with those in the case of constant marginal holding cost. In the two-period horizon,

$$(34) \qquad f_2(x) = \max_{y_2 \geq x} \left\{ - c(y_2-x) + \int_0^\infty R_2(y_2;p)\varphi(p)dp \right\} ,$$

where

$$(35) \qquad R_2(y_2;p) = \max_{y_2 \geq s_2 \geq 0} \left\{ p \cdot s_2 - h(y_2-s_2) + \alpha f_1(y_2-s_2) \right\} .$$

Since $f_1(y_2-s_2) = \overline{p}(y_2-s_2) + [\overline{p} \cdot \overline{q} - c(\overline{q})]$, the optimal sales policy is the same as in the two-period sales model with increasing marginal holding cost and the expectation $\overline{R}_2(y_2) = \overline{r}_2(y_2) + D_2$. As $\overline{R}_2'(y_2)$ varies with y_2, it is apparent that the optimal starting stock and the optimal output now depend on the initial stock. If $c'(0) < \overline{p}$, by differentiating $c'(y_2^*-x) = \overline{R}_2'(y_2^*)$ and $y_2^* = q_2^* + x$, we obtain,

$$\text{(36)} \qquad \frac{dy_2^*}{dx} = \frac{c''(y_2^*-x)}{c''(y_2^*-x)-\overline{R}_2''(y_2^*)} \leq 1 , \quad \text{and} \quad \frac{dq^*}{dx} \leq 0 ,$$

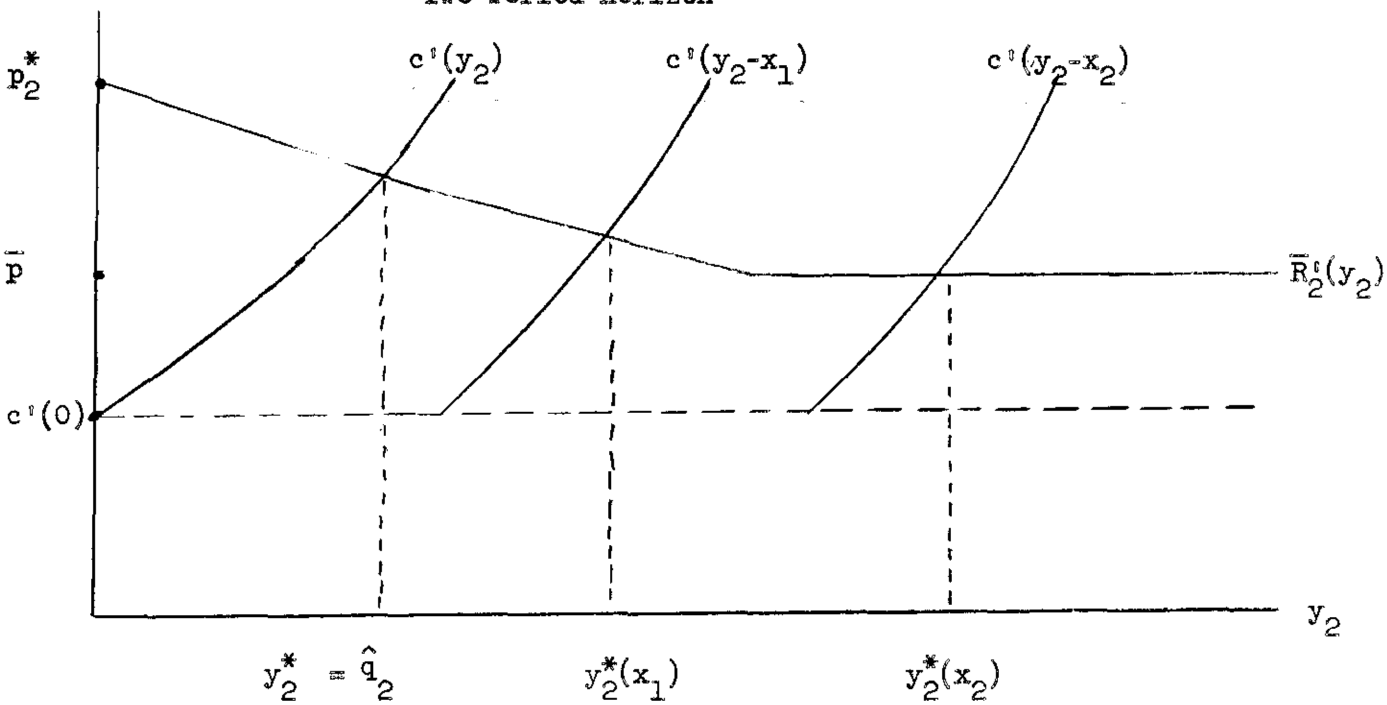
the equalities holding only if $\overline{R}_2' = \overline{p}$, which implies $\overline{R}_2'' = 0$. In this latter event output is the same as in the static theory; otherwise output exceeds this amount $\overline{q}$ but falls short of output produced if the marginal holding cost is constant. Since $\overline{R}_2' = \overline{p}$ only if $\alpha\overline{p} \leq h'(y_2)$, output equals the minimum amount $\overline{q}$ only if $\alpha\overline{p} \leq h'(x+\overline{q})$. These and other characteristics of optimal output and the optimal starting stock are seen more clearly in Figure 3 where the equation $c'(y_2^*-x) = \overline{R}_2'(y_2^*)$ is satisfied at three different initial stocks. For example, if $q_2^*(x)$ is the optimal output, the maximum output is given by $q_2^*(0) = \hat{q}_2$ where $c'(\hat{q}_2) = \overline{R}_2'(\hat{q}_2)$. It is also apparent from Figure 3 that if $p_2^* > c'(0) > \overline{p}$, output becomes zero if the initial stock is sufficiently large.

From $f_2(x) = - c(q_2^*) + \overline{R}_2(y_2^*)$ we obtain

$$\text{(37)} \qquad \begin{aligned} f_2'(x) &= [\overline{R}_2'(y_2^*) - c'(q_2^*)] \frac{dq_2^*}{dx} + \overline{R}_2'(y_2^*) \\ &= \overline{R}_2'(y_2^*) , \end{aligned}$$

Figure 3

Starting Stocks and Output
Two-Period Horizon



since either $[\bar{R}_2'(y_2^*) - c'(q_2^*)]$ or $\frac{dq_2^*}{dx}$ equals zero. Thus, the expected return from an additional unit of initial inventory equals the expected return from an additional unit of the starting stock (at the level of the optimal starting stock). Differentiating again yields,

(38) $$f_2''(x) = \bar{R}_2''(y_2^*) \frac{dy_2^*}{dx} \leqq 0 \ .$$

Since the derivatives of $f_2(x)$ are significant in characterizing the sales policy if the horizon is three periods, an important particular value of the first derivative is $f_2'(0) = \bar{R}_2'(\hat{q}_2) = \bar{r}_2'(\hat{q}_2) < p_2^*$. Passing over the details of deriving this sales policy, which the reader could now readily supply, these results imply that, in anticipation of future production with possible increases in holding costs, the firm will tend to sell more of its stock at each price than in the case of the pure sales model of three periods. It is also easy to show that $\bar{R}_2'(x) \leq \bar{R}_3'(x) \leq \bar{r}_3'(x)$; $\bar{R}_3'(0) < \bar{r}_3'(0) = p_3^*$ if $\alpha\bar{p} > h$; and $\bar{R}_3''(x) \leq 0$. In consequence of the increasing marginal storage cost, the optimal output thus tends to decrease with increases in initial inventory, as in the two-period model. In contrast to the two-period model, however, the prospect of storing both current and future production affects the amount of current output ($\bar{R}_3'(0) < p_3^*$ as compared to $\bar{R}_2'(0) = p_2^*$) . In comparing the two and three-period models with production, we may also show that the firm tends to sell less at each price and to produce more at each initial inventory in the three-period model.

By an induction similar conclusions hold in an n-period horizon. Again, we omit the details.

One final matter to consider briefly is the effect of parameter changes on optimal policies and the derivation of short-run supply curves. The conclusions in cases of constant marginal storage cost are directly useful. An upward shift of the storage cost function, in general, will cause the firm to sell more at each price and produce less at each initial inventory. The direction of change is equally obvious if the marginal storage cost, the discount factor, or the variance changes. If the mean price increases, then at each initial inventory the results in cases of constant marginal storage cost will be reinforced, i.e., in general, the expected marginal profits will increase but at a slower rate because of the increase in expected marginal storage costs occasioned by future production. Thus, at each initial inventory the short-run supply curves will lie closer to the static curve, approaching and eventually coinciding with this curve as the initial inventory increases and finally becomes sufficiently large.

VI. Increasing Marginal Holding Cost and Infinite Horizons

Rather than taking limits of optimal policies and expected profits as a method of establishing optimal behavior in infinite horizons, which would tend to be tedious and not particularly instructive, we propose to consider the infinite horizon directly, after making several plausible assumptions. First, we take for granted requisite existence and uniqueness properties of solutions to the functional equations. Second, we suppose the expected profits functions are twice (piecewise) differentiable and that the second derivatives of certain expected profits functions are nonpositive.

In the sales model let $\bar{r}(x) = \int_0^\infty r(x;p)\varphi(p)dp$ and rewrite equation (1) as,

$$(39) \qquad r(x;p) = \max_{x \geq s \geq 0} \left\{ p \cdot s - h(x-s) + \alpha\bar{r}(x-s) \right\} .$$

Supposing $\bar{r}''(x) \leq 0$, it follows that $\alpha\bar{r}''(x) - h''(x) < 0$, which implies $\alpha\bar{r}'(0) - h'(0) > \alpha\bar{r}'(x) - h'(x)$. If we assume that $\alpha\bar{r}'(x) > h'(x)$, the optimal sales policy is now readily obtained as,

$$(40) \qquad s^*(x;p) = \begin{cases} x & \text{if } p \geq \alpha\bar{r}'(0) - h'(0) \\ \pi(x;p) & \text{if } \alpha\bar{r}'(0) - h'(0) \geq p \geq \alpha\bar{r}'(x) - h'(x) \\ 0 & \text{otherwise} \end{cases}$$

where π satisfies: $p + h'(x-\pi) - \alpha\bar{r}'(x-\pi) = 0$. Differentiating, this equation yields,

$$(41) \qquad \frac{\partial\pi}{\partial x} = 1 , \quad \text{and} \quad \frac{\partial\pi}{\partial p} = \frac{1}{h''(x-\pi)-\alpha\bar{r}''(x-\pi)} > 0 .$$

Substituting the optimal sales policy into equation (39), taking the expectation, and differentiating, we obtain,

$$(42) \qquad \bar{r}'(x) = \int_{\alpha\bar{r}'(x)-h'(x)}^{\infty} p\,\varphi(p)dp + [\alpha\bar{r}'(x)-h'(x)]\Phi \geq \bar{p} .$$ [8]

From this equation, and using the notation and argument of Section III in the infinite horizon case, $\bar{r}'(0) = T(\bar{r}'(0) = p^*$ where p^* is the appropriate critical price for $h = h'(0)$. We may also show that $\alpha\bar{r}'(x) > h'(x)$ if and only if $\alpha\bar{p} > h'(x)$, as follows. If $\alpha\bar{p} > h'(x)$, it is immediate that $\alpha\bar{r}'(x) > h'(x)$ since $\bar{r}'(x) \geq \bar{p}$ from equation (42). If $\alpha\bar{r}'(x) > h'(x)$, then, again, equation (42) and the argument of Section III require that $\alpha\bar{p} > h'(x)$. Taking the second derivative of $\bar{r}(x)$ yields,

$$(43) \qquad \begin{aligned} \bar{r}''(x) &= [\alpha\bar{r}''(x)-h''(x)]\Phi \\ &= \frac{-h''(x)\Phi}{1-\alpha\Phi} . \end{aligned}$$

Substituting (43) into (41) we now obtain a simpler expression for the derivative of π with respect to p,

$$(44) \qquad \frac{\partial\pi}{\partial p} = \frac{1-\alpha\Phi}{h''(x-\pi)} .$$

If $\alpha\bar{p} < h'(x)$, then $\bar{r}'(x) = \bar{p}$ and the sales policy would need to be modified to allow positive sales even at zero price. Geometrically, if $\alpha\bar{p} > h'(0)$, a diagram like Figure 2 would represent the momentary supply curve for various initial inventories.

In the production model rewrite equation (2) as,

$$f(x) = \max_{y \geq x} \left\{- c(y-x) + \int_0^\infty R(y;p)\varphi(p)dp\right\} , \tag{45}$$

where

$$R(y;p) = \max_{y \geq s \geq 0} \left\{p \cdot s - h(y-s) + \alpha f(y-s)\right\} , \tag{46}$$

and let $\bar{R}(y) = \int_0^\infty R(y;p)\varphi(p)dp$. Supposing $f''(y) \leq 0$, it follows that $\alpha f''(y) - h''(y) < 0$, which implies $\alpha f'(0) - h'(0) > \alpha f'(y) - h'(y)$. If we assume that $\alpha f'(y) > h'(y)$, the optimal sales policy is simply,

$$s^*(y;p) = \begin{cases} y & \text{if } p > \alpha f'(0) - h'(0) \\ \pi(y;p) & \text{if } \alpha f'(0) - h'(0) \geq p \geq \alpha f'(y) - h'(y) \\ 0 & \text{otherwise ,} \end{cases} \tag{47}$$

where π now satisfies: $p + h'(y-\pi) - \alpha f'(y-\pi) = 0$. Differentiating,

$$\frac{\partial\pi}{\partial x} = 1 , \quad \text{and} \quad \frac{\partial\pi}{\partial p} = \frac{1}{h''(y-\pi) - \alpha f''(y-\pi)} > 0 . \tag{48}$$

Substituting the optimal sales policy into equation (46), taking the expectation, and differentiating twice,

$$\bar{R}'(y) = \int_{\alpha f'(y)-h'(y)}^{\infty} p\ \varphi(p)dp + [\alpha f'(y) - h'(y)]\ \Phi \geq \bar{p}\ ,$$

(49)

$$\bar{R}''(y) = [\alpha f''(y) - h''(y)]\ \Phi \leq 0\ .$$

If $c'(0) < \bar{R}'(0)$, the optimal starting stock $y^*(x)$ satisfies: $c'(y^*-x) = \bar{R}'(y^*)$, and by differentiating this equation and $y^* = q^* + x$, we obtain,

$$\text{(50)} \qquad \frac{dy^*}{dx} = \frac{c''(y^*-x)}{c''(y^*-x)-\bar{R}''(y^*)} \leq 1\ , \quad \text{and} \quad \frac{dq^*}{dx} \leq 0\ .$$

If $c'(0) > \bar{R}'(0)$, then $y^*(x) = x$ and $q^*(x) = 0$.

Additional characteristics of optimal policies are derived by differentiating $f(x) = -c(q^*) + \bar{R}(y^*)$,

$$f'(x) = [\bar{R}'(y^*) - c'(q^*)]\ \frac{dq^*}{dx} + \bar{R}'(y^*) = \bar{R}'(y^*)\ ,$$

(51)

$$f''(x) = \bar{R}''(y^*)\ \frac{dy^*}{dx}\ ,$$

since either $[\bar{R}'(y^*) - c'(q^*)]$ or $\frac{dq^*}{dx}$ equals zero. Now, since $f''(y) \leq 0$, $\bar{R}''(y) \leq 0$, $f'(x) = \bar{R}'(y^*)$, and $y^* \geq x$, it follows that $f'(y) \leq \bar{R}'(y)$ and

$$\text{(52)} \qquad \bar{R}'(y) \leq \int_{\alpha\bar{R}'(y)-h'(y)}^{\infty} p\ \varphi(p)dp + [\alpha\ \bar{R}'(y) - h'(y)]\Phi\ .$$

From equation (52) and the argument of Section III in the infinite horizon case, we derive,

$$f'(y) \leq \bar{R}'(y) \leq \bar{r}'(y) \quad . \tag{53}$$

In particular, from equation (53), $f'(0) \leq \bar{R}'(0) \leq \bar{r}'(0) = p^*$, where p^* is the appropriate critical price for $h = h'(0)$. In the example with $\alpha\bar{p} > h'(0)$ and $c'(0) < \bar{p}$, we leave it to the reader to show that $\bar{R}'(0) < p^*$ and $f'(0) = \bar{R}'(\hat{q}) < p^*$ where $c'(\hat{q}) = \bar{R}'(\hat{q})$ and $\hat{q} > \bar{q}$. It is also not difficult to show that $\alpha f'(y) > h'(y)$ if and only if $\alpha\bar{p} > h'(y)$. If $\alpha\bar{p} \leq h'(y)$, optimal behavior is modified in a familiar manner and we omit the details.

In general, compared to the behavior in the pure sales model with increas- in marginal holding cost, the firm now tends to sell more of a given stock at any given price. Compared to behavior in the production model with constant marginal holding cost and an infinite horizon, the firm will produce less and production now depends on the initial stock. If $c'(0) < \bar{p}$, then the firm will always produce at least the amount $\bar{q}$, but if $p^* > c'(0) > \bar{p}$, we have the possibility that the firm will not ever produce or will produce intermittenly depending on the initial stock. A diagram like Figure 3 could be used to describe output behavior.

VII. Conclusions

In this paper we have obtained characteristics of optimal behavior for the competitive firm producing a single good under the assumptions of a finite or infinite horizon, constant or increasing marginal storage cost, and an unchanging probability distribution of prices. A number of extensions of the model are apparent. One direction of development, which would probably not cause too great difficulty, is to allow inventory holdings of inputs and to assume that the firm produces multiple products. A more important development would consider the problem of price expectation formation for the firm. In particular, it would be of interest to trace the impact on optimal policies of having the density of prices depend on what actual prices turn out to be. Good starting points in this direction would be to assume "cob-web" expectations or "adaptive" expectations in which the mean price becomes the current price or some weighted average of the current price and the past mean price.

FOOTNOTES

1. This research was supported partly by a National Science Foundation Grant, GS-385, to the University of Rochester and partly by the Ford Foundation in the author's tenure as a Ford Foundation Faculty Research Fellow at the Cowles Foundation for Research in Economics.

2. As the reader will surmise later on, U-shaped marginal production cost can be handled readily enough but the argument is less tedious by supposing $c'' > 0$.

3. Nothing essential changes if we suppose p has a finite upper bound, but the cost, again, would be a more tedious argument.

4. Following a method developed by Modigliani and Hohn [5], Mills presents an algorithm for obtaining optimal behavior if the horizon is finite and future prices are represented by "certainty equivalents" [Ch. 4, 4] . In cases where future prices are stochastic Mills attempts to develop an approximation to a solution rather than the solution itself. [Ch. 4, 4].

5. We note that whenever a $\hat{p}$ or Φ appears with or without subscripts, these terms will be defined by integrals like those in equation (6) with the respective lower and upper limits given by some $(\alpha p^* - h)$ where the p^* is specified in an appropriate expectation. In the two-period example $\bar{p} = p_1^*$.

6. It is possible that the supply curves will "jump" to the vertical line since, as the mean price first becomes large enough to ensure $\alpha\bar{p} > h$, the critical prices jump from the mean price to new critical prices. To simplify the exposition we have also overlooked a minor difficulty. If $\bar{p} < b$ holds, the density $\varphi(p)$ would extend over some negative prices, unless otherwise modified. In such a circumstance we could truncate the density at zero and suppose a positive probability of a zero price, or otherwise modify the density, without significantly complicating matters.

7. The second partial derivative of π_2 with respect to price clearly depends on the third derivative of h. We have depicted a case in which $h''' < 0$.

8. If $\alpha\bar{r}'(x) \leq h'(x)$, it is easy to see that $\bar{r}'(x) = \bar{p}$ so that $\bar{r}'(x) \geq \bar{p}$ in any event.

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