COWLES FOUNDATION DISCUSSION PAPER NO. 57

Cardinal Utility for Even-chance Mixtures of Pairs of Sure Prospects*

Gerard Debreu

September 17, 1958

* Research undertaken by the Cowles Commission for Research in Economics under Contract Nonr-358(01) with the Office of Naval Research.

Cardinal Utility for Even-chance Mixtures of Pairs of Sure Prospects.*

Gerard Debreu

This short note had its origin in conversations that the author was privileged to have with Jacob Marschak. It is respectfully dedicated to him on the occasion of his sixtieth birthday.

In the utility theory of J. von Neumann and O. Morgenstern [8] a subject is faced with a set of prospects on which a mixture operation is defined, i.e., two prospects x,y and a probability p define a prospect denoted $(px,(1-p)y)$ and interpreted as the prospect of having prospect x with probability p and prospect y with probability $1-p$. Given a complete preference preordering** on the set of prospects, axioms (on the mixture operation and on the preference preordering) which insure the existence on that set of a real-valued, order-preserving function v satisfying the relation

$$v(px,(1-p)y) = pv(x) + (1-p)v(y)$$

have been offered by J. von Neumann and O. Morgenstern [8], J. Marschak [7], I. N. Herstein and J. Milnor [6] and others.

Some of the difficulties encountered in the testing of that theory originate in the inability of subjects to grasp the significance of complex prospects. This has led D. Davidson and J. Marschak [3] to study a situation

** A preordering is a reflexive and transitive binary relation.

* A technical report of research undertaken by the Cowles Foundation for Research in Economics under contract with the Office of Naval Research.

where the subject is faced only with the simplest type of uncertain prospect. Let S be a set of sure prospects (e.g. commodity bundles) which will be assumed, for the sake of conceptual simplicity, to be a subset of R^n. Given two elements a and b of S, the symbol ab denotes the prospect of having a with probability 1/2 and b with probability 1/2. The set SxS of prospects is completely preordered by the relation $\precsim$ which is read "is not preferred to." As usual, $\sim$ is read "is indifferent to," and $\succ$ is read "is preferred to." In this context,

A utility function is a real-valued, order-preserving function u on SxS such that

$$u(ab) = 1/2[u(aa) + u(bb)] \text{ for every } a \text{ and } b \text{ in } S.$$

The object of this note is to give sufficient conditions for the existence of such a function.

The first axiom requires that the set S be of one piece:

(1) S is a connected subset of R^n.

The second imposes continuity properties on the preference preordering:

(2) $\precsim$ is a complete preordering of SxS such that $\{ab \in SxS \mid ab \succsim a'b'\}$ and $\{ab \in SxS \mid ab \precsim a'b'\}$ are closed in SxS for every $a'b'$ in SxS.

The last axiom is:

(3) $[a_1b_2 \precsim a_2b_1 \text{ and } a_2b_3 \precsim a_3b_2] \Rightarrow [b_3a_1 \precsim b_1a_3]$.

It is clearly a necessary condition for the existence of a utility function.

Theorem. Under assumptions (1), (2), (3), there is a continuous utility function determined up to an increasing linear transformation.

First a few consequences of (3) will be established. Making a_1 , b_1 equal to a and a_2, b_2, a_3, b_3 equal to b in (3), one obtains [ab $\precsim$ ba and aa $\precsim$ aa]$\Rightarrow$[ba $\precsim$ ab]. Therefore[ab $\precsim$ ba]$\Rightarrow$[ba $\precsim$ ab]. Hence

(4) <u>ab ~ ba for every a and b in S</u>.

From this and (3) one can easily show that

(5) $\underline{[a_1b_2 \sim a_2b_1 \text{ and } a_2b_3 \sim a_3b_2] \Rightarrow [a_3b_1 \sim a_1b_3]}$.

Making now a_1, b_2 equal to a ; a_2,b_1 equal to b ; a_3,b_3 equal to c in (3), one obtains [aa $\precsim$ bb and bc $\precsim$ ca]$\Rightarrow$[ca $\precsim$ bc]. Hence [aa $\precsim$ bb] $\Rightarrow$ [ca $\precsim$ bc]. Making then a_1,b_3 equal to a ; a_3,b_1 equal to b ; a_2,b_2 equal to c in (3), one obtains [ac $\precsim$ cb and ca $\precsim$ bc]$\Rightarrow$[aa $\precsim$ bb]. Hence, by (4), [ca $\precsim$ bc]$\Rightarrow$[aa $\precsim$ bb]. Summing up, and using (4) again,

(6) <u>[aa $\precsim$ bb]$\Rightarrow$[ca $\precsim$ cb]</u>.

Finally, from (6) and (4), [aa ~ a'a']$\Rightarrow$[ab ~ a'b] and [bb ~ b'b']$\Rightarrow$ [a'b ~ a'b']. Hence

(7) <u>[aa ~ a'a' and bb ~ b'b']$\Rightarrow$[ab ~ a'b'].</u>

The theorem will be proved by means of a representation of SxS in R^2. According to [4], there is, by (2), a continuous real-valued, order-preserving function f on SxS. Let ab be a generic element of SxS. Using the notation $\alpha = f(aa)$ and $\beta = f(bb)$, one defines the representation by $ab \rightarrow (\alpha,\beta)$. Since S is connected, the range of α is a real interval Σ . The real number f(ab) depends only on α and β on account of (7). Let φ be the function defined on $\Sigma x \Sigma$ in that way:

$$f(ab) = \varphi(\alpha,\beta).$$

(4) implies that $\varphi(\alpha,\beta) = \varphi(\beta,\alpha)$. And (6) implies that φ is increasing in each one of its two variables. It follows, without great difficulty,

that φ is continuous. One may also notice that $\varphi(\alpha,\alpha) = \alpha$. Two curves corresponding to two different values of φ have been drawn on fig. 1.

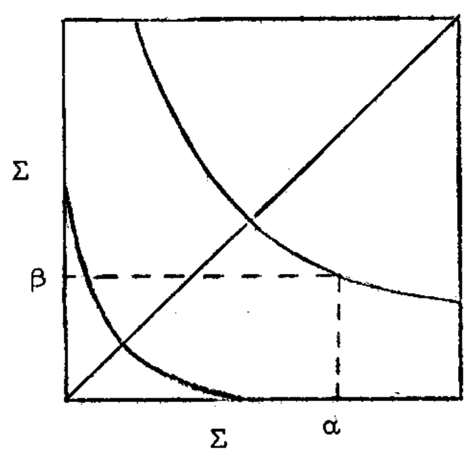

fig. 1

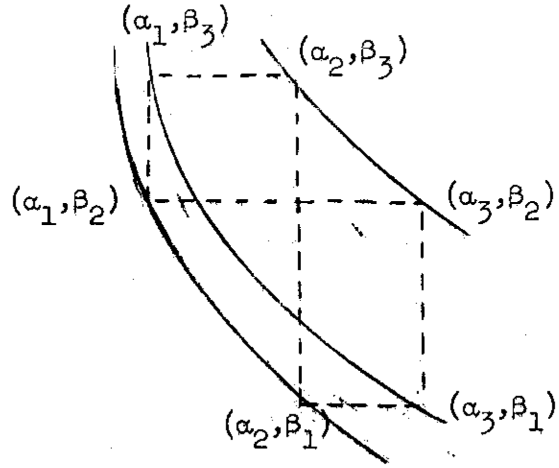

fig. 2

Finally (5) implies that

$$[\varphi(\alpha_1, \beta_2) = \varphi(\alpha_2, \beta_1) \text{ and } \varphi(\alpha_2,\beta_3) = \varphi(\alpha_3,\beta_2)] \Longrightarrow [\varphi(\alpha_3,\beta_1) = \varphi(\alpha_1,\beta_3)].$$

Fig. 2 illustrates this relation.

The existence of a utility function u on SxS is clearly equivalent to the existence of a real-valued function ν on $\Sigma x \Sigma$, derived from φ by an increasing transformation Θ, and such that

$$\nu(\alpha,\beta) = 1/2[\nu(\alpha,\alpha) + \nu(\beta,\beta)].$$

This, in turn, is equivalent, because of the relation $\varphi(\alpha,\alpha) = \alpha$, to the existence of an increasing transformation Θ on both coordinates on fig. 1 which carries the level curves of φ into straight line segments

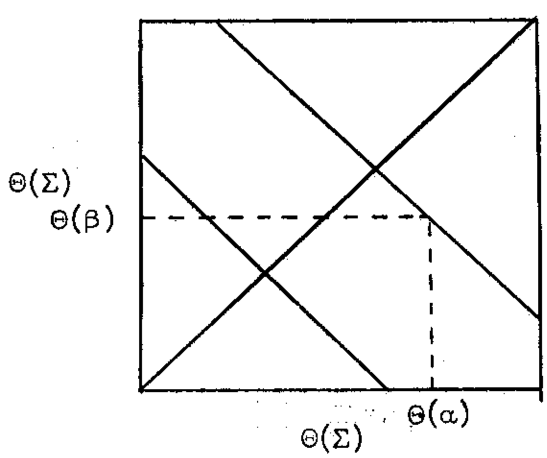

fig. 3

perpendicular to the diagonal as on fig. 3. The close analogy of the latter problem with the one treated in [5], both being particular cases of a problem of plane topology* studied by G. Thomsen [9] and W. Blaschke [1]

---

* The problem can be briefly described as follows. Given three families of curves in a plane, when does there exist a topological transformation carrying them into three families of parallel straight lines? On fig. 1 the three families are the level curves of φ , the verticals, the horizontals. After the transformation, on fig. 3, they are the perpendiculars to the diagonal, the verticals, the horizontals.

(See also W. Blaschke and G. Bol [2]) makes it unnecessary to go into the details of the proof. The hexagonal configuration of fig. 2 allows one to show that there exist a <u>continuous</u> transformation Θ of the wanted type. The continuous utility function u is defined by

$$u(ab) = \Theta[\varphi(\alpha,\beta)].$$

It is obviously determined up to an increasing linear transformation.

References

[1] Blaschke, W., "Topologische Fragen der Differentialgeometrie. I," Mathematische Zeitschrift, 28, 150-157, 1928.

[2] Blaschke, W., und Bol, G., Geometrie der Gewebe, Berlin, Springer, 1938.

[3] Davidson, D. and Marschak, J., "Experimental Tests of Stochastic Decision Theory," in Measurement: Definitions and Theories, C. West Churchman, ed., New-York, John Wiley and Sons, 1959.

[4] Debreu, G., "Representation of a Preference Ordering by a Numerical Function," Decision Processes, R.M. Thrall, C.H. Coombs, R.L. Davis, eds, New York, John Wiley and Sons, 159-165, 1954.

[5] Debreu, G., "Stochastic Choice and Cardinal Utility," Econometrica, 26, 1958.

[6] Herstein, I. N., and Milnor, J., "An Axiomatic Approach to Measurable Utility," Econometrica, 21, 291-297, 1953.

[7] Marschak, J., "Rational Behavior, Uncertain Prospects, and Measurable Utility," Econometrica, 18, 111-141, 1950.

[8] von Neumann, J., and Morgenstern, O., Theory of Games and Economic Behavior, Princeton University Press, 1947.

[9] Thomsen, G., "Un teorema topologico sulle schiere di curve e una caratterizzazione geometrica delle superficie isotermo-asintotiche," Bollettino della Unione Matematica Italiana, 6, 80-85, 1927.