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A Few Remarks on the Assortment Problem

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## A Few Remarks on the Assortment Problem

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Abstract: The paper deals with the assortment problem in a particular case - that of structural steel beams. Using the dynamic programming technique we show that this assortment problem is equivalent to the economic lot size problem and also to the routing problem.

### 1. Introduction

In a previous paper [1] the author considered the problem of the optimal assortment of structural steel beams. The optimal assortment was one which gave the greatest economy of steel, subject to certain constraints. This paper is intended to remedy certain defects in the computational procedure suggested previously, and also to take account of some new factors such as the cost of the machines used to produce beams and the capacities of those machines.

The method applied here is Bellman's dynamic programming technique. The reader will see that the model formulated in section 2 is equivalent to an economic lot size problem and also to a routing problem.

### 2. The problem of the optimal assortment of steel beams.

Most steel-producing countries of the world face the problem of selecting an optimal assortment of structural beams, and of cutting down on the many

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possible kinds of beams which can be produced. The most important difference between beams is their strength, and this is different for each kind of beam. For a given profile and length, there is a very strict relationship between the strength of a beam and its weight. Very often this dependence is a parabolic one (fig. 1). According to this relation, all beams may be represented as points of the curve  $OP$ . Hence an assortment can be described by an  $n$ -tuple of strengths  $(s_1, s_2, \dots, s_n)$  or by an equivalent  $n$ -tuple of weights per unit length  $(w_1, w_2, \dots, w_n)$ .

It is quite obvious that one of the ways to economize on the use of iron would consist of finding beam profiles that guarantee maximum strength given the weight of a beam of unit length. Such considerations are of a technological nature, and we shall not take them into account. We shall instead assume that the functional dependence between the weight and strength of the beam is known, e.g. given by a formula

$$(1) \quad w = g(s)$$

where  $w$  is the weight of unit length beam, and  $s$  is its strength.\*

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\* We shall assume that the function (1) is increasing.

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Let  $F(s)$  be the demand per unit time (e.g. one year) for beams with strength  $s$  or less; we assume  $F(s)$  differentiable and denote by  $f(s) = F'(s)$  the "demand density." Then the demand for beams whose strength lies in the interval  $(s_{i-1}, s_i)$  is given by the formula

$$(2) \quad D_{s_{i-1}, s_i} = \int_{s_{i-1}}^{s_i} f(s) ds$$

where  $f(s)$  is given (see Fig. 2) over the entire interval  $(S_1, S_2)$ .

Any actual assortment of beams will contain only a finite number of sizes, e.g.  $(s_1, s_2, \dots, s_n)$ . In all such cases, if the beam demanded has a strength  $s_x$  (with  $s_{i-1} < s_x \leq s_i$ ;  $i = 2, 3, \dots, n$ ), one must use the beam with greater strength, i.e., with strength  $s_i$ . Since the function  $g(s)$  is increasing, this means that we are using a heavier beam. In this way, there is a certain waste of steel that comes about. Our object now will be to find an assortment that guarantees the smallest possible waste.

As we remarked, the function  $f(s)$  represents the demand density for beams of given strength  $s$ . From (1), we may then construct the demand density for beams of given weight,  $w$ .

Let us denote the inverse function of (1) by

$$(3) \quad s = h(w) .$$

The demand density for beams of given weight  $w$ , will then be

$$(4) \quad l(w) = f[h(w)] \frac{dh}{dw}$$

Fig. 3 gives an example of the function  $l(w)$ , where  $W_1 = g(S_1)$ ,  $W_2 = g(S_2)$ . For simplicity let us denote by  $A_n = (w_1, w_2, \dots, w_n)$  an assortment of  $n$  different types (weights) of beams. Assuming  $n$  fixed, our task is to find an optimal assortment  $A_n^*$  -- one which will minimize the steel loss, subject to the stipulation that this assortment will satisfy the prescribed demand.

It is clear that the beams with weight  $W_2 = g(S_2)$  will have to be produced in any event. Hence  $w_n = W_2$ . We must then find the  $n - 1$  remaining types of beams.

The total loss of steel, given the assortment  $A_n$ , will be as follows

$$(5) \quad L_1 = \sum_{i=1}^n \int_{w_{i-1}}^{w_i} (w_i - w) \ell(w) dw, \quad (i = 1, 2, \dots, n)$$

assuming  $w_0 = W_1 = g(S_1)$  and  $w_n = W_2 = g(S_2)$ , where  $W_1$  and  $W_2$  are given. The formula (5) may be represented by

$$(6) \quad L_1 = \sum_{i=1}^n w_i \int_{w_{i-1}}^{w_i} \ell(w) dw - \sum_{i=1}^n \int_{w_{i-1}}^{w_i} w \ell(w) dw .$$

The second term on the right hand in (6) is a constant. To minimize (6), it is sufficient to minimize the first term. Let us denote this term by  $L_2$

$$(7) \quad L_2 = \sum_{i=1}^n w_i \int_{w_{i-1}}^{w_i} \ell(w) dw .$$

Using the differential calculus it is possible (at least in principle) to find  $(w_1^*, w_2^*, \dots, w_{n-1}^*)$  such that

$$(8) \quad \text{Min } L_2(w_1, w_2, \dots, w_{n-1}) = L_2(w_1^*, w_2^*, \dots, w_{n-1}^*)$$

Taking into account the fact that  $w_n^* = w_n = W_2$ , we have an optimal  $n$ -beam assortment  $A_n^* = (w_1^*, w_2^*, \dots, w_n^*)$ . Using (3) one obtains the optimal assortment of beams with various strengths

$$A_n^* = (s_1^*, s_2^*, \dots, s_n^*) \quad \text{where } s_i^* = h(w_i^*) .$$

### 3. Computational difficulties.

The effective determination of an optimal assortment  $A_n^*$  even for values of  $n$  of the order of magnitude of 10, is not an easy matter.

This requires the minimization of  $L_2$ , given by (7). Denoting by

$$(8) \quad \Phi(w) = \int_{w_1}^w l(w)dw$$

$L_2$  may be written as

$$(9) \quad L_2 = \sum_{i=1}^n w_i [\Phi(w_i) - \Phi(w_{i-1})] .$$

Differentiating  $L_2$  with regard  $w_1, w_2, \dots, w_{n-1}$  we obtain:

$$\frac{\partial L_2}{\partial w_1} = \Phi(w_1) - \Phi(w_0) - (w_2 - w_1) \Phi'(w_1)$$

$$(10) \quad \frac{\partial L_2}{\partial w_2} = \Phi(w_2) - \Phi(w_1) - (w_3 - w_2) \Phi'(w_2)$$

$$\frac{\partial L_2}{\partial w_{n-1}} = \Phi(w_{n-1}) - \Phi(w_{n-2}) - (w_n - w_{n-1}) \Phi'(w_{n-1})$$

Equating these derivatives to zero, and recalling that

$$\Phi'(w_i) = l(w_i) \quad \text{and} \quad w_n = w_2$$

we obtain,

$$(w_2 - w_1) l(w_1) = \Phi(w_1) - \Phi(w_0)$$

$$(w_3 - w_2) l(w_2) = \Phi(w_2) - \Phi(w_1)$$

$$(11) \quad \begin{array}{cccccccccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

$$(w_2 - w_{n-1}) l(w_{n-1}) = \Phi(w_{n-1}) - \Phi(w_{n-2})$$

or

$$w_2 = w_1 + \frac{\Phi(w_1) - \Phi(w_0)}{\ell(w_1)}$$

$$w_3 = w_2 + \frac{\Phi(w_2) - \Phi(w_1)}{\ell(w_2)}$$

(12)    . . . . .  
         . . . . .

$$w_n = w_{n-1} + \frac{\Phi(w_{n-1}) - \Phi(w_{n-2})}{\ell(w_{n-1})}$$

To solve the system of  $n-1$  equations (12), one could apply an iterative procedure, starting off with an arbitrary value of  $w_1$ . From the first equation one finds  $w_2$  which may then be used in the second equation to obtain  $w_3$ . This value of  $w_3$  may be used in the next equation, etc. On the right side of the last equation we will end up either with a value greater than or smaller than  $w_2$ . In the first case we have to repeat our procedure starting with a smaller value of  $w_1$ ; in the second case we should start with the greater value of  $w_1$ , etc.\*

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\* Such a procedure is suggested by F. Hanssmann [2] who found exactly the same system of equations as our system (12).

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However, it is not at all obvious that the above procedure will be successful. Convergence is by no means guaranteed. Furthermore, it is quite hazardous to use this procedure. There is always the possibility of hitting upon a local optimum, rather than a global one.

There is also another inconvenience connected with this procedure. Thus far, we have taken account of only one kind of cost, namely that of steel loss, and have held constant the number of different types of beams. However, it is clear that the greater the number of kinds of beams, the greater become the industry's costs of production. (For each size chosen, it will be necessary to install and to operate an individual piece of equipment.) Let us assume then that the cost of a new machine for beams of weight  $w_i$  is  $c(w_i)$  where  $c(w)$  is an increasing function of  $w$ . (For simplicity these costs are expressed in units of equivalent steel value.) Taking into account this additional category of costs, the cost associated with a given assortment involves not only the steel loss, but also the cost of the rolling equipment. In other words instead of  $L_1$  given by (6), we now have  $L_3$  :

$$(13) \quad L_3 = L_1 + \sum_{i=1}^n c(w_i) = \sum_{i=1}^n w_i \int_{w_{i-1}}^{w_i} l(w)dw - \sum_{i=1}^n \int_{w_{i-1}}^{w_i} w l(w)dw + \sum_{i=1}^n c(w_i)$$

To minimize (13) it is enough to minimize  $L_4$ , namely

$$L_4 = L_2 + \sum_{i=1}^n c(w_i) = \sum_{i=1}^n w_i \int_{w_{i-1}}^{w_i} l(w)dw + \sum_{i=1}^n c(w_i)$$

Differentiating  $L_4$  with regard to  $w_1, w_2, \dots, w_{n-1}$  and equating to zero, we obtain a system of  $n-1$  equations analogous to (12):

$$(14) \quad \begin{aligned} w_2 &= w_1 + \frac{\Phi(w_1) - \Phi(w_0) - c'(w_1)}{l(w_1)} \\ w_3 &= w_2 + \frac{\Phi(w_2) - \Phi(w_1) - c'(w_2)}{l(w_2)} \\ &\dots \end{aligned}$$



. . . . .

$$W_2 = w_{n-1} + \frac{\Phi(w_{n-1}) - \Phi(w_{n-2}) - c'(w_{n-1})}{l(w_{n-1})}$$

From a theoretical point of view, it is possible to find the solution of this system of equations (as well as for (12)) for every value of  $n$ . In other words, it is possible to find an optimal assortment for given  $n$ . To find the proper  $n$  it would be necessary to calculate the optimal value of  $L_3$  for many values of  $n$ , and to choose that one which guarantees the minimum costs  $L_3$ . However, as we remarked previously, the solution of (14) is not a simple matter. Further, it is not so clear that the chosen  $n$  is the best one ( $L_3$  may have several local minima as a function of  $n$ ).

All of these difficulties may be avoided by the use of Bellman's dynamic programming approach.\*

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\* See R. Bellman [3].

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#### 4. Dynamic programming approach.

If we stipulate that the production must satisfy the entire demand, and if we have the demand density function  $l(w)$ , it is quite obvious that we must have at least one machine producing the beams with weight  $W_2 = b$  (see fig. 4). An additional machine installed at point  $y$ ,  $a < y < b$  may then lead to a decrease in <sup>total</sup> costs. The reason for this possibility is that the reduction in steel waste may be greater than the cost of the new machine

producing beams with a weight of  $y$  . This additional economy may be expressed by the formula

$$(15) \quad \int_a^y bl(w)dw - \int_a^y yl(w)dw - c(y) = S(a,y)$$

The interpretation of this formula is simple enough. Without any machine at point  $y$  (see fig. 4), in the whole interval  $(a,y)$  we would use the beams of weight  $b$  . After the installation of equipment capable of producing at point  $y$  , we will use beams of weight  $y$  in this same interval. However, there still remain the additional costs of installing and operating the rolling equipment,  $c(y)$ . In this way,  $S(a,y)$  represents the additional saving that arises from the installation of a new machine at point  $y$  . This economy is a function of  $y$  alone ( $a$  is a parameter), and so we must find a value of  $y$  for which the saving  $S(a,y)$  is maximum.

$$(16) \quad \text{Max}_{a < y < b} \left\{ \int_a^y bl(w)dw - \int_a^y yl(w)dw - c(y) \right\} = f_1(a) .$$

The maximum value of  $S(a,y)$  is some constant,  $S^*$  . However, it will be more convenient to regard this constant as a function  $f_1$  of the lower end-point parameter  $a$  . Now we may look for the best point  $y_1$  at which to install the next machine. The reasoning is exactly the same as previously. Let us assume for a moment that  $y$  is fixed, and that we are looking for the best point  $y_1$  for installing a new machine in the interval  $(y,b)$  . The additional economy\* connected with installing a new machine at point  $y_1$

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\* Of course this "economy" may be negative as well as positive.

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will be

$$(17) \quad S(y, y_1) = \int_y^{y_1} b\ell(w)dw - \int_y^{y_1} y_1\ell(w)dw - c(y_1)$$

Obviously we shall choose the point  $y_1$  in such a way as to maximize this additional economy  $S(y, y_1)$ . Due to (16), we obtain

$$(18) \quad \text{Max}_{y < y_1 < b} S(y, y_1) = f_1(y)$$

The entire economy, taking account of both machines,  $y$  and  $y_1$ , is

$$(19) \quad \text{Max}_{a < y < b} \{S(a, y) + f_1(y)\} = f_2(a)$$

Assuming that the next machine will be installed at point  $y_2$  we may use our previous results (19). Fixing for a moment the point  $y$  we have the interval  $(y, b)$  in which we are to locate two machines  $y_1$  and  $y_2$ . From (19), we know that the saving in this interval will be

$$(20) \quad \text{Max}_{y < y_1 < b} \{S(y, y_1) + f_1(y_1)\} = f_2(y)$$

and the whole saving, including machine  $y$ , will be

$$(21) \quad \text{Max}_{a < y < b} \{S(a, y) + f_2(y)\} = f_3(a)$$

In general, applying the principle of recursive optimality\* we have

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\* See R. Bellman [3], p. 83.

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$$(22) \quad \text{Max}_{a < y < b} \{S(a, y) + f_{i-1}(y)\} = f_i(a) \quad i = 2, 3, \dots$$

(16) and (22) define the sequence  $\{f_i(a)\}$  for  $i = 1, 2, \dots$

This procedure does not contain any such pitfalls as the one proposed in section 3. For any arbitrary  $i = 1, 2, 3, \dots$  in a relatively simple way we may find the optimal assortment for  $n = 2, 3, \dots$  kinds of beams.

Computing step by step  $f_1(a), f_2(a), \dots$  we obtain not only the value of the maximal saving connected with installing  $i = 1, 2, \dots$  new machines (the machine at point  $b$  is installed in any case) but also  $Y_1(a), Y_2(a), \dots$  namely the best point for the smallest machine\* with the total number of

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\* This means the machine producing beams with the smallest unit strength and weight.

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machines  $n = i+1$ . Having the fixed  $n = N + 1$  we find the best assortment  $y^*, y_1^*, y_2^*, \dots, y_N^*$

$$\begin{aligned} Y_N(a) &= y^* \\ Y_{N-1}(y^*) &= y_1^* \\ Y_{N-2}(y_1^*) &= y_2^* \\ &\vdots \\ Y_1(y_{N-2}^*) &= y_{N-1}^* \quad \text{and} \quad y_N^* = b \end{aligned}$$

Further, it is worth pointing out that our procedure has some convenient properties. The best number of machines may be found very easily without any additional calculations.

Step by step, we calculate the values  $f_1(a), f_2(a), \dots$  and at the same time the functions  $Y_1(a), Y_2(a), \dots$ . We go ahead with this procedure as long as  $f_i(a) \geq f_{i-1}(a)$ , but we stop as soon as  $f_i(a) < f_{i+1}(a)$ . This stop-rule ensures that by going further we will not realize any additional economies.

5. Economic lot size interpretation of the assortment problem.

Our solution of the assortment problem obtained in section 4 is equivalent to a solution of the well-known economic lot size problem.\*

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\* The literature on the economic lot size problem is vast. For our present purposes, we may refer the reader to the Bellman book [3], chap. V, pp. 152-182.

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Let us assume that the point  $b$  (see fig. 5) is the present point of time and that we are dealing with the interval of time  $(b, a)$ . In this period of time the time path of demand for an item is given by the function  $l(t)$ . Assuming that we want to meet the whole demand and that we have no stock at time  $b$  we are looking for the best points of time at which to initiate orders for the item. The machine installation sizes correspond exactly to the ordering points for the following reason: In any case, we must initiate an order at point  $b$  (the cost of ordering is a function of time  $c(t)$ ). Then we look for the next best ordering point  $y$ . The steel economy in the present case must be interpreted as a saving in inventory costs. The costs of installation and operation of a new machine must be interpreted as the cost of ordering.

The method of solution for this economic lot size problem would be the same as in section 4. The only difference is that in the inventory problem it might be necessary to introduce a discount factor to deal with the present value of all future costs.

#### 6. Capacity of machines.

To be more realistic it would be necessary to take account of an additional complication -- the fact that every machine has a limited capacity.

Up to this point, we have tacitly assumed, that there was no effective limit upon the amount of beams that a machine might produce per unit of time. This assumption is by no means a realistic one, and might be changed in at least two ways.

First we might assume that we could install machines with fixed initial capacities. This means that the cost of installation of a machine at a certain point depends not only on the unit length and weight of beams produced by those machines, but also upon the quantity of beams produced. It may happen that at some point it will not be enough to install just a single machine, and in this case the cost of the proper number of machines will be greater.

Secondly we may assume that each machine may have a different capacity, and the greater the capacity the greater the cost of such a machine.

From the point of view of further analysis there is no real difference between those two cases. The only trouble is that taking into account the problem of capacity, it is no longer quite so easy to calculate the numerical solution to our problem.

However, dealing with the case of a discrete demand distribution (instead of a continuous density function), the solution may still be obtained using a dynamic programming approach. We shall now show how dynamic programming may be applied to this more difficult problem.

Let us assume that the discrete demand function is given by  $l_i (i = 1, 2, \dots)$  (see fig. 6).

Knowing the capacity of the machines which can be installed and their costs, it is possible -- taking account of the steel loss also -- to calculate the costs,  $t_{ij} (j = 1, \dots, n)$ , over the interval  $(i, j)$ . For example,  $t_{1j}$  represents the cost of installation of the proper number of machines at point 1 (i.e.  $i = 1$ ), plus the costs of the additional steel needed with this size of machine in order to satisfy the demand  $l_1 + l_2 + \dots + l_j$ . Similarly, for any other  $i = 2, 3, \dots, n$ , it is possible to calculate the cost  $t_{ij} (i \leq j)$  -- namely the cost of installation of the machine itself at point  $i$  plus the cost of the additional steel used in order to satisfy the demand  $l_i + l_{i+1} + \dots + l_j$ . Those costs may be presented in the form of table 1. The entries above the diagonal are empty, because we assume that it is impossible to use a lighter beam in order to satisfy the demand for a heavier one. Our table also contains one artificial point "n + 1" (This may be assigned at any arbitrary point -- just as long as it corresponds to a smaller unit weight than point  $\underline{n}$ .) By convention, the demand  $l_{n+1} = 0$ . This artificial point is employed solely to make our computational layout more symmetrical. Of course, we have  $t_{n+1, n+1} = 0$ .

Table 1

Points of demand	Points of Production					
	1	2	3	.	n	n+1
1	$t_{11}$			.		
2	$t_{12}$	$t_{22}$		.		
3	$t_{13}$	$t_{23}$	$t_{33}$	.		
.	.	.	.	.	.	.
n	$t_{1n}$	$t_{2n}$	$t_{3n}$	.	$t_{nn}$	
n+1	$t_{1,n+1}$	$t_{2,n+1}$	$t_{3,n+1}$	$\dots$	$t_{n,n+1}$	$t_{n+1,n+1}$

The problem of the optimal assortment (the best points at which to install machines) is, in this special case, very similar to the so called "routing problem."\* The routing problem may be described as follows:

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\* see [4]

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We have  $n$  cities, and each pair of them is linked by a road. The cost of travelling (or the time of travel) from town  $i$  to  $j$  is given by  $t_{ij}$ . The problem consists of finding the best path between two given cities e.g. between 1 and  $n$ . The best path is one which provides the minimum cost (or time) of travelling. The routing problem is a little more general than our assortment problem because there is no assumption that  $i \leq j$ . Furthermore in the routing problem  $t_{ii} = 0$  whereas in our assortment problem this is true only in the case of  $i = n+1$ . In the assortment problem,  $t_{ij}$  represents the cost of the equipment and the steel needed in order to satisfy  $l_i$  by beams of strength  $i$ . The fact that we are starting from city 1 may be interpreted as follows: At point 1, we necessarily install some machines in order to satisfy the demand  $l_1$ . On the other hand we must arrive at point  $n+1$  because the entire demand must be satisfied. For example, if we find that the best path from 1 to  $n$  is (1,3,  $n-2$ ) it means that we shall produce beams at points 1 in order to satisfy  $l_1$  and  $l_2$ , at point 3 to satisfy  $l_3, l_4, \dots, l_{n-3}$  and at point  $n-2$  to satisfy  $l_{n-1}, l_n$ , and  $l_{n+1}$ . (And formally,  $l_{n+1}$ .)



The solution of our problem could be found directly by enumeration of all possible paths (all possible assortments) because there are only a finite number of possible paths. However, by using a dynamic programming approach proposed by Bellman, the computational work may be reduced quite considerable.

Following Bellman's reasoning and using his notation let us denote by

$f_i$  = the cost of satisfying the demand from point  $i$  to  $n+1$  ( $i = 1, 2, \dots, n$ ) using an optimal policy

with  $f_{n+1} = t_{n+1, n+1} = 0$ .

Using the principle of optimality we have

$$f_i = \text{Min}_{j \geq i} [t_{ij} + f_{j+1}] \quad i = 1, 2, \dots, n$$

(25)

$$f_{n+1} = 0$$

The definition of  $f_i$  is slightly different than that given by Bellman in his routing problem. Instead of  $f_j$  we have in formula (25),  $f_{j+1}$ . This involves the necessity of the artificial assumption

$$(26) \quad f_{n+1} = 0$$

With this one minor difference, the procedure is identical with Bellman's. Only  $(n-1)$  iterations are required in order to produce the optimal solution.

7. Final remarks.

This paper is intended to provide no more than an introduction to the assortment problem. There are some further questions -- most of them much more difficult than those treated in this paper. I should like to point out two questions which deserve further investigation. The most important of these is the time structure of the optimal assortment policy. In that case, we must deal with  $n$  future time periods of time,  $n$  demand distributions, and we must decide upon an optimal assortment over the entire time span.

In such a model, it would be necessary to take into account not only the problem of capacity (more complicated than in the static case) but also the fact that the life of a machine is a random variable.

It would also be interesting to generalize the assortment problem to the multidimensional case, i.e., to deal with the case in which the individual items were characterized by more than one physical property.

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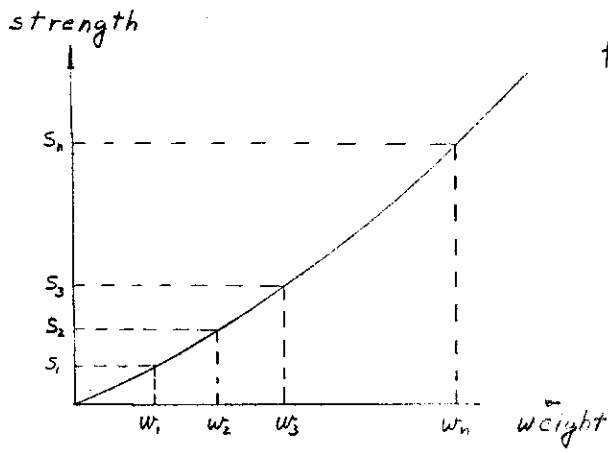


Figure 1

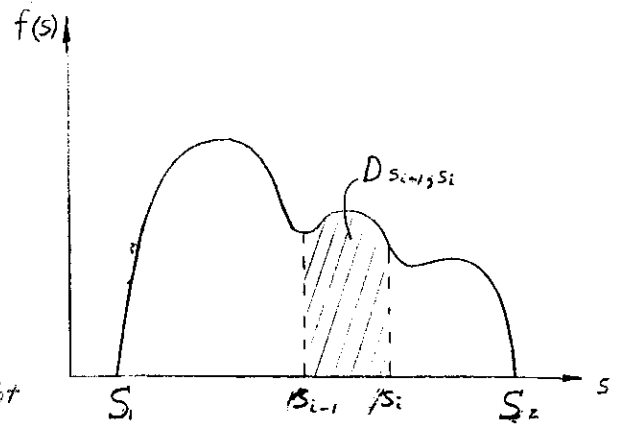


Figure 2

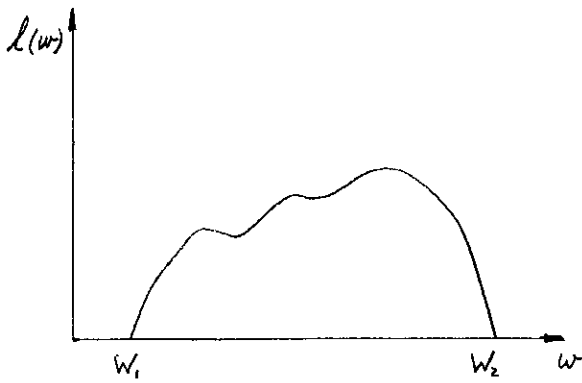


Figure 3

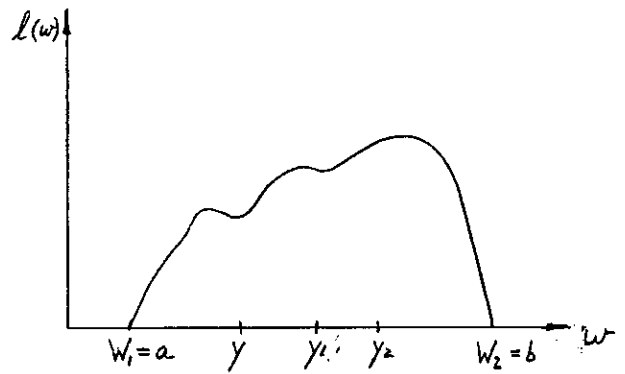


Figure 4

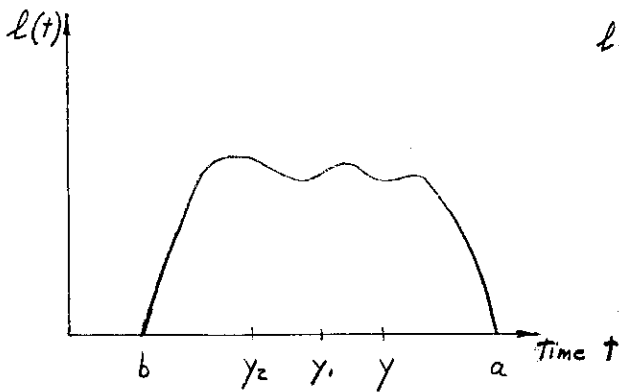


Figure 5

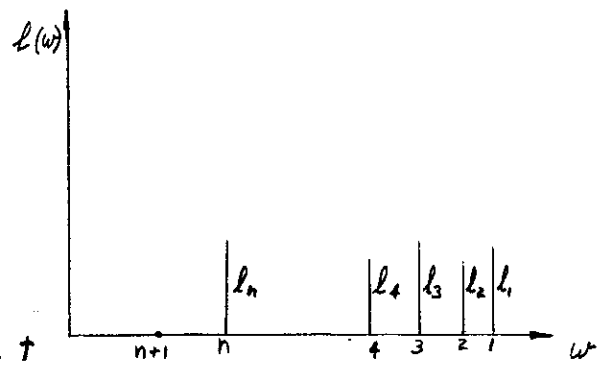


Figure 6