COWLES FOUNDATION DISCUSSION PAPER NO. 37

A Target Assignment Problem*

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July 9, 1957

* Research undertaken by the Cowles Foundation for Research in Economics under Contract Nonr-358(01) with the Office of Naval Research.

1. Introduction

At a recent conference, Merrill Flood described a target assignment model which he considered to be of some military relevance. He pointed out that although the mathematical form involved optimization subject to constraints similar to those of the "personnel assignment" problem [5], the minimand was of distinctly non-linear form, and from this fact he felt tempted to conclude that linear programming would be of no avail to him.

This note is to suggest that by a fairly minor modification of the original problem, and then a transformation of variables, it is possible to recast the model into linear programming form - and indeed, into a special case of linear programming under uncertainty. In this form, specific numerical solutions should require little more than clerical talent.

2. The assignment problem

It is assumed that there are <u>m</u> guns, labelled <u>i</u> = 1,2, . . ., m, and that these are to be assigned against <u>n</u> targets, indicated by the subscript <u>j</u> = 1,2, . . . n. The objective is stated as one of minimizing the value of the expected number of surviving targets. More formally, if we let x_{ij} represent the probability with which the i <u>th</u> gun will be assigned to the j <u>th</u> target, p_{ij} the probability that the i <u>th</u> gun will destroy the j <u>th</u> target, and a_j be the unit worth of the j <u>th</u> target, Flood's problem becomes one of selecting values for the x_{ij} so as to minimize: *

$$(1) \qquad \sum_{j=1}^{n} a_j \prod_{i=1}^{m} (1 - p_{ij}x_{ij})$$

subject to:

(2) $$\sum_{j=1}^{n} x_{ij} = 1 \qquad (\underline{i} = 1, \ldots m)$$

and (3) $$x_{ij} \geq 0$$

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* Note that if the i th gun is assigned to the j th target with a probability of x_{ij}, the probability of surviving this gun is:

$$(1 - x_{ij}) + x_{ij}\,(1 - p_{ij}) = (1 - p_{ij}x_{ij})$$

Minimand (1) is clearly of a non-linear nature, and I know of no way to obtain an exact solution to the stated problem. What is proposed is to make the following not-so-heroic approximations: (I) If a gun has a non-zero probability of destroying the j th target, the individual kill probability is p_j - a value which is identical with that for all other guns which can be brought to bear upon the particular target. And (II), the approximate survival probabilities are to coincide with the original ones for integral assignments - i.e., for x_{ij} values of zero and one. If these two shortcuts are legitimate, expression (1) becomes equivalent to the following:

Minimize (1.A) $$\sum_{j=1}^{n} a_j\,(1 - p_j)^{y_j}$$

where y_j represents the total number of guns assigned to the j th target.

In its new form, the minimand still appears distinctly non-linear, but at this point the idea can be invoked that Dantzig, Charnes and Lemke have suggested for dealing with "separable convex functions". [1 and 3] The individual terms $(1 - p_j)^{y_j}$ are each convex decreasing functions of y_j, and there is no difficulty in providing a linear programming analogue to such functions.* All that needs to be done is to replace y_j

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* Convexity results from the fact that $1 > p_j > 0$.

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with a sum of individual terms y_{kj}, to impose upper bounds of unity upon these individual terms, and to label them in decreasing sequence of their "marginal productivity".*

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* Since p_j represents the probability of destroying the j th target with a single gun, $1 > p_j > 0$. Now if k guns are assigned against this target, the probability of survival is $(1 - p_j)^k$. Hence the "marginal productivity" of the k th gun (or change in total survival probability attributable to the k th gun) is:

$$[(1 - p_j)^k - (1 - p_j)^{k-1}] = -p_j (1 - p_j)^{k-1} < 0.$$

Clearly the k th term in such a series is smaller absolutely than the k-1 st term. Hence in a minimizing solution, $y_{kj} \leq y_{k-1, j}$. The process of minimization ensures that the components y_{kj} will be assigned positive values in ascending order of their index k, so that the function (1 B) will provide a close approximation to (1.A). Indeed, in any optimal linear programming solution, the value of the two minimands will be identical.

Why is this so? Because the model is now of the same form as the "transportation" problem, and in an optimal solution each of the x_{ij} and y_{kj} variables will take on the value of either zero or unity. [2] Not only does the integral nature of such solutions mean that (1.A) and (1.B) will coincide at all relevant points. This fact also eliminates any embarrassing questions about the physical interpretation of a solution that requires a gun to be assigned against each of two targets with a 50-50 probability. Fractional assignment values are automatically excluded.

The model now looks as follows:

Minimize (1.B) $$\sum_{j=1}^{n} a_j \left[1 - \sum_{k=1}^{m} p_j (1-p_j)^{k-1} y_{kj}\right]$$

subject to:

(4) $$\sum_{i=1}^{m} x_{ij} - \sum_{k=1}^{m} y_{kj} = 0 \qquad (\underline{j}=1, \ldots, n)$$

(5) $$\sum_{j=1}^{n} x_{ij} = 1 \qquad (\underline{i}=1, \ldots, m)$$

(6) $$x_{ij} \geq 0$$

and (7) $$1 \geq y_{kj} \geq 0$$

3. <u>A numerical illustration</u>

To demonstrate the equivalence between this and the "transportation" problem,* it seems easiest to make use of a numerical example. Hereafter

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* It is true that this is a "transportation" problem that also involves upper bounds upon the individual y_{kj} variables. But see Dantzig [3] for a proof that such upper bounds do not alter the essential character of the problem.

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we shall confine the discussion to the case of m = n = 4 (four guns and four targets). Further, to illustrate the case of zero kill probabilities, we shall assume that gun 1 is incapable of destroying target 3, and that gun 2 cannot hit target 1. The array of variables now appears as follows:

target <u>j</u>

	1	2	3	4	Availabilities
gun 1	x_{11}	x_{12}	X	x_{14}	= 1
gun 2	X	x_{22}	x_{23}	x_{24}	= 1
gun 3	x_{31}	x_{32}	x_{33}	x_{34}	= 1
gun 4	x_{41}	x_{42}	x_{43}	x_{44}	= 1
guns of effectiveness 1	$-y_{11} \geq -1$	$-y_{12} \geq -1$	$-y_{13} \geq -1$	$-y_{14} \geq -1$	
guns of effectiveness 2	$-y_{21} \geq -1$	$-y_{22} \geq -1$	$-y_{23} \geq -1$	$-y_{24} \geq -1$	
guns of effectiveness 3	$-y_{31} \geq -1$	$-y_{32} \geq -1$	$-y_{33} \geq -1$	$-y_{34} \geq -1$	
guns of effectiveness 4	X	$-y_{42} \geq -1$	X	$-y_{44} \geq -1$	
Requirements	= 0	= 0	= 0	= 0	

In this "transportation" tableau, the column sums correspond to equations (4), the first four row sums to equations (5), and the restrictions on y_{kj}, to the inequalities (7). Both the x_{ij} and the y_{kj} are understood to be nonnegative.

All that remains is to state the "cost" coefficients. From the minimand (1.B), we observe that those coefficients associated with the x_{ij} variables are all zero, and that those connected with the y_{kj} depend only upon two parameters for each target j: the kill probability p_j and the unit worth a_j. Table 1 contains a set of assumed numerical values for these parameters, and then the corresponding unit cost coefficients.

Table 1. Calculation of the y_{kj} "cost" coefficients

Index k	Target j	1	2	3	4
	Kill probability, p_j	.2	.9	.5	.4
	Target value, a_j	100	40	400	60
1	$-a_j\ p_j\ (1-p_j)^0$	-20	-36	-200	-24
2	$-a_j\ p_j\ (1-p_j)^1$	-16	-3.6	-100	-14.4
3	$-a_j\ p_j\ (1-p_j)^2$	-12.8	-.36	-50	-8.64
4	$-a_j\ p_j\ (1-p_j)^3$	X	-.036	X	-5.184

The optimal solution for this problem can be read off by inspection: Assign gun 1 to target 2 and all the others to target 3. In terms of our variables, $x_{12} = x_{23} = x_{33} = x_{43} = y_{12} = y_{13} = y_{23} = y_{33} = 1$.

All remaining variables are set at zero.

This assignment pattern looks peculiar only if one's attention is confined to the target value coefficients, a_j . Target 2 (with the lowest value of a_j) is attacked in preference to targets 1 and 4. The paradox is easily cleared up, however, by referring to the kill probabilities, p_j. The fact that p_1 and p_4 are so low makes it worthwhile to concentrate on the other targets, and to leave these two alone. Drawing a reckless generalization from this single example, we conclude that if the kill probability for a particular target is low, either the target should not be attacked at all, or else a considerable effort should be expended against it. An intermediate policy is unlikely to minimize the total expected value of the survivors.

4. A generalization

George Dantzig has proposed a generalization of the computing method that has just been described. Since his proposal is free from the assumption made here about uniformity of all non-zero kill probabilities, it is a considerably more powerful one. His suggestion is to replace minimand (1) with the following:

$$\text{Minimize} \qquad (1.C) \qquad \sum_{j=1}^{n} a_j \prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}}$$

Dantzig points out that the functions (1) and (1.C) take on identical values whenever the x_{ij} are integral, and that they should not be too dissimilar for fractional x_{ij}. There is, of course, no assurance that the set of x_{ij} which minimizes (1.C) will also minimize (1). The only guarantee that can be made is that the minimum value of (1.C) represents a lower bound upon that of (1).

The problem of minimizing (1.C) subject to constraints (2) and (3) is still not a linear programming problem, but it may be converted into that form by defining new variables y_j as follows:

$$(8) \qquad -y_j = \sum_{i=1}^{m} x_{ij} \log_e (1-p_{ij})$$

Then (1.C) may be written:

$$\text{Minimize} \qquad (1.D) \quad \sum_{j=1}^{n} a_j e^{-y_j}$$

Since e^{-y_j} is convex, the new minimand may be approximated by a convex broken-line function, and the argument proceeds as previously in the conversion of (1.A) to (1.B). The problem has been reduced to the "transportation" form with upper bounds upon the individual components of y_j. The only major difference between the two models is that the simple column sum equations (4) are replaced by the more general linear equations (8). Numerical solutions to this class of problems are not difficult to obtain. The computing layout is now identical with that encountered by Ferguson and Dantzig in their model of aircraft allocation to routes with uncertain demands. Aside from some increase in numerical effort, the only objection to Dantzig's suggestion is, as he points out, the fact that the resulting linear programming solution cannot be guaranteed to be free from fractional values for the x_{ij} variables. Not only does this create difficulties of physical interpretation. It also means that the minimum value of (1.C) may lie considerably below that which is attainable for the original expression (1).

References

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