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A Note on the Modigliani-Hohn Production Smoothing Model*

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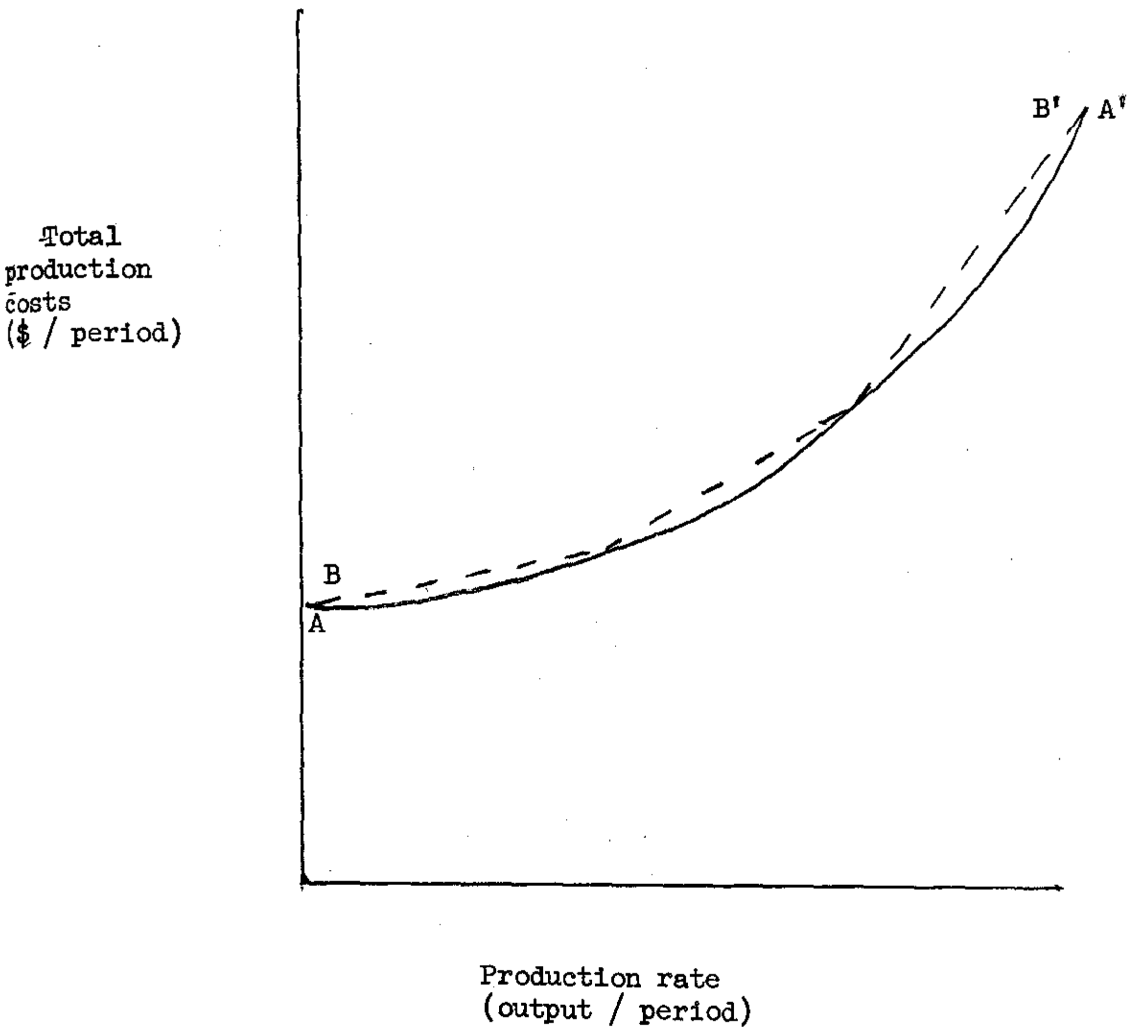
1. Introduction

Modigliani and Hohn [5] have formulated a production planning and inventory control model that appears relevant to an important class of non-stochastic decision-making problems. In their own words:

> We consider the problem of scheduling the production of a given commodity $\underline{x}$ over $\underline{T}$ equal, successive periods of time in such a way as (1) to meet initially known requirements $s_1, s_2, \ldots, s_T$ in these periods while (2) incurring the lowest possible cost. [5, p. 46]

It is the purpose of this note to suggest: (1) that the Modigliani-Hohn problem may be formulated in terms of linear programming as well as through the calculus model originally employed; (2) that since the linear programming version is equivalent to a "transportation problem" [3], even large systems may be solved readily by hand; and (3) that the linear programming version is especially well-suited to trace out the cost implications of stabilizing the work force at alternative levels. Like the "caterer problem" [4] and the "warehousing problem" [1], this linear programming model represents another instance in which every basis is pure triangular and contains no elements but zero, +1, and -1.


Total
production
costs
($ / period)
B' A'
B
A
Production rate
(output / period)

an economic interpretation in terms of the labor premiums paid for successive shifts within a given plant - night work, overtime, Saturdays, Sundays and holidays, etc. - and for this reason we shall refer to the individual segments as though they corresponded to shifts 1, 2, . . . , J.* Within any one shift as so defined, marginal production costs remain constant, but in going from one shift to the next higher cost category, the marginal costs increase discontinuously.

In order to simplify subsequent work, a few changes have been made in the notation used by Modigliani and Hohn. Ours is as follows:

(a) unknowns

x_{jt} = output during the j th shift within time period t.
($j = 1, 2, \ldots, J$)
($t = 1, 2, \ldots, T$)

y_{jt} = excess of output limit over actual output during the j th shift within time period t ($j = 1, 2, \ldots, J$)
($t = 1, 2, \ldots, T$)

h_t = inventory on hand at end of period t ($t = 1, 2, \ldots, T-1$)

H_t = inventory on hand at beginning of period t ($t = 2, 3, \ldots, T$)

(b) constants

s_t = sales requirement occurring within period t ($t = 1, 2, \ldots, T$)

* The rise in marginal production costs along curve BB' need not have anything to do with pay premiums, but might result from the diversion of a given material (say, heating oil) away from successively more valuable alternate uses. See Charnes, Cooper, and Symonds [2]. The identical form for BB' would still remain applicable.

(b) constants (cont'd.)

a_{jt} = upper limit on output during the j th shift within period t.
(j = 1, 2, . . ., J)
(t = 1, 2, . . ., T)

H_1 = inventory available at beginning of period 1.

h_T = inventory requirement for end of period T.

(c) cost coefficients

α = cost of bringing one unit of inventory forward from one period to the next.

c_j = marginal production cost within the j th shift. (j = 1, 2, . . . J)

The linear programming problem may be phrased as one of choosing values for the unknowns in such a way as to minimize expression (2.1), subject to meeting conditions (2.2) - (2.5). The minimand represents the sum of the variable inventory costs plus production costs during each shift within each time period:

$$\alpha \sum_{t=1}^{T-1} h_t + \sum_{j=1}^{J} c_j \sum_{t=1}^{T} x_{jt} \tag{2.1}$$

Condition (2.2) states that within each period, the production plus net inventory change equals the sales requirement for that period:

$$\sum_j x_{jt} + H_t - h_t = s_t \qquad (t = 1, 2, \ldots, T) \tag{2.2}$$

Condition (2.3) indicates the upper bound on output within each shift for each time period:

$$(2.3)\quad x_{jt} + y_{jt} = a_{jt} \qquad \begin{cases} j = 1, 2, \ldots, J \\ t = 1, 2, \ldots, T \end{cases}$$

And condition (2.4) indicates the identity between the inventory carried over from the end of one period and that on hand at the beginning of the next: *

$$(2.4)\quad h_t = H_{t+1} \qquad (t = 1, 2, \ldots, T-1)$$

Finally, there are the usual non-negativity restrictions on all unknowns:

$$(2.5)\quad \begin{aligned} x_{jt} &\geq 0 \\ y_{jt} &\geq 0 \end{aligned} \qquad \left\{ \begin{matrix} j = 1, 2, \ldots, J \\ t = 1, 2, \ldots, T \end{matrix} \right\}$$

$$h_t \geq 0 \qquad (t = 1, 2, \ldots, T-1)$$

3. Equivalence to the transportation model

To show that the problem just defined is equivalent to a transportation model, it suffices to illustrate the array for J=T= 3. (See Table 1.) In this instance, 22 variables altogether are present: x_{11}, x_{21}, . . . x_{33}; y_{11}, y_{21}, . . ., y_{33}; h_1, h_2, H_2, and H_3. The remaining cells of the 4 x 11 array are empty. There are 14 restrictions on the row and column totals of Table 1: the three row totals corresponding to equation group (2.2), the first nine column totals corresponding to equation group (2.3), and the last two column totals to group (2.4). No restriction is placed upon the sum of the row containing the "slack variables", y_{jt}.

* The "costless" variables H_t, along with the identity conditions (2.4), are really superfluous. They are introduced here only to facilitate setting up the linear programming problem in the format of the "transportation"problem.

										Inventory carried over		Requirements
period t	1			2			3			period 1 to 2	period 2 to 3	
shift j	1	2	3	1	2	3	1	2	3			
	x_{11} [c_1]	x_{21} [c_2]	x_{31} [c_3]							$-h_1$ [$-\alpha$]	[α]	$s_1 - H_1$
				x_{12} [c_1]	x_{22} [c_2]	x_{32} [c_3]				$+H_2$	$-h_2$ [$-\alpha$]	s_2
							x_{13} [c_1]	x_{23} [c_1]	x_{33} [c_3]		$+H_3$ [0]	$s_3 + h_3$
	y_{11} [0]	y_{21} [0]	y_{31} [0]	y_{12} [0]	y_{22} [0]	y_{32} [0]	y_{13} [0]	y_{23} [0]	y_{33} [0]			d_4
upper limits on output, a_{jt}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	a_{13}	a_{23}	a_{33}	0	0	

General form of LP problem:

Minimize $\sum_h c_h z_h$

subject to $z_h \geqq 0$

$\sum_h b_{gh} z_h = d_g$

Transportation form

Minimize $\sum_{hi} c_{hi} z_{hi}$

subject to $z_{hi} \geqq 0$

$\sum_i z_{hi} = d_h$

$\sum_h z_{hi} = e_i$ } where $\sum_h d_h = \sum_i e_i$

The above problem can be put in quasi-transportation form by specifying c_{hi} as inserted above in pencil, z_{hi} as printed above (of which two, $-h_1$ & $-h_2$ are nonpositive!)

$[e_i] = [a_{11}, \ldots, a_{33}, 0, 0]$

$[d_h] = [s_1 - H_1, s_2, s_3 + h_3, d_4]$ $\quad d_4 \equiv \sum_i e_i - \sum_{h=1}^{3} d_h$

In the case of an array such as the one shown in Table 1, Dantzig has proved the following theorem: At each iteration, it will be possible to find either a row or a column containing exactly one basis variable. Solving for this variable and then working with the reduced array formed by deleting the corresponding row or column, it is proved by induction that every basis must be strictly triangular. Furthermore, since the only coefficients entering into the equations of the initial linear programming problem are zero, +1, and -1, the only arithmetic operations required for a solution consist of addition and subtraction.

[Dantzig, 3, p. 365].

4. An illustrative example

Suppose that the constants and the cost coefficients for a 3-shift, 3 time period problem are as given in Table 2. For these data, Table 3 then contains the minimum cost solution as computed by the "transportation" method. Values for the dual variables u_t, the "shadow prices" associated with each of the sales requirement equations, are also entered in Table 3. In this, as in any optimal solution to this class of linear programming problems, the following conditions are necessary, and may be employed to facilitate computations: *

* Note that the conditions of (4.1) correspond to the Modigliani-Hohn calculus inequalities (3.10), [5 , p.50].

[redacted] 3-shift, 3 time period smoothing problem.

Constants (units of output)		Cost coefficients ($$10^3$/unit of output)
$s_1 - H_1 = 80$		$c_1 = 10$
$s_2 = 160$		$c_2 = 15$
$s_3 + h_3 = 240$		$c_3 = 20$
$a_{jt} = 100$	(all j, t)	$\alpha = 4$

Table 3. Optimal solution to hypothetical smoothing problem

time period $\underline{t}$ \ shift $\underline{j}$	x_{jt} 1	2	3	H_t	h_t	Requirements	u_t ($$10^3$)
1	100	0	0	--	20	80	11
2	100	80	0	20	40	160	15
3	100	100	0	40	--	240	19
a_{jt}	100	100	100				
c_j ($$10^3$)	10	15	20				

$$\text{Inventory} + \text{labor costs} = \alpha \sum_{t=1}^{T-1} h_t + \sum_{j=1}^{J} c_j \sum_{t=1}^{T} x_{jt} = \$5{,}940{,}000.$$

(4.1) $u_{t+1} \leq u_t + \alpha$ $(t = 1, 2, \ldots, T-1)$

(4.2) $(u_t + \alpha - u_{t+1})(h_t) = 0$ $(t = 1, 2, \ldots, T-1)$ *

(4.3) If, for any given shift j and period t,

$x_{jt} > 0$, then $x_{j-1,t} = a_{j-1,t}$ $(j = 2, 3, \ldots, J;\ t = 1, 2, \ldots, T)$

(4.4) If, for any given shift j and period t,

$a_{jt} > x_{jt} > 0$, then $u_t = c_j$. If $x_{jt} = a_{jt}$, then $u_t \geq c_j$.
And if $x_{jt} = 0$, then $u_t \leq c_j$. (all j, t)

(4.5) The "costless" variables $H_2, H_3, \ldots, H_T$ will all be contained in an optimal basis - although some of them may take on the value of zero.

5. Work stabilization policies

The model with which we have been dealing includes no explicit costs for altering the rate of production or the size of the work force from one period to the next. Indeed, in any concrete application, it may be extremely difficult for a firm to place dollar cost estimates on the loss of employee and community good will that results from deliberate fluctuations in employment. As an alternative to making arbitrary estimates of these "rate of change" costs, it will frequently be desirable to calculate several alternative inventory and production plans - each corresponding to a different assumption as to the "work force commitment", i.e., the level below which employment will not be reduced during any of the time periods covered by the production plan.

* In the terminology of Modiglian and Hohn, if $u_t + \alpha > u_{t+1}$, period t and $t + 1$ belong to two different "intervals". "Small" changes in $s_{t+1}, s_{t+2}, \ldots, s_T + h_T$ would not alter the optimal solution for periods $1, 2, \ldots, t$.

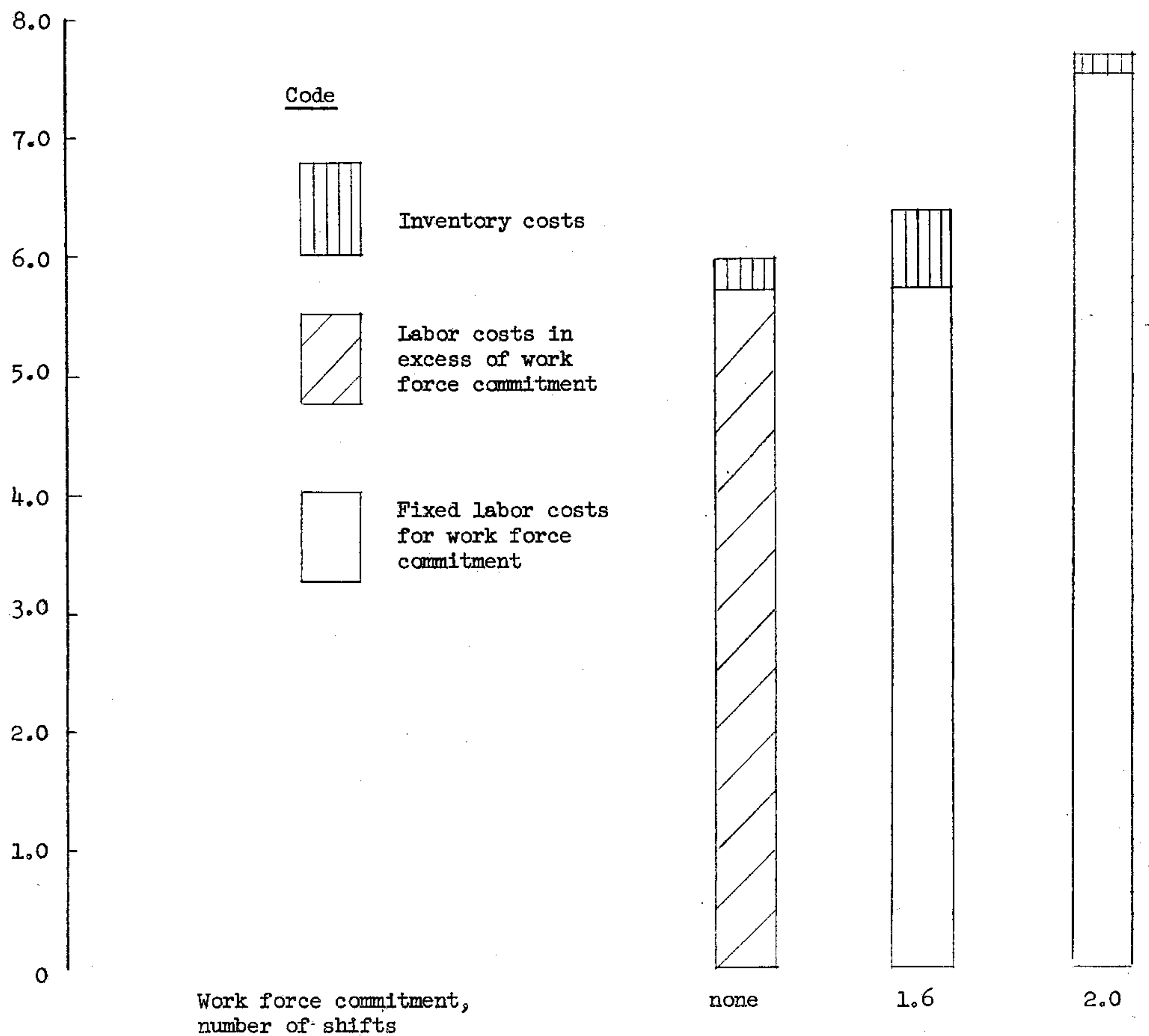
In the case of our hypothetical example, the optimum solution shown in Table 3 already provides a minimum of one-shift employment during each of the three periods. The bar chart of Figure 2 compares the cost of this plan with those of two further linear programming solutions - one that assumes a work force commitment of 1.6 shifts and one of 2.0. * Interestingly enough, the total of inventory plus labor costs increases very little from a 1.0 to a 1.6 shift commitment - from $5.94 millions to 6.34. In going from the 1.6 to the 2.0 shift level, however, some of the employees must remain idle, and a substantial cost increase occurs. The total goes up to $7.66 millions. When presented with an explicit choice of this sort, few managements would hesitate to choose the 1.6 shift policy - despite their initial reluctance to place a dollar value on the worth of stabilizing employment.

In order to obtain a linear programming solution for the cases of 1.6 and 2.0 shifts, only a few changes need be made in the constants and cost coefficients shown in Tables 2 and 3:

Work force commitment, number of shifts	none (Tables 2 and 3)	1.6	2.0
a_{1t}	100	160	0
a_{2t}	100	40	200
c_1	10	0	0
c_2	15	15	0

* Note that 1.6 shifts yield 480 units of output during the three periods - exactly the cumulated production requirements.

Figure 2


Three-period costs, millions of dollars
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
Code
Inventory costs
Labor costs in excess of work force commitment
Fixed labor costs for work force commitment
Work force commitment, number of shifts
none
1.6
2.0

These alterations consist simply of recognizing that the effect of a work force stabilization policy is to make the marginal labor costs zero for output rates that fall short of using up the fixed labor commitment, but that the marginal costs increase discontinuously at production rates in excess of this level. Here it is apparent that a linear programming idealization approaches economic reality much more closely than a calculus model which assumes that the marginal cost curve is a continuous one. Both from the viewpoint of problem formulation and of numerical computations, the linear programming version seems to be as useable as its calculus counterpart.

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