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A MODEL OF INDUCED INVENTION, GROWTH AND DISTRIBUTION

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One of the Great Ratios of contemporary economics is the ratio of wages (and of profits) to national income. Notwithstanding some correlation with slack in the economy and perceptible trends in some countries, distributive shares have been remarkably constant in most western economies. Yet the modern economist has almost ceased to wonder at Bowley's Law. He is familiar with the kind of growth model[1] in which a path of golden age growth -- in which every variable changes, if at all, at a constant proportionate rate so that output, consumption, investment and capital all grow at the same rate -- is approached from most or all initial states. In a golden age, factor shares are constant, their magnitudes a function -- except in the special Cobb-Douglas case -- of the parameters determining the particular golden-age path the economy takes: the saving-income ratio, the population growth rate and the technological parameters.

But this kind of growth model does not really solve the puzzle of factor share constancy. For a golden age growth path can exist only if technical progress is Harrod neutral along that path;[2] and Harrod neutrality entails that progress have a special factor saving character. Further, in demonstrating

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1 See especially R. M. Solow, "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics, Vol. 70 (February 1956) and H. Uzawa, "Neutral Inventions and the Stability of Growth Equilibrium," Review of Economic Studies, Vol. 28 (February 1961).

2 Progress is said to be Harrod neutral if, when the marginal product of capital is constant, the average product of capital and hence capital's share is also constant.

the stability of the equilibrium golden age path -- the tendency of the economy to approach that golden age path corresponding to the parameters of the model -- it is usually postulated[3] that progress is Harrod neutral for all capital-labor ratios and therefore that progress can be expressed as (purely) labor augmenting.[4] Thus, after scrutinizing this growth model, one is led to ask why progress should be assumed to be Harrod neutral, either in or out of equilibrium.[5]

We conclude that a satisfactory model of the evolution of factor shares -- and such a model must be at the same time a model of economic growth -- depends on a satisfactory theory of the factor saving character of technical progress. This conclusion is not new. Hicks, Fellner and others,[6]

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3 See Uzawa, *op. cit.*

4 Technical progress is said to be factor augmenting if the production function $F(X_1, X_2, \ldots, X_n; t)$ can be put into the form $G(A_1(t)X_1, A_2(t)X_2, \ldots, A_n(t)X_n)$ where the X_i's are inputs and t denotes time. In effect, progress "augments" the inputs. The proportionate rate of increase of $A_i(t)$ is said to be the "rate of augmentation of the i-th input." Progress is (purely) "i-th factor augmenting" when only $A_i(t)$ increases over time, all other coefficients constant.

5 It is not only golden age equilibria that can exhibit constant factor shares. For example, if the saving-income ratio is exponentially declining, there may exist an equilibrium growth path on which shares are constant, capital and output grow exponentially (at different rates) and consumption grows non-exponentially. But the existence of such a growth path, like a golden age path, requires that progress have a special factor saving character.

6 J. R. Hicks, *The Theory of Wages* (London: Macmillan and Co., 1932), Chapter 6; W. J. Fellner, "Two Propositions in the Theory of Induced Innovations," *Economic Journal*, Vol. 71 (June 1961) and "Does the Market Direct the Relative Factor-Saving Effects of Technological Progress?" in the National Bureau of Economic Research volume, *The Rate and Direction of Inventive Activity* (Princeton, N.J.: Princeton University Press, 1962).

observing the "constancy" of factor shares, presuming that the aggregate elasticity of substitution is less than one and deducing that progress is labor saving (in the Hicksian sense), have asked whether there is some market mechanism which slants progress in the labor saving direction (in the Hicksian sense). Hicks asserted that there is, without specifying the mechanism. Fellner has argued that competitive firms will lean toward a relatively labor saving invention only if they expect wages to rise faster than capital-good rentals -- but even then, Fellner argued, the optimal invention may be capital saving. To our knowledge, this was as far as the theory of induced invention had gone until the publication of Charles Kennedy's seminal paper on the subject.[7]

Kennedy takes a great step forward by introducing what we shall call the <u>invention possibility frontier</u>.[8] Previous writers failed to specify in their models, to conjecture as it were, the family of alternative new technologies (isoquants) which inventors can produce and from which firms must choose, given the original technology. It was primarily for this reason that the theory lacked any very useful results. Kennedy's postulated frontier fully characterizes (at least on one interpretation of his model) the alternative new isoquants that are producible, given the original isoquant. He combines this

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7 Charles Kennedy, "Induced Innovation and the Theory of Distributive Shares," <u>Economic Journal</u>, Vol. 74 (September 1964). Earlier unpublished work by Christian von Weizsäcker was remarkably similar.

8 Kennedy calls it the "innovation possibility function" and prefers generally to speak of induced "innovation." By "innovation" we are accustomed to think of the introduction of known techniques not previously utilized, the costs of whose discovery have already been incurred. On that definition, there is no problem of choosing among innovations; the firm should accept all innovations that reduce unit costs. It seems to us more reasonable to speak of induced <u>invention</u>, as Hicks first had it.

frontier concept with a maximization postulate that may be a good first approximation, namely that firms seek to maximize, subject to the frontier, the current rate of cost reduction (hence, the current rate or intensity of technical progress), taking no interest in the factor saving bias of technical progress *per se*.

On these and other postulates, Kennedy shows that there may exist golden age equilibria, all of which yield the same factor shares. In these equilibria, he shows, factor shares depend only upon the shape of the invention possibility frontier (at a particular point), not upon relative factor supplies (hence the saving-income ratio) nor the elasticity of substitution as conventional growth theory holds. Thus we are given a new theory of distributive shares in golden age equilibrium.

But more needs to be done. Kennedy failed to show the stability of the factor share equilibrium. If factor shares do not approach their equilibrium values for many initial conditions, the new theory of equilibrium shares is uninteresting. Further, a constant saving-income ratio is implicit in Kennedy's golden age analysis. We believe it is useful to consider the behavior of the model under additional saving postulates.

We present here a model of induced invention, based on an interpretation of Kennedy's invention possibility hypothesis. In this model, technical progress is factor augmenting. The invention possibility frontier indicates the maximum rate of labor augmentation corresponding to a given rate of capital augmentation. We then investigate, under two kinds of postulated saving behavior, the existence, uniqueness and stability of a growth equilibrium (not necessarily a golden age) in which factor shares are constant. A brief summary of the principal findings and some suggestions for improving the model conclude the paper.

I. BASIC CONCEPTS AND RELATIONS

We shall consider a one-sector economy composed of identical, purely competitive firms. The firm's production function, which is also the aggregate production function for the economy, is supposed to be twice differentiable (smooth marginal productivities), homogeneous of the first degree in capital and labor (constant returns to scale), with positive marginal products and diminishing marginal rate of substitution everywhere:

$$Q = F(K, L; t) \tag{1}$$

where Q denotes the rate of output, K the stock of capital, L the labor force and t is time. While capital and labor are each homogeneous -- there is no capital-embodied or labor-embodied technical progress -- there is technical progress of the "disembodied" kind if $\frac{\partial F}{\partial t} \equiv F_t$ is positive.

Two characteristics of technical progress that are important to a neo-classical analysis of the evolution of factor shares are the <u>rate</u> or <u>intensity</u> of progress, R, and the factor-saving <u>bias</u> or <u>direction</u> of progress, D. We define the rate of progress as the (proportionate) rate of growth of output for fixed inputs:

$$R = \frac{F_t}{F} \tag{2}$$

Our measure of bias, in the Hicksian sense, is the proportionate rate of change of the ratio of the marginal product of capital (F_K) to the marginal product of labor (F_L), i.e., the marginal rate of substitution, for a given capital-labor ratio:

(3) $$D = m_K - m_L$$

where $$m_K = \frac{F_{Kt}}{F_K}, \quad m_L = \frac{F_{Lt}}{F_L}$$

Both R and D may be functions of the capital-labor ratio and time.[9] At a particular capital-labor ratio and at a particular time, technical progress is labor-saving in the Hicksian sense if $D > 0$, Hicks neutral if $D = 0$, and capital-saving if $D < 0$.

Another important concept is the elasticity of substitution:

(4) $$\sigma = -\frac{d(K/L)}{d(F_K/F_L)} \cdot \frac{F_K/F_L}{K/L}$$

The substitution elasticity may vary with the capital-labor ratio and time; we shall ultimately restrict (1) to be such that the quantity $\sigma - 1$ is of constant algebraic sign for all capital-labor ratios and time.[10]

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[9] These measures are in the same spirit as those defined by W. E. G. Salter in his Productivity and Technical Change (Cambridge: Cambridge University Press, 1960), Chapter 3. He defined the rate of progress as the proportionate rate of reduction of unit costs for given factor prices. His measure equals ours if there are constant returns to scale and factor prices equal the respective factors' marginal products. His measure of bias is the proportionate rate of increase of the least-cost capital-labor ratio for fixed factor prices and output. His measure is equal to ours times the elasticity of substitution; thus the two measures are of the same algebraic sign. The measures employed here, which are the most convenient for the purposes at hand, have been used before. See J. C. H. Fei and G. Ranis, Development of the Labor Surplus Economy (Homewood, Illinois: Irwin, 1964), A. A. Amano, "Neoclassical Biased Technical Progress and a Neoclassical Theory of Economic Growth," Quarterly Journal of Economics, Vol. 78 (February 1964), and P. A. Diamond, "Disembodied Technical Change in a One Sector Model," International Economic Review, Vol. 6 (May 1965).

[10] The "constant elasticity of substitution" production function of K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow is an example of such a function. See their paper, "Capital-Labor Substitution and Economic Efficiency," Review of Economics and Statistics, Vol. 43 (August 1961).

Following Amano [11] and Diamond[12], we can now derive an equation for the growth of capital's competitive share. This share, denoted $\underline{a}$, is

(5) $$a = \frac{F_K K}{Q}$$

whence, letting $\hat{x}$ denote the proportionate rate of change of a variable x , $\frac{\dot{x}}{x}$,

(6) $$\hat{a} = \hat{F}_K + \hat{K} - \hat{Q}$$

Amano and Diamond have shown, from equation (1) - (4), that

(7) $$\hat{F}_K = m_K - \frac{1-a}{\sigma} (\hat{K} - \hat{L})$$

Also,

(7a) $$\hat{F}_L = m_L + \frac{a}{\sigma} (\hat{K} - \hat{L})$$

Total differentiation of (1) with respect to time shows that

(8) $$\hat{Q} = a \hat{K} + (1-a) \hat{L} + R$$

By constant returns to scale,

(9) $$Q = F_K K + F_L L$$

Partial differentiation of this with respect to time, holding K and L constant, yields

(10) $$R = a\, m_K + (1-a)\, m_L$$

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[11] Amano, op. cit.

[12] Diamond, op. cit.

(The rate of progress equals a share-weighted average of the rates of increase of the marginal productivities for fixed K and L .)

Upon substituting (7), (8) and (10) into (6), one obtains the required equation for the growth of capital's share as a function of the bias, the substitution elasticity and the rates of increase of the factors:

$$\dot{a} = a(1-a)\,[D - \frac{1-\sigma}{\sigma}(\hat{K} - \hat{L})] \tag{11}$$

This equation confirms a familiar proposition in distributive share theory. Constancy of nonzero shares ($\dot{a} = 0$) in the face of, say, a rising capital-labor ratio ($\hat{K} > \hat{L}$) requires that technical progress be labor saving ($D > 0$) if $\sigma < 1$, Hicks neutral if $\sigma = 1$, and capital saving if $\sigma > 1$.

If σ , D , $\hat{K}$ and $\hat{L}$ are exogenous or functions only of factor shares, then these functions and (11) are all that is required for the analysis of the evolution of capital's share. For qualitative analysis, it is sufficient, under the particular invention possibility hypothesis studied here, to suppose that $\sigma - 1$ is of constant algebraic sign.

We suppose that L is exogenous and constant:

$$L(t) = L_0 e^{\gamma t} \quad \text{or} \quad \hat{L} = \gamma \geq 0 \ , \quad L_0 > 0 \ . \tag{12}$$

Concerning $\hat{K}$, we shall consider two kinds of saving functions. Case 1 is a constant saving-capital ratio, ρ :

$$\dot{K} = \rho K \quad \text{or} \quad \hat{K} = \rho \ , \quad K(0) > 0 \ . \tag{13a}$$

Case 2 is a constant or exponentially declining saving-income ratio:

$$(13b)\qquad \dot{K}(t) = \Theta\, e^{-\eta t}\, Q(t) \quad \text{or} \quad \hat{K}(t) = \Theta\, e^{-\eta t} \frac{Q(t)}{K(t)} ,$$

$$0 < \Theta < 1 \;,\quad \eta \geq 0 \;,\quad K(0) > 0 \;.$$

Finally, the Kennedy-based theory of induced invention makes the rate and bias of progress functions only of factor shares, hence $D = D(a)$, $R = R(a)$.

In Case 1, therefore, (11), (12), (13a) and the relation $D = D(a)$ yield a single differential equation of the form

$$(14)\qquad \dot{a} = a(1-a)\ [D(a) - \frac{1-\sigma}{\sigma}(\rho-\gamma)]$$

In this case we shall study the existence, uniqueness and stability of an equilibrium in which $a(t) = a^*$ for all t . By our assumption of everywhere positive marginal products and always positive factor supplies, both factor shares must be positive: $0 < a < 1$ for all t . Hence $0 < a^* < 1$ if a^* exists. By (14), such an equilibrium exists if and only if there is an a^* such that

$$(15)\qquad D(a^*) = \frac{1-\sigma}{\sigma}(\rho-\gamma)$$

for this is necessary and sufficient for $\mathring{a} = 0$ when $0 < a < 1$.
Of course, we shall also study the behavior of a when there exists no equilibrium.

In Case 2, $\hat{K}$ is an endogenous variable for which we need another differential equation. From (13b) we have, upon differentiation,

$$(16)\qquad \frac{\dot{\hat{K}}}{\hat{K}} = -\eta + \hat{Q} - \hat{K}$$

where $\dot{\hat{K}} = \frac{d\hat{K}}{dt}$, the absolute time rate of change of the (proportionate) growth rate of capital. Substituting the formula for the growth rate of output, (8), into (16) and using the relation $R = R(a)$ to be derived yields the second differential equation required:

$$(17) \qquad \dot{\hat{K}} = \hat{K}[R(a) - \eta - (1-a)(\hat{K}-\gamma)]$$

(Amano[13] previously derived this equation with $\eta = 0$.)

Equations (11) and (17) form a complete system for the analysis of Case 2. Our assumptions in (13b) that $\Theta > 0$ and initial $K(0) > 0$ guarantee that $\hat{K} > 0$ for all $\underline{t}$. Hence an equilibrium in Case 2 means a growth path such that $\hat{K}(t) = \hat{K}^* > 0$, $a(t) = a^*$, $0 < a^* < 1$ for all $\underline{t}$. Such an equilibrium is defined by the following equations, derived from setting the bracketed expressions in (11) and (17) equal to zero:

$$(18) \qquad D(a^*) = \frac{1-\sigma}{\sigma} (\hat{K}^* - \gamma)$$

$$(19) \qquad R(a^*) - \eta = (1-a^*)(\hat{K}^* - \gamma)$$

We shall consider now a model of induced invention that provides a theory of the rate and bias of technical progress.

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13 Amano, op. cit.

II. THE INDUCED INVENTION HYPOTHESIS

First, we suppose that technical progress is factor augmenting, meaning that the production function (1) is of the form[14]

$$(1') \qquad Q(t) = F[B(t)K(t)\ ,\ A(t)L(t)]\ ,$$

where the coefficients $B(t)$ and $A(t)$ are constants at any moment of time, i.e., independent of the capital-labor ratio.

Over time, firms can contrive to increase B or A or both by employing exogenously supplied inventors. The rates of factor augmentation, $\hat{B}(t)$ and $\hat{A}(t)$, are endogenous variables, being subject to choice by firms; the production function (1') merely records the implications for output growth of the paths of $B(t)$ and $A(t)$ selected by firms.

Second, we suppose the existence of an invention possibility frontier, known to firms, that give the maximum rate of labor augmentation obtainable for a given rate of capital augmentation.[15] This frontier, illustrated in Figure 1, is postulated to be constant over time and invariant to the capital-labor ratio (hence to factor prices and shares). The frontier is strictly convex: ever increasing amounts of labor augmentation must be sacrificed to obtain equal successive increments of capital augmentation. Positive factor

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[14] We exclude the Cobb-Douglas function ($\sigma = 1$ everywhere) for, by the multiplicative character of that function, $A(t)$ and $B(t)$ are then not defined.

[15] Such a frontier was postulated by Christian von Weizsäcker in unpublished work in 1962-63. The model he formulated was different from the model studied here.

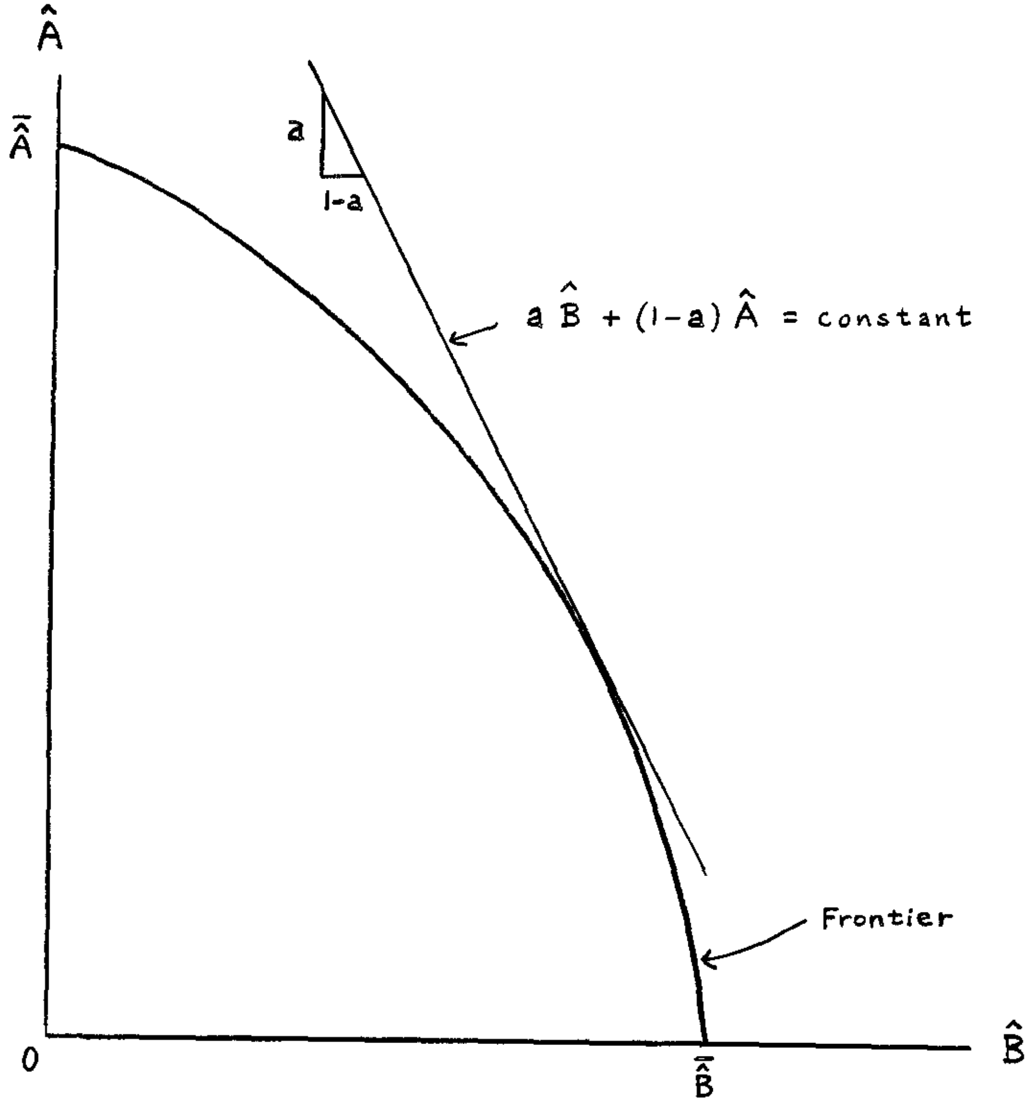

Figure 1: Invention Possibility Frontier

augmentation is feasible. Finally, the efficient portion of the frontier lies everywhere in the positive quadrant; that is, it is infeasible for inventors to "buy" additional labor (capital) augmentation at the cost of capital (labor) diminution -- in short, it never pays to give up cumulative capital (labor) augmentation achieved in the past. (Further, we wish the "new" isoquant not to intersect the "old" which means that the rate of technical progress, $a\hat{B} + (1-a)\hat{A}$, is non-negative for all capital-labor ratios; to guarantee that progress is everywhere non-negative for all production functions, it must be supposed that the rates of augmentation chosen are non-negative.)

Summarizing mathematically,

$$(20) \qquad \hat{A} = \Phi(\hat{B}) \,, \quad \Phi(0) > 0 \,, \quad \Phi'(\hat{B}) < 0 \,, \; \Phi'(\hat{B}) < 0 \,, \; \hat{A}, \hat{B} \geq 0 \,.$$

The maximum rates of augmentation will be denoted $\bar{\hat{A}}$ and $\bar{\hat{B}}$.

The last part of the present hypothesis is that firms will choose $\hat{A}(t)$ and $\hat{B}(t)$ so as to maximize the current rate of technical progress (because they wish to maximize the current rate of cost reduction) subject to the frontier (20).

Differentiation of (1') partially with respect to time shows that the rate of progress is a share-weighted average of the rates of augmentation:

$$(21) \qquad R = a\hat{B} + (1-a)\hat{A}$$

Using (20) we can express the firms' maximization problem as

$$(22) \qquad \max_{\hat{B}} \; R = a\hat{B} + (1-a)\,\Phi(\hat{B})$$

For a certain range of $\underline{a}$, an interior maximum exists in which the derivative with respect to $\hat{B}$ of the maximand in (22) equals zero:

$$\frac{\partial R}{\partial \hat{B}} = a + (1-a)\,\Phi'(\hat{B}) = 0$$

(23)

$$\Phi'(\hat{B}) = \frac{-a}{1-a}$$

This solution is illustrated in Figure 1 by the tangency of the frontier with a straight line of slope $\frac{-a}{1-a}$; this line is the locus of points giving the maximum feasible rate of progress.

By geometrical analysis of Figure 1, one can see immediately that an increase of $\underline{a}$ increases optimal $\hat{B}$ and decreases optimal $\hat{A}$ in the range of $\underline{a}$ for which the maximum is an interior one. Total differentiation of (23) with respect to $\underline{a}$ confirms this:

$$\hat{B}'(a) \equiv \frac{d\hat{B}}{da} = \frac{-1}{(1-a)^2\Phi''(\hat{B})} > 0 \quad \text{since} \quad \Phi'' < 0$$

(24)

$$\hat{A}'(a) \equiv \frac{d\hat{A}}{da} = \Phi'(\hat{B})\,\hat{B}'(a) < 0 \quad \text{since} \quad \Phi' < 0 .$$

Thus, technical progress will be more capital augmenting on balance ($\hat{B} - \hat{A}$ larger) the larger is capital's share in this range.

It is clear from Figure 1 that if $\underline{a}$ is so large that the straight line is everywhere steeper than the frontier then there is a corner maximum at the horizontal axis. That is, if $a + (1-a)\,\Phi'(\bar{\hat{B}}) \geq 0$ or equivalently

$a \geq a_2 \equiv \dfrac{-\Phi'(\bar{\hat{B}})}{1-\Phi'(\bar{\hat{B}})}$ then $\hat{B} = \bar{\hat{B}}$, $\hat{A} = 0$. Similarly, if $\underline{a}$ is so small that $a + (1-a)\,\Phi'(0) \leq 0$ or equivalently $a \leq a_1 \equiv \frac{-\Phi'(0)}{1-\Phi'(0)}$ then $\hat{B} = 0$, $\hat{A} = \bar{\hat{A}}$.

Summarizing, the optimal $\hat{B}$ and $\hat{A}$ are continuous functions of capital's share with the following properties:

$$(25)\qquad \begin{aligned} \hat{B} &= \hat{B}(a)\,, \\ \hat{A} &= \hat{A}(a)\,, \end{aligned} \quad \begin{cases} \hat{A}(a) = \bar{\hat{A}}\,, \quad \hat{B}(a) = 0\,, \quad 0 \leq a \leq a_1 \equiv \dfrac{-\Phi'(0)}{1-\Phi'(0)} \\ \hat{A}'(a) < 0\,, \quad \hat{B}'(a) > 0\,, \quad \dfrac{-\Phi'(0)}{1-\Phi'(0)} \equiv a_1 < a < a_2 \equiv \dfrac{-\Phi'(\bar{\hat{B}})}{1-\Phi'(\bar{\hat{B}})} \\ \hat{A}(a) = 0\,, \quad \hat{B}(a) = \bar{\hat{B}}\,, \quad \dfrac{-\Phi'(\bar{\hat{B}})}{1-\Phi'(\bar{\hat{B}})} \equiv a_2 \leq a \leq 1 \end{cases}$$

We need now to express the bias of progress in terms of $\hat{B}$ and $\hat{A}$. In the appendix we derive the following formulas for the rates of increase of marginal productivities, holding factors fixed:

$$(26)\qquad \begin{aligned} m_K &= \hat{B} - \frac{1-a}{\sigma}\,(\hat{B} - \hat{A}) \\ m_L &= \hat{A} + \frac{a}{\sigma}\,(\hat{B} - \hat{A}) \end{aligned}$$

From these formulas and the definition (3) we obtain the bias:

$$(27)\qquad D = \frac{1-\sigma}{\sigma}\,(\hat{A} - \hat{B})$$

This result implies that technical progress which is labor augmenting on balance $(\hat{A} > \hat{B})$ will be labor saving $(D > 0)$ or capital saving $(D < 0)$ according as σ is less than or greater than one. Similarly, predominantly capital augmenting progress will be capital saving if $\sigma < 1$ and labor saving if $\sigma > 1$. If $\hat{B} = \hat{A}$, progress is Hicks neutral for all σ .[16]

We are ready now to characterize the functions R(a) and D(a) . But it will be convenient to do this in the course of analyzing Case 1 and Case 2. But the following remark about the dependence of bias on capital's share may be useful. We saw that (in a certain range) an increase (decrease) of capital's share increases (decreases) the rate of capital augmentation on

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[16] This last result raises a question in our minds as to whether Kennedy intended his invention frontier to measure rates of factor augmentation. He asserted that progress is labor saving, neutral or capital saving according as his p (our $\hat{A}$) is greater, equal, or less than his q (our $\hat{B}$); but if it is factor augmentation rates on the axes of the frontier diagram, this statement of Kennedy's is true if and only if $\sigma < 1$, as we have just shown. Further, Kennedy makes his frontier extend into the second and fourth quadrants; as we pointed out earlier, the rates of augmentation must be non-negative if for all production functions, progress is to be <u>everywhere</u> factor augmenting and non-negative. And he states that "there is a good deal to be gained" by holding factor prices constant; but there is nothing to be gained on the above interpretation of the frontier. These considerations have led us to suspect that his frontier measures the rates of decrease of least-cost labor for alternative rates of decrease of least-cost capital -- the output rate and the factor-price ratio being taken as given. This interpretation accords perfectly with the way Kennedy pictured the frontier and with his statement about bias. We have analyzed such a model but space limitations prevent us from presenting it here.

The evidence on the other side is that in the second part of Kennedy's paper he constructs growth models in which technical progress is "factor-price diminishing" and this is equivalent to factor augmenting progress.

In any case, Kennedy with von Weizsäcker deserves credit for suggesting the model under analysis here.

balance; and that an increase (decrease) of capital augmentation on balance will make technical more (less) capital saving if $\sigma < 1$. But as (11) shows, an increase (decrease) of "capital savingness" depresses (raises) capital's share. Hence, if $\sigma < 1$, the mechanism of induced invention hypothesized here tends to stabilize capital's share around some equilibrium value. The subsequent analysis demonstrates, among other things, that $\sigma < 1$ is sufficient for stability of factor-share equilibrium.

III. THE BEHAVIOR OF FACTOR SHARES

(i) Case 1. Here we have the single differential equation, from (14) and (27)

$$\mathring{a} = a(1-a)\left[\frac{1-\sigma}{\sigma}\left(\hat{A}(a) - \hat{B}(a)\right) - \frac{1-\sigma}{\sigma}(\rho-\gamma)\right] \tag{28}$$

or

$$\mathring{a} = a(1-a)\left(\frac{1-\sigma}{\sigma}\right)\left[\hat{A}(a) - \hat{B}(a) - (\rho-\gamma)\right] \tag{28a}$$

which, together with (25) which specifies $\hat{B}(a)$ and $\hat{A}(a)$, governs the evolution of capital's share. An equilibrium a^*, $0 < a^* < 1$, is therefore determined by

$$\hat{A}(a^*) - \hat{B}(a^*) = \rho - \gamma \tag{29}$$

We now construct a phase diagram to study the existence, uniqueness and stability of such an equilibrium.

In Figure 2 we plot $\hat{A}(a)$, $\hat{B}(a)$ and $\hat{A}(a) - \hat{B}(a)$ as given by (25). These curves have three segments corresponding to the three ranges of $\underline{a}$.

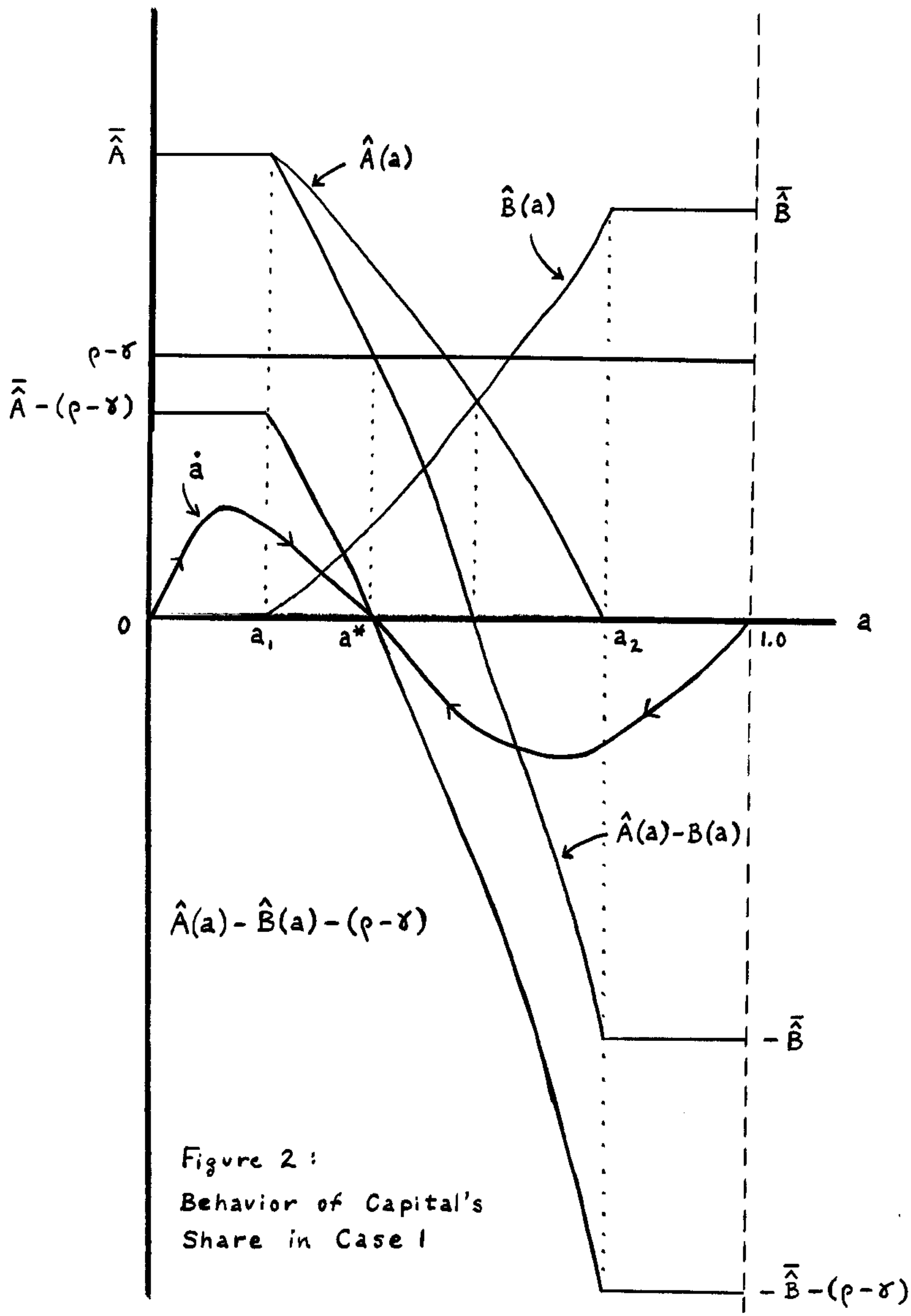

Figure 2:
Behavior of Capital's Share in Case 1

Next we draw in a horizontal line with height $\rho - \gamma$ to represent the proportionate rate of increase of the capital-labor ratio, $\rho - \gamma$; for purposes of illustration we take $\rho - \gamma > 0$.

The intersection of the $\rho - \gamma$ line with the $\hat{A}(a) - \hat{B}(a)$ curve determines the equilibrium a^* . For if $\hat{A}(a) - \hat{B}(a) = \rho - \gamma$ then $\dot{a} = 0$ for all σ so a is constant and hence in equilibrium. Clearly, such an equilibrium exists and is unique if $\bar{\hat{A}} > \rho - \gamma > - \bar{\hat{B}}$.

To study the stability of such an equilibrium, we next construct the curve $\hat{A}(a) - \hat{B}(a) - (\rho - \gamma)$, which is seen to be positive for $0 \leq a < a^*$ and negative for $a^* < a \leq 1$. Then we plot $\dot{a}$, which, as (28') shows, will have the same sign as $\hat{A}(a) - \hat{B}(a) - (\rho - \gamma)$ if $\sigma < 1$ and the opposite sign if $\sigma > 1$; the $\sigma < 1$ case is illustrated in the diagram. As (28') indicates, $\dot{a} = 0$ at $a = 0$ and $a = 1$. Thus there appear to be two "corner" equilibria at $a = 0$ and $a = 1$. But if initial factor supplies are positive and marginal products are everywhere positive, as postulated, then these values of a cannot be attained. Hence there is just one equilibrium, a^* . This equilibrium is globally stable if and only if $\sigma < 1$ for then $\dot{a} > 0$ when $0 < a < a^*$ and $\dot{a} < 0$ when $a^* < a < 1$. This is because it is only if $\sigma < 1$ that the bias becomes more capital saving (or less labor saving) when $a > a^*$ and vice-versa when $a < a^*$.

If we continue to suppose that $\bar{\hat{A}} > \rho - \gamma > - \bar{\hat{B}}$, we see from Figure 2 that a^* is a decreasing function of the rate of capital deepening, $\rho - \gamma$, *independently of* σ . An interesting point suggested to us by Professor Samuelson is that if the invention frontier is symmetrical, $a^* = \frac{1}{2}$ when $\rho - \gamma = 0$; for then equilibrium $\hat{A}$ equals equilibrium $\hat{B}$ and the a which makes optimal A equal optimal B is $\frac{1}{2}$. But by the same reasoning, when $\rho - \gamma > 0$, $a^* < \frac{1}{2}$. Hence the theory of long-run distributive shares here is consistent, on the assumption of a symmetrical frontier, with the observed tendency for capital's share to be less than one-half.

To complete the analysis, we note first that if $\rho - \gamma = \bar{\hat{A}}$ then $\dot{a} = 0$ for all $0 < a \leq a_1$. Any a in this range is a possible equilibrium a^* , the actually resulting equilibrium being dependent upon initial conditions. Capital's share will approach this equilibrium range when initial $a > a_1$ if and only if $\sigma < 1$. These equilibria are golden ages. For since $\hat{B} = 0$ in this range, $\rho + \hat{B} = \gamma + \hat{A} = \hat{Q}$ by virtue of (1') so that capital, investment, output and consumption all grow at the "natural" golden-age rate, $\bar{\hat{A}} + \gamma$. The saving-output ratio is therefore constant.

If $\rho - \gamma > \bar{\hat{A}}$, there is no equilibrium. If $\sigma < 1$, a will approach zero and if $\sigma > 1$, a will approach unity. But such a large ρ may be infeasible, given γ . The production function, the growth rate of labor, the frontier and the constraint that investment not exceed income determine a maximum sustainable rate of capital growth which ρ cannot exceed. We omit analysis of this problem.[17]

If $\rho - \gamma = -\bar{\hat{B}}$, an indeterminacy of equilibrium also results. And if $\rho - \gamma < -\bar{\hat{B}}$, a will approach unity if $\sigma < 1$ and approach zero if $\sigma > 1$.

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17 A path in which $\sigma < 1$ and $\rho > \bar{\hat{A}} + \gamma$ for all t is impossible since then $a \to 0$; but once $a < a_1$, $\hat{A}^* = \bar{\hat{A}}$ and $\hat{B}^* = 0$ which implies that, since labor is required for positive output if $\sigma < 1$, $\bar{\hat{A}} + \gamma$ is the maximum sustainable growth rate of output and capital.

(ii) Case 2. When the saving-income ratio is constant or exponentially declining, we have the two differential equations, from (11), (12), (17), (21) and (27):

$$\dot{a} = a\,(1-a)\,\left(\frac{1-\sigma}{\sigma}\right)\,[\hat{A}(a) - \hat{B}(a) - (\hat{K} - \gamma)] \tag{30}$$

$$\dot{\hat{K}} = \hat{K}\,[a\,\hat{B}(a) + (1-a)\,\hat{A}\,(a) - \eta - (1-a)(\hat{K} - \gamma)] \tag{31}$$

which, given (25), govern the behavior of the share and growth rate of capital.

From (30) and (31) we see that an equilibrium, $\hat{K}^* > 0$, $0 < a^* < 1$, must satisfy

$$\hat{K}^* = \hat{A}(a^*) - \hat{B}(a^*) + \gamma \tag{32}$$

and

$$\hat{K}^* = \frac{a^*}{1-a^*}\,\hat{B}(a^*) + \hat{A}(a^*) - \frac{\eta}{1-a^*} + \gamma \tag{33}$$

Equating the righthand sides of (32) and (33) yields

$$\hat{B}(a^*) = \eta \tag{34}$$

which determines a^*. Recalling that $\hat{A} = \Phi(\hat{B})$ and substituting (34) into (32), we obtain

$$\hat{K}^* = \Phi(\eta) - \eta + \gamma \tag{35}$$

Thus (34) and (35) characterize an equilibrium of this system, if one (or more) exists. We shall now investigate the existence, uniqueness and stability of this equilibrium by geometric analysis.

In Figure 3 (and also Figures 4 and 5) there are two curves, labelled $\dot{a} = 0$ and $\dot{\hat{K}} = 0$. The first of these curves is the locus of points $(a, \hat{K})$ which make a constant, $0 < a < 1$. The equation of this curve, derived from (30) is

$$(36) \qquad \hat{K} = \hat{A}(a) - \hat{B}(a) + \gamma$$

Differentiating,

$$(37) \qquad \frac{d\hat{K}}{da} = \hat{A}'(a) - \hat{B}'(a) \begin{cases} < 0 & \text{if } a_1 < a < a_2 \\ 0 & \text{otherwise } [\text{by } 25] \end{cases}$$

Using (25), we see that this curve has three segments corresponding to the three ranges of a . In Figure 3 it is assumed that $\bar{\hat{B}} < \gamma$ so that the curve lies everywhere above the horizontal axis. (Figure 4 illustrates the curve when $\bar{\hat{B}} > \gamma$.)

Now, by (30) and (36), if $\sigma < 1$, then $\dot{a} < 0$ (a decreasing) when $(a, \hat{K})$ lies above this curve and $\dot{a} > 0$ (a increasing) when $(a, \hat{K})$ lies below this curve. The opposite is true if $\sigma > 1$. Supposing that $\sigma < 1$, we therefore show arrows pointing in some easterly direction below the $\dot{a} = 0$ curve and westerly above.

The $\dot{\hat{K}} = 0$ curve is the locus of points $(a, \hat{K})$ which make $\hat{K}$ constant, $\hat{K} > 0$. The equation of this curve, derived from (31) is

$$(37) \qquad 0 = a\,\hat{B}(a) + (1-a)\,\hat{A}(a) - \eta - (1-a)\,\hat{K} + (1-a)\gamma$$

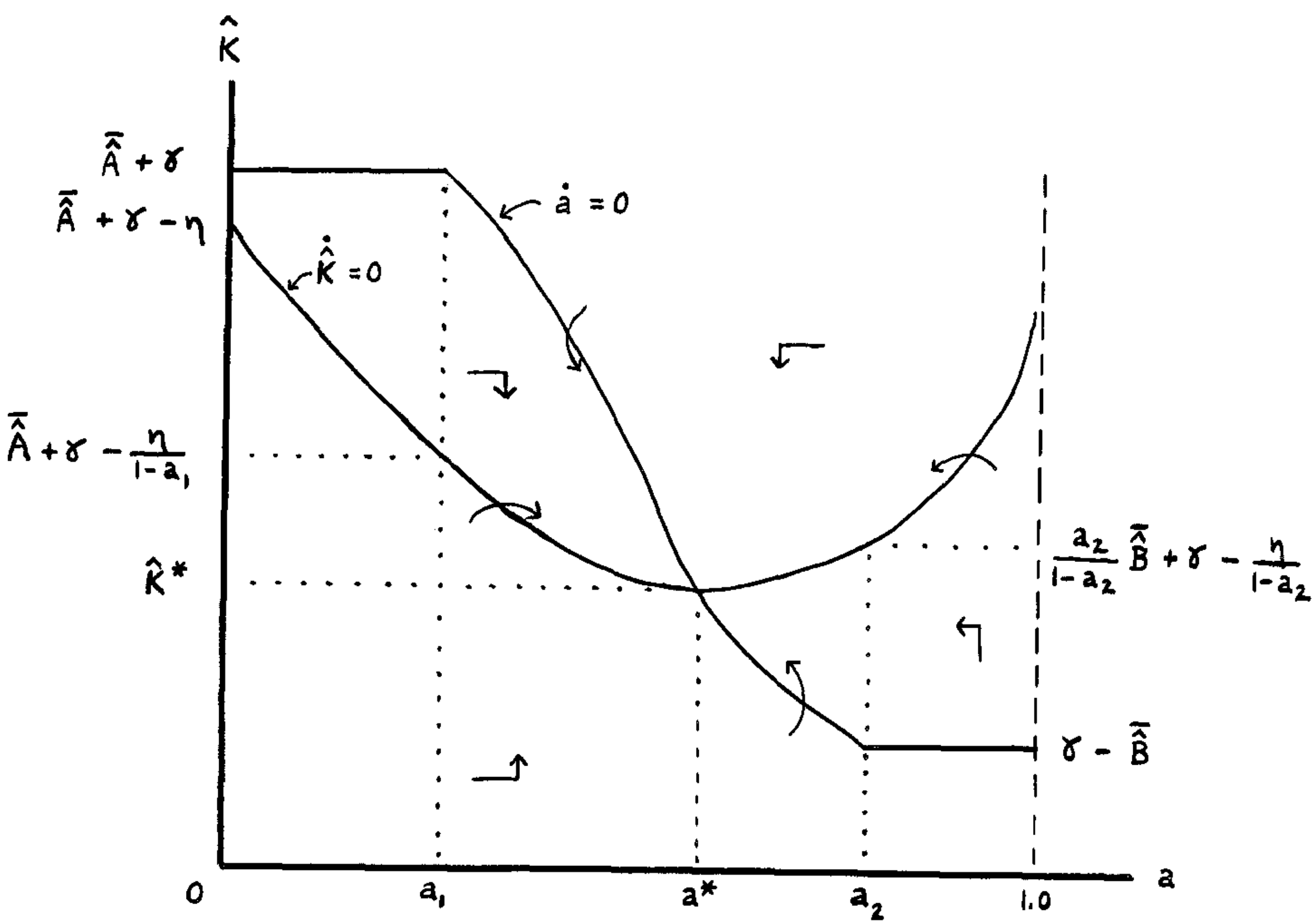

Figure 3: Behavior of the Share and Growth Rate of Capital in Case 2: $0 < \eta < \hat{\bar{B}} < \gamma$

or

$$(37a) \qquad \hat{K} = \frac{a}{1-a}\ \hat{B}(a) + \hat{A}(a) - \frac{\eta}{1-a} + \gamma$$

Differentiating (37), we obtain[18]

$$(38) \qquad \frac{d\hat{K}}{da} = \frac{1}{1-a}\left\{ [\hat{B}(a) - \hat{A}(a) + \hat{K} - \gamma] + [a\ \hat{B}'(a) + (1-a)\hat{A}'\ (a)] \right\}$$

$$= \frac{1}{1-a}\ [\hat{B}(a) - \hat{A}(a) + \hat{K} - \gamma] \qquad [\text{by } (23) \text{ and } (25)]$$

Differentiating (37a) we obtain an equivalent result:

$$(38a) \qquad \frac{d\hat{K}}{da} = \frac{\hat{B}(a) - \eta}{1 - a}$$

Equations (37a), (38) and (38a), together with (25), give the information required to plot the $\dot{\hat{K}}$ curve.

It is supposed that $0 < \eta < \bar{\hat{B}}$ in both Figures 3 and 4. Since $\eta > 0$, the $\dot{\hat{K}} = 0$ curve starts below the $\dot{a} = 0$ curve. Since the latter curve satisfies $\hat{B} - \hat{A} + \hat{K} - \gamma = 0$, the $\dot{\hat{K}}$ curve must, by (38), be downward sloping until it reaches the $\dot{a} = 0$ curve, have a zero slope at the intersection, and be upward sloping thereafter. (As $a \to 1$, the slope of the $\dot{\hat{K}} = 0$ curve becomes infinite by virtue of (38a) and (25), if $\eta < \dot{\hat{B}}$ as supposed in Figures 3, 4 and 5.) The two curves <u>must</u> intersect in this fashion when $0 < \eta < \bar{\hat{B}}$ for then

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[18] The last step in (38) follows from (23) and (25) which imply that in the end regions of <u>a</u>, $\hat{A}'(a) = \hat{B}'(a) = 0$ and elsewhere (for $a_1 < a < a_2$) there is an interior maximum, $\frac{\partial R}{\partial \hat{B}} = 0$ so that $a\hat{B}'(a) + (1-a)\hat{A}'(a)$

$$= [a + (1-a)\,\frac{d\hat{A}}{d\hat{B}}\,]\,\frac{d\hat{B}}{da} = \frac{\partial R}{\partial \hat{B}} \cdot \frac{\partial \hat{B}}{da} = 0\ .$$

$\frac{a_2}{1-a_2}\ \bar{\hat{B}} - \frac{\eta}{1-a_2} + \gamma > \gamma - \bar{\hat{B}}$, so that the $\mathring{\hat{K}} = 0$ curve lies above the $\dot{a} = 0$ curve at $a = a_2$.

If $(a, \hat{K})$ lies above the $\mathring{\hat{K}} = 0$ curve, then $\mathring{\hat{K}} < 0$ ($\hat{K}$ decreasing); if $(a, \hat{K})$ lies below this curve, $\mathring{\hat{K}} > 0$ ($\hat{K}$ increasing); this is independent of σ since (31) does not contain σ . Thus the arrows above the $\mathring{\hat{K}} = 0$ curve point south and the arrows below point north.

Figure 3, in which it is supposed that $0 < \eta < \bar{\hat{B}} < \gamma$, shows that there exists in this case a unique equilibrium, $\hat{K}^* > 0$, $0 < a^* < 1$. If $\sigma < 1$ then this equilibrium is globally stable. The arrows show the direction of the path of $(a, \hat{K})$ in any zone for $\sigma < 1$. At worst, $(a, \hat{K})$ can begin a counter-clockwise cycle around the equilibrium, landing in the zone bounded by the two curves in the interval $0 \leq a \leq a^*$. The arrows show that this zone traps $(a, \hat{K})$ and leads it to the equilibrium.

But if $\sigma > 1$, then it can be shown that the equilibrium is not stable even in the neighborhood of equilibrium: $\mathring{a}$ has the wrong sign for $(a, \hat{K})$ off the $\dot{a} = 0$ curve, with the result that <u>a</u> approaches zero or one asymptotically.

An equilibrium necessarily exists in the case just analyzed where $\eta < \bar{\hat{B}} < \gamma$. Suppose now that $\bar{\hat{B}} \geq \gamma$, which is more reasonable in progressive economies. Figure 4 illustrates the case $\bar{\hat{B}} > \gamma$ so that the $\dot{a} = 0$ curve enters the lower (fourth) quadrant.

We continue to suppose that $0 < \eta < \bar{\hat{B}}$. This condition, as we saw, guarantees that the $\mathring{\hat{K}} = 0$ curve will intersect the $\mathring{a} = 0$ curve in the downward sloping range of the latter curve. If the intersection occurs in the upper quadrant, so that $\hat{K}^* > 0$, then the situation is inconsequentially different from Figure 3: a unique equilibrium exists and is globally stable if and only if $\sigma < 1$. But from (35) we see that this intersection will occur in the upper quadrant (so $\hat{K}^* > 0$) if and only if $\Phi(\eta) - \eta + \gamma > 0$ or $\eta < \Phi(\eta) + \gamma$; thus if and only if η is not too large.

In Figure 4 we analyze a case where there is no equilibrium because $\eta > \Phi(\eta) + \gamma$; the curves intersect in the lower quadrant. In this case there is, however, what might be called a quasi-equilibrium, $(a^*, \hat{K}^*)$ given by the intersection of the $\dot{a} = 0$ curve with the horizontal axis. This quasi-equilibrium satisfies the equations

(39) $$\hat{K}^* = 0$$

(40) $$\hat{A}(a^*) - \hat{B}(a^*) + \gamma = 0 \qquad \text{[by (30)]}$$

This state is not a true equilibrium for it cannot be attained -- it is not a feasible initial state -- since $\Theta > 0$ in (13b) and initial $K(0) > 0$ imply that $\hat{K}(t) > 0$ for all $\underline{t}$. (If one wants to set $\Theta = 0$ then we are back in Case 1 with $\rho = 0$.) But this quasi-equilibrium can be approached asymptotically. It will be approached (i.e., it is globally stable) if and only if $\sigma < 1$. The arrows in Figure 4 indicate the direction of $(a, \hat{K})$ when $\sigma < 1$. The zone bounded by the $\dot{\hat{K}} = 0$ curve, the horizontal axis and the $\dot{a} = 0$ curve forces $(a, \hat{K})$ toward $(a^*, \hat{K}^*)$. As the arrows show, either $(a, \hat{K})$ goes directly to $(a^*, \hat{K}^*)$ or it cycles into this zone and thence to $(a^*, \hat{K}^*)$. Of course, as with the approach to a true equilibrium, $\hat{K}$ approaches zero only asymptotically

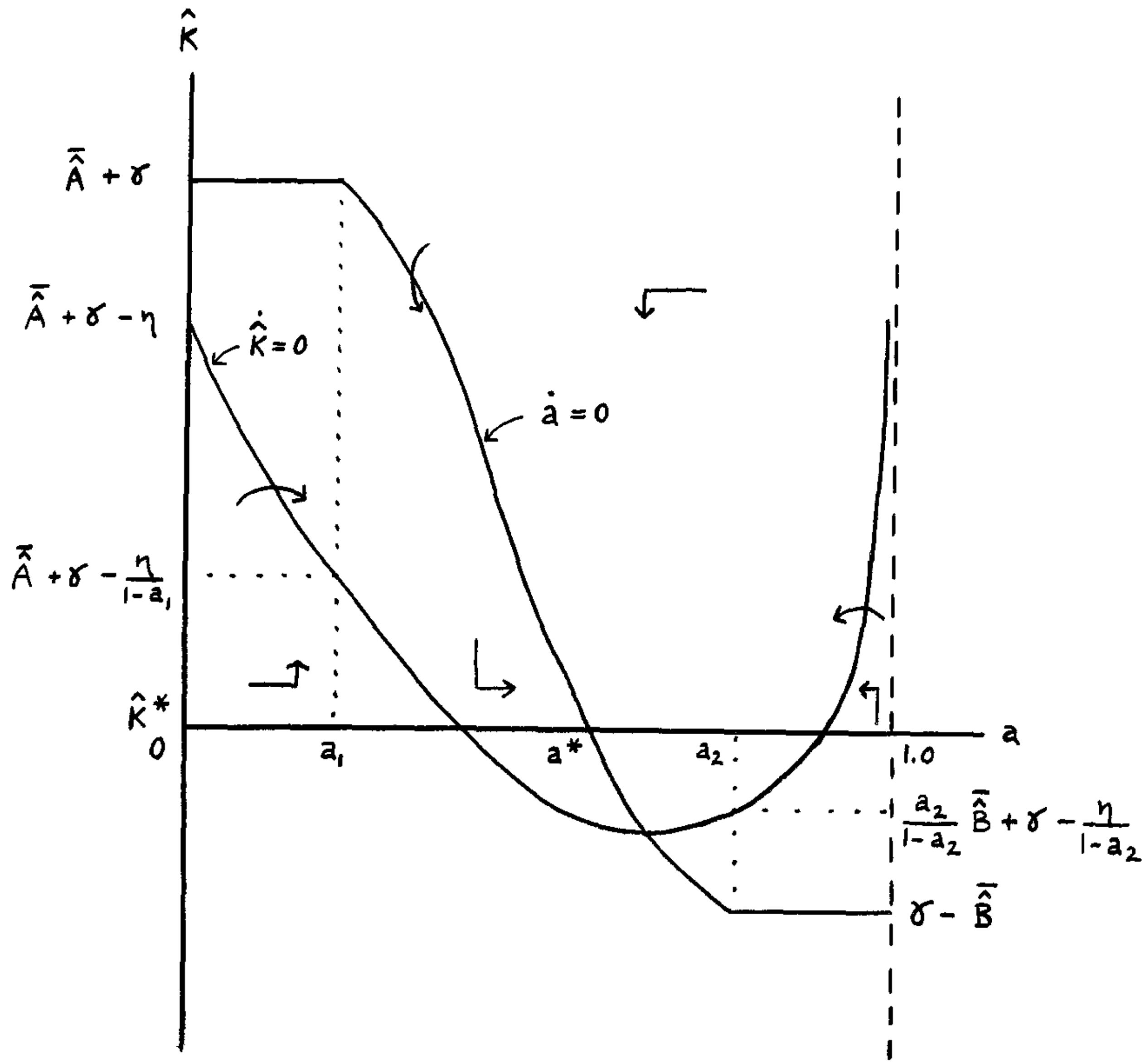

Figure 4: Behavior of the Share and Growth Rate of Capital in Case 2: $0 < \eta < \bar{\hat{B}}$, $\bar{\hat{B}} \geq \eta$

for as $\hat{K} \to 0$, $\dot{\hat{K}} \to 0$ by (31).

We have treated exhaustively the case $0 < \eta < \bar{\hat{B}}$. We shall omit a detailed analysis of the case $\eta \geq \bar{\hat{B}}$. If $\eta = \bar{\hat{B}}$, the $\dot{\hat{K}} = 0$ curve coincides with the $\dot{a} = 0$ curve over the lower horizontal stretch of the latter curve (where $a_2 \leq a \leq 1$) . If $\eta > \bar{\hat{B}}$ the curves cannot intersect (there is no equilibrium therefore) and the $\dot{\hat{K}} = 0$ curve is everywhere downward sloping with infinite slope as $a \to 1$.

Now if $\bar{\hat{B}} \geq \gamma$, then the case $\eta \geq \bar{\hat{B}}$ is inconsequentially different from Figure 4: the quasi-equilibrium will be approached if and only if $\sigma < 1$. If $\hat{B} < \gamma$ (as in Figure 3), then, when $\eta = \bar{\hat{B}}$, there will be a continuum of equilibria, $a_2 \leq a^* < 1$, $\hat{K}^* = \gamma - \bar{\hat{B}}$ which will be approached if and only if $\sigma < 1$; when $\eta > \bar{\hat{B}}$, there being no equilibrium, a approaches one or zero according as σ is less than or greater than one.

We turn now to the last remaining case, $\eta = 0$, i.e., a constant saving-income ratio. Figure 5 illustrates this case. The $\dot{\hat{K}} = 0$ curve coincides with the $\dot{a} = 0$ curve along the horizontal stretch, $0 \leq a \leq a_1$ [by (37a)], and is upward sloping thereafter [by (38a)]. Hence there is a continuum of equilibria, $0 < a^* \leq a_1$, $\hat{K}^* = \bar{\hat{A}} + \gamma$. These equilibria are golden ages: output, capital, investment and consumption all grow at the "natural" rate, $\bar{\hat{A}} + \gamma$.

The arrows in Figure 5, like those in all our diagrams here, are based on the assumption that $\sigma < 1$ everywhere. They show that, at worst, $(a, \hat{K})$ will cycle counter-clockwise around the point $(a_1, \hat{K}^*)$ and approach a point on the horizontal equilibrium line. Thus if $\sigma < 1$, some point on the horizontal line will be approached asymptotically from every feasible initial state.

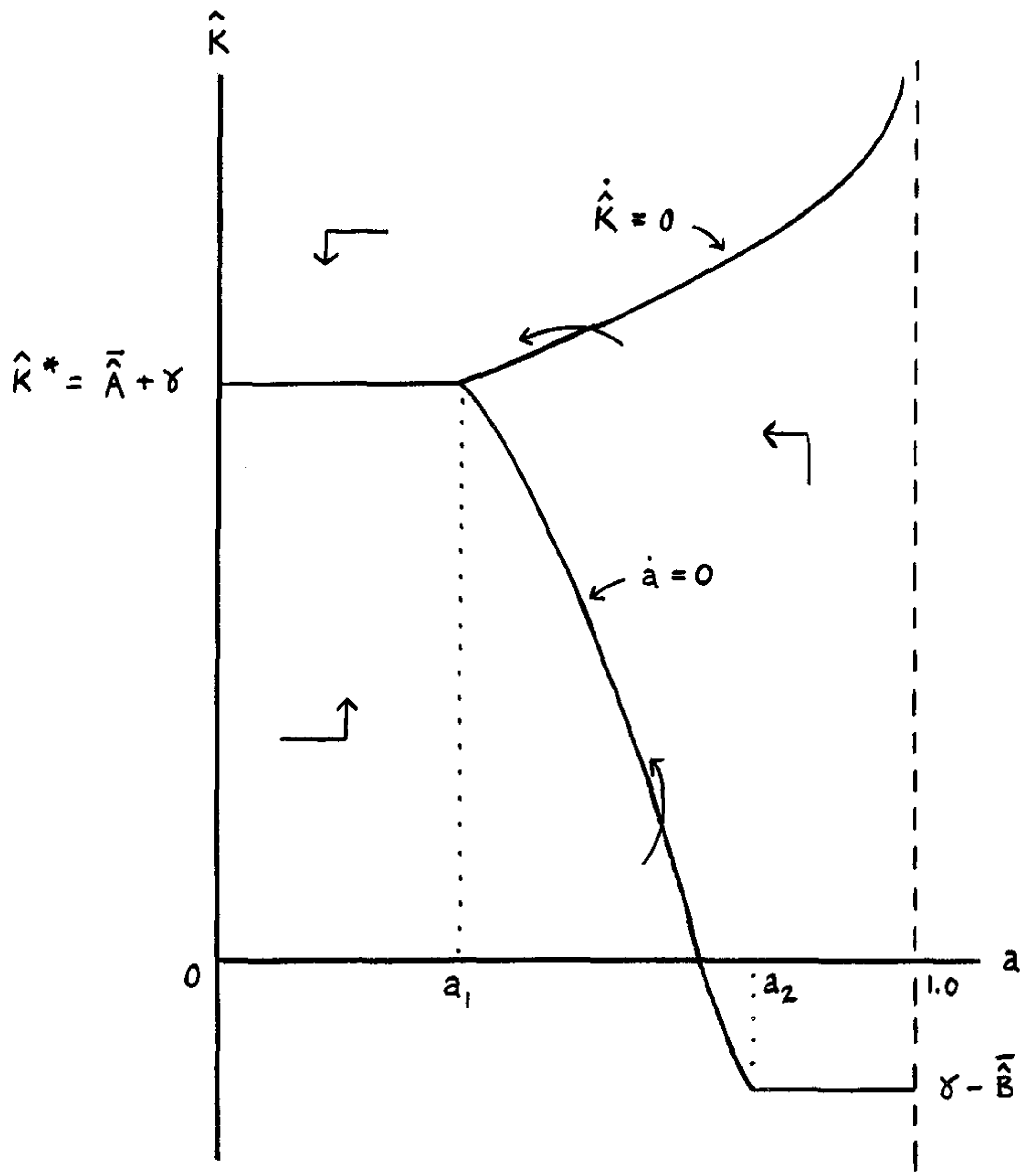

Figure 5: Behavior of the Share and Growth Rate of Capital in Case 2: $\eta = 0$

More precisely, if the initial state is in the southwest zone, $(a, \hat{K})$ moves northeast and may approach some equilibrium point monotonically or it may enter the east zone -- the course depending on the exact initial state. From the east zone the particular equilibrium state $(a_1, \hat{K}^*)$ may be approached or $(a, \hat{K})$ will enter the north zone. From the north zone, where the motion is southwest, some point on the horizontal line will be approached -- which one depending upon the initial state. The point $(0, \hat{K}^*)$ is one such point but any other point on the horizontal line is a possibility.[19]

This "indeterminacy" -- the asymptotic shares depend upon initial conditions -- means that the conventional growth-theoretic explanation of capital's share has some relevance. The initial conditions will involve the saving ratio, the labor growth rate, the rate of labor augmentation and the elasticity of substitution, precisely those parameters to which conventional theory gives an explanatory role.

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19 Contrariwise, it might be thought that $(0, \hat{K}^*)$ is the only possible resting point from the north zone. Our colleague David Cass has suggested a way of showing, by a linear approximation of the two differential equations, that there are paths of $(a, \hat{K})$ which approach any interior point of the horizontal line with a nonzero slope.

Incidentally, no equilibrium point in this $\eta = 0$ case meets the necessary and sufficient conditions for local stability. But this is to be expected when there is a continuum of equilibrium points: If $(a, \hat{K})$ departs from one equilibrium point, it is not necessarily pulled back to that same point but likely pulled to another equilibrium point.

The analysis of this special $\eta = 0$ case can be conducted along a different line, in which the variables are the ratio of augmented capital to augmented labor, BK/AL, and the accumulated degree of capital augmentation, B. When $a < a_1$ and $\hat{K} > \hat{K}^*$, this analysis reduces to the usual growth model with purely labor augmenting progress. An equilibrium a^*, $0 < a^* \leq a_1$, can be shown to be approached by the usual kind of analysis.

The case $\sigma > 1$ is especially interesting. Should the economy be in the southwest zone, the motion is northwest and a point on the horizontal line will be approached, possibly $(0, \hat{K}^*)$. In the north zone, the motion is southeast and either an equilibrium point is approached or the east zone is entered. In the east zone; $a \to 1$ and $\hat{K} \to \infty$. The result $\hat{K} \to \infty$ conforms to our knowledge that when the saving-income ratio is constant and progress is at least partially capital augmenting, as it will be once $a > a_1$, output and capital will grow "super-exponentially" -- no finite, equilibrium, exponential growth rate of capital is approached..

It is unnecessary to consider the case $\eta < 0$ (exponentially increasing saving ratio) as saving would then eventually exceed income, which is impossible without foreign aid.

We turn briefly to comparative statics. If a <u>unique</u> equilibrium exists -- the conditions for which are $0 < \eta < \max [\bar{\hat{B}}, \Phi(\eta) + \gamma]$ -- then (34) and (35) show a^* and $K^* - \gamma$ to be functions only of η . Since $\hat{B}(a^*)$ is increasing in a^* , a^* is increasing in η . And since $\Phi(\eta) - \eta$ is decreasing in η , $\hat{K}^* - \gamma$ is decreasing in η , as expected. Since $\hat{B}^* + \hat{K}^* = \hat{A}^* + \gamma = \hat{Q}^*$ in equilibrium [by (32)], and $\hat{A}^* = \Phi(\eta)$ is decreasing in η , $\hat{Q}^*$ is also decreasing in η . It may be of interest that since σ can, by (1'), be a function only of the ratio of augmented capital to augmented labor, BK/AL , and this is constant in equilibrium [by (32)], σ will be constant in equilibrium. Finally, our earlier observation that if the invention frontier is symmetrical, then $a^* < \frac{1}{2}$ when $\hat{K}^* - \gamma > 0$ is valid in Case 2 also. Hence, if there is symmetry, then $a^* < \frac{1}{2}$ when $\hat{K}^* - \gamma = \Phi(\eta) - \eta > 0$; this inequality is satisfied for sufficiently small $\eta \geq 0$.

IV. CONCLUDING REMARKS

By way of summary, the following results stand out. If the saving-income ratio is constant, there will exist a continuum of equilibria with Harrod-neutral progress, constant shares and golden age growth. The particular equilibrium that is approached (if there is an approach to some equilibrium) will depend upon conventional factors such as the magnitude of the saving-income ratio and the parameters of the production function. If the saving ratio is exponentially declining, but not too fast, there will exist a unique factor-share equilibrium with exponential growth of capital and output. In this case, equilibrium shares depend only upon the shape of the invention possibility frontier and the equilibrium rate of capital deepening (which depends in turn upon the rate of decrease of the saving ratio) -- not upon the initial level of the saving ratio nor the elasticity of substitution. However, the substitution elasticity was found to be critical for the stability of equilibrium. If the invention frontier is assumed to be symmetrical, the model will predict capital's share to be less than one-half in an economy enjoying steady capital deepening (capital outpacing labor).

We have many reservations about the model presented here. The vehicle for our analysis, a non-vintage, one-sector model of production, is undoubtedly unrealistic. Inventors produce new hardware, hence <u>capital embodied</u> progress, so that a "vintage" model of production is appropriate. In a multi-sector model one could study the problem of the optimal allocation of inventive effort among sectors. But one's greatest doubts center on the invention hypothesis.

Maximization of the current rate of technical progress (or rate of cost reduction) can be only a crude approximation to the optimal invention policy. Such maximization may be shortsighted for two reasons. First, even if the invention frontier be stationary and even if expected wages and rentals be stationary, maximization of current cost reduction may alter the shares of labor and capital in future unit costs and thereby, possibly, diminish the maximum rate of cost reduction attainable in the future. Second, even if there is to be just one invention, so that one does not need to consider the effect of current invention on the pay-off to optimal future invention, the firm will want to evaluate alternative inventions not only at current but also at future expected factor prices.[20] (The appropriate maximand under certainty and a perfect credit market is the reduction of the present discounted value of the stream of expected unit costs.)

Most critical of all is the postulate of a stationary invention frontier of the form $\psi(\hat{A}, \hat{B}) = 0$. The postulate of factor augmenting progress is very restrictive. Second, such a function will vary with research effort and the latter should be endogenous in the model. Finally, a controversial objection to the stationarity of the frontier has been raised by Professor Gary Becker. He suggests that if the rate of labor augmentation has been "abnormally" high for a long time, maintenance of such a rate of labor augmentation will become increasingly expensive in terms of capital augmentation -- that inventors might even exhaust (temporarily?) the possibilities for further labor augmentation. Clearly the notion, if accepted, that there are "normal" paths of labor and capital augmentation ($A(t)$ and $B(t)$) from which the actual paths cannot far deviate, while not fatal to a mechanism of partially induced invention, may severely limit the scope of such a mechanism for explaining the behavior of factor shares.

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20 See Fellner, "Two Propositions," op. cit.

APPENDIX

Let

(1) $$F(K, L ; t) = G(BK, AL)$$

Then

(2) $$F_K = G_1 B \; ; \quad a = \frac{F_K K}{F} = \frac{G_1 BK}{G} \; ; \quad 1-a = \frac{G_2 AL}{G}$$

(3) $$\sigma = \frac{F_K F_L}{F\, F_{KL}} = \frac{G_1 B\; G_2 A}{G\; A\; G_{12} B} = \frac{G_1 G_2}{G\; G_{12}}$$

where the subscripts denote partial derivatives. Hence

(4) $$m_K = \frac{\partial F_K}{\partial t} \cdot \frac{1}{F_K} = \hat{B} + \frac{\partial G_1}{\partial t}\, \frac{1}{G_1}$$

$$= \hat{B} + (G_{11}\, K\, \dot{B} + G_{12}\, L\, \dot{A})\, \frac{1}{G_1}$$

$$= \hat{B} + [(G_{11} BK)\hat{B} + (G_{12}\, AL)\hat{A}]\, \frac{1}{G_1}$$

But $G_{11} BK = - G_{12} AL$ since G_1 is homogeneous of degree zero in BK and AL.

Hence

(5) $$m_K = \hat{B} + \frac{G_{12} AL}{G_1} (\hat{A} - \hat{B})$$

$$= \hat{B} + \frac{G_2 AL\; G\; G_{12}}{G\; G_1 G_2} (\hat{A} - \hat{B})$$

$$= \hat{B} + \frac{1-a}{\sigma} (\hat{A} - \hat{B}) \; .$$

The formula for m_L is derived similarly.