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MODELS OF TECHNICAL PROGRESS

AND THE GOLDEN RULE OF RESEARCH

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The primary purpose of this paper is to present a model of technical progress and economic growth from which one can derive a Golden Rule of Research quite analogous to the Golden Rule of Accumulation. The secondary purpose is to discuss various models of technical progress.

That there might exist a Golden Rule for research is not surprising since research is a kind of investment -- investment in technology -- just as "accumulation" denotes investment in tangible capital. Nevertheless "capital" and "technology" differ in important respects. If, given the technology, the production of commodities is homogeneous of degree one in labor and capital goods, as is frequently postulated, then production cannot be homogeneous of degree one in labor, capital and "technology" (or the appropriate index of past research effort). Further, the production of technical progress, unlike the production of investment goods, seems unlikely to exhibit constant returns to scale in capital and labor, given the technology. Hence technology cannot be treated as an ordinary capital good. Because of this novelty that our subject presents, this paper must be highly tentative and conjectural.

MODELS OF TECHNICAL PROGRESS

In order to speak about the "level of technology" in an operational way without specifying all details of the production function and in order to make possible the generation of golden ages we shall postulate that technical progress is solely labor-augmenting, that is, Harrod-neutral. Hence

$$Q(t) = F\Big(K(t)\ ,\quad T(t)\ N(t)\Big) \tag{1}$$

Where $Q(t)$ denotes the rate of output of the single consumer-capital good of the economy, $K(t)$ the stock of capital, $N(t)$ the rate of employment of ordinary production workers (as distinct from researchers) and $T(t)$ the level of technology, all at time $\underline{t}$. Technical progress here is of the "disembodied" kind: old capital and new capital are identical, as are old workers and young workers. It would not affect our results if we were to make our labor-augmenting technical progress "capital-embodied"; and while, to our knowledge, "labor-embodied" technical progress has never been modelled, it seems likely that such embodiment could also be introduced without critical effect.

Second, we suppose that the labor force is homogeneous. No one has an absolute advantage in either research or the production of commodities. Later we relax this assumption.

Finally, we suppose initially that only labor is productive in the research industry; capital, therefore, is not employed to produce technical progress. Thus the level of technology at any particular point of time depends in some way upon past research by the labor force. We address ourselves now to the nature of that functional relationship.

There are certain properties which we wish the "technology function" to possess. The first of these properties is diminishing returns. Given the current level of technology, we require that the marginal effectiveness of researchers decrease with the amount of research done at a particular time. Consider the following technology function, inspired by the frequent assumption that $T(t) = T(0)\, e^{\lambda t}$:

$$T(t) = T(0)\, e^{\beta \int_0^t R(v)dv} \tag{a}$$

where $R(v)$ denotes the amount of employment in the research industry. This function implies increasing marginal effectiveness of research at time v since $\frac{\partial T(t)}{\partial R(v)} = \beta\, T(t)$ and $T(t)$ is an increasing function of $R(v)$. So we rule out this function.

Second, we require that there should be a loss from bunching a given amount of total research effort in a short interval of time. This is the condition for diminishing marginal rate of substitution. Consider the function proposed by Mansfield:

$$T(t) = \beta \left[\int_{-\infty}^t R(v)dv \right]^\alpha , \quad \alpha < 1 . \tag{b}$$

This function exhibits diminishing returns but it implies that the marginal rate of substitution between research at one time and at another for producing the technology level at time t is constant (and unitary) so that there is no premium on a smooth research trend.

The third property we require is that the marginal effectiveness of current research be an increasing function of the level of technology recently attained. We may call this property technical progress in research. This is perhaps controversial but it seems plausible that today's researchers have a higher marginal effectiveness than those of a century ago. This does not imply, of course, that the relative rate of technical progress must increase over time with a stationary research trend or even with an exponentially rising research trend. Consider, then, the following technology function which, if $G(R)$ is increasing and concave, implies both diminishing marginal returns and diminishing marginal rate of substitution:

$$T(t) = \int_{-\infty}^{t} G\Big(R(v)\Big)\, dv \tag{c}$$

This function implies that the marginal effectiveness function, $G'(R)$, is stationary so that researchers do not get more productive over time. It would be easy to introduce exogenous technical progress in the research industry (and perhaps natural to do so in an open economy) but we prefer to suppose that such technical progress is endogenously produced by the researchers themselves.

The fourth property we require is that exponential growth of researchers will produce an exponential increase of the level of technology, provided research has always been growing exponentially at the same rate. Clearly this assumption is motivated by our desire to generate golden ages. We have no evidence to attack or defend this assumption; suffice it to say that it is not so far-fetched as to merit no analysis. The heuristic value of analyzing golden ages is plain. Consider, then, the following technology function, which meets the first three of our requirements if H(R, T) is an increasing, concave function:

(d) $$T(t) = \int_{-\infty}^{t} H\Big(R(v)\ ,\ T(v-\omega)\Big)\, dv\ , \quad \omega > 0\ .$$

Here the effectiveness of researchers at time $\underline{t}$ in producing technical progress at time $\underline{t}$ is an increasing function of the level of technology at time $t - \omega$. The "retardation," ω, might be interpreted as the "publication lag." But whether this function satisfies our fourth requirement depends upon the nature of H(R, T).

Suppose that the productivity of researchers in producing technical progress at time $\underline{t}$ is proportional to the level of technology at $t - \omega$. That is, suppose that H is homogeneous of degree one in $T(v-\omega)$:

(d') $$H\Big(R(v),\ T(v-\omega)\Big) = T(v-\omega)\ G\Big(R(v)\Big)$$

Then, since $\dot{T}(t) = H\Big(R(t),\ T(t-\omega)\Big)$,

$$\dot{T}(t) = T(t-\omega)\quad G\Big(R(t)\Big)$$

or

$$\frac{\dot{T}(t)}{T(t)} = \frac{T(t-\omega)}{T(t)}\ G\Big(R(t)\Big)\ .$$

This implies that a constant rate of research permits a constant relative rate of increase of the technology level. For, as may be easily verified from the previous equation, if $R(v) = c$ for all v then

$\dot{T}(t)/T(t) = r$ in the transcendental equation

$$r = e^{-r\omega}\, G(c)\ .$$

Further, it is only if $R(v) = c$ that technology will grow exponentially; if $R(v)$ should grow exponentially, $T(t)$ will rise faster than exponentially. Thus (d') fails to satisfy our fourth requirement that exponential growth of research will produce a constant relative rate of technological progress.

Fortunately there is at least one function which will satisfy our four requirements. This is the technology function in (d) where it is assumed now that $H(R, T)$ is homogeneous of degree one in both $R(v)$ and $T(v-\omega)$. This means that if the technology level should double we would require exactly twice the amount of research to double the absolute time rate of increase of the technology, $\dot{T}(t) = H\Big(R(t), T(t-\omega)\Big)$. Thus we shall suppose, in supplement to (d), the following:

$$\text{(e)} \qquad H\Big(R(v), T(v-\omega)\Big) = T(v-\omega) \quad H\left(\frac{R(v)}{T(v-\omega)}, 1\right)$$

Letting $h(x) = H(x, 1)$ and using $\dot{T}(t) = H\Big(R(t), T(t-\omega)\Big)$ we obtain

$$\text{(e')} \qquad \frac{\dot{T}(t)}{T(t)} = \frac{T(t-\omega)}{T(t)} \; h\left(\frac{R(t)}{T(t-\omega)}\right)$$

It can be shown that constant research effort will cause the relative rate of technical progress to approach zero.*

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*Proof: Let $R(v) = c$ for all v. If T approaches an upper limit then obviously $\dot{T}/T$ approaches zero. Assume now that T increases without limit. Then

$$\lim_{t\to\infty} \frac{\dot{T}(t)}{T(t-\omega)} = \lim_{t\to\infty} h\left(\frac{c}{T(t-\omega)}\right) = 0$$

so that

$$\lim_{t\to\infty} \frac{\dot{T}(t)}{T(t)} \frac{T(t)}{T(t-\omega)} = \lim_{t\to\infty} \frac{\dot{T}(t)}{T(t)} \cdot \lim_{t\to\infty} \frac{T(t)}{T(t-\omega)} = 0$$

But $\lim_{t\to\infty} \frac{T(t)}{T(t-\omega)} > 0$. Hence $\lim_{t\to\infty} \frac{\dot{T}(t)}{T(t)} = 0$.

Constancy of the relative rate of growth of technology, that is, exponential technical progress, occurs if research is exponentially rising and has always done so. Suppose that $R(t) = R(0)\, e^{\gamma t}$. Then $T(t) = T(0)\, e^{\gamma t}$ satisfies (e'). That is, $\dot{T}/T = \gamma$ is the solution to (e') . If we make these substitutions in (e') we obtain the transcendental equation

$$(e'') \qquad \gamma = e^{-\gamma\omega}\, h \left(\frac{R(0)}{T(0)}\, e^{\gamma\omega} \right)$$

which yields the steady-growth "level" of technology at the reference time point, time zero, as a function of the level of research at that time. Equation (e") implies that a doubling of the "level" of research, without any change of the relative rate of increase of research, will produce a doubling of the "level" of ~~research~~ technology (without changing the relative rate of growth of the technology). This in no way contradicts the diminishing returns requirement for that refers to a <u>ceteris paribus</u> variation of research at a single moment of time. Such proportionality is analogous to the proportionality, in golden ages, of the "level" of exponentially growing output to the "level" of the exponentially growing labor force, despite diminishing marginal productivity of labor in the usual sense.

GOLDEN AGES AND GOLDEN RULES

To complete the model we introduce first a full-employment condition

$$R(t) + N(t) = L(t) \tag{2}$$

where $L(t)$ is the total labor force and $N(t)$ is employment in the production of commodities.

Second, all unconsumed output is invested and there is no depreciation. Hence

$$C(t) + \dot{K}(t) = Q(t) \tag{3}$$

where $C(t)$ denotes consumption and $\dot{K}(t)$ the rate of increase of capital (hence, the rate of investment).

By definition, in a golden age, the total labor force, researchers and commodity producing workers all grow at constant relative rates. This condition together with (2) imply that they must grow at the same relative rate. Hence

$$L(t) = L_o e^{\gamma t} , \tag{4}$$

$$R(t) = R(0) e^{\gamma t} , \qquad \gamma > 0 . \tag{5}$$

Equations (2), (4) and (5) imply that $N(t) = N(0) e^{\gamma t}$ so that the latter is not an independent equation. Note that while $L(0)$ is an historical datum given by the parameter L_o , $R(0)$ is not; the "level" of research effort (even in the infinite past) we imagine to be open to choice.

Equations (5) and (d)-(e) imply, as we showed in the previous section, that technology grows exponentially at rate γ and that the "level" of technology at time zero is proportional to the level of research at that time. Hence we may write

$$(6) \qquad T(t) = \eta\, R(0)\, e^{\gamma t} , \qquad \eta > 0 .$$

Note that if there exist technology functions other than (d)-(e) which, with (5), imply (6) then the following analysis will apply to them as well as the technology function we have adopted.

Equations (1)-(6) contain seven unknown variables: $L(t)$, $R(t)$, $N(t)$, $T(t)$, $Q(t)$, $K(t)$ and $C(t)$. To determine all these variables we require a seventh equation specifying the behavior of the capital stock. For golden ages -- equilibria in which all variables change at constant relative rates -- it is, given the other equations, necessary and sufficient that the ratio of capital to effective labor be constant:

$$(7) \qquad \frac{K(t)}{T(t)N(t)} = k .$$

If we assume constant returns to scale in (1) then

$$(8) \qquad Q(t) = T(t)\, N(t)\, F\left(\frac{K(t)}{T(t)\, N(t)} , 1\right) ,$$

whence, by (4), (5), (6), and (7)

$$(9) \qquad Q(t) = T(0)\, e^{\gamma t}\, N(0)\, e^{\gamma t}\, F(k, 1)$$

$$= T(0)\, N(0)\, e^{2\gamma t}\, F(k, 1) .$$

Hence, output will grow exponentially at the rate 2γ if the ratio of capital to effective labor is fixed.

By virtue of (4), (5), (6) and (7), the capital stock will grow exponentially at the same rate:

$$K(t) = T(0)\ N(0)\ e^{2\gamma t}\ k \tag{10}$$

Therefore, investment will do the same:

$$\mathring{K}(t) = 2\gamma\ T(0)\ N(0)\ e^{2\gamma t}\ k \tag{11}$$

Hence, by (3), consumption will also grow at the rate 2γ and is given by

$$C(t) = \left[F(k,\ 1) - 2\gamma k \right]\ T(0)\ N(0)\ e^{2\gamma t}\ . \tag{12}$$

For every k there corresponds a constant tangible investment-output ratio,

$$s = \frac{\mathring{K}(t)}{Q(t)} = \frac{2\gamma k}{F(k,\ 1)}\ , \tag{13}$$

and a constant marginal productivity of capital,

$$\frac{\partial Q(t)}{\partial K(t)} = F_K\ (k,\ 1)\ . \tag{14}$$

If capital receives its marginal product, therefore, its relative share of output will also be constant:

$$a = \frac{\partial Q(t)}{\partial K(t)}\ \frac{K(t)}{Q(t)} = \frac{F_K(k,\ 1)k}{F(k,\ 1)}\ . \tag{15}$$

Thus constancy of the capital-effective labor ratio together with (1)-(6) imply a golden age. Conversely, it can be shown that a golden age implies constancy of the ratio of capital to effective labor.

That the golden-age or "natural" growth rate is simply 2γ is a striking result and one which suggests an empirical test of the model. But it is not likely the model would perform well until the technology function is made to accommodate "imported" technical progress.

We have shown that a golden age exists for every value of k and $R(0)$. Let us fix $R(0)$ and ask in which golden age -- one golden age corresponds to every k -- the path of consumption is maximal. In short, let us derive the Golden Rule of Accumulation in this model where technical progress is endogenous.

Assuming an interior maximum to be attained, we need merely differentiate $C(t)$ with respect to k in (12) and equate the derivative to zero:

(16) $$F_K(k, 1) - 2\gamma = 0$$

Hence, the Golden Rule of Accumulation prescribes choosing k so as to equate the marginal productivity of capital to the golden-age rate of growth. Another way to express the Golden Rule of Accumulation results from multiplying both marginal productivity and growth rate by the capital-output ratio, whence

(17) $$s = a$$

This states that, on the maximal consumption path, the investment-output ratio equals capital's competitive share. Or to put it another way, of the workers producing commodities, a proportion <u>a</u> of them earmark their output for investment along the maximal path.

This is the familiar rule. The endogenity of technical progress does not alter the rule. Nor does it matter at which level we fix $R(0)$, provided only that $0 < R(0) < L_0$.

Let us now vary the ratio of researchers to commodity-producing workers, this time fixing the capital-effective labor ratio and hence the tangible investment-output ratio. Corresponding to each value of $R(0)$ $\Big($and hence $N(0)\Big)$ is a different golden age and a certain consumption path. The higher $R(0)$ the greater will be the level of technology, $T(0)$, hence the greater will be output and consumption for any $N(0)$; but the higher $R(0)$ the smaller must $N(0)$ be, hence the smaller will output and consumption be for any given level of technology. It is immediately clear that neither $R(0) = L_o$ nor $R(0) = 0$ is consumption-maximizing so that we may expect an interior maximum. What is the consumption-maximizing research level? That is, what is the Golden Rule of Research?

Looking again at (12), we see that if we wish to maximize $C(t)$ we need only maximize "effective labor," $T(0)\,N(0)$, subject to (2). If this is surprising, note that since k and hence the tangible investment-output ratio is fixed, maximization of consumption is equivalent to maximization of output, $Q(t) = F\Big(K(0)\,,\ T(0)\,N(0)\Big)\,e^{2\gamma t}$; and that since $K(0) = k\,T(0)\,N(0)$ we are maximizing $F\Big(k\,T(0)\,N(0)\,,\,T(0)\,N(0)\Big)$ which reduces to maximizing $T(0)\,N(0)$.

Differentiating $T(0)\,N(0)$ with respect to $R(0)$, subject to (2), and equating the derivative to zero yields

$$(18) \qquad 0 = \frac{\partial T}{\partial R} N + \frac{\partial N}{\partial R} T$$

$$= \eta\, N - T$$

But from (6) we have $T(0) = \eta\, R(0)$ so that

$$(19) \qquad N = R$$

is the solution.* Thus we see that the Golden Rule of Research prescribes

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* Or, in (18), replace $\frac{\partial T}{\partial R}$ by T/R (by virtue of the proportionality between T and R) which yields $N/R = 1$.

engaging exactly one half of the labor force in research. Note that just as the Golden Rule of Accumulation specifies that a fraction <u>a</u> of the commodity-producing workers should earmark their output for investment independently of the ratio of researchers to the labor force, the Golden Rule of Research specifies that one half the labor force should engage in research independently of the tangible-investment-output ratio or capital-effective labor ratio.

Fifty percent is surprisingly large. Our assumptions that the labor force is homogeneous and that no capital is required in research have biased our result toward high research. Later we shall relax both these assumptions.

It might be thought that our result would differ if we fixed not the tangible investment-output <u>ratio</u> or, equivalently, the capital-effective labor <u>ratio</u> but rather the "absolute" path of the capital stock itself. But this is not so. Of course, we require exponential growth of capital at the rate 2γ for a golden age. So let us replace (7) by the condition

$$K(t) = K_o \, e^{2\gamma t} \tag{7'}$$

Then the golden-age consumption path, which is a function of $R(0)$, is

$$\begin{aligned} C(t) &= F\left(K_o \, e^{2\gamma t} \, , \; T(0) \, N(0) \, e^{2\gamma t}\right) - 2\gamma \, K_o \, e^{2\gamma t} \\ &= \left[F\left(K_o \, , \, T(0) \, N(0)\right) - 2\gamma \, K_o\right] e^{2\gamma t} \end{aligned} \tag{20}$$

We see that maximization of $C(t)$ with respect to $R(0)$ once again entails only the maximization of "effective labor." It makes no difference whether we fix the "absolute" capital stock path or the path of the capital-effective labor ratio (hence the tangible investment-output ratio).

The Golden Rule of Accumulation was expressed, among others, in terms of the marginal productivity of capital and the rate of growth. The former is the rate of return to investment in this model. Indeed, in more general models, it is always the rate of return which is equated to the growth rate along the Golden Rule path and not necessarily the marginal productivity of capital. The question may be asked, therefore, whether the Golden Rule of Research calls for equating the rate of return from research to the growth rate. We show now that it does.

To do this we cast the model into a discrete-time framework, where

$$C_t + I_t = Q_t \ , \qquad I_t = K_{t+1} - K_t \ . \tag{3'}$$

$$L_t = L_0 \ (1 + \gamma)^t \ , \tag{4'}$$

$$R_t = R_0 \ (1 + \gamma)^t \ , \tag{5'}$$

and

$$T_t = \Phi(R_{t-1}, R_{t-2}, \ldots\ldots) \tag{6'}$$

where

$$T_t = T_0 \ (1 + \gamma)^t \qquad \text{if} \quad (5') \text{ holds.}$$

Output can be expressed as follows:

$$Q_t = P(R_{t-1}, R_{t-2}, \ldots \ ; \ I_{t-1}, \ I_{t-2}, \ldots\ldots \ ; \ L_t - R_t) \tag{1'}$$

In a golden age where $Q_t = Q_0(1 + g)^t$, $I_t = I_0(1 + g)^t$ we have

(9')
$$Q_t = P\left(R_t(1 + \gamma)^{-1} , R_t(1 + \gamma)^{-2} , \ldots ; I_t(1 + g)^{-1} , I_t(1 + g)^{-2} , \ldots ; L_t - R_t\right)$$

Hence consumption in a golden age can be written

(12')
$$C_t = P\left(R_t(1 + \gamma)^{-1}, R_t(1 + \gamma)^{-2}, \ldots ; I_t(1 + g)^{-1}, I_t(1 + g)^{-2}, \ldots ; L_t-R_t\right)-I_t$$

As was demonstrated in the continuous time model, in a golden age Q_t , C_t and I_t grow like $T_t N_t = T_0(1 + \gamma)^t N_0(1 + \gamma)^t$ so that $(1 + g) = (1 + \gamma)^2$.

As a preliminary exercise we show that (assuming an interior maximum) the Golden Rule of Accumulation prescribes equality of the rate of return from investment and the rate of growth: Fix R_0 and maximize C_t with respect to I_t . This yields

$$\frac{\partial C_t}{\partial I_t} = 0 = \sum_1^\infty \frac{\partial P}{\partial I_{t-i}} (1 + g)^{-i} - 1$$

or

(21)
$$\sum_1^\infty \frac{\partial P}{\partial I_{t-i}} (1 + g)^{-i} = 1 .$$

Now we define the rate of return to investment at time $\underline{t}$ as that value of $\underline{r}$ such that

(22)
$$\sum_1^\infty \frac{\partial P_{t+i}}{\partial I_t} (1 + r)^{-i} = 1$$

Next we observe that in this non-vintage model, as well as in the vintage model, the marginal productivity at <u>t</u> of investment made at t-i is equal to the marginal productivity at t+i of investment at <u>t</u> <u>in a golden age</u>. (In fact, the model implies constancy of the marginal productivity of investment.)

But if $\frac{\partial P_{t+i}}{\partial I_t} = \frac{\partial P_t}{\partial I_{t-i}}$ then, by (22),

$$(23) \qquad \sum_{1}^{\infty} \frac{\partial P_t}{\partial I_{t-i}} (1 + r)^{-i} = 1$$

Equations (21) and (23) imply that $r=g$ when $\frac{\partial C_t}{\partial I_t} = 0$.

Let us now fix the investment path, that is, I_0 , and maximize C_t with respect to R_t . This yields

$$\frac{\partial C_t}{\partial R_t} = 0 = \sum_{1}^{\infty} \frac{\partial P}{\partial R_{t-i}} (1 + \gamma)^{-i} - \frac{\partial P}{\partial N_t}$$

or

$$\sum_{1}^{\infty} \frac{\partial P}{\partial R_{t-i}} (1 + \gamma)^{-i} = \frac{\partial P}{\partial N_t}$$

or

$$(24) \qquad \sum_{1}^{\infty} \frac{\partial P}{\partial R_{t-i}} \frac{\partial N_t}{\partial P} (1 + \gamma)^{-i} = 1 .$$

Note that the marginal productivity of labor <u>i</u> periods ago is

$$(25) \quad \left(\frac{\partial P}{\partial N}\right)_{t-i} = \frac{\partial P}{\partial N_t} (1 + \gamma)^{-i} \quad \text{or} \quad \frac{\partial N_t}{\partial P} = \left(\frac{\partial N}{\partial P}\right)_{t-i} (1 + \gamma)^{-i} ,$$

since the average and marginal productivity of labor is growing like $(1 + \gamma)^t$

in a golden age. Thus we may write

$$(26)\qquad \sum_{1}^{\infty}\left[\frac{\partial P}{\partial R_{t-i}}\left(\frac{\partial N}{\partial P}\right)_{t-i}\right]\left[(1+\gamma)^{2}\right]^{-i}=1$$

The expression in the first bracket measures the increase in product at time t resulting from the sacrifice of one unit of output at time $t-i$ in order to release some commodity-producing workers into research. It is $-\frac{\partial P_t}{\partial P_{t-i}}$ or what may be called the marginal productivity at time t of "investment in technology" at time $t-i$.

By analogy to the previous exercise we assume now that _in a golden age_ the marginal productivity at time $t+i$ of investment in technology at time t is equal to the marginal productivity at t of investment in technology at time $t-i$. This stationarity of the current marginal productivity of investment in technology i periods earlier can be presumed from the constancy of the input proportions in both the production and technology functions. Thus if the rate of return to investment in technology at time t is defined as

$$(27)\qquad \sum_{1}^{\infty}\frac{-\partial P_{t+i}}{\partial P_t}(1+r)^{-i}=1\ ,$$

where $\frac{-\partial P_{t+i}}{\partial P_t}=\frac{\partial P_{t+i}}{\partial R_t}\,\frac{\partial N_t}{\partial P_t}$ and denotes the marginal productivity at $t+i$ of investment in technology at t , then, by our stationarity assumption, we may also write

$$(28)\qquad \sum_{1}^{\infty}-\frac{\partial P_t}{\partial P_{t-i}}(1+r)^{-i}=1$$

Equation (26) and (28) imply that, when $\frac{\partial C_t}{\partial R_t} = 0$, $1 + r = (1 + \gamma)^2$; but $(1 + \gamma)^2 = 1 + g$ so that $r = g$. Thus the Golden Rule of Research prescribes equating the rate of return from investment in technology to the rate of growth.

We have been speaking of maximizing golden-age consumption with respect to research effort while holding constant the capital stock or the ratio of capital to effective labor; and of maximizing golden-age consumption with respect to capital intensity while holding research effort constant. It is perfectly clear that if we wish to maximize golden-age consumption with respect both to capital intensity and research effort we must equate the rate of return of both investment in technology and investment in tangible capital to the rate of growth. We may call this truly maximal consumption path the Golden Rule path. This Golden Rule path is dynamically efficient: Like some but not all golden age paths, the rates of return to the two kinds of investment are equal; equality of the rates of return is clearly one necessary condition for dynamical efficiency. And, secondly, this common rate of return -- which is the rate of return to saving -- is not smaller than the rate of growth.

In the particular, highly simple model we have been analyzing, exactly $\frac{1}{2} L(t)$ workers will produce commodities in this Golden Rule state while $\frac{1}{2} aL(t)$ workers will earmark their commodity output for investment. But these characteristics, unlike the equalities pertaining to rates of return, are due to the critical assumptions that the labor force is homogenous and that capital is unproductive in the research sector. We now relax these assumptions.

A nonhomogenous labor force

We shall continue to suppose, as an approximation, that no worker has an absolute advantage in the production of commodities. But we suppose now that all workers differ in their effectiveness in research. If we require just one researcher, then, for efficiency, we must assign that worker with the greatest comparative, and hence absolute, advantage in research. If we require an additional worker we assign that worker with the next greatest absolute advantage; but the increase of "effective research" will not be proportional because the second worker is inferior to the first. Thus we suppose that, with efficient allocation of labor, "effective research" increases with the number of researchers at a decreasing rate; we take as fixed here the size of the labor force.

Second, we suppose that the skill mix is in some sense stationary over time in that if both the labor force and the number of researchers required is doubled, we can, by suitable assignment of workers, double effective research.

Third, we suppose that without research there is no effective research.

To express these assumptions mathematically we replace $R(v)$ by the "effective research function," $E\Big(R(v),\ L(v)\Big)$, in the technology function,

$$(29) \qquad T(t) = \int_{-\infty}^{t} H\Big(E\big(R(v)\ L(v)\big),\ T(v-\omega)\Big)\, dv\ ,$$

imposing on the effective research function the following restrictions:

(a) $$E_R\left(R(v),\ L(v)\right) > 0\ ,\quad E_{RR}\left(R(v),\ L(v)\right) < 0$$

(b) $$E\left(R(v),\ L(v)\right) = L(v)\ \ E\left(\frac{R(v)}{L(v)}\ ,\ 1\right)$$

(c) $$E\left(0,\ L(v)\right) = 0\ .$$

We suppose, as before, that $H(E, T)$ is homogenous of degree one; i.e., a doubling of effective research at v and of technology at $v-\omega$ will cause a doubling of the absolute rate of increase of technology at v , $\mathring{T}(v) = H\left(E\left(R(v),\ L(v)\right), T(v-\omega)\right)$. And, as before, we suppose that $H(0, T) = 0$.

It is now easy to see that in a golden age, in which $R(t)$ and $L(t)$ both grow exponentially at rate γ , the technology path will also grow exponentially at rate γ and will satisfy the transcendental equation, analogous to (e") ,

(30) $$\gamma = e^{-\gamma\omega}\ h\left(\frac{E\left(R(0),\ L_o\right)}{T(0)}\ \ e^{\gamma\omega}\right)$$

Thus the golden-age "level" of technology is proportional to "effective research," so that we obtain

(31) $$T(t) = \beta\ E\left(R(0),\ L_o\right)\ e^{\gamma t}$$

To maximize golden-age consumption with respect to research effort, it suffices, as before, to maximize "effective labor" subject to the constraint [in (2)] $R(0) + N(0) = L_o$. Equating to zero the derivative of $T(0)\ N(0)$ with respect to $R(0)$ yields

$$(32) \qquad 0 = \frac{\partial T(0)}{\partial R(0)} N(0) + \frac{\partial N(0)}{\partial R(0)} T(0)$$

$$= \eta \frac{\partial E}{\partial R} N(0) - T(0) \qquad \text{[by (2) and (31)]}$$

$$= \eta \frac{\partial E}{\partial R} N(0) - \eta E\Big(R(0), L_o\Big) . \qquad \text{[by (31)]}$$

This equation can be written in the form

$$(33) \qquad \frac{R(0)}{N(0)} = \frac{\partial E\Big(R(0), L_o\Big)/\partial R(0)}{E\Big(R(0), L_o\Big)/R(0)} .$$

The right hand side of (33) is the elasticity of effective research with respect to research. (It is stationary over time for fixed R/L .) Thus the Golden Rule of Research prescribes equating the ratio of researchers to non-researchers to the research elasticity of effective research. If E were proportional to R , as we supposed in the first part of this paper, this elasticity would be unitary; then $R/N = 1$ which was the result obtained before. But on our new assumptions on $E(R, L)$ this elasticity is unitary only at $R=0$; on our assumption that $E_{RR} < 0$ this elasticity must be smaller than one for all $R > 0$. Hence, as R/N is increased toward one, there must come a point, $\widehat{(R/N)}$, where (33) is satisfied. Clearly, $\widehat{(R/N)} < 1$ since $\widehat{(R/N)}$ could equal one only if the elasticity equalled one at $R = \frac{1}{2} L$ which is impossible for $L > 0$. Thus we see that if workers have differing absolute advantage in research but are equally productive in the commodity-producing sector, the Golden Rule of Research dictates assigning less than half of all workers to the research sector.

Capital in the research sector

Heretofore we have supposed that capital could not contribute to research effort. Let us suppose now that it can and that, as seems reasonable, "effective research" is homogenous of degree one in $R(t)$, $L(t)$ and $M(t)$ where $M(t)$ denotes the stock of capital employed in the research sector:

$$(34) \qquad T(t) = \int_{-\infty}^{t} H\Big(E\big(R(v),\ L(v),\ M(v)\big),\ \ T(v-\omega)\Big)dv$$

where

$$(a) \qquad E_R > 0\ , \quad E_{RR} < 0\ , \quad E_M > 0\ , \quad E_{MM} < 0\ .$$

$$(b) \qquad E = L\,E\left(\frac{R}{L}\,,\,1\,,\,\frac{M}{L}\right) = M\,E\left(\frac{R}{M}\,,\,\frac{L}{M}\,,\,1\right)$$

$$(c) \qquad E(0,\ L,\ M) = E(R,\ L,\ 0) = 0\ .$$

As before, we suppose that $H(E,\ T)$ is homogenous of degree one in E and T and that $E(0,\ T) = 0$.

From (34) we have

$$(35) \qquad \frac{\dot{T}(t)}{T(t)} = \frac{T(t-\omega)}{T(t)}\ H\left(\frac{E\big(R(t),\ L(t),\ M(t)\big)}{T(t-\omega)}\ ,\ 1\right)$$

Hence, if the technology grows exponentially and has always done so it must be that "effective research" also grows exponentially at the same rate, say j , so as to satisfy (35):

$$(36) \qquad j = e^{-j\omega}\ H\left(\frac{E\big(R(0),\ L_o,\ M(0)\big)e^{jt}}{T(0)e^{j(t-\omega)}}\ ,\ 1\right)$$

$$= e^{-j\omega}\ H\left(\frac{E\big(R(0),\ L_o,\ M(0)\big)}{T(0)}\,e^{j\omega}\ ,\ 1\right)$$

Now effective research would indeed grow exponentially if $R(t)$, $L(t)$ and $M(t)$ all grew at the same rate j. Then "effective labor" in the commodity-producing sector would, in a golden age, grow at the rate $j + \gamma = 2j$ since $R(t)$ and $N(t)$ must grow at the same rate. But then capital in the commodity sector would have to grow at the rate $2j$. But $M(t)$ grows only at the rate j so that the two capital stocks would not grow at the same rate; therefore total capital would not grow exponentially, nor would investment and therefore consumption. So a golden age is impossible if $R(t)$, $L(t)$ and $M(t)$ grow at the same rate.

But if $R(t)$, $L(t)$ and $M(t)$ grow at different rates $\left(M(t) \text{ growing like } j + \gamma\right)$ then can $E(R, L, M)$ grow exponentially? The answer is no unless "effective research" is a Cobb-Douglas function! Let us assume it is. Then if both $R(t)$ and $L(t)$ grow at rate γ and $M(t)$ grows at rate $\underline{m}$ the relative rate of growth of "effective research," and hence of the technology, j, will be given by

$$(37) \qquad j = \alpha m + (\beta' + \beta'')\gamma$$

where α is the capital elasticity of "effective research" and β' and β'' are the elasticities of effective research with respect to $R(t)$ and $L(t)$ respectively. By the homogeneity postulate, $\beta' + \beta'' = 1 - \alpha$.

Now "effective labor" will, in a golden age, grow at rate $j + \gamma$ so that the rate of growth of output and capital in the commodity sector, say g, must also be $j + \gamma$. Hence

$$(38) \qquad g = \alpha m + (1-\alpha)\gamma + \gamma$$

But, for a golden age, capital must grow at the same rate in both sectors, otherwise total capital cannot grow exponentially. Hence, substituting

(39) $$g = m$$

into (38) we obtain

(40) $$g = \frac{(2 - \alpha)\gamma}{1 - \alpha}$$

as the golden-age growth rate of capital in both sectors and of output. In the special case in which capital is unproductive, so that $\alpha = 0$, we obtain once again the result that $g = 2\gamma$. But if $\alpha > 0$, $g > 2\gamma$. Since capital grows faster than labor, by virtue of technical progress, it is helpful to the technical progress rate, and hence the growth rate, that capital can be employed in the research sector. For $g > \gamma$ so that when $m = g$, $j = \alpha m + (1-\alpha)\gamma$ exceeds γ for all $\alpha > 0$.

We turn now to the question of how the productivity of capital in research affects the proportion of the labor force and the proportion of the capital stock engaged in research on the Golden Rule path. First, given that total capital has been allocated between the research and commodity sectors in a consumption-maximizing way, how must labor be allocated on the Golden Rule path? As before, labor is allocated on the Golden Rule path so as to maximize "effective labor" in the commodity sector. Thus $T(0)\ N(0)$ is at a maximum with respect to $R(0)$. Hence

$$(41) \qquad 0 = \frac{\partial T(0)}{\partial R(0)} N(0) + \frac{\partial N(0)}{\partial R(0)} T(0)$$

$$= \eta \frac{\partial E\Big(R(0),\ L_o,\ M(0)\Big)}{\partial R(0)} N(0) - T(0) \ .$$

But

$$(42) \qquad T(t) = \eta\ E\Big(R(0),\ L_o,\ M(0)\Big)\ e^{[\alpha g + (1-\alpha)\gamma]t} \ .$$

Therefore

$$(43) \qquad \frac{R(0)}{N(0)} = \frac{\partial E\Big(R(0),\ L_o,\ M(0)\Big)/\partial R(0)}{E\Big(R(0),\ L_o,\ M(0)\Big)/R(0)} = \beta' \ .$$

Once again we find that the ratio of researchers to nonresearchers equals the elasticity of effective research with respect to researchers. In the present, Cobb-Douglas case, this elasticity is the constant β', which is less than one by our homogeneity assumption.

We see that the presence of capital in the effective research function is another reason why less than half the labor force is assigned to research on the Golden Rule path. For even if the labor force is homogeneous $(\beta'' = 0)$, the elasticity of effective research with respect to the number of researchers, β', must be less than one if capital is productive $(\alpha > 0)$ and there are constant returns to scale in effective research $(\alpha + \beta' = 1)$.

What of the allocation of total capital between the research and commodity sectors on the Golden Rule path? Given the exponentially increasing path of total capital and the allocation of labor appropriate to the Golden Rule path, $M(0)$ is such as to maximize

$$(44) \qquad C(t) = F\Big(K(0),\ T(0)\ N(0)\Big) - \Big(\dot{K}(0) + \dot{M}(0)\Big)\ e^{gt}$$

on the Golden Rule path. Hence, equating to zero the derivative of $C(t)$ with respect to $K(0)$ and noting that $dM/dK = -1$ we obtain

$$\begin{aligned}
(45) \qquad 0 &= F_K + F_N\, N(0)\ \frac{dT(0)}{dK} \\
&= F_K - F_N\, N(0)\ \eta\ \frac{\partial E\Big(R(0),\ L_o,\ M(0)\Big)}{\partial M(0)} \\
&= \left(\frac{F_K\ K(0)}{Q(0)}\right) - \frac{K(0)}{T(0)}\left(\frac{T(0)F_N\ N(0)}{Q}\right)\eta\ \frac{\partial E}{\partial M} \\
&= (a) - \frac{K}{\eta E}\,(b)\ \eta\ \frac{\partial E}{\partial M} \\
&= \frac{a}{b} - \frac{K}{M}\ \frac{\partial E/\partial M}{E/M}\ ,
\end{aligned}$$

whence

$$(46) \qquad \frac{M(0)}{K(0)} = \frac{b}{a}\ \frac{\partial E\Big(R(0),\ L_o,\ M(0)\Big)/\partial M(0)}{T\Big(R(0),\ L_o,\ M(0)\Big)/M(0)} = \frac{1-a}{a}\,\alpha\ .$$

On the Golden Rule path, therefore, the ratio of capital in research to capital in the commodity sector depends not only on the capital elasticity of effective research but also upon $\underline{a}$, the capital elasticity of output. Note that $M(0) = 0$ if $\alpha = 0$. Note too that the research sector will be more capital intensive than the commodity sector -- that is, $\frac{M}{K} > \frac{R}{N}$ -- if and only if $\frac{b}{a}\alpha > \beta'$ or $\frac{\alpha}{\beta'} > \frac{a}{b}$ which was to be expected.

Other theorems which could be established are that, in the present model as well as in the model where capital is unproductive in the research sector, the rates of return to investment in technology and to investment in the commodity sector are equal to the rate of growth on the Golden Rule path. Proofs of these theorems would be essentially a repetition of proofs given earlier.

CONCLUSIONS

We postulated a certain technology function which is conducive to golden ages. We then showed that when golden-age consumption is maximal with respect to research effort, the rate of return to "investment in technology" equals the rate of growth. We referred to this equalization as the Golden Rule of Research. It was also shown that when golden-age consumption is maximized with respect to tangible investment in the commodity sector, the rate of return from tangible investment equals the rate of growth. This is the familiar Golden Rule of Accumulation. When golden-age consumption is maximal with respect to both kinds of investment, the two rates of return are therefore both equal to each other and to the rate of growth. This suggests the General Golden Rule of Investment: To maximize golden-age consumption, assuming an interior maximum to exist, equate the rate of return from each kind of investment to the rate of growth.

We also characterized the Golden Rule state in terms of labor allocation. It was shown that if the labor force is homogeneous and capital is unproductive in the research sector then exactly one half the labor force is assigned to research on the Golden Rule path. But if capital is productive in research or if the skill at research of the marginal researcher falls with increasing numbers of researchers, due to a nonhomogeneous skill mix, then, on certain assumptions, less then one half of the labor force does research on the Golden Rule path.