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Topological Methods in Cardinal Utility Theory*

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Topological Methods in Cardinal Utility Theory[1]

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The concept of cardinal utility is studied in three different situations (stochastic objects of choice, stochastic act of choice, independent factors of the action set) in this article by means of the same mathematical result giving a topological characterization of three families of parallel straight lines in a plane. This result, proved first by G. Thomsen [24] under differentiability assumptions, and later by W. Blaschke [2] in its present general form (see also W. Blaschke and G. Bol [3]), can be briefly described as follows. Consider the topological image G of a 2-dimensional convex set and in it three families of curves (in each family a curve is the topological image of a real interval and depends continuously in a one-to-one fashion on a parameter varying in a real interval) such that (a) through a point of G goes exactly one curve of each family, (b) two curves of different families have at most one common point. Is there a topological transformation carrying these three families of curves into three families of parallel straight lines? In the affirmative, the hexagonal configuration of Fig. 1.a will be observed: let P be an arbitrary point of G , draw through it a curve of each family, and take on one of these curves an arbitrary point A ; by drawing through A the curves of the other two families one may obtain B , B'; from them one may obtain C , C'; it is clear that if two of the curves marked by arrows

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intersect, the third one must concur with them, for the same construction carried out for three families of parallel straight lines yields three concurrent lines. Thus a necessary condition for the existence of the desired

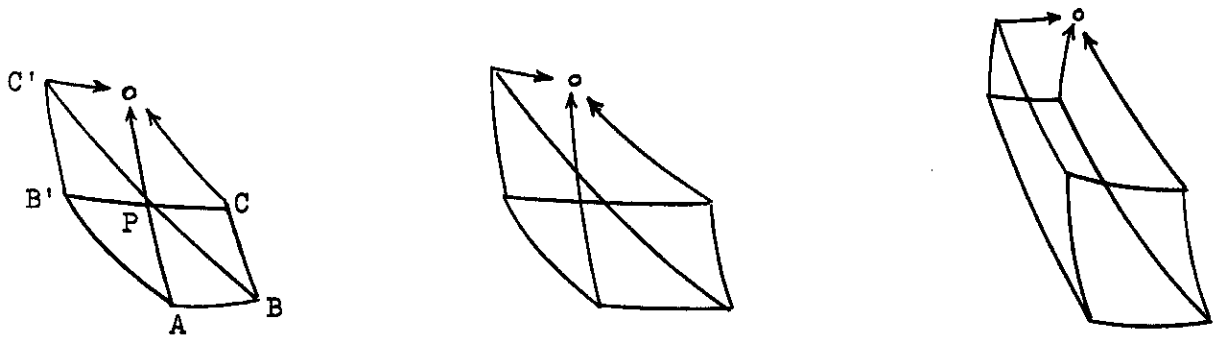

Fig. 1.a Fig. 1.b Fig. 1.c

topological transformation is that the hexagon of Fig. 1.a can be completed for every P and A such that the curves involved in the construction intersect. The theorem of Thomsen-Blaschke asserts that this is also a sufficient condition. Two equivalent forms of that condition are represented in Fig. 1.b and Fig. 1.c which are self-explanatory. They are necessary for the same reason as above. They are sufficient since they obviously imply the condition of Fig. 1.a. Actually a stronger theorem is true: if every point of G has a neighborhood in which one of the three configurations of Fig. 1 holds, then there is on G a topological transformation of the desired type.

Of the three applications which will now be made of this theorem to utility theory, the first two ones have been presented in detail elsewhere [10], [9], and for them only a brief account will be given.

1. Stochastic Objects of Choice

Difficulties have been met with in the testing of the axioms offered by J. von Neumann and O. Morgenstern [17] (or of axioms equivalent to them) for the existence of a cardinal utility in this situation. Some of them may be ascribed to the inability of subjects to grasp the meaning of complex prospects. This has led D. Davidson and P. Suppes [6][2] to suggest that the subjects be presented only with the simplest type of uncertain prospect, namely even-chance mixtures of pairs of sure prospects. An axiomatization of this case will be given here. Let S be a set of pure prospects (e.g., commodity bundles). Given two elements a and b of S , the symbol ab denotes the prospect of having a with probability $\frac{1}{2}$ or b with probability $\frac{1}{2}$. The set S x S of prospects is completely preordered by the relation $\precsim$ which is read "is not preferred to." As usual, $\sim$ is read "is indifferent to," and $\succ$ is read "is preferred to." In this context one puts the

Definition: A utility function is a real-valued, order-preserving function u on S x S such that

$$u(ab) = \frac{1}{2}\ [u(aa) + u(bb)] \text{ for every } a \text{ and } b \text{ in } S .$$

The problem of finding conditions on S and $\precsim$ which guarantee the existence of a utility function defined in this fashion has been considered by F. P. Ramsey [19] and, more recently, by D. Davidson and P. Suppes [6], D. Davidson, P. Suppes and S. Siegel [7], P. Suppes [23]. The object of this section is to present a simple solution. The axioms will be:

(1) S is connected and separable

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2. And not D. Davidson and J. Marschak [5] as I asserted in [10].

(2) $\precsim$ is a complete preordering of $S \times S$ such that $\{ab \in S \times S \mid ab \succsim a'b'\}$ and $\{ab \in S \times S \mid ab \precsim a'b'\}$ are closed for every $a'b'$ in $S \times S$.

(3) $[a_1b_2 \precsim a_2b_1 \text{ and } a_2b_3 \precsim a_3b_2] \Longrightarrow [b_3a_1 \precsim b_1a_3]$.

The last one being clearly a necessary condition for the existence of a utility function. One can then prove:

Theorem: Under assumptions (1), (2), (3), there is a continuous utility function determined up to an increasing linear transformation.

The proof uses a representation of $S \times S$ in R^2. According to [8], there is a continuous real-valued, order-preserving function f on $S \times S$. Let ab be a generic element of $S \times S$. Using the notation $\alpha = f(aa)$ and $\beta = f(bb)$, one defines the representation by $ab \longrightarrow (\alpha, \beta)$. Since S is connected, the range of α is a real interval Σ. The indifference classes of $S \times S$ are represented by curves in $\Sigma \times \Sigma$, two of which are drawn in Fig. 2.a. These indifference curves have the marked diagonal as an axis of symmetry, and on any one of them one variable is a

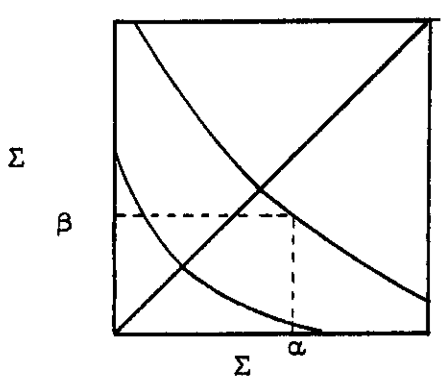

Fig. 2.a

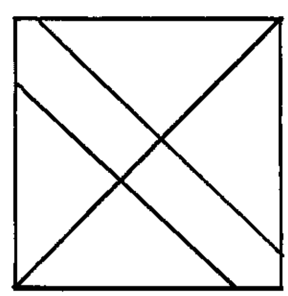
Fig. 2.b

decreasing function of the other. If the function f happened to be a utility function, the indifference curves would satisfy the relation $\alpha+\beta$ = constant and would thus be straight lines perpendicular to the diagonal as in Fig. 2.b. Since two real-valued, order-preserving functions on $S \times S$ are derived from one another by an increasing transformation, the proof amounts to showing that there is an increasing transformation on both coordinates carrying the indifference curves of Fig. 2.a into the straight lines perpendicular to the diagonal of Fig. 2.b. Such a transformation carries the following three families of curves, the verticals, the horizontals, the indifference curves into the following three families of parallel straight lines, the verticals, the horizontals, the perpendiculars to the diagonal. It exists if and only if the condition of Thomsen-Blaschke is satisfied. And it is easy to check, using (3), that the hexagonal configuration of Fig. 1.b holds in Fig. 2.a.

2. Stochastic Act of Choice.[3]

Instead of introducing a stochastic element in the object of choice, one can introduce it in the act of choice. Let S be a set of actions. The subject is presented with a pair (a,b) of actions in S and asked to choose one. He is assumed to choose a with probability p(a,b) and b with probability $p(b,a) = 1 - p(a,b)$. Formally:

(1) S is a set, p is a function from $S \times S$ to [0,1] such that $p(a,b) + p(b,a) = 1$ for every (a,b) in $S \times S$.

3. I wish to add to the bibliography of [9] the following items which appeared too late to be included in it: J. S. Chipman [4], N. Georgescu-Roegen [12], R. D. Luce [16], and J. Pfanzagl [18].

It is natural to give to the inequality $p(a,b) > p(c,d)$ the interpretation "a is preferred to b more than c is preferred to d ," i.e., to put the

<u>Definition</u>: <u>A utility function for (S,p) is a real-valued function u on S such that</u>

$$[p(a,b) \leqq p(c,d)] \Longleftrightarrow [u(a) - u(b) \leqq u(c) - u(d)].$$

D. Davidson and J. Marschak [5], who have studied this aspect of cardinal utility, remark that $u(a) - u(b) \leqq u(c) - u(d)$ is equivalent to $u(a) - u(c) \leqq u(b) - u(d)$, hence that the existence of a utility function for (S,p) implies

(2) $[p(a,b) \leqq p(c,d)] \Longleftrightarrow [p(a,c) \leqq p(b,d)]$.

This will be taken as the second axiom. The last one is a continuity condition:

(3) <u>If $p(b,a) \leqq q \leqq p(c,a)$, then there is d in S such that $p(d,a) = q$.</u>

One can prove:

<u>Theorem</u>: <u>Under assumptions (1), (2), (3), there is for (S,p) a utility function determined up to an increasing linear transformation</u>.

The proof uses a representation of S in $[0,1]$. Let k be an arbitrary element of S which will be kept fixed. The generic element a of S is represented by the number $\alpha = p(a,k)$. According to (3), the range of α is an interval Σ in $[0,1]$. The number $p(a,b)$ is readily seen, on account of (2), to depend only on the images α, β of a,b in the representation. Let π be the function defined on $\Sigma \times \Sigma$ in this way,

$$p(a,b) = \pi(\alpha,\beta)$$

It is clear that finding a utility function u for (S,p) is equivalent to finding a utility function ν for (Σ,π), the two utility functions being related by

$$u(a) = \nu(\alpha).$$

In Fig. 3.a five isoprobability curves have been drawn. The marked diagonal corresponds to the probability $\frac{1}{2}$, two curves corresponding to

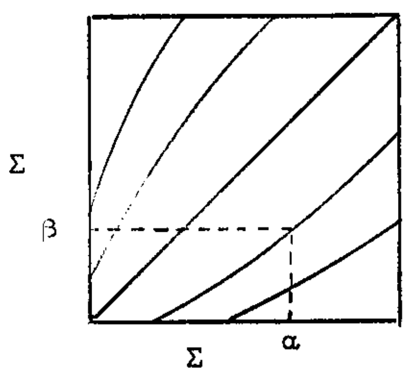

Fig. 3.a

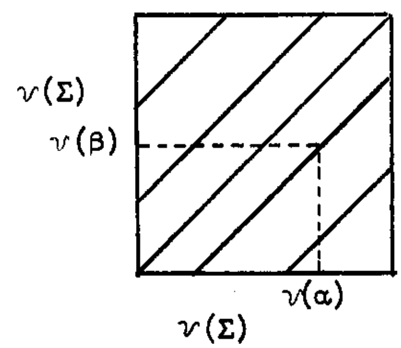

Fig. 3.b

probabilities adding up to 1 are symmetric of each other with respect to that diagonal, and on any isoprobability curve one variable is an increasing function of the other. The proof of the theorem amounts to showing that there is an increasing transformation ν on both coordinates carrying the isoprobability curves of Fig. 3.a into the straight lines $\nu(\beta) - \nu(\alpha) = \text{constant}$ of Fig. 3.b. Such a transformation carries the following three families of curves, the verticals, the horizontals, the isoprobability curves into the following three families of parallel straight lines, the verticals, the horizontals, the parallels to the diagonal. It exists if and only if the condition of Thomsen-Blaschke is satisfied. And it is easy to check, using (2), that the hexagonal configuration of Fig. 1.a holds in Fig. 3.a.

3. Independent Factors of the Action Set

The last situation where a cardinal utility will be defined is a generalization of the classical economic problem of independent commodities. A calculus solution and references to the literature will be found in P. A. Samuelson [20] Chap. 7 (other questions closely related to the present topic have been studied by W. Leontief in [14], [15] equally with a calculus technique). Differentiability assumptions will be dropped here. This will result, as usual, not only in a more general but also in a more natural answer.

Consider a consumer making a consumption plan represented by a m-tuple x of real numbers, where m is the number of commodities. If the class of commodities is partitioned into n subclasses indicated by an index i running from 1 to n, a consumption plan can also be represented by the n-tuple (x_i) where x_i is the tuple of real components of x corresponding to the ith subclass of commodities. For certain partitions of the class of commodities one is led to try to represent the preferences of the consumer for x by a real-valued function u of the form

$$u(x) = \sum_{i=1}^{n} u_i(x_i) .$$

Examples are: 1) the partition according to basic needs, food, housing, clothing, etc.; 2) when the consumption plan covers several consecutive time-intervals and the definition of a commodity includes the time-interval in which it is available, the partition according to the time-interval (see, for instance, R. H. Strotz [21], [22], W. M. Gorman [13]).

The main concepts of the analysis can now be formally introduced:

(1) Given n connected, separable spaces $S_1, \ldots, S_n$, $\precsim$ is a complete preordering of their product $S = \prod_{i=1}^{n} S_i$ such that $\{x \in S \mid x \succsim x'\}$ and $\{x \in S \mid x \precsim x'\}$ are closed for every x' in S.

Definition: A utility function is a real-valued, order-preserving function u on S such that for every $x = (x_i)$ in S

$$u(x) = \sum_{i=1}^{n} u_i(x_i) \, ,$$

where u_i is a real-valued function on S_i for every $i = 1, \ldots, n$.

The concept which is basic to the solution is that of independence. Let N be the set of the first n integers, and let I be an arbitrary subset of N. Imagine that the x_i where $i \in I$ are given, then the preordering $\precsim$ on S induces on the product $\prod_{i \notin I} S_i$ a preordering which will be called the preordering given $(x_i)_{i \in I}$. It is clear that this preordering is independent of the particular tuple $(x_i)_{i \in I}$ chosen if there is a utility function on S. Thus a necessary condition for the existence of a utility has been obtained; it will be shown to be sufficient provided that S has more than two essential factors. The factor S_i will be said to be inessential if for every $(x_j)_{j \neq i}$ all the elements of S_i are indifferent for the preordering given $(x_j)_{j \neq i}$; otherwise it will be said to be essential. Summing up:

Definitions: Let I be a subset of $N = \{1, \ldots, n\}$, and for every $i \in I$ let x_i be an element of S_i. The preordering given $(x_i)_{i \in I}$ is the preordering induced by $\precsim$ on $\prod_{i \notin I} S_i$ when the element of S_i is equal to x_i for every $i \in I$. The n factors of S are independent if for every subset I of N the preordering given $(x_i)_{i \in I}$ is independent of $(x_i)_{i \in I}$. The factor S_i is essential if for some $(x_j)_{j \neq i}$ not all its elements are indifferent for the preordering given $(x_j)_{j \neq i}$.

Theorem: Under assumption (1), if the n factors of S are independent, and if more than two of them are essential, there is a continuous utility function determined up to an increasing linear transformation.

The case of two essential factors of S, which has been discussed by E. Adams and R. Fagot [1], and W. Edwards [11], is an immediate generalization of the first situation studied above from which it differs only by the absence of the symmetry displayed by Fig. 2.a. The solution of this case will appear here only implicitly as a step in the following proof.

Proof of the theorem: Denote by $\underset{i}{\precsim}$ the preordering given $(x_j)_{j \neq i}$ (which is independent of $(x_j)_{j \neq i}$ by assumption). It is easily seen that

(2) $\ll x_i \underset{i}{\sim} x'_i$ for every $i \gg$ implies $\ll (x_i) \sim (x'_i) \gg$.

According to [8] there is on S a continuous real-valued, order-preserving function v and similarly there is on each S_i a real-valued, order-preserving function v_i . By (2) the image y of x by v depends only on the images y_i of x_i by v_i ; let f be the function defined in this fashion.

(3) $$y = f(y_1, \ldots, y_n) .$$

The image T_i of S_i by v_i is a real interval since S_i is connected and v_i is continuous. This interval degenerates to a point if and only if S_i is inessential. The function f from $T = \prod_{i=1}^{n} T_i$ to the reals is increasing in each variable; it is also continuous in each variable. It follows, without difficulty, that f is continuous.

The initial problem which consists in finding the $n+1$ real-valued functions $u_1, \ldots, u_n, u$ defined respectively on $S_1, \ldots, S_n, S$ is equivalent to the notably simpler one of finding $n+1$ real-valued, increasing transformations $t_1, \ldots, t_n, t$ defined respectively on $T_1, \ldots, T_n, f(T)$ such that (3) becomes

$$t(y) = \sum_{i=1}^{n} t_i(y_i).$$

It is this second problem which will now be solved. It will be assumed that there are no inessential sets S_i , i.e., no degenerate intervals T_i , since their role is trivial. The terminology and the notation adopted for the preordering of S will be freely used for the preordering obtained by carrying it over to T in the obvious fashion.

By fixing the values of $y_3, \ldots, y_n$ in the <u>interiors</u> of $T_3, \ldots T_n$ (the reason for this restriction to the interiors will appear later) one obtains a plane P of R^n and, in this plane, a preordering of the points (y_1, y_2) of $T_1 \times T_2$. It will be proved that the indifference curves of this preordering and the parallels to the axes satisfy the condition of Fig. 1.b in the small. Given a point of $T_1 \times T_2$ in the plane P , one can always find in P a closed rectangular neighborhood U of that point having its sides parallel to the axes and such that the indifference hyper-surface going through the greatest (according to the preordering) vertex of U intersects the linear variety orthogonal to P through the least (according to the preordering) vertex of U. The above restriction to interiors was designed to insure this possibility. Consider the two indifferent points a and b of U defined respectively by the pairs of their two first co-ordinates (y_1^2, y_2^1) and (y_1^1, y_2^2); consider similarly the two indifferent points c and d of U defined respectively by (y_1^3, y_2^1) and (y_1^1, y_2^3) (the reasoning can be followed in Fig. 4 drawn for the case $n = 3$). To prove

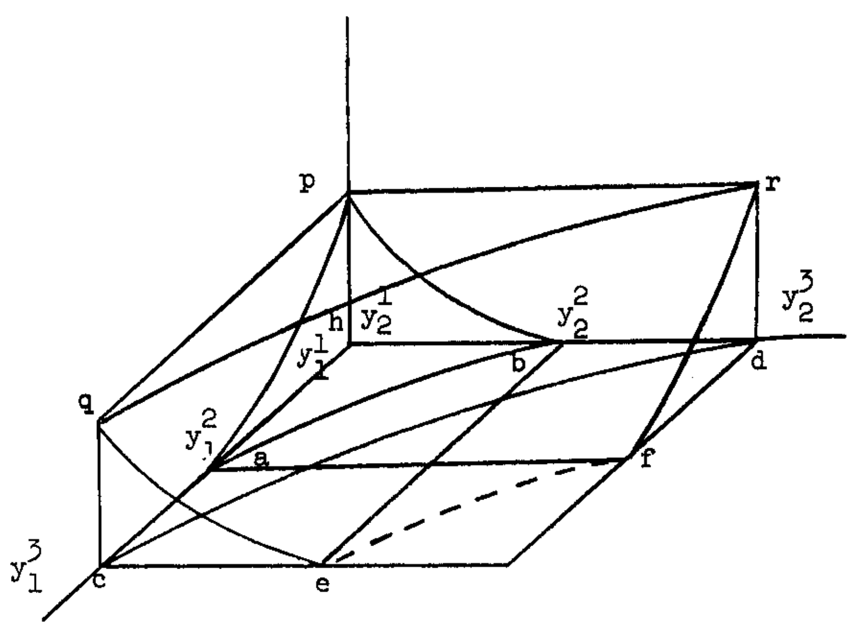

Figure 4

that the condition of Fig. 1.b holds is to prove that the two points e and f of U defined respectively by (y_1^3, y_2^2) and (y_1^2, y_2^3) are indifferent. The criterion according to which U was chosen implies immediately that the indifference hypersurface through a and b intersects the linear variety orthogonal to P through the point h of U defined by (y_1^1, y_2^1). Let p be a point in that intersection (in the case $n = 3$, p is unique). Let also q (resp. r) be the point derived from p by the translation hc (resp. hd). Since p and b are indifferent, so are q and e by the independence assumption. Similarly the indifference of p and a implies that of r and f. Finally the indifference of c and d implies that of q and r. Summing up, $e \sim q$, $q \sim r$, $r \sim f$; hence $e \sim f$.

Thus there are two continuous increasing transformations t_1, t_2 defined on T_1, T_2 respectively carrying the indifference curves in $T_1 \times T_2$ into the straight lines $t_1(y_1) + t_2(y_2) = \text{constant}$.

A reasoning by induction will complete the proof. Assume that there are continuous increasing transformations $t_1, \ldots, t_{k-1}$ on $T_1, \ldots, T_{k-1}$ such that the indifference hypersurfaces in $\prod_{i=1}^{k-1} T_i$ are represented by $\sum_{i=1}^{k-1} t_i(y_i) = \text{constant}$. This additive representation will be extended to $\prod_{i=1}^{k} T_i$. Denote $t_i(y_i)$ by z_i; the y indifference hypersurface in $\prod_{i=1}^{k} T_i$ can be represented by

$$(4) \qquad z_1 + \ldots + z_{k-1} = g_k(y_k, y),$$

where g_k is a continuous function of (y_k, y), decreasing in y_k and increasing in y . Consider a point (y_k^o, y^o) <u>interior</u> to the domain of g_k . It will be proved that this point has a neighborhood V in which g_k is the sum of a function of y_k and a function of y . For this, take $(z_1^o, \ldots, z_{k-1}^o)$ in R^{k-1} in the <u>interior</u> of the set of $(z_1, \ldots, z_{k-1})$ defined by: $z_i \in t_i\ (T_i)$ for every $i = 1, \ldots, k-1$ and $\sum_{i=1}^{k-1} z_i = g_k\ (y_k^o, y^o)$. Thus, in particular,

$$(5) \qquad \sum_{i=1}^{k-2} z_i^o + z_{k-1}^o = g_k(y_k^o, y^o) .$$

Select then a closed rectangular neighborhood V of (y_k^o, y^o) having its sides parallel to the axes, small enough for the operations connected with (6) and (7) to be possible, and let (y_k^1, y^1) be an arbitrary point of V .

Define z_{k-1}^1 in $t_{k-1}\ (T_{k-1})$ by

$$(6) \qquad \sum_{i=1}^{k-2} z_i^o + z_{k-1}^1 = g_k(y_k^1, y^o) .$$

Choose $z_1^1, \ldots, z_{k-2}^1$ in $t_1(T_1), \ldots, t_{k-2}\ (T_{k-2})$ such that

$$(7) \qquad \sum_{i=1}^{k-2} z_i^1 + z_{k-1}^o = g_k(y_k^o, y^1) .$$

The two points $(z_1^o, \ldots, z_{k-2}^o, z_{k-1}^o, y_k^o)$ and $(z_1^o, \ldots, z_{k-2}^o, z_{k-1}^1, y_k^1)$ are on the y^o indifference hypersurface according to (5) and (6). Hence, by the independence assumption, the two points $(z_1^1, \ldots, z_{k-2}^1, z_{k-1}^o, y_k^o)$ and $(z_1^1, \ldots, z_{k-2}^1, z_{k-1}^1, y_k^1)$ are indifferent. Since the first is on the y^1 indifference hypersurface according to (7), one has

$$(8) \qquad \sum_{i=1}^{k-2} z_i^1 + z_{k-1}^1 = g_k(y_k^1, y^1) .$$

Subtracting (7) from (8) and (5) from (6), one obtains

$$g_k(y_k^1, y^1) - g_k(y_k^o, y^1) = z_{k-1}^1 - z_{k-1}^o = g_k(y_k^1, y^o) - g_k(y_k^o, y^o) .$$

The relation

$$g_k(y_k^1, y^1) = g_k(y_k^1, y^o) + g_k(y_k^o, y^1) - g_k(y_k^o, y^o)$$

proves that g_k decomposes in V as desired.

The property in the large follows from the property in the small: throughout its domain, g_k is the sum of a decreasing function of y_k and an increasing function of y, and can therefore be written in the form

$$g_k(y_k, y) = - t_k(y_k) + h_k(y) .$$

It suffices to substitute this for g_k in (4) to see that t_k is a transformation on T_k allowing one to extend the additive representation to $\prod_{i=1}^{k} T_k$.

References

[1] Adams, E. and R. Fagot, "A Model of Riskless Choice," Behavioral Science, 4, 1-10, 1959.

[2] Blaschke, W., "Topologische Fragen der Differentialgeometrie. I" Mathematische Zeitschrift, 28, 150-157, 1928.

[3] Blaschke, W. und G. Bol, Geometrie der Gewebe, Berlin, Springer, 1938.

[4] Chipman, J. S., "Stochastic Choice and Subjective Probability," (abstract), Econometrica, 26, 613, 1958.

[5] Davidson, D. and J. Marschak, "Experimental Tests of Stochastic Decision Theory" in Measurement: Definitions and Theories, C. West Churchman and Ph. Ratoosh, eds., New York, John Wiley and Sons, 1959.

[6] Davidson, D. and P. Suppes, "A Finitistic Axiomatization of Subjective Probability and Utility,: Econometrica, 24, 264-275, 1956.

[7] Davidson, D and P. Suppes and S. Siegel, Decision Making, Stanford University Press, 1957.

[8] Debreu, G., "Representation of a Preference Ordering by a Numerical Function," Decision Processes, R. M. Thrall, C. H. Coombs, R. L. Davis, eds., New York, John Wiley and Sons, 159-165, 1954.

[9] Debreu, G., "Stochastic Choice and Cardinal Utility," Econometrica, 26, 440-444, 1958.

[10] Debreu, G., "Cardinal Utility for Even-Chance Mixtures of Pairs of Sure Prospects," Review of Economic Studies, 26, 1959.

[11] Edwards, W., "Subjective Probability in Decision Theories," hectographed, Willow Run Laboratories, University of Michigan, 1959.

[12] Georgescu-Roegen, N., "Threshold in Choice and the Theory of Demand," Econometrica, 26, 157-168, 1958.

[13] Gorman, W. M., "Separable Utility and Aggregation," mimeographed, University of Birmingham, 1959.

[14] Leontief, W., "A Note on the Interrelation of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives," Bulletin of the American Mathematical Society, 53, 343-450, 1947.

[15] Leontief, W., "Introduction to a Theory of the Internal Structure of Functional Relationships," Econometrica, 15, 361-373, 1947.

[16] Luce, R. D., "A Probabilistic Theory of Utility," Econometrica, 26, 193-224, 1958.

[17] von Neumann, J. and O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1947.

[18] Pfanzagl, J., Die axiomatischen Grundlagen einer allgemeinen Theorie des Messens, Würzburg, Physica, 1959.

[19] Ramsey, F. P., The Foundations of Mathematics, New York, Harcourt, Brace, 156-198, 1931.

[20] Samuelson, P. A., Foundations of Economic Analysis, Harvard University Press, 1947.

[21] Strotz, R. H., "Myopia and Inconsistency in Dynamic Utility Maximization," Review of Economic Studies, 23, 165-180, 1956.

[22] Strotz, R. H., "The Empirical Implications of a Utility Tree," Econometrica, 25, 269-280, 1957.

[23] Suppes, P., "The Role of Subjective Probability and Utility in Decision-Making," Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Vol. V, J. Neyman, ed., University of California Press, 61-73, 1956.

[24] Thomsen, G., "Un teorema topologico sulle schiere di curve e una caratterizzazione geometrica delle superficie isotermo-asintotiche" Bollettino della Unione Matematica Italiana, 6, 80-85, 1927.