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FACTOR SUBSTITUTION IN THE TWO-SECTOR GROWTH MODEL

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1. INTRODUCTION

1.1. In a series of recent papers by Shinkai [6], Uzawa [10], [11], Solow [8], Inada [2], the familiar two-factors two-goods production model has been systematically applied to the analysis of the growth process of capital accumulation.[1] In particular, the stability properties of balanced growth paths have been examined under various simplifying assumptions. It has been shown by Uzawa in [10], [11], that a sufficient condition for the stability of the balanced growth path is that the consumption-goods industry is more capital-intensive than the capital-goods industry. Shinkai [6] has shown that the above condition is a necessary and sufficient condition for stability in the case of fixed coefficients of production.

1.2. In this paper we examine the problem of stability of the balanced growth path paying special attention to the properties of the relevant production functions. We find that if the elasticity of factor substitution of both production functions approaches zero, then the greater capital-intensity

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[1] We note that technical progress is not considered in these papers.

of the consumption-goods industry becomes a necessary and sufficient condition for stability. In any other case it remains a sufficient condition. As however we consider production functions with greater elasticities of substitution, this condition becomes an overly strong one. Finally, if the elasticity of factor substitution of either one of the production functions is greater than or equal to one, then the stability of the balanced growth path is obtained without regard to other conditions.

It can easily be explained why the balanced growth path, and the corresponding to it capital-labor ratio, are asymptotically stable whenever the elasticity of factor-substitution in either one of the industries is sufficiently high. The main point to be emphasized is that a relative 'abundance' -- relative to the balanced growth path -- of any one of the factors at any time will be accommodated through two channels. The first is an inter-industry movement towards the industry which uses relatively more of that factor. The second, which will be operating in the case where alternative techniques for the production of one or of both of the goods exist, is an intra-industry movement. If e.g., the available quantity of one of the factors changes exogenously then this will cause an alteration of the equilibrium factor prices; then with factor substitutability a change in the factor proportions used in the relevant industry will occur; if factor substitutability is high enough, this will insure the absorption of the relatively 'abundant' factor with only small inter-industry movements.

2. THE MODEL

2.1. The model examined below is the familiar two-factors, two-goods neoclassical model as it has been formulated by Uzawa [10]. Consequently, only a brief description of the economy under consideration will suffice.

On the production side we have the capital-goods and the consumption-goods sector. The participants in each market convention are distinguished into workers (holders of labor services) and capitalists (holders of the capital goods). Both offer in each convention these services in exchange for consumption and capital goods which will become available in the next period. Competitive equilibrium is achieved in each market and by means of the exogenous growth of the labor force and the utilization of the newly produced capital goods a process of successive competitive equilibria through time is generated.

2.2. The main assumptions are the following:

(A_1) In each time period there exist only four distinct commodities: labour services, capital services, capital goods, and consumption goods.2/ Consumption goods are not used for production. Labour and capital goods do not yield direct services for consumption. Each good is produced separately by the use of capital and labour services.

2/ This is equivalent to the assumption that the price ratios among the goods within each one of the four categories remain fixed throughout the accumulation process. Thus we can confine our attention only to these categories of goods.

(A_2) The production structure is described by two neo-classical production functions which are twice continuously differentiable, homogeneous of degree one, concave, and have nonnegative marginal rates of substitution. Production takes one time period to be completed.

(A_3) A given percentage of the capital goods existing in each period is withdrawn from production on account of depreciation. Otherwise all capital goods are homogeneous and they are freely transferable between the two sectors.

(A_4) The marginal (and average) propensity to save out of the current income of workers and capitalists are given constants.

Assumption (A_4) specifies the saving patterns of the economy and completes the model by introducing the demand side of the economy.

2.3. Let us give a brief description of the workings of the economy during each convention. The economy consists of entrepreneurs, consumers, and an auctioneer. The auctioneer announces the factor prices. The entrepreneurs minimize their unit-costs of production. The auctioneer then announces output prices equal to these unit-costs. The consumers knowing their incomes announce their demands.[3/] Finally, the auctioneer ascertains whether

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[3/] We assume that the consumers have already determined in the previous market the quantities of the goods currently required for consumption. Hence the consumption goods available at t do not appear in the market at t .

the production of the consumption and capital goods which is necessary to meet these demands can be accommodated. Changes in the factor prices take place until the factor markets are cleared.[4/]

The assumption that workers and capitalists have formed rigid saving patterns, which are not influenced by the accumulation process itself, is a really strong assumption. (A_4) in effect separates each market convention from all successive ones. Price expectations and interest rates do not play any independent role in our model because of (A_4).[5/]

2.4. Our notation follows mainly that in [10]. The capital-goods sector is denoted by the subscript 1 and the consumption-goods sector by the subscript 2. Considering the market convention at t we have:

[4/] The existence of such an equilibrium position will be examined in section 3. The stability of the tâtonnement process is however assumed.

[5/] An alternative approach is that in which saving patterns are considered as derived from utility maximization over time; see e.g. Srinivasan [9]. However, in such a model perfect foresight for the whole future has to be assumed. The present model is exactly the opposite in this respect; the economic horizon extends over only one time period.

L_t = Total quantity of labour services at t.

K_t = Total quantity of capital at t.

L_{it} = Labour input in sector i at t, $i = 1, 2$.

K_{it} = Capital input in sector i at t, $i = 1, 2$.

$Y_{i,t+1}$ = Output of sector i at $t+1$.

w_t = Wage rate at t.

r_t = (Gross) rental of a unit of the capital good at t.

$P_{i,t+1}$ = Price (at t) of a unit of the capital goods ($i = 1$), and of the consumption goods ($i = 2$), which will become available at $t+1$.

μ = Rate of depreciation of capital goods.

s_r, s_w = Marginal (= average) propensity to save of capitalists and workers, respectively; we have

$$1 \geqq s_r, s_w \geqq 0 .$$ 6/ 7/

2.5. Competitive Equilibrium. The production functions of the two sectors are denoted by $F^i(K_i, L_i)$, $K_i, L_i \geqq 0$, $i = 1, 2$. (A_2) is given by the following:

6/ s_r is the gross propensity to save out of the gross capitalists' income, i.e., $s_r r_t K_t$ represents the gross savings of the capitalists.

7/ We exclude the trivial cases $1 = s_r = s_w$, $0 = s_r = s_w$.

(1)

(a) $F^i(K_i, L_i) > 0$ for all $K_i, L_i > 0$;

(b) F^i is twice continuously differentiable;

(c) $F^i(\lambda K_i, \lambda L_i) = \lambda F^i(K_i, L_i)$, for all $\lambda > 0$;

(d) $F^i_L(K_i, L_i) > 0$, $F^i_K(K_i, L_i) > 0$, $d^2 F^i \leq 0$, for all $K_i, L_i > 0$.

The competitive equilibrium conditions during each convention are given by:

(2) $Y_{i,t+1} = F^i(K_{it}, L_{it}), \quad i = 1, 2,$

(3) $w_t = P_{1t+1} F^1_L(K_{1t}, L_{1t}) = P_{2t+1} F^2_L(K_{2t}, L_{2t})$,

(4) $r_t = P_{1t+1} F^1_K(K_{1t}, L_{1t}) = P_{2t+1} F^2_K(K_{2t}, L_{2t})$,

These are the production equilibrium conditions; i.e., the wage rate w_t is equal to the value of the marginal product of labour, and the rental of the capital goods r_t is equal to the value of the marginal product of capital in both sectors.

We also have:

(5) $K_{1t} + K_{2t} = K_t$,

(6) $L_{1t} + L_{2t} = L_t$,

Labour and capital services are <u>fully</u> employed and freely transferable between sectors.

(7) $$s_r \, r_t \, K_t + s_w \, w_t \, L_t = P_{1t+1} \, Y_{1t+1} \quad ,$$

(8) $$(1-s_r) \, r_t \, K_t + (1-s_w) \, w_t \, L_t = P_{2t+1} \, Y_{2t+1} \quad ,$$

The last two equations express the equilibrium conditions in the markets for capital and consumption goods respectively.

In the market at t , $K_t = \bar{K} \geqq 0$, $L_t = \bar{L} \geqq 0$, are given. We are interested in specifying the conditions under which a unique competitive equilibrium at t exists.[8/] For a solution to (2) - (8) to be economically meaningful the following restrictions are necessary:

(9) $$K_{it}, \; L_{it}, \; Y_{it+1}, \; w_t, \; r_t, \; P_{it+1} \geqq 0 \; .$$

3. THE ELASTICITY OF FACTOR SUBSTITUTION AND THE EXISTENCE OF COMPETITIVE EQUILIBRIUM IN EACH CONVENTION[9/, 10/]

3.1. The constant-returns-to-scale feature of the model enable us to deal mainly with the ratios of inputs and of prices.

8/ We note that only relative prices are determined in the model, and also that one equation among (2)-(8) follows from the rest.

9/ It can be easily seen that no essential differences are created whether we assume that production is instantaneous or not. We will follow the first alternative, and thus the analysis will be carried out in terms of differential equations.

10/ Time subscripts are suppressed in the following derivations: Proofs are placed in starred sections separate from the statements of the corresponding theorems, so that all proofs may be omitted without loss of continuity.

Let $k = \frac{K}{L}$, $k_i = \frac{K_i}{L_i}$, $\ell_i = \frac{L_i}{L}$, $y_i = \frac{Y_i}{L_i}$, $\omega = \frac{w}{r}$, $p = \frac{P_1}{P_2}$. Also let

(10) $$\sigma_i = \frac{\omega}{k_i} \frac{dk_i}{d\omega}$$

be the elasticity of substitution between capital and labour in sector $i = 1, 2$.

Then, $Y_i = F^i(K_i, L_i) = L_i F^i(k_i, 1) = L_i f_i(k_i)$ or $y_i = f_i(k_i)$.[11]

3.2. The equilibrium conditions (2) - (9) now become:

(11) $y_i = f_i(k_i)$, $i = 1, 2$.

(12) $\omega = f_i(k_i)/f_i'(k_i) - k_i$, $i = 1, 2$,

(13) $k_1 \ell_1 + k_2 \ell_2 = k$,

(14) $\ell_1 + \ell_2 = 1$,

(15) $s_r k + s_w \omega = \ell_1 f_1(k_1)/f_1'(k_1)$,

(16) $k_i, \ell_i, y_i, \omega \geqq 0$,

and $k = \bar{k} \geqq 0$.

We want to show that equations (11)-(15) have a unique solution for y_i, k_i, ℓ_i, ω , which is positive for $\bar{k} > 0$.

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[11] Conditions (1) imply that: (a) $f_i(k_i) > 0$ for $k_i > 0$, (b) f_i is twice continuously differentiable, (c) $f_i'(k_i) > 0$, $f_i(k_i) - k_i f_i'(k_i) > 0$, $f_i''(k_i) < 0$ for $k_i > 0$.

3.3.* First, $\alpha_i(k_i) \equiv \dfrac{f_i(k_i)}{f_i'(k_i)} - k_i$ is a strictly increasing function of k_i, for $k_i > 0$. Therefore, for any positive wage-rentals ratio ω, (12) can be solved for a unique positive capital-labour ratio $k_i = k_i(\omega)$ in each sector, provided

(17) $$\underline{\omega} < \omega < \overline{\omega},$$

where $\underline{\omega} = \max[\underline{\omega}_1, \underline{\omega}_2]$, and $\overline{\omega} = \min[\overline{\omega}_1, \overline{\omega}_2]$, and

(18) $$\underline{\omega}_i = \lim_{k_i \to 0} \alpha_i(k_i), \qquad \overline{\omega}_i = \lim_{k_i \to \infty} \alpha_i(k_i)$$

Of course $0 \leqq \underline{\omega}_i < \overline{\omega}_i \leqq +\infty$.

Second, for any ω satisfying (17) and such that $k_1(\omega) \neq k_2(\omega)$, equations (13)-(14) give us

(19) $$\ell_1(\omega) = \frac{k_2(\omega) - k}{k_2(\omega) - k_1(\omega)}, \quad \text{and} \quad \ell_2(\omega) = \frac{k - k_1(\omega)}{k_2(\omega) - k_1(\omega)}.$$

We will see below that $\ell_1(\omega)$, $\ell_2(\omega)$ are positive.

3.4.* Finally, (15) can be written using (12) and (19), as

(20) $$s_r k + s_w \omega = [\omega + k_1(\omega)] \frac{k_2(\omega) - k}{k_2(\omega) - k_1(\omega)}$$

or as

$$(21) \qquad k = \psi(\omega) \equiv \frac{k_1(\omega)\ k_2(\omega) + [s_w k_1(\omega) + (1-s_w)k_2(\omega)]\omega}{(1-s_r)\ k_1(\omega) + s_r k_2(\omega) + \omega} = \frac{a(\omega)}{b(\omega)} .$$

Now, $\psi(\omega) > 0$ for $\underline{\omega} < \omega < \overline{\omega}$. Thus $k = \psi(\omega)$ has a solution for any k such that

$$(22) \qquad \psi(\underline{\omega}) < k < \psi(\overline{\omega}) .$$

A unique solution for ω is insured if $\psi(\omega)$ is a strictly monotonic function of ω.

3.5.* The elasticity of factor substitution in each sector is also a function of ω, $\underline{\omega} < \omega < \overline{\omega}$, denoted by $\sigma_i = \sigma_i(\omega)$, $i = 1, 2$. Let $c(\omega) = k_1(\omega)\ [\omega + k_2(\omega)]\ [s_r k_2(\omega) + s_w\omega]$, $d(\omega) = k_2(\omega)$. $\cdot\ [\omega + k_1(\omega)]\ [(1-s_r)k_1(\omega) + (1-s_w)\omega]$, $e(\omega) = \omega[k_2(\omega) - k_1(\omega)]$. $[s_r(1-s_w)\ k_2(\omega) - (1-s_r)\ s_w\ k_1(\omega)]$.

We can show that

$$(23) \quad \frac{1}{\psi(\omega)}\ \psi'(\omega) = \frac{1}{\omega}\ \frac{c(\omega)\sigma_1(\omega) + d(\omega)\sigma_2(\omega) + e(\omega)}{a(\omega)\ b(\omega)} \equiv \frac{1}{\omega}\ \sigma(\omega) .$$

We will examine below several cases for which $\sigma(\omega)$ is positive for $\underline{\omega} < \omega < \overline{\omega}$. In all these cases equation (21) either expresses k as a function of ω, $k = \psi(\omega)$ for $\underline{\omega} < \omega < \overline{\omega}$, or ω as a function of k, namely,

$$(21') \qquad \omega = \psi^{-1}(k) , \quad \text{for } k \text{ satisfying (22).}$$

Of course $\omega'(k) = \frac{1}{\psi'(\omega)} = \frac{1}{\sigma[\omega(k)]} \quad \frac{\omega(k)}{k}$, and

$\sigma(\omega) = \frac{\omega}{\psi(\omega)} \psi'(\omega)$ is thus equal to the elasticity of $\psi(\omega)$.

Hence, provided that $\sigma(\omega)$ is positive for $\underline{\omega} < \omega < \bar{\omega}$, (21) uniquely relates the equilibrium wage-rental ratio and the equilibrium aggregate capital-labour ratio throughout the process of capital accumulation.

3.6.* Since $a(\omega)$, $b(\omega)$, $c(\omega)$, $d(\omega)$, and $\sigma_1(\omega)$, are all positive for $\underline{\omega} < \omega < \bar{\omega}$, we will first examine the cases in which $e(\omega)$ is also positive. $e(\omega)$ is positive if and only if

$$k_2(\omega) \left\{ \begin{matrix} \geq \\ < \end{matrix} \right\} k_1(\omega) \quad \text{and} \quad \frac{s_r(1-s_w)}{s_w(1-s_r)} \left\{ \begin{matrix} \geq \\ < \end{matrix} \right\} \frac{k_1(\omega)}{k_2(\omega)} \ .$$

Hence $\psi'(\omega)$ is positive for all $\underline{\omega} < \omega < \bar{\omega}$ if

(a) $s_r > s_w$, $k_2(\omega) > k_1(\omega)$; (b) $s_w > s_r$, $k_1(\omega) > k_2(\omega)$;

and (c) $s_r = s_w$. For these cases <u>any given capital-labour ratio</u> $\bar{k}$, <u>such that</u> $\psi(\underline{\omega}) < \bar{k} < \psi(\bar{\omega})$, <u>uniquely determines the equilibrium wage-rental ratio</u> $\omega(\bar{k}) > 0$.

3.7.* Through $\omega(k)$ all other quantities and prices in the convention at t are determined. We can easily show that they are all positive for k satisfying (22). From (21) we get

(24) $$k - k_1 = (k_2 - k_1)\ \frac{(1-s_r)k_1 + (1-s_w)\omega}{(1-s_r)k_1 + s_r k_2 + \omega}$$

(25) $$k - k_2 = (k_1 - k_2)\ \frac{s_r k_2 + s_w\ \omega}{(1-s_r)k_1 + s_r k_2 + \omega}$$

Thus we see that for any k satisfying (22)

(26) $$\max_{i=1,2}\ k_i[\omega(k)] > k > \min_{i=1,2}\ k_i[\omega(k)]$$

holds. This is naturally a consequence of the feasibility of factor substitution. For any k satisfying (22) equilibrium wage-rental ratio $\omega(k)$ will be such that (26) is satisfied. From (26) and (19) we immediately see that $\ell_i[\omega(k)]$ are positive. Also, w, r, p, are positive in equilibrium.

3.8.* Till now we have assumed that for all relevant k, $k_1[\omega(k)] \neq k_2[\omega(k)]$ holds. However, the production functions may be such that there exist ω, $\underline{\omega} < \omega < \overline{\omega}$, for which $k_1(\omega) = k_2(\omega)$. Then (13)-(14) give us

(27) $$k_1(\omega) = k_2(\omega) = k\ .$$

Since both $k_i(\omega)$ are strictly increasing functions of ω, $\underline{\omega} < \omega < \overline{\omega}$, (27) is satisfied by a unique ω, $\omega(k)$, for given $k > 0$. Then from (12) and (15) we get

(28) $$\ell_1[\omega(k)] = \frac{s_r k + s_w\ \omega(k)}{k + \omega(k)}\ .$$

Hence $0 < \ell_1[\omega(k)] < 1$, and finally $0 < \ell_2[\omega(k)] < 1$ is determined from (14) .

3.9. We have proved the following:

EXISTENCE THEOREM 1: For any aggregate capital-labour ratio k satisfying (22), the equilibrium ω, k_i, y_i, w, r, p, are uniquely determined and they are positive, in any of the following cases:

1. If $k_1[\omega(k)] \neq k_2[\omega(k)]$, whenever

(a) $s_r > s_w$, $k_2[\omega(k)] > k_1[\omega(k)]$, or

(b) $s_w > s_r$, $k_1[\omega(k)] > k_2[\omega(k)]$, or

(c) $s_r = s_w$; and

2. If $k_1[\omega(k)] = k_2[\omega(k)]$.

3.10. The above theorem is a generalization of the existence theorems in [10] and [11]. Also Inada [2] has independently proved Theorem 1, along with the corresponding stability theorem of Section 4.4 below.

We note that in the proof of this theorem we completely ignore the possibilities of factor substitution in any of the two sectors. Instead we rely on the difference between the capital-labour ratios in the two sectors. Thus e.g., in the most important case, i.e., that where $s_r > s_w$, we have to assume that the consumption-goods sector is always[12]

12/ Namely, $k_2[\omega(k)] > k_1[\omega(k)]$ for all k satisfying (22).

more capital intensive than the capital-goods sector. However, many objections can be raised concerning the reasonableness of this capital intensity condition.[13/]

Actually, a closer consideration of the possibilities of factor substitution in both sectors makes apparent that such capital intensity conditions are, in most cases, unnecessarily strong sufficient conditions.

3.11.* Let us first consider the case where $\sigma_1(\omega) = \sigma_2(\omega) = 1$ for all $\underline{\omega} < \omega < \bar{\omega}$, i.e., the case where both production functions are of the Cobb-Douglas type. Then, we can easily see from (23) that $\sigma(\omega) = 1$ holds and thus that $\psi'(\omega)$ is positive for all $\underline{\omega} < \omega < \bar{\omega}$. Also if $\sigma_1(\omega)$, $\sigma_2(\omega) \geqq 1$, for all $\underline{\omega} < \omega < \bar{\omega}$,[14/] then $\sigma(\omega) \geqq 1$ and $\psi'(\omega) > 0$ holds. Therefore, if the factor elasticity of substitution in both sectors is greater than or equal to one, for all $\underline{\omega} < \omega < \bar{\omega}$, then the equilibrium wage-rental ratio is uniquely determined by (21), for any k satisfying (22).

3.12.* Weaker conditions for $\sigma(\omega) > 0$, for all $\underline{\omega} < \omega < \bar{\omega}$, are possible.

Let us examine the sign of $g(\omega) \equiv \alpha\, c(\omega) + (1-\alpha)\, d(\omega) + e(\omega)$

13/ See e.g., Solow [8, p. 48].

14/ Constancy of $\sigma_i(\omega)$ is not required.

for $1 \geqq \alpha \geqq 0$. We find that

$$g(\omega) = 2\,[(1-\alpha)(1-s_r)(1-s_w) + \alpha\, s_r\, s_w]\omega\, k_1\, k_2 + \alpha\, s_r\, k_1\, k_2^2 + \alpha\, s_w\, \omega^2 k_1$$

$$+ (1-s_r)[(1-\alpha)k_2 + s_w\omega]k_1^2 + (1-s_w)[(1-\alpha)\omega + s_r k_2]\,\omega\, k_2 > 0 \ .$$

Therefore, if $\sigma_1(\omega) + \sigma_2(\omega) \geqq 1$ holds for all $\underline{\omega} < \omega < \bar{\omega}$, then again $\sigma(\omega) > 0$, $\underline{\omega} < \omega < \bar{\omega}$, is satisfied. If e.g., either one of $\sigma_i(\omega)$ is greater than or equal to one for all $\underline{\omega} < \omega < \bar{\omega}$, $\sigma(\omega) > 0$ holds.

In general, if we ignore the possibilities of factor substitution in the capital-goods sector, or in the consumption-goods sector, respectively, then[15]

$$(29) \qquad \sigma_1(\omega) > \max\left\{ -\frac{e(\omega)}{c(\omega)}\ ,\ 0 \right\}, \quad \text{or}$$

$$(30) \qquad \sigma_2(\omega) > \max\left\{ -\frac{e(\omega)}{d(\omega)}\ ,\ 0 \right\}, \quad \text{respectively,}$$

must hold, for all $\underline{\omega} < \omega < \bar{\omega}$, in order to have $\sigma(\omega) > 0$ and thus $\psi'(\omega) > 0$ for all $\underline{\omega} < \omega < \bar{\omega}$.

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[15] Of course $d(\omega) + e(\omega) > 0$, $c(\omega) + e(\omega) > 0$, for all $\underline{\omega} < \omega < \bar{\omega}$.

3.13. We have proved the following:

EXISTENCE THEOREM 2: Let the factor elasticity of substitution in either one of the two sectors be greater than or equal to one (or satisfy (29) or (30)). Then, for any aggregate capital-labour ratio k satisfying (22), the equilibrium ω, k_1, y_i, w, r, p, are uniquely determined and they are positive.

4. EXISTENCE AND STABILITY OF THE BALANCED GROWTH PATH

4.1. We now examine the behaviour through time of the aggregate capital-labour ratio, and by means of it that of all other variables, given that the economy is continuously at equilibrium.

We assume:

(A_5) The rate of growth of the labour force is exogenously determined and remains constant throughout the accumulation process.

Let λ be this rate. Then $\hat{L} = \lambda$ 16/.

4.2. The process of capital accumulation is given by

$$\dot{K} = Y_1 - \mu K , \quad \text{or} \tag{31}$$

$$\hat{K} = \frac{Y_1}{K} - \mu = \frac{y_1 \ell_1}{k} - \mu \tag{32}$$

16/ . and ^ denote the time rate of change and the relative time rate of change of a variable.

Then

$$\hat{k} = \hat{K} - \hat{L} = \frac{y_1 \ell_1}{k} - (\lambda + \mu) . \tag{33}$$

Since the economy is always at equilibrium, $\frac{y_1 \ell_1}{k}$ must satisfy (15) . Thus 17/

$$\hat{k} = \frac{s_r k + s_w \, \omega(k)}{k} \; f_1'[k_1[\omega(k)]] \; - (\lambda + \mu) . \tag{34}$$

k^* is a balanced capital-labour ratio if

$$\frac{s_r k^* + s_w \, \omega^*}{k^*} f_1'(k_1^*) = \lambda + \mu \quad , \tag{35}$$

where ω^* , k_1^* are the equilibrium wage rental ratio and capital-labour ratio in sector 1, respectively, which correspond to k^* .

4.3.* Let

$$\varphi(k) \equiv \frac{s_r k + s_w \, \omega(k)}{k} \; f_1' \; [k_1[\omega(k)]] . \tag{36}$$

We can show that

$$\frac{1}{\varphi(k)} \varphi'(k) = - \frac{s_w \; \omega(k)}{[s_r k + s_w \, \omega(k)]k} + \frac{s_w \, k_1[\, \omega(k)] - s_r k}{[s_r k + s_w \, \omega(k)][\, \omega(k) + k_1[\, \omega(k)]]} \cdot \omega'(k) \tag{37}$$

17/ (34) generalizes the corresponding equation appearing in [10] and [11]. Solow [8] derives a relation which is identical to [34]. [34] is also examined by Inada [2].

We will examine below several cases in which $\varphi'(k) < 0$ holds. Therefore, in all these cases a unique k^* exists, provided that

$$(38) \qquad \lim_{k \to \psi(\underline{\omega})} \varphi(k) > \lambda + \mu > \lim_{k \to \psi(\bar{\omega})} \varphi(k) .$$

4.4.* First, let us suppose that the capital-intensity condition holds, i.e., that the consumption-goods industry is always at least as capital intensive as the capital-goods industry. Thus, $k_2[\omega(k)] \geqq k_1[\omega(k)]$, for all k satisfying (38). We also assume that $s_r \geqq s_w$.

Under the present conditions, $s_r k \geqq s_w k_1[\omega(k)]$, and $\omega'(k) = 1/\psi'(\omega) > 0$, are seen to hold for all k satisfying (38). Therefore, $\varphi'(k) < 0$ for all such k, and there exists a unique k^* such that $\varphi(k^*) = \lambda + \mu$. Further, as $\varphi(k)$ is a strictly decreasing function of k, we easily see from (35) that this unique k^* is globally stable.

4.5. We have proved the following:

STABILITY THEOREM 1: Under the capital intensity condition and if $s_r \geqq s_w$ holds, then along any path of intertemporal equilibria, for which (38) holds, the capital-labour ratio k approaches asymptotically the unique balanced capital-labour ratio k^*.

4.6.* Let us examine again (36) and (37). (37) can be written as

$$(37') \quad \frac{1}{\varphi(k)} \varphi'(k) = \frac{1}{\sigma[\omega(k)]} \frac{\omega(k)}{k[\omega(k)] + k_1[\omega(k)]]} \frac{1}{[s_r k + s_w \omega(k)]}$$

$$\cdot \left\{ s_w k_1[\omega(k)] - s_r k - s_w \, \sigma[\omega(k)] \, . \, \Big(\omega(k) + k_1[\omega(k)]\Big) \right\},$$

since $\omega'(k) = 1/\psi'(\omega) = \frac{1}{\sigma[\omega(k)]} \frac{\omega(k)}{k}$.

This equation gives us the necessary and sufficient conditions for stability of the balanced capital-labour ratio k^* . Global stability requires that $\varphi'(k) < 0$ holds for all k satisfying (38). Hence it requires that if, for k satisfying (38), $\sigma[\omega(k)] \left\{ \gtrless \right\} 0$, then $\left\{ s_w k_1[\omega(k)] - s_r k - s_w \, \sigma[\omega(k)]\Big(\omega(k) + k_1[\omega(k)]\Big) \right\}$ $\left\{ \lesseqgtr \right\} 0$ holds. In the former case we must have

$$(39) \qquad \sigma[\omega(k)] > \max \left\{ \frac{s_w k_1[\omega(k)] - s_r k}{s_w(\omega(k) + k_1[\omega(k)])} , \; 0 \right\}$$

while in the latter

$$(40) \qquad \sigma[\omega(k)] < \min \left\{ \frac{s_w k_1[\omega(k)] - s_r k}{s_w(\omega(k) + k_1[\omega(k)])} , \; 0 \right\} .$$

This is as far as we can go with respect to the necessary and sufficient conditions for stability. We may remark that if $s_w = 0$, then direct consideration of (37) shows that stability is insured if and only if $\sigma[\omega(k)] > 0$, i.e., if and only if $\psi(\omega)$ is an increasing function of ω. Finally, if $s_r = s_w$, then we know that $\sigma[\omega(k)]$ is positive and thus only (39) is relevant for stability.

4.7.* With respect to sufficient conditions for stability several results are now readily available.

First, $\sigma_1[\omega(k)]$, $\sigma_2[\omega(k)] \geqq 1$ is a sufficient condition for the stability of the balanced capital-labour ratio k^*. For then $\sigma[\omega(k)] \geqq 1$,

and $\dfrac{s_w k_1[\omega(k)] - s_r k}{s_w(\omega(k) + k_1[\omega(k)])} < 1$ holds for all k satisfying (38).

Thus (39) is satisfied.

Second, let us ignore any possibilities for factor substitution in the capital-goods industry, and also assume that $\sigma_1[\omega(k)] = 1$, for all k satisfying (38).

Then, $\sigma[\omega(k)] = \dfrac{d[\omega(k)] + e[\omega(k)]}{a[\omega(k)]\; b[\omega(k)]} > 0$. We wish to show that

$$\sigma[\omega(k)] > \frac{s_w k_1[\omega(k)] - s_r k}{s_w(\omega(k) + k_1[\omega(k)])} = \frac{s_w b[\omega(k)]\; k_1[\omega(k)] - s_r a[\omega(k)]}{s_w b[\omega(k)]\; (\omega(k) + k_1[\omega(k)])} \quad \text{or}$$

$$s_w \frac{d[\omega(k)] + e[\omega(k)]}{a[\omega(k)]} > s_w(1 - s_r)\; k_1[\omega(k)] - s_r(1 - s_w)\; k_2[\omega(k)] .$$

This last inequality is easily seen to be true. Thus even when the capital-goods industry operates under (almost) fixed technical coefficients, we may still have stability of the balanced growth path if factor substitutability in the consumption-goods industry is high enough, <u>without regard to the factor intensities in the two industries.</u>[18]

We may also examine the opposite case, i.e., the case where $\sigma_1[\omega(k)] = 1$ for all k satisfying (38), while we ignore any possibilities of factor substitution in the consumption-goods industry.[19] Again $\sigma[\omega(k)] = \dfrac{c[\omega(k)] + e[\omega(k)]}{a[\omega(k)]\ b[\omega(k)]} > 0$, and we finally come to the inequality

$$s_w c[\omega(k)] > k_2[\omega(k)]\ (\omega(k) + k\ [\omega(k)])\ (s_w(1 - s_r)\ k_1[\omega(k)] - s_r(1 - s_w)\ k_2[\omega(k)]) .$$

However this inequality cannot be unequivocally established.[20] Nevertheless, the global stability of the balanced capital-labour ratio k^* can be shown for some particular values of s_w and s_r . Thus if $s_w = 0$, then we have seen in page 21 that global stability is obtained since $\sigma[\omega(k)] > 0$.

18/ This result can be more easily shown if we consider directly a Cobb-Douglas production function in the consumption-goods industry and fixed technical coefficients in the capital-goods industry.

19/ This may throw some light on the impact of the asymmetricity exhibited in our model by the fact that only capital goods are directly fed back into the process of capital accumulation. Otherwise, the case is of little practical interest.

20/ The inequality of course holds if $s_r \geq s_w > 0$, $k_2[\omega(k)] > k_1[\omega(k)]$, which was shown in Section 4.4.*

Also if $s_r = 1$, the above inequality holds. Since $\varphi'(k)$ is a continuous function of the parameters s_r, s_w, global stability is obtained for s_r near one and s_w near zero.

4.8. We have proved the following:

STABILITY THEOREM 2: Let

(a) $\sigma_2[\omega(k)] \geqq 1$ hold for all k satisfying (38); or

(b) $\sigma_1[\omega(k)] \geqq 1$ hold for all k satisfying (38) and s_r be sufficiently near one or s_w be sufficiently near zero.

Then along any path of intertemporal equilibria, the capital-labour ratio k approaches asymptotically the unique balanced capital-labour ratio k^* .

5. SPECIAL CASES

5.1. Fixed Technical Coefficients in both Sectors.[21]

The model can be outlined by the following equations:[22]

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[21] The main theorem in the present case has been proved by Shinkai [6]. He showed that the "Simple model" of J. Robinson [5], as formulated by Morishima [4], exhibits global stability of the balanced growth path if and only if the capital intensity condition is met.

[22] In general, the profit and market equilibrium conditions will be expressed as inequalities with the usual nonnegativity restraints on all variables. Shinkai also examines only this case, [6, p. 110].

(41) $$Y_1 = \eta K_1 = \theta L_1 ,$$

(42) $$Y_2 = \gamma K_2 = \delta L_2 ,$$

(43) $$1 = \frac{1}{\gamma} r + \frac{1}{\delta} w ,$$

(44) $$p = \frac{1}{\eta} r + \frac{1}{\theta} w ,$$ as well as equations (5) - (9).

(43) and (44) express the profit conditions in the two sectors. We have $k_1 = \frac{\theta}{\eta}$, $k_2 = \frac{\delta}{\gamma}$. Thus, (5) - (7) may be given by (13), (14), and by

(21') $$k = \psi(\omega) \equiv \frac{k_1 k_2 + k_2 \omega - s_w (k_2 - k_1)\omega}{(1-s_r) k_1 + s_r k_2 + \omega}$$

We must point out that $\max(k_1, k_2) > k > \min(k_1, k_2)$ must be satisfied if we wish to examine the possibilities of full employment equilibrium only.

5.2. The existence of a competitive equilibrium is readily established. We have:

$$\psi'(\omega) = \frac{(k_2 - k_1)\left(s_r (1-s_w) k_2 - s_w (1-s_r) k_1\right)}{\left((1-s_r) k_1 + s_r k_2 + \omega\right)^2}$$

Thus, (a) if $s_r > 0$, $s_w = 0$, then $k_2 - k_1 > 0$ is a necessary and sufficient condition for $\psi'(\omega) > 0$; (b) if $1 > s_r = s_w > 0$, then $\psi'(\omega) > 0$; and (c) if $1 > s_r > s_w$ and $k_2 > k_1$, then $\psi'(\omega) > 0$.

In all these cases if a competitive equilibrium exists, it is uniquely determined by $\bar{k}$. However, the limits within which $\bar{k}$ must lie for a competitive equilibrium to exist are even narrower than those indicated in the previous paragraph. For,

$$\lim_{\omega \to 0} \psi(\omega) = \frac{k_1 k_2}{s_r(k_2-k_1) + k_1} > 0 , \quad \lim_{\omega \to \infty} \psi(\omega) = \frac{k_2-s_w(k_2-k_1)}{1} > 0$$

for $k_2 > k_1$. These limits coincide with the limits referred to in section 5.1 only if $s_r = 1$, $s_w = 0$. Otherwise they are narrower. Full employment equilibrium is seriously limited by the saving propensities of capitalists and workers.[23]

5.3. With respect to the accumulation path in such an economy we have the following:

$$(45) \qquad \hat{k} = \frac{y_1 \ell_1}{k} - (\lambda + \mu) .$$

But $y_1 = \theta$ and $\ell_1 = \dfrac{k_2 - \bar{k}}{k_2 - k_1}$ are both constants. Thus the accumulation path does not depend on the price system and on the saving patterns. If the initial position $k(o)$ is given, the path is completely determined.

The balanced capital-labour ratio k^* is given by $\dfrac{y_1}{k^*} \dfrac{k_2-k^*}{k_2-k_1} = \lambda + \mu$,

or by $k^* = \dfrac{y_1 k_2}{(k_2-k_1)(\lambda+\mu)+y_1}$. Let $\varphi(k) = \dfrac{y_1}{k} \dfrac{k_2-k}{k_2-k_1}$. Then,

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[23] This is to be expected. If e.g., $\bar{k}$ is near k_1 and if the capitalists wish to consume a lot, then more resources have to be allocated to the consumption sector than is feasible.

$\frac{1}{\varphi(k)} \varphi'(k) = - \frac{k_2}{k(k_2-k)}$, which is negative if and only if $k_2 > k_1$.

This also insures that $k^* > 0$.

5.4. We can now appreciate the distinguishing features of a two-sector model with fixed technical coefficients and full employment of all productive factors. In such a model the equilibrium price system passively adjusts to the changing quantities of the primary factors without having any influence on the accumulation process. Furthermore, such an equilibrium price system may very well not be feasible.

5.5. Capital as the only Input in the Capital-goods Sector.

We finally examine the special case where the capital-goods sector uses only capital goods.[24] This is a very special assumption indeed. See however Kurtz [3].

The changes which are now introduced are expressed by:

(46) $$Y_1 = \eta K_1 , \quad Y_2 = F^2 (K_2, L_2) ,$$

(47) $$p = \frac{1}{\eta} r ,$$

(48) $$K_1 + K_2 = K ,$$

(49) $$L_2 = L ,$$

---

[24] If the capital-goods sector uses only capital goods, then $k_1[\omega(k)] = +\infty$ for all relevant k. However, under conditions (1) above, $k_1[\omega(k)] < +\infty$ for all k satisfying (38). Therefore, our Stability Theorem 2 does not cover the present case.

with the rest equations the same as in our original model.

We have $\frac{K_1}{L} = k - k_2$, $y_2 = f_2(k_2)$, $\omega = \left(f_2(k_2)/f_2'(k_2)\right) - k_2$.

5.6. Then (15) becomes

$$(1-s_r)k = s_w\omega + k_2(\omega) \; .\tag{50}$$ 25/

Then $\psi(\omega) = (1/1-s_r)\,(s_w\omega + k_2(\omega))$,

and $\psi'(\omega) = \frac{1}{1-s_r}\left(s_w + k_2'(\omega)\right) > 0$, and thus a unique and positive equilibrium exists for all k satisfying

$$\psi(\underline{\omega}_2) < k < \psi(\overline{\omega}_2) \; .\tag{51}$$

All other quantities and prices are thereby determined and they are positive. Further,

$$\omega'(k) = \frac{1}{\psi'(\omega)} = \frac{1 - s_r}{s_w + k_2'(\omega)} = \frac{(1-s_r)\,\omega}{s_w\omega + k_2'(\omega)\;\sigma_2(\omega)} \; .\tag{52}$$

5.7. The process of capital accumulation is given by

$$\hat{k} = \frac{Y_1}{K} - (\lambda + \mu) = \eta\;\frac{k - k_2[\omega(k)]}{k} - (\lambda + \mu) \; .\tag{53}$$

25/ We note that if $s_r = 1$, $\omega = 0$. Actually, if the capitalists do not consume at all, they do not have any use for labour.

Let $\varphi(k) \equiv \eta \dfrac{k-k_2[\omega(k)]}{k}$.

Then, using (50) and (52), we can show that

$$\varphi'(k) = -\eta \frac{s_w\ \omega(k)\ k_2[\omega(k)]}{k^2\Big(s_w\omega(k) + k_2[\omega(k)]\sigma_2[\omega(k)]\Big)} \cdot \Big(\sigma_2[\omega(k)] - 1\Big)$$

Thus if $\sigma_2[\omega(k)] > 1$, for all k satisfying (51), $\varphi'(k) < 0$, and we still get the global stability of the balanced capital-labour ratio k^* .

If however $\sigma_2[\omega(k)] = 1$, i.e., the production function for the consumption goods is of the Cobb-Douglas type, $Y_2 = K_2^{\beta} L_2^{1-\beta}$, then $\varphi'(k) = 0$. Then $\varphi(k) = g$ = constant. $(g - \mu)$ is the rate of growth of the capital stock. In this case we can show that

$$\hat{K} = \hat{K}_1 = \hat{K}_2 = g - \mu \ , \quad \text{where} \quad g = \frac{\beta\ s_r + (1 - \beta)\ s_w}{\beta + (1 - \beta)\ s_w} \ .$$

Hence, $\hat{k} = g - (\lambda + \mu)$ holds, and only if λ happens to be equal to the rate of growth of the capital stock, will balanced growth subsist. The unique balanced capital-labour ratio k^* , determined by putting $g - \mu = \lambda$, is thus found to be unstable. The rate of growth of the capital stock is independent of the rate of growth of labour and no mechanism is provided by the market to bring the two together.

Finally, if $\sigma_2[\omega(k)] < 1$, then $\varphi'(k) > 0$. Again the unique balanced capital-labour ratio k^* is unstable.

5.8. Hence, corresponding to the Stability Theorem 2 we have:

STABILITY THEOREM 3: Let the capital-goods sector use only capital goods. If $\sigma_2[\omega(k)] > 1$ holds for all k satisfying (51), then along any path of intertemporal equilibria, the capital-labour ratio $k(t)$ approaches asymptotically the unique balanced capital-labour ratio k^*, if the latter exists.

6. FINAL REMARKS

6.1. The uniqueness of a competitive equilibrium at each time period, as well as the existence of a unique balanced capital-labour ratio, was shown in sections 3 and 4 on the basis of conditions (17) and (18), (22) and (38). These conditions indicate what the ranges of the functions $\alpha_i(k_i)$, i=1, 2, $\psi(\omega)$, and $\varphi(k)$, respectively, must be for existence of solutions of the equations concerned.

It is possible to specify the ranges of the above functions by strengthening the conditions imposed on the production functions $f_i(k_i)$, i=1, 2 . E.g., in [11], and [2], the following conditions are imposed:

$$f_i(0) = 0 \quad , \quad f_i(\infty) = \infty$$

(54)

$$f_i'(0) = \infty \quad , \quad f_i'(\infty) = 0 \quad .$$

Then, it can be shown that

(a) $\lim_{k_i \to 0} \alpha_i(k_i) = 0$, $\lim_{k_i \to \infty} \alpha_i(k_i) = \infty$;

thus in (17) $\underline{\omega} = 0$, $\overline{\omega} = +\infty$. Further,

(b) $\lim_{\omega \to 0} \psi(\omega) = 0$, and $\lim_{\omega \to \infty} \psi(\omega) = \infty$.

Finally,

(c) $\lim_{k \to 0} \varphi(k) = \infty$, $\lim_{k \to \infty} \varphi(k) = 0$.

However, conditions (54), suggested by production functions of the Cobb-Douglas type, are not met for production functions of other familiar types, as e.g., for constant-elasticity-of-substitution production functions.

In the present paper we prefer to impose directly (17), (18), (22), and (38), instead of (54), mainly because we consider various cases in which the elasticities of factor substitution exhibited by the production functions are within certain limits. It is not apparent that for such production functions the values of $\alpha_i(k_i)$, $\psi(\omega)$, and $\varphi(k)$, at the end-points of their domain of definitions, are those indicated by (54).

6.2. We have deliberately refrained from introducing an interest rate into the model. In the present model, in which our assumptions in effect separate one market convention from all successive ones, an interest rate does not play an independent role. It can simply be defined by an equation referring to the yield of the capital goods throughout their life span. Thus, if for simplicity we assume static expectations, we have:

$$p = \int_0^\infty r(s)\, e^{-(\rho+\mu)s}\, ds = \frac{r}{\rho + \mu} \tag{55}$$

which serves to define ρ . Equivalently, we may think of p in (55) as being the "demand price" of capital goods, determined when r and ρ are given. On the other hand the maximizing behavior of the entrepreneurs (i.e., equations (3) and (4)) provides us with the "supply price" of the capital goods. Equality of the two is needed for an equilibrium position.

The rate of profit is $\frac{r - \mu p}{p} = \rho$. With static expectations the rate of profit is equal to the rate of interest. Further, the rate of growth of the capital stock is

$$\hat{K} = (\rho + \mu)\ \frac{s_r k + s_w \omega}{k} - \mu = g \tag{56}$$

. If $\frac{s\ k + s\ \omega}{k} \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 1$ or if $s\ wL \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} (1-s)\ rK$, then $g \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} \rho$. Namely, if the total wage bill exceeds the value of consumption of both workers and capitalists, then the rate of growth of the capital stock is greater than the rate of profit and vice versa.

6.3. The assumptions of Section 2.2 can be relaxed in some respects. E.g., a part of (A_1) can be substantially relaxed. Namely, in each market convention the total existing quantity of labour services, $\bar{L}$, may not be ncessarily offered to the market at any price. On the contrary we assume that labour services yield directly consumable services. We assume however that the supply of labour sarvices for production purposes is an increasing function of the real wage w . Since $w'(\omega) > 0$ can easily be shown to hold, we denote this function by $L(\omega)$. $L(\omega)$ may be of the form:

$$L(o) = 0 \ , \quad L(\omega) \leqq \bar{L} \text{ , for all } \omega \text{ satisfying (17), and } L'(\omega) \geqq 0 \ .$$

This case arises if e.g., the workers decide on the basis of their real wage what proportion of their labour services will be offered to the market and what will be withheld for direct consumption by themselves.

Further, we may assume that the rate of growth of the labour force λ is an increasing function of the real wage, or $\lambda = \lambda(\omega)$, within some exogenously given (physiological) limits. This is a reasonable assumption in the present context.

We can easily show that none of our conclusions change by this generalization.

Equation (6) now becomes

$$(6') \qquad L_1 + L_2 = L(\omega) \ , \quad L(\omega) \leqq \bar{L} \ .$$

The capital-labour ratio at any time period is now a decreasing function of ω , $k(\omega)$. Thus (15) again provides us with a unique solution for ω under exactly the same conditions as before. The only case which is still excluded is that of a 'backward-rising' supply function of labour services, in which case a unique solution ω is of course not necessarily insured.

With respect to intertemporal equilibria again our present assumptions do not affect our conclusions. We have:

(57) $\hat{L} = \lambda[\omega(k)]$. Then,

(34') $\hat{k} = \frac{s_r k + s_w \omega(k)}{k} - \lambda[\omega(k)] - \mu$.

We again obtain the global stability of k^* under the same conditions as in Stability Theorems 1 and 2. As a matter of fact, global stability of k^* must be more prevalent here than before. This is of course natural for an instability of k^* would mean a continuous increase or decrease of k and would in due time generate corrective counteractions on the part of the labour force.

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