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Inconsistency and Indeterminacy in Classical Economics

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by Karl Brunner

Patinkin's two articles on the properties of classical systems proved to be highly stimulating. The second article, "The Indeterminacy of Absolute Prices in Classical Economic Theory" gave rise to a lively controversy. Patinkin's analysis was honored by the publication of 3 critical contributions to Econometrica in January 1950. Five additional manuscripts were submitted to Econometrica. This paper is an attempt to develop systematically the issues involved in the "Patinkin problem." A careful study of the manuscripts submitted to Econometrica helped considerably to clarify these issues and they have to be regarded as an important basis of the present paper.

I. The Issue

1. The central problem raised by Patinkin is the proper construction of a theory of a monetary system. All these systems may be conceived as a set. Before we evaluate the members of this set from the point of view of their "degree of realism," we want to know, if there exists a subset of systems which might be discarded on purely analytical considerations as a scientific hypothesis on monetary systems. If such a subset can effectively be described,
the analytical basis for the empirical work will be the more definite after its elimination.

2. The "Paretikin discussion" is not concerned with classifying any conceivable monetary system. Its purpose is more limited. There exists a well-defined subset of monetary systems which form the material under consideration. This subset of analytically fallacious theories is characterized by a property common to each member system. The feature uniting the group shall be termed the "complementing property" of the theory of a monetary system. The following meaning will be attached to this term: Economic theory traditionally started with a study of market behavior and market processes. Thus was developed an analysis of price formation on commodity markets—and to focus the analysis money problems were completely eliminated. In this way classical theory built up a (conceivably) consistent and determinate theory of a real economy. To proceed from such a real to a monetary system it was thought sufficient to "complement" the real system with an equation relating the monetary variables with the real variables incorporated as parameters. Thus the total system was composed of two subsystems—a real and a monetary—each one closed with respect to its relevant variables. The simultaneous existence of these two subsystems gives rise to what has been termed the dichotomization of the pricing process. Behavior in the real part depends solely on the real variables, which are (if at all) uniquely determined by these behavior patterns only. In the monetary sphere behavior is linked to money values which result thus from behavior patterns relevant only for this sector of the economy.

The "complementing property" is thus equivalent to the logical dichotomy of the pricing process. The analytical value of both depends on the consistency of two postulates: namely, the postulate that market functions in the real subsystem are homogeneous of degree zero in the prices and that there exists a
meaningful money equation).

3. An objective of the present paper is to prove that the monetary systems characterized by the complementing property are analytically fallacious and cannot be utilized as explanatory devices of observable phenomena. It will be shown that the systems under consideration will not satisfy the minimum requirements of a theory of a monetary system. As the controversy around the "Tatinkin problem" started with a macro-economic variant out of the subset of dichotomized monetary systems (the "Casselian system"), the analysis will proceed at first on a macro-level. It will then be shown by a macro-analytical argument that systems with complementing property break down as they imply serious contradictions. But in order to clarify the nature of these inconsistencies it is necessary to resort to micro-analysis. As a result we shall be able to formulate the conditions for the proper construction of a theory of a monetary system.

II. Explicit and Implicit Money Functions of a Monetary System

1. The money equation of the monetary subsystem formulates a relation between the money prices, the quantities demanded and supplied of money. This equation is usually developed as a relation between stocks—and if presented as a flow relation some elementary transformation gives us the stock equation. Thus the two formulations of the monetary subsystem are equivalent and we may concentrate on the form:

\[
M_D(P_1 \cdots P_{n-1}) - M_S(P_1^{*} \cdots P_{n-1}^{*}) = M_X(P_1 \cdots P_{n-1})
\]

(2.1)

where \(M_D\) = demand for money as stock, \(M_S\) = amount of money outstanding, \(P_1^{*} \cdots P_{n-1}^{*}\) = commodity prices, \(M_X\) = excess demand for money in stock terms.

This relation implies that for any price vector \(p\) such that \(M_D > M_S\), the totality of economic units desires to substitute money for commodities. This
decision will be reflected on the "commodity" markets. Money can only be acquired or disposed of by selling or buying "commodities" (= non-money objects). Thus the situation in the monetary subsystem has definite implications for the commodity markets. On the other hand, money flows are generated by the demand and supply decisions on these markets and the net result of these flows must affect the stock relationships. This fact suggests that the explicitly formulated money equation of the monetary subsystem finds its counterpart in a money equation which is implied by the working of the real subsystem:

\[(2.2) \quad D_n = \sum_{i=1}^{n-1} p_i S_i, \quad S_n = \sum_{i=1}^{n-1} p_i D_i\]

\[(2.3) \quad D_n - S_n = X_n\]

when \(S_i\) = supply function of \(i^{th}\) commodity, \(D_i\) = demand function of \(i^{th}\) commodity, and \(D_n\) and \(S_n\) are the flows of money generated by the decisions to supply and demand commodities, \(X_n\) = excess demand for money in terms of flows.

2. The logical relation between the two money functions has to be formulated. A simple budget restriction which takes account of the existence of money may look like this:

\[(2.4) \quad \sum_{i=1}^{n-1} p_i (z_{i,\alpha} - \bar{z}_{i,\alpha}) + z_{n,\alpha} - \bar{z}_{n,\alpha} = 0\]

where the subscript \(n\) designates money and \(\alpha\) the individual, the barred quantities are resources disposable before marketing, and unbarred ones the collection of desired commodities. By rearranging terms we find:

\[(2.5) \quad z_{n,\alpha} - \bar{z}_{n,\alpha} = -\sum_{i=1}^{n-1} p_i (z_{i,\alpha} - \bar{z}_{i,\alpha})\]

The sum of the right may be divided into two groups. One group is composed of all terms satisfying the relation

\[z_{j,\alpha} - \bar{z}_{j,\alpha} < 0\]
while the members of the other group satisfy
\[ z_{k\alpha} - \bar{z}_{k\alpha} > 0 \]

Let us suppose that there are \( r \) terms in the first and \( s \) terms in the second group. Then we know that \( r + s = n-1 \). Further suppose that the commodities are ordered in such a way that 1, 2...\( r \) belong to the first group and \( r+1...n-1 \) to the second one. Then (2.5) can be written in the following way:

(2.6) \[ \bar{z}_{n\alpha} - \bar{z}_{n\alpha} = \sum_{j=1}^{r} p_j (\bar{z}_{j\alpha} - z_{j\alpha}) - \sum_{k=r+1}^{n-1} p_k (z_{k\alpha} - \bar{z}_{k\alpha}) \]

The first term on the right defines (for any given price vector) the total inflow into the \( \alpha \)'th individual's cash-balance

(2.7) \[ D_{n\alpha} = \sum_{j=1}^{r} p_j (\bar{z}_{j\alpha} - z_{j\alpha}) \quad \text{when} \quad \bar{z}_{j\alpha} - z_{j\alpha} > 0 \]

and the second term defines (for given price vectors) the total outflow

(2.8) \[ S_{n\alpha} = \sum_{k=r+1}^{n-1} p_k (z_{k\alpha} - \bar{z}_{k\alpha}) \quad \text{when} \quad z_{k\alpha} - \bar{z}_{k\alpha} > 0 \]

Thus (2.6) may be written in the form:

(2.9) \[ \bar{z}_{n\alpha} - z_{n\alpha} = D_{n\alpha} - S_{n\alpha} \]

which expresses an accounting identity that the net inflow into a cash-balance and the change in the cash-balance coincide, whatever the price-vector may be. By summing (2.9) over all the \( \alpha \)'s we have

(2.10) \[ \sum_{\alpha=1}^{m} \bar{z}_{n\alpha} - z_{n\alpha} = \sum_{\alpha=1}^{m} D_{n\alpha} - \sum_{\alpha=1}^{m} S_{n\alpha} \]

when \( \sum_{\alpha=1}^{m} \bar{z}_{n\alpha} = \sum_{\alpha=1}^{m} D_{n\alpha} = \sum_{\alpha=1}^{m} S_{n\alpha} \)

3. To proceed further, let us divide the economy into two sectors, the money creating sector B and the money utilizing sector P. The identity (2.10) takes evidently only account of the units in the P sector. For this reason a P
superscript has been inserted. To complete the presentation of the demand side of the monetary system, an accounting identity for the B sector will be formulated.

Let us designate by \( \Delta D^B \) the amount of money desired by the B sector; \( D_n^B \) the amount held by B before operations started; \( D_n^B \) the inflow and \( S_n^B \) the outflow of money (both at given price vectors). Then we find

\[
(2.11) \quad \Delta D^B = D_n^B + S_n^B
\]

To clarify further the nature of \( \Delta D^B \) we define the sum

\[
(2.12) \quad \Delta D = \Delta D^B + \Delta D^P
\]

where-by \( \Delta D \) is designated the total potential sum of money which could be created by a given monetary system. Thus

\[
(2.13) \quad \Delta D^B = \Delta D - \Delta D^P
\]

so \( \Delta D^B \) is the non-utilised margin of this potential sum \( \Delta D \).

4. Formula (2.11) leads immediately to the supply side. The term \( (D_n^B - S_n^B) \) describes the "B-desired change in B held stock of money." But the out-and-inflow of money from and into the B sector must affect the "P held stock of money." So let us designate by \( (S_n^B - D_n^B) \) the net inflow into the P sector from the B sector. Then we see

\[
(2.14) \quad (S_n^B - D_n^B) = - (D_n^B - S_n^B)
\]

that is, the "B-desired change in P held stock of money" is the negative of the "B-desired change in B-held stock of money." And the supply identity is given by

\[
(2.15) \quad \Delta S^B - \Delta S^P = S_n^B - D_n^B
\]
where $S^B_{B}$ denotes the "desired level of B-held stock of money"—or applying (2.12), the actual quantity of money planned by the B sector.

5. Some further remarks on the supply identity are necessary. Creation and destruction of money may proceed in two forms:

a) by fiat: there is no counterflow of commodities or claims against the money flow. This "fiat component" of $S^B_{B}$ and $D^B_{B}$ will be designated by $S^B_{nf}$ and $D^B_{nf}$.

b) by market transaction. In this case there exists a counterflow to the money flow. To simplify, we suppose that this counterflow consists only of claims. The "claims component" of $S^B_{B}$ and $D^B_{B}$ will be denoted by $D^B_{nc}$ and $S^B_{nc}$.

Thus the relations hold

\[(2.16) \quad S^B_{B} = S^B_{nc} + S^B_{nf}, \quad D^B_{B} = D^B_{nc} + D^B_{nf}\]

Because of the fiat components $D^P_{B}$ and $S^P_{B}$ do not describe the total in- and outflow for the cash-balances of the P sector. These quantities were related to the sale and purchase of commodities. Then, total inflow into the cash-balance of the P-sector in case of fiat flows $S^B_{nf}$ is given by

\[(2.17) \quad S^P_{B} = D^P_{B} + S^B_{nf}\]

and total outflow

\[(2.18) \quad S^P_{B} = S^P_{B} + D^B_{nf}\]

If we subtract (2.15) from (2.10) after substituting $D^P_{B}$ and $S^P_{B}$ for $D^P_{B}$ and $S^P_{B}$ in (2.10) and of (2.16) in (2.15) we find that

\[(2.19) \quad \Delta^P_{B} = \Delta^B_{B} = [D^P_{B} + (D^B_{nc} + D^B_{nf})] - [S^P_{B} + (S^B_{nc} + S^B_{nf})]\]
This relation which proclaims the identity of the excess demand for money in terms of stocks and flows will be labelled the monetary identity. As all the quantities involved are supposed to be functions of the price vector, the monetary identity expresses further the fact that the explicit and implicit money functions of a monetary system can be defined in terms of each other.\(^9\)

6. By rearranging terms in (2.19) we can derive

\[
-M \times D^P_n + D^B_{nc} - S^P_n - S^B_{nc} = 0
\]

All the flow quantities in this relation are generated by supply and demand decisions of commodities and claims (bonds). The supply and demand function of bonds are denoted by the subscript \((n-1)\). Applying (2.2) we find

\[
D^P_n = \sum_{i=1}^{n-2} p_i S_i + p_{n-1} S^P_{n-1}
\]

\[
S^P_n = \sum_{i=1}^{n-2} p_i D_i + p_{n-1} D^P_{n-1}
\]

so these two flow terms are linear combinations of supply and demand functions \(S_i\) and \(D_i\). Further, by hypotheses \(S^B_{nc}\) and \(D^B_{nc}\) are related to the transactions on the bonds market. \(D^B_{nc}\) is the flow demand of the B sector for money resulting from the supply of bonds to the P sector. Thus

\[
D^B_{nc} = p_{n-1} S^B_{n-1}
\]

and \(S^B_{nc}\) is the B-sector's flow of money emanating from its flow demand for bonds from the P sector. Thus

\[
S^B_{nc} = p_{n-1} D^B_{n-1}
\]

The four quantities \(S^B_{n-1}\), \(S^P_{n-1}\), \(D^P_{n-1}\), \(D^B_{n-1}\) are components of a general excess demand function for bonds, in such a way that

\[
D_{n-1} = D^P_{n-1} + D^B_{n-1} \quad S_{n-1} = S^P_{n-1} + S^B_{n-1}
\]

when \(D_{n-1}\) and \(S_{n-1}\) denote the supply and demand of bonds aggregated over all units of the P and the B sector.
Substituting (2.21)-(2.25) into (2.20) and by expressing excess demand

\[(2.26) \quad x_i^* = d_i - s_i \quad i = 1, \ldots, n-1\]

\[(2.27) \quad \sum_{i=1}^{n-1} p_i x_i^* = 0\]

This expression defines a functional dependence of all the excess demand functions of a given system. If we suppose that \(p_i(i=1, \ldots, n-1) \neq 0\) then we can obviously express any excess demand function as a linear combination of all the others and so eliminate it from the system. It is important to note that Walras' law and the monetary identity are equivalent statements.

Walras' law states an obvious limitation of the economic universe: if all objects of transaction (including money) are accounted for, it follows that the sum total of the excess-demand of all objects must be zero, whatever the price-vector. By the nature of the market problem we cannot conceive of the contrary situation^10).

Our main instrument in the analysis of macro-systems with complementing property are the concepts presented in this section: the relationship between the explicit and the implicit money function in a given monetary system and some implications of Walras' law furnish us the criterion for the analytical test.

III. The Logical Character of Monetary Systems with Complementing Property.

A. The \(M_x\) Systems:

1. The Case of Invalidity I

1. The general purpose of section III A is an analysis of the implications of two postulates:

a) the "real" functions of the system are homogeneous of degree 0 in the same subset of p's.

b) The \(M_x\) function is non-homogeneous in p's11).

A variety of illustrative systems were utilized in the discussion of the "Fatinkin problem." According to the postulated form of the excess demand function for money
we shall differentiate within the subset under consideration between two groups, namely, the $M_x$ systems and the $X_n$ systems. The $M_x$ function is assumed to be homogeneous of degree 1 in the price and the stock of money but non-homogeneous in $p$ alone. These properties are satisfied by the Cambridge function

$$ (3.1) \quad K \cdot p_{n-1} \sum_{r=1}^{n-1} \frac{p_r}{p_{n-1}} S_r\left(\frac{p_1}{p_{n-1}}, ..., \frac{p_{n-2}}{p_{n-1}}\right) - M = M_x^* $$

where $K$ and $M$ are given constants, $S_i$ denotes the supply functions of commodities and $M_x^*$ is a specific creature of the genus $M_x$. $X_n$ is defined as a linear combination of commodity excess demand functions, as described by (2.2) and (2.26).

We shall assume further, for this section, that the net injection of the B-sector vanishes identically, thus

$$ (3.2) \quad S^B_n - p^B_n = 0 $$

This reduces the monetary identity to the form

$$ (3.3) \quad M^*_n - M^B_n = p^B_n - S^B_n $$

We consider a system composed of $n-1$ commodities and money. The supply and demand functions of commodities conform to the first postulate, and the excess function for money has the form described by (3.1). The system of excess demand equations is presented by (3.4)

$$ (3.4) \quad \sum_{i=1}^{n-1} \frac{p_i}{p_{n-1}} S_i\left(\frac{p_1}{p_{n-1}}, ..., \frac{p_{n-2}}{p_{n-1}}\right) = 0 \quad i = 1, ..., n-1 $$

$$ \quad M_x^*\left(p_1, ..., p_{n-1}\right) = 0. $$

By a simple proof it can be shown that the linear combination of commodity excess demand functions $X_n$ will be hdf if all the $X_i(i=1, ..., n-1)$ are hdf in all $p$'s. Thus, we observe that by the first postulate $X_n$ is hdf in $p$'s, while by the second $M_x^*$ is made nh in $p$'s. So the monetary identity

$$ \quad M_x = X_n $$

cannot hold. Such a statement implies that there exists a set of price-vectors for which the net-inflow into cash-balances will not be equal to the change in cash-balances.
Nothing essential will be changed if it is supposed that $S_{n-1}$ and $D_{n-1}$ denote supply and demand for bonds and that these functions are hdl in $p_1, \ldots, p_{n-2}$ while all $S_i, D_i$ ($i=1, \ldots, n-2$) are hdo in the same subset of $p$'s. As shown by Patinkin\textsuperscript{13}, the linear combination $X$ is still hdl. Thus the system implies again a relation between the explicit and implicit money function which contradicts the monetary identity\textsuperscript{14}.

2. The result established for the two systems was completely independent of any considerations of consistency or inconsistency for the real parts. So whatever is done to assure consistency the two basic postulates lead to contradictory statements and the systems fail. But the analysis of consistent systems provides additional analytical insight into the properties of the postulational structures under investigation in this paper. With the information given, it is not possible to decide if the real subsystems above do have a solution or not. So long as we deal with general classes of functions and not with restricted parametric situations, we need assumptions which will assure us the consistency of real subsystems. The two following postulates will be useful\textsuperscript{15}:

aa) the particular systems under consideration have a unique solution, if and only if the number of variables and independent equations coincide.

bb) for the particular systems under consideration any function of the real subsystem can be expressed as a linear combination of powers of all the other functions of the same subsystem\textsuperscript{16,17}.

A variety of forms will satisfy the second postulate. A general formula which covers it is given by (3.5)\textsuperscript{18}:

\[
(3.5) \quad \sum \phi_i(p) x_i^k(p) \frac{\partial}{\partial p} 0
\]

where $p$ denotes the price-vector and where at least one of the $\phi_i$ does not vanish. This linear combination reflects a functional dependence of the $x_i$ functions. As has been shown by Hickmann\textsuperscript{19}, the Jacobian of $(n-1)$ commodity excess demand functions hdo in $p$ vanishes identically, thus implying a function
(3.6) \[ F(X_1, X_2, \ldots, X_{n-1}, 0) = 0 \]

But this result is not sufficient to decide, whether the functional dependence defined by (3.6) has the special properties of the combination of powers in (3.5) which is essential in order to establish consistency. This form states that for a given price vector such that \( n-2 \) of the functions have a zero, the last of the commodity excess functions will have a zero as well. The second postulate above successfully eliminates an \( F \) function of the form \( F(X^*_1, 0, \ldots, 0) = 0 \), where \( X^*_1 \neq 0 \).

Classical theory provides both of these postulates. While the first was rigorously introduced by Walras the second goes back to the founders of classical theory. Say's law formulates a dependence of the commodity excess functions which appears as a special case of (3.5). It states

\[
(3.7) \quad \sum_{i=1}^{n-1} p_i X^i = 0
\]

that whatever the price-vector, the total money value of all commodities' excess demand will be zero. Say's law expresses that money operates only as a medium of exchange. The inflow of money from commodities supplies is completely utilized for purchasing other commodities. (3.7) excludes any substitution process between money and commodities. Stating (3.7) in addition to the homogeneity postulates does not eliminate the inconsistent propositions created by the negation of the monetary identity. But in addition, a system with Say's law produces another set of inconsistent propositions.

By Say's law it follows that

\[
(3.8) \quad X_n = 0
\]

while the \( M_x \) function implies

\[
(3.9) \quad X_n = 0
\]

(3.8) declares that the excess flow of money will vanish for any price-vector and (3.9) formulates that there exists a unique price vector at which the money excess flow will be zero \( ^{21,22} \).
3. Say's law is only one among many ways to create consistency. Relation (3.5) makes this fact clear. Other possibilities are present. One of them is developed in the system analysed by Hickmann.

\[
\begin{pmatrix}
-s_1 & d_1 \\
-d_2 & -s_2
\end{pmatrix}
\begin{pmatrix}
s_1 \\
d_2
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

where \( z = \frac{p_1}{p_2} \) the ratio of the two commodity prices in Hickmann's system. \( s_1 \) and \( d_1 \) are the parameters of the excess-demand function of commodity 1 and \( s_2 \) and \( d_2 \) are the parameters of \( X_2 \). Consistency is assured if

\[
\begin{vmatrix}
-s_1 & d_1 \\
d_2 & -s_2
\end{vmatrix} = 0
\]

The solutions \( z = \sqrt{\frac{d_1}{s_1}} = \sqrt{\frac{s_2}{d_2}} \) will then simultaneously satisfy both equations of the system. (3.11) implies that there exist \( k_1 \) and \( k_2 \) not both zero, such that

\[
\begin{align*}
k_1 (s_1) + k_2 (d_2) &= 0 \\
k_1 (d_1) + k_2 (-s_2) &= 0
\end{align*}
\]

expressing the linear dependence of the coefficients of the excess-demand functions.

Suppose \( k_2 \neq 0 \), and \( \frac{k_1}{k_2} = c \). Then we derive from (3.11)

\[
\begin{align*}
d_2 &= c \cdot s_1 \\
s_2 &= c \cdot d_1
\end{align*}
\]

These relationships imply \( c \cdot X_1 = X_2 \). Thus the linear combination of \( X_1 \) and \( X_2 \) with coefficients 1 and \( c \) vanishes identically with respect to \( p_1 \) and \( p_2 \).

\[
X = c \cdot X_1 + X_2 = 0 \quad \text{where} \quad c = \frac{d_2}{s_1} = \frac{s_2}{d_1}
\]

Consider the case where \( c = z \), such that \( z = \sqrt{\frac{d_2}{s_1}} = \sqrt{\frac{s_2}{d_1}} \). For this instance (3.14) may be written

\[
X_n = p_1 \cdot X_1 + p_2 \cdot X_2 = 0
\]

Thus only for the specific structure considered will Hickmann's system imply Say's law and is thus subject to the criticism formulated above. So by excluding this
structure Hickmann is able to show that

(3.16) \( \bar{x}_n = 0 \) and \( \bar{m}_x = 0 \)

in spite of \( x = 0 \), because \( x \neq x_n \). But Hickmann evidently overlooks the inconsistency which is completely independent from such considerations. Applying the criterion furnished by the monetary identity, we find

(3.17) \( \bar{x}_n \) is a function hdi in p’s and

(3.18) \( \bar{m}_x \) is a function nh in p’s

Thus it follows for the Hickmann system that \( \bar{m}_x \neq \bar{x}_n \).

11. The Implications of Walras Law.

1. According to Patinkin’s analysis the various systems with complementing properties must first be considered by the monetary identity as to the existence of invalidity I. If they do not pass this test, they are already proved to be fallacious, and in case they satisfy this identity after proper adjustment, they flounder in general on the rocks of invalidity II. The development of Patinkin’s argument makes it clear that he does not consider the non-existence of both invalidities in a given system as a sufficient condition for its analytical soundness. We shall actually present two systems with complementing properties which are free from both invalidities. Thus it seems that the criterions employed by Patinkin are not sufficiently powerful. We are therefore interested in finding some analytical measure with a greater range of applicability.

2. The "monetary identity criterion" was essentially based on a comparison of the homogeneity properties of the explicit and implicit money functions of any given monetary system. The method employed for the analytical test of the first system is a generalization of this idea based on Walras’ law. Given this relation, there exists for any function of the system (provided \( p_i \neq 0 \), \( i-1, \ldots, n-1 \)) an implicit function in terms of all other functions in the system. The problem will then be: Do the postulates imply coincident properties of the implicit and explicit function of any commodity—if not Walras’ law cannot hold and the system under consideration produces inconsistent propositions.
Patinkin II defined a system which with appropriate modifications will be presented here \(^\text{25}\).

\[
(3.19) \quad X_1 (p_2, \frac{p_2}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}) = 0 \quad i = 1, \ldots, n-1 \tag{3.20}
\]

The \(X_i\) functions are postulated to be \(hd0\) in \(p_r\) \((r = 2, \ldots, n-1)\) and \(nh\) in \(p_1\). \(M_x\) is \(nh\) in \(p_i\) \((i = 1, \ldots, n-1)\). As shown in Patinkin II \(^\text{26}\), the linear combination \(\sum_{i=1}^{n-1} p_i X_i\) is \(nh\) in \(p_i\)'s. Thus we are not able to disprove the monetary identity by the considerations applied in the last section. The system contains two subsystems: the real part \((3.19)\) and the monetary part \((3.20)\). In \((3.19)\) there are \((n-1)\) equations in \((n-3)\) real prices and the sole money price \(p_1\) -- altogether \((n-2)\) variables. By introducing the two postulates of III.A1.3. \((3.19)-(3.20)\) evidently describe a dichotomized pricing process which is free from both of Patinkin's invalidities.

But trouble arises in the following way: Walras' law is defined as

\[
(3.21) \quad M_x + \sum_{i=1}^{n-1} p_i X_i (p) = 0
\]

Provided \(p_i \neq 0\) we can define any \(X_i\) in terms of the other \(n-1\) excess-functions.

Let us single out \(X_1\), such that

\[
(3.22) \quad X_1 = - \frac{M_x}{p_1} - \sum_{r=2}^{n-2} \frac{p_r}{p_1} \cdot X_r
\]

The right side of the relation is built up by a first component \(nh\) in \(p_i\)'s and \(p_r\) and a second component \(hd1\) in \(p_r\). Both together form an expression \(nh\) in \(p_i\)'s and \(p_r\).

The postulates of the system stated that \(X_i\) functions are \(hd0\) in \(p_r\), so the relation \((3.22)\) cannot hold. Similar arguments hold for any other of the \(X_i\) functions. (They will differ by the fact that the second component on the right will be \(nh\) in \(p_r\) and \(p_i\).) It follows then, if \((3.22)\) does not hold, that \((3.21)\) contradicts the systems postulates—so Walras' law is excluded by \((3.19)-(3.20)\). It was shown that Walras' law and the monetary identity are equivalent statements—thus the monetary identity fails as well. The proof for this failure had to be more roundabout than in the case of invalidity \(^\text{27}\).
3. An important feature characterized all the cases analysed in the last sections: They covered a class of systems with only one excess demand function nh in p's while the others were all hdo or hdl in the same subset of p's. Within this class, Aras' law cannot hold, and thus the monetary identity will fail as well. 

As soon as a second nh excess-demand function becomes part of the system the situation is essentially changed. The above argument becomes invalid for such a situation. A linear combination of h functions with at least two nh functions may be h or nh. This indeterminacy provides the necessary degree of freedom for the addition of a further postulate, namely to state that Aras' law and so the monetary identity actually holds. A set of postulates constructed in this way is evidently not only immune against the charges of Patinkin's invalidity theorems, they hold their place against the attack to which the "modified Casselian system" succumbed.

Let us consider a system with two nh excess-demand functions:

\[
\begin{align*}
I_i \left( \frac{p_1}{p_{n-2}}, \ldots, \frac{p_{i-3}}{p_{n-2}}, p_{n-1} \right) &= 0 & i = 1, \ldots, n-2 \\
I_{n-1} \left( p_1, \ldots, p_{n-1} \right) &= 0 \\
M_x \left( p_1, \ldots, p_{n-1} \right) &= 0
\end{align*}
\]

where \( I_{n-1} \) and \( M_x \) are nh in p. We are of course free to assume any one of the \( I_i \) to be nh in p. A good case can be constructed in favor of attributing nh to the bond excess-function. By utilising either one or both of the assumptions, namely:

a) The net injection \( B_n - D_n \) does not vanish, and the following relations are accepted:
\[
\begin{align*}
B_n &= B_{nc} \\
D_n &= D_{nc} \\
M_S &= F(p, \alpha)
\end{align*}
\]

\( F \) is a nh function in p and \( \alpha \), designates a set of exogenous-institutional factors. \( F \) expresses the fact that there exists a finite upper boundary to the money-creation process which is independent from the working of the price-mechanism.

b) There exist imperfections in the capital market. They may take the form of special "costs of transactions".

4. The first rationalization of the "modified Lange system" seems to be preferable as it arises from a connection between the two monetary functions in the form of
transactions between the two sectors, while the second rationalization introduces considerations of "costs of transactions," but limited to some arbitrary sphere of the economy. Irrespective of procedure the real subsystem contains \((n-2)\) equations in \((n-3)\) real commodity prices and the rate of interest \(r = \frac{1}{p_{n-1}}\). Further, we postulate Walras' law as the form of the functional dependence, to which all the functions are subject. By this postulate a definite restriction is imposed on the functional patterns. Utilizing Walras' law to eliminate \(X_{n-1} = 0\), we insert \(p_r = Z_r \cdot p_{n-2}\) and \(r\) into \(M_x = 0\), and solve for \(p_{n-2}\). In this way the "modified Lange system" presents a dichotomized pricing system, which seems to be analytically unassailable.

Closer inquiry reveals a curious type of inconsistency in the system. By eliminating either \(X_{n-1} = 0\), or \(M_x = 0\) we obtain a straightforward dichotomy with respect to real variables and money variables. Suppose now that a commodity equation has been eliminated\(^3\). Then we are left with \((n-3)\) equations in the real subsystem in \((n-2)\) variables. We can at best solve for all the real commodity prices in terms of \(r\). Inserting these solutions in \(X_{n-1} = M_x = 0\) we have two equations in \(r\) and \(p_{n-2}\).

The first elimination procedure via the monetary subsystem results thus in a dichotomized process, while the elimination procedure via the real part defines a system without any dichotomy, which requires all relations to find a solution for any proper subset of variables. Thus the two ways of eliminating the redundant equation picture two essentially different notions about the working of the economic mechanism.

This divergence becomes very marked when we consider the properties of the interest mechanism in the "modified Lange system." If we choose to eliminate \(X_{n-1} = 0\) or \(M_x = 0\), then we are bound to state that \(r\) is determined within the real subsystem. On the other hand, to delete some \(X_g = 0\) (3 any one out of \(1, \ldots, n-2\) implies that we have to say: \(r\) is determined by the operations of the monetary mechanisms\(^3\)).

These statements reveal inconsistent economic conceptions about the working of the interest mechanism.

On the level of simplified static analysis it has been normally assumed that
each excess-demand function is essentially related to one variable acting as an adjuster. Thus procedure 1 would imply that \( r \) is an adjustment variables for one of the "real" relations, such that

\[
\frac{dX_i}{dt} = H(X_i) \quad i = \text{any one out of } 1, \ldots, n-2, \text{ where } 0 = H(0)
\]

Proceeding the other way we are allowed to claim \( r \) as an essential adjustment variable for the bond excess-function

\[
\frac{dr}{dt} = F(X_{n-1}) \quad \text{where } 0 = F(0).
\]

Utilizing Walras' law to substitute \( X_1 \) in (3.20) by an expression in the other functions we establish

\[
\frac{dr}{dt} = H[-\frac{X}{P_1} - \frac{p_{n-1}}{P_1} X_{n-1} - \frac{i-1}{r} \sum_{r=1}^{n-2} \frac{p_r}{P_1} X_r - \sum_{s=1-r}^{n-2} \frac{p_s}{P_1} X_s]
\]

Comparing (3.30) and (3.29) we observe that a vanishing bond excess demand implies according to (3.29) a zero rate of change of \( r \). This connection does not hold according to (3.30). Even \( X_{n-1} = 0 \), interest rate may still fluctuate.

Unlike systems with only one function \( nh \) in \( p \) the "modified Lange system" does not produce the type of logical inconsistencies observed above; but we may state that it creates an inconsistency in economic propositions which are derived from its special logical properties.

B. The \( X_n \) System:

1. Let us now investigate the result of the following step: We accept as before the homogeneity postulate as a restriction imposed on the class of "real" functions considered. For the \( X_n \) postulate the assumption is substituted that the money function is expressed as a linear combination of the non-money functions such that

\[
-X_n = \sum_{i=1}^{n-1} P_i X_i
\]

The discussion around the "Patinkin problem" produced the following cases for consideration:

a) the \( X_i \)-functions \((i=1, \ldots, n-1)\) are \( h\#0 \) in the \( p_i \).

b) the \( X_i \)-functions are \( h\#0 \) in the same \( p_s \), where \( p_s \) is a proper subset of \( p_i \).
c) the $X_{r}$ (r=1,...,n-2) are hdl in $p_{r}$ (r=1,...,n-2), while $X_{n-1}$ is hdl in $p_{r}$.
d) the $X_{r}$ are hdl in $p_{r}$, while $X_{n-1}$ is nh in $p_{r}$.

Let us arrange the possible outcomes according to the homogeneity properties of $X_{n}$. Then we find:

A. Cases a and c imply $X_{n}$ hdl in a set of p's.
B. Cases b and d imply $X_{n}$ nh in $p$.\(^{33}\)

2. By their very construction the $X_{n}$ systems satisfy the monetary identity.

By this same construction Walras' law is of no relevance any more for these systems. They have the property

\[(3.28)\quad \sum_{i=1}^{n} p_{i} X_{i}(x) = 0\]

which is an identity in the $X_{i}$, not only in p. This simply means that Walras' law has already been utilized to express $X_{n}$ in terms of $X_{i}$, so that nothing further is left for elimination on this title. The terms just cancel each other in \((3.28)\).

The $X_{n}$ systems are essentially systems in (n-1) equations and (n-1) variables. For each single system of this type there is a fundamental set of (n-1) equations, which is regularly obtained whatever equation is deleted. The complete $X_{n}$ system is equivalent to its fundamental system—as it is derived from it by some elementary transformations.

Our problem is thus an evaluation of the fundamental set of equations of each $X_{n}$ system. The criterion will be if the set of relationships present a consistent and determinate theory of a monetary system. An investigation of the two situations will furnish the following results:

A. The fundamental system is defined by:

\[
X_{1}(\frac{p_{1}}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}) = 0 \quad \text{or by } X_{r}(\frac{p_{1}}{p_{n-2}}, \ldots, \frac{p_{n-3}}{p_{n-2}}, \frac{p_{n-2}}{p_{n-2}}, \frac{p_{n-1}}{p_{n-1}}) = 0 \quad r = 1,\ldots,n-2
\]

\[i=1,\ldots,n-1 \quad \frac{1}{p_{n-2}} X_{n-1}(\frac{p_{1}}{p_{n-2}}, \ldots, \frac{p_{n-3}}{p_{n-2}}, \frac{p_{n-2}}{p_{n-2}}, \frac{p_{n-1}}{p_{n-1}})\]

Without postulating the property $F(0,\ldots,0) = 0$ for $F(X_{1},\ldots,X_{n-1}) = 0$ we state with Patinkin that the system is either inconsistent or indeterminate. If $F(0,\ldots,0) = 0$ is assumed, the system is consistent, but will be satisfied by an infinite set of
money prices.

B. Case b adds nothing to those already treated under A. But an interesting feature is presented by case d) which is described by

\[ X_1 \left( \frac{p_1}{p_{n-2}}, \ldots, \frac{p_{n-3}}{p_{n-2}}, p_{n-1} \right) = 0 \quad i = 1, \ldots, n-2 \]

\[ X_{n-1} (p_1, \ldots, p_{n-1}) = 0 \]

This situation describes evidently a dichotomized pricing process. The set of \( X_i \) defines a real subsystem producing solutions for real prices and the rate of interest \( r = \frac{1}{p_{n-1}} \), while in the monetary subsystem after inserting for \( p_r = \frac{1}{r} \cdot p_{n-2} \)
the money-price of the numeraire commodity is determined. Like the Hickmann system, these conditions provide us a solution in money prices. But two objections remain:

First: The theory of interest involved states that \( r \) is an essential adjustment variable in one of the "real" relations. Thus, if there is a \( r^0 \) and \( e^r \), such that the corresponding \( X_{r^0} = 0 \), then \( r^0 \) is unaffected by \( X_{n-1} \neq 0 \). Such a conception of interest mechanism strains our imagination.

Second: The set of postulates under consideration does not satisfy the minimum requirements of a theory of a monetary system. It is able to account for finite prices, but fails completely to account for a non-zero stock of money. As the system stands any stock is compatible with the value-flow generated by the system.

Thus, we conclude our review of the \( X_n \) systems by stating that this set fails, even if consistency is assured, because it fails to meet the minimum requirements of a theory of a monetary system.

IV. Assets and Homogeneity of Market Functions

1. A macro-analytical argument has shown us that the subset of monetary systems characterized by a complementing property is fallacious. It is possible to clarify further the nature of the fallacy by utilizing the possibilities inherent in micro-analysis. In this way we shall prove the fundamental theorem of monetary systems:
The demand and supply functions of a monetary system with non-zero assets (including money stocks) and finite money prices will be non-homogeneous in the set of money prices.

It has been stated in Patinkin I that a utility function with money included is a necessary condition for solubility in money prices\(^35\). Actually as conceded in Patinkin III, the inclusion of money in the utility function is not a necessary condition. Before this problem is elaborated, let us prove in a simple and general way the above theorem.

Whenever a system of demand and supply functions is derived from some utility function, a budget restriction is combined with the first order conditions of a relative maximum utility to solve for the desired quantities in terms of the postulated parameters. Such a budget restriction reflects some general properties of the system considered. Thus a monetary system will have a budget equation containing terms for cash-balances. Let us consider now a system with money and bonds. In such a case the budget equation after division by \(p_{n-2}\) will look like

\[
(4.1) \quad \sum_{i=1}^{n-2} \frac{p_i}{p_{n-2}} \left( z_{i,\gamma} - \bar{z}_{i,\gamma} \right) + \frac{1}{r} \left( \frac{z_{n-1,\gamma} - \bar{z}_{n-1,\gamma}}{p_{n-2}} \right) + \frac{z_{n,\gamma}}{p_{n-2}} - \frac{\bar{z}_{n,\gamma}}{p_{n-2}} = 0
\]

In any monetary theory an expression like this combined with all first order conditions will give us the solutions. In solving out the system of relations the terms

\[
(4.2) \quad - \frac{\bar{z}_{n,\gamma}}{p_{n-2}} - (1 + r) \frac{\bar{z}_{n-1,\gamma}}{p_{n-2}}
\]

of the budget equation will become a part of all the solution functions. These terms will only disappear, if either \(\bar{z}_{n,\gamma} = \bar{z}_{n-1,\gamma} = 0\), or if \(p_{n-1} = \infty\) for any given set of finite real prices. Both of the quantities \(\bar{z}_{n-1,\gamma}\) and \(\bar{z}_{n,\gamma}\) are given parameters of the system and so the two terms will not be homogeneous in the prices.

So long as \(\bar{z}_{n,\gamma} \neq 0 \neq \bar{z}_{n-1,\gamma}\) and \(|p_{n-2}| < \infty\) market functions must depend on these terms, and thus cannot in general be homogeneous of any degree in prices\(^36\).
2. Using the results of Patinkin I it is easy to prove that a system with some utility function not including money and without any side relation in addition to the budget restraint will have for any non-zero money stock infinite money prices—with corresponding infinite circulation velocity.\footnote{37} If we start analysis with a non-zero stock of money and wish to derive finite prices, then the system has to provide some mechanism which keeps velocity at some finite level. This can be achieved by various processes. Two versions of one of the analytical possibilities have been worked out in Patinkin I and Patinkin III. Both versions are characterized by the fact that positive total and marginal utility is attributed to cash-balances. This postulate is considered to reflect some significant economic facts. The first variant includes nominal cash-balances while the second considers real cash-balances. In this way a mechanism is introduced which induces economic units to hold circulation velocity at a finite level.

3. Marschak has shown by an interesting argument\footnote{38} that we can derive a demand for (non-zero) money stocks without postulating money to be a part of the utility function. If the units have a horizon extending over a multiplicity of marketing periods, it will be rational to hold money-balances at any given moment as soon as money prices are expected to be lowered. So in such a system circulation velocity will be kept at a finite level so long as at least some units have this type of expectations.

Marschak was able to develop still another mechanism trapping velocity without putting money into the utility function. The starting point was given by the important notion of "cost of transaction." This phenomenon was expressed by a very ingenious device. While keeping the assumption of perfectly competitive markets in the sense that the exchange-price is considered by the individual to be unaffected by his supply and demand decisions the net-price is regarded as a decreasing function of the amount transacted. The net price differs from the exchange price by the costs of transactions. By this construction Marschak is able to utilize some parts of the apparatus of imperfect competition for the proof that velocity will be kept at a finite level without inserting money into the utility function.
4. Let us suppose that money is not in the utility function, but that there exists apart from the budget restraint another equation relating money to the objects actually entering the utility function. Such an equation has to be considered as a second restraint subject to which utility will be maximized\(^{39}\). This procedure allows different variants with varying degrees of empirical content and usefulness. At this point the simplest will be chosen in order to clarify the essential structure of this type of argument\(^{40}\).

The utility function contains all objects with the exception of money

\[(4.3) \quad U^{\alpha'} = U^{\alpha'}(z_{1,\alpha'}, \ldots, z_{n-1,\alpha'})\]

and the budget equation is given by

\[(4.4) \quad \sum_{i=1}^{n} p_i (z_{i,\alpha'} - \bar{z}_{i,\alpha'}) = 0 \quad \quad \quad \quad p_n = 1\]

To realize our objective \(U\) must be maximized not only subject to (4.4) but as well subject to some restriction with respect to the rate of utilization of given stocks of money. Thus we postulate

\[(4.5) \quad z_{n,\alpha'} = k_{\alpha'} \cdot \sum_{r=1}^{s} p_r (z_{r,\alpha'} - \bar{z}_{r,\alpha'}) \quad \quad \quad \quad \quad where \quad s < n - 1\]

The summation on the right defines the total value of the individuals net purchases. \(k_{\alpha'}\) is a given constant determined outside the economic nexus\(^{41}\). Solving the budget equation for \(z_{n,\alpha'}\) and combining the resulting expression with (4.5), we may establish

\[(4.6) \quad \bar{z}_{n,\alpha'} = \sum_{j=1}^{n-1} p_j (z_{j,\alpha'} - \bar{z}_{j,\alpha'}) = k_{\alpha'} \sum_{r=1}^{s} p_r (z_{r,\alpha'} - \bar{z}_{r,\alpha'}) \quad \quad \quad \quad \quad z_{r,\alpha'} - \bar{z}_{r,\alpha'} > 0 \quad \quad \quad \quad \quad s < n - 1\]

(4.4) has to be maximized subject to (4.6).

The first order conditions are

\[(4.7) \quad U_{r,\alpha'}^{\alpha'} + \lambda p_r (1 + k_{\alpha'}) = 0 \quad \quad \quad \quad r = s < n - 1\]

\[(4.8) \quad U_{k,\alpha'}^{\alpha'} + \lambda p_k = 0 \quad \quad \quad \quad k = s + 1, \ldots, n-1\]

Let us order the commodities in such a way that all cases \(z_{r,\alpha'} - z_{r,\alpha'} > 0\) comprise the first \(s\) of \(n-l\). Then let us choose \((n-l)\) as numéraire and divide (4.8) through
by \( p_{n-1} \). Thus we obtain

\[(4.8) \quad \frac{\bar{z}_{n,\alpha}}{p_{n-1}} = \frac{n-1}{\sum_{j=1}^{n-1} \frac{1}{p_{n-1}}} (z_{j,\alpha} - \bar{z}_{j,\alpha}) = k \frac{s}{\sum_{r=1}^{s} \frac{p_r}{p_{n-1}}} (z_{r,\alpha} - \bar{z}_{r,\alpha}) \]

and from (4.7) we derive

\[(4.9) \quad \frac{\bar{y}_r}{p_{n-1}} = \frac{p_r}{p_{n-1}} (1 + k_{\alpha}) \quad \text{when } r = 1, \ldots, s \]

\[(4.10) \quad \frac{\bar{y}_k}{p_{n-1}} = \frac{p_k}{p_{n-1}} \quad k = s + 1, \ldots, n-2 \]

(4.8) and (4.9) form together \((n-1)\) equations in \((n-1)\) unknown \(z_{i,\alpha} \) \((i=1, \ldots, n-1)\) for the \(\alpha^{th}\) individual. By inspection of the system it follows that \(p_j\) \((j=1, \ldots, n-2)\), \(p_{n-1}\), and \(\bar{z}_{n,\alpha}\) always appear in ratios. Clearly, the solutions of \(z_{1,\alpha}\) in terms of \(p_j\), \(p_{n-1}\), \(\bar{z}_{j,\alpha}\), and \(\bar{z}_{n,\alpha}\) are identical to the money stock \(\bar{z}_{n,\alpha}\) but not in \(p\) alone.

**Net demand and supply are then defined**

\[(4.10) \quad \frac{p_r}{p_{n-1}} = z_{r,\alpha} - \bar{z}_{r,\alpha} \quad r = 1, \ldots, s \leq n-1 \text{ and } z_{r,\alpha} - \bar{z}_{r,\alpha} > 0 \]

\[(4.11) \quad \frac{p_k}{p_{n-1}} = \bar{z}_{k,\alpha} - z_{k,\alpha} \quad k = s+1, \ldots, n-1 \text{ and } \bar{z}_{k,\alpha} - z_{k,\alpha} > 0 \]

By adding over all \(\alpha^{th}\) and all commodities we derive the market (net) demand and (net) supply functions. By adding over all money restrictions, the market function of money is found. These results are organized in the following system:

\[(4.12) \quad D_i = D_i \left( \frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}, z_n \right) \]

\[(4.13) \quad S_i = S_i \left( \frac{p_1}{p_{n-1}}, \ldots, \frac{p_{n-2}}{p_{n-1}}, z_n \right) \]

\[(4.14) \quad D_1 = S_1 \]

\[(4.15) \quad z_n = k \cdot p_{n-1} \sum_{i=1}^{n-1} \frac{p_i}{p_{n-1}} \cdot s_i \]

\[(4.16) \quad \bar{z}_n = \bar{z}_n \]

This system eliminates any homogeneity in prices together with the dichotomy in the pricing process. The fact of a "meaningful" money equation is sufficient to make the commodity functions, as well as properly derived, dependent on the real value of assets. Systems of this type are free from both invalidities, they are consistent with Walras' law and further do not present conflicting conceptions about the basic mecha-
nisms of the economic process.

V. Conclusion

1. Patinkin raised an interesting and important problem affecting our fundamental notions of the pricing mechanism. The critical reactions to his analysis showed that there was some need for inquiry into these problems. Patinkin himself contributed a very thoughtful and compact restatement of his position. The following summary of the arguments developed in this paper is to be viewed essentially as a "complementary commodity" to his main thesis.

We observed that dichotomisation arises in systems combining one or two functions nh in p's with functions h in some subset of p's: For the set of systems combining (n-1) excess-functions h in some subset of p's with one function nh in p we observed that the postulates were inconsistent with Walras' law. According to the specific system considered, it was shown that apart from denying Walras' law further inconsistencies were met. Systems combining (n-2) excess demand functions h in some subset of p with two nh functions were free of this type of logical fallacies. But they imply at one and the same time that the pricing mechanism is dichotomized and that there exists no dichotomy. Thus, this type creates inconsistent propositions about the very essence of dichotomisation. Such a conflict in basic notions about the nature of the economic process within one system is not conducive to satisfactory analysis. The n systems on the other hand were shown to be simply unable to build up a monetary theory. Hence we may formulate the general statement.

Monetary Systems with complementing property cannot be accepted as explanatory devices for the following two reasons:

a) either they are inconsistent with Walras' law or

b) they do not satisfy the minimum requirements of a theory of a monetary system.

2. Macro-analysis is able to prove the fallacy of the systems under consideration.

But we have to resort to micro-analysis to derive proper guiding principles for the construction of monetary systems. We realized that there is no need for inserting money into the utility function in order to build up a consistent monetary theory.
Considerable more investigations are required to clarify the relative usefulness of various approaches and their economic implications. If any properly constructed monetary system is considered, it will be seen that the explicit and implicit money function will coincide. This is a necessary condition of proper construction. The clarification of this fact is an important contribution of micro-analysis.

3. How is classical theory affected by our results? We have to realize that classical theory is composed of two unrelated segments. One contains a theory of a real economy and the other the theory of a monetary economy—and there is no simple step from the one to the other. They are two completely different theories with different implications. Their essential difference is best reflected by a comparison of the interest-mechanism involved. For the real theory the thesis may hold that the rate of interest is the main adjustment variable for savings and investment. But for the monetary theory—with liquid assets—the statement holds that the interest rate is the main variable for adjusting supply and demand of the stock of securities while the savings-investment mechanism is dominated by the level of money prices.
FOOTNOTES

1) "Relative Prices, Say's Law and the Demand for Money," *Econometrica*, Vol. 16, April 1948. It will be referred to as Patinkin I.


2) Among the five was Patinkin III, the paper published in the current issue of *Econometrica*. The three contributions in the January number of *Econometrica* 1950 are: W.S. Hicks, "The Determinacy of Absolute Prices in Classical Economic Theory"; W. Leontief, "The Consistency of the Classical Theory of Money and Prices"; Cecil G. Phipps, "A Note on Patinkin's Relative Prices." The manuscripts were submitted by Kenneth Arrow, Gary S. Becker, Donald Font and Cecil G. Phipps.

3) I have to acknowledge further indebtedness to H. Arrow, C. Christ, D. Font, C. Hildreth, L. Hurwitz, T.C. Koopmans, N. Mankowitz, J. Marschak, and L. Slater for the discussion of various aspects of the problem. Of especial importance has been the discussion with D. Patinkin to whose patient clarification of most points I feel very much indebted. Needless to say, the responsibility for all errors remains unfortunately mine.

4) By this I mean an equation involving finite money prices and a positive stock of money. It has to be added that the formulation above is only valid for macro-analysis. In the case of micro-analysis, homogeneity is no postulate; it is a theorem. For this case we may say that the complementing property implies that a meaningful money equation may be superimposed on the equilibrium conditions without affecting the properties of these conditions.

5) These minimum requirements are: the system must "explain": (a) non-zero money-stocks, and (b) finite money prices—without producing inconsistent propositions.

6) Micro-analytical considerations are introduced into the "Patinkin problem" only for the reasons given above. Whether some macro-models are consistent with some micro-models is of secondary importance.

7) The notation \( \mathbf{p} \) means an identity in the price-vector. After the type of identity has been defined, it will not be repeated again in later formulations.

8) I am indebted to C. Christ for the suggestion to complete the demand side of the accounting identity with a consideration of the B-sector.

9) These sections are essentially an elaboration of section 8 in Patinkin II.

10) It is necessary to remind the reader that the whole discussion is kept within a static framework. That is, we suppose that the relevant phenomena can be analyzed as if they were solutions of static systems. Especially is it assumed that the timing, structure of market behavior, actual transactions and payments is irrelevant.
11) To abbreviate the exposition the following notation is introduced
hdt = homogeneous of degree t = h in p = homogeneity (or homogeneous)
in money prices = nh in p = non-homogeneity (or non-homogeneous) in
money-prices.

12) The proof is on p. 13 in Patinkin II. The particular system discussed
is presented on the same page. We shall term it the "Casselian system 1."


14) This system might be termed the "Lange system." It is not equivalent
to the Lange system presented in section 12 of Patinkin II. The last
one is an $X_n$ system and not an $U_n$ system.

15) The development of the argument in Patinkin II and III and in Hickmann's
paper implies very clearly the thesis that consistency cannot be derived
from homogeneity in macro-analysis. A. Becker provided in a written
discussion a very interesting proof of this statement. So most partici-
pants to the discussion of the "Patinkin problem" agree that in macro-
analysis some additional postulates are required.

16) These two postulates will eliminate cases a and b in Patinkin III, p. 3
of the current issue. It must be noted that Patinkin's two invalidity
theorems are independent of such postulates. He achieved in this way
a major improvement as against Patinkin II.

17) It would be sufficient to postulate that the functional dependence of
the $X_i$-functions (i=1...n-1)
$F(X_1, X_2, ..., X_{n-1}) = 0$ is of the form $F(0, 0, ..., 0) = 0$

This postulate is less restrictive than the above one. I suppose
Hickmann was aware of this—consider footnote 19 on p. 19 of his paper.

18) This is essentially Hickmann's formula on p. 18 of his paper. It differs
only by not supposing all the $X_i$'s to be identical.

19) op. cit. p. 16. Actually, as was acknowledged by Hickmann, this result
was already stated in Patinkin II, p. 15.

20) Hickmann's paper is very clear on this point. Unfortunately his
presentation suggests a critique of Patinkin which is not well taken.
He states on p. 19 that a relation of the type $F(X_1, X_2, ..., X_{n-1}) = 0$
is implied by the homogeneity postulate. If a solution-vector exists for
$X_2 = X_3 = ... = X_{n-1} = 0$, then $F(X, 0, ..., 0) = 0$, where $X$ is the value of $X_1$
for the given price-vector. Then he states: "For a wide class of
functions the equation of dependence will imply that $X_1$ is zero."
Because a class of functions exists, such that $F(0, 0, ..., 0) = 0$, Hickmann
thinks it "unsire to assume a priori that the commodity equations are
independent and inconsistent." As was already mentioned, no such
assumption is involved to establish Patinkin's central theorems. There
is actually no need for a dispute between Patinkin and Hickmann at this
point. Both state that given the homogeneity postulate there exists a
class of functions such that $\Psi(X_1, 0, ..., 0) = 0$, $X_1 \neq 0$ and another
class of \( X_i \) functions such that \( \psi_j(0,\ldots,0) = 0 \). The difference between Patinkin and Hickmann in this context must be seen in the fact that Patinkin elaborates the implications of the homogeneity postulate without introducing any additional assumption, while Hickmann assumes in addition a proper restriction on the class of functions. Thus Patinkin is able to derive theorems which cover both consistency and inconsistency. Hickmann seems to think that by providing consistency he may escape the fallacy formulated by Patinkin's invalidity-theorems.

A further point must be mentioned: In conformity with Patinkin III Hickmann states "the dependence of classical functions by no means eliminates the possibility that the equations may be inconsistent." Nevertheless he writes on p. 17: "Dependent equations are always consistent..."

21) The difference between the monetary identity fallacy and the Say's law fallacy must be clearly noted. The first case involves properties of equivalent functions—and the second one the comparative properties of equations and identities.

22) The fact that a system with Say's law provides a solution of relative prices in the real part and of money prices in the monetary subsystem seems to indicate its analytical soundness. But such a procedure utilizes an irrelevant criterion. We have to consider the logical propositions by which a solution is realized. It may be that the fact of a solution has blinded some economists against the contradictions involved in establishing a solution. The nature of this inconsistency eliminates its being reflected in the mathematical solution—thus the latter is completely useless as a criterion.

Consider the procedure of the system: First, we have a proposition, stating that for any price-vector \( p \), \( M_D \) and \( M_S \) will coincide. So, if a specific \( p^0 \) is selected, it will always fit in with the solutions of first proposition. The real problem is then concerned with the question: Are the logical properties of such a selection procedure consistent or not with the first proposition. If we state

\[
(1) \quad x_i \left( \frac{p_1}{p_{n-1}} \ldots \frac{p_n}{p_{n-1}} \right) = 0 \quad i = 1 \ldots n-1 \quad \text{and postulate}
\]

\[
(2) \quad \sum_{i=1}^{n-1} p_i x_i = 0
\]

we have a consistent system determinate in \( z_r = \frac{p_r}{p_{n-1}} \) \( r = 1 \ldots n-2 \).

And if we add

\[
(3) \quad p_{n-1} = p^0
\]

where \( p^0 \) is a given parameter, then a unique value of \( p_{n-1} \) has been selected from the infinite set of \( p_{n-1} \) by a proposition fully consistent with (1). On the other hand a unique value \( p^0 \) of \( p_{n-1} \) may be selected by stating

\[
(4) \quad K \cdot p_{n-1} \sum_{r=1}^{n-1} \frac{p_r}{p_{n-1}} s_r - M = 0
\]
But in this case, proposition (h) defining \( d^0 \) cannot be interpreted with a proposition implied by (1). Both (3) and (h) select one \( n_{i-1} \) out of an infinite number acceptable to (1)—but the propositions by which this is done are not equivalent: (3) is compatible with (1), while (h) is not.

In this context Leontief's contribution to the discussion has to be considered. Patinkin stated that the classical systems with complementing property are either inconsistent or indeterminate (after the money function has been adjusted to the homogeneity postulate). Leontief retorted by stating the consistency of the real subsystems, as soon as Say's law is introduced. His case has been fully recognized in Patinkin II (section 13). But Patinkin's theorem states further: given Say's law, indeterminacy cannot be removed. On this Leontief remains silent. Thus, his statement is correct, but irrelevant. Why does he entitle his contribution "The Consistency of the Classical Theory of Money and Prices," while he analyzes a consistent theory of a real economy? If I have to interpret him as implying that determinacy is introduced by "simply" complementing the real system with a money equation, then the above criticism applies. In this case his proof of the consistency of the "classical theory" fails. And in case he removes indeterminacy by some device like (3) above, then the system presented fails to satisfy the "minimum requirements" of a monetary theory.

Leontief's statement that Patinkin's criticism is aimed at the logical foundations of the non-Keynesian theory of general equilibrium (p. 21) has a suggestive connotation which is not correct. The issue raised by the "Patinkin discussion" has no relation with the question "Keynesianism" versus "Classicism." The latter is essentially characterized by a set of theorems which are not disputed in this context; our question is then: which system properly constructed, produces such theorems?

I want to acknowledge that S. Becker's paper contained a very thoughtful and elegant criticism of Leontief's contribution to the discussion. It is further necessary to remind the reader of Oscar Lange's contribution to the present issue in his paper, "Say's Law: A Restatement and Criticism," in Studies in Mathematical Economics and Econometrics.

23) It may be worthwhile to express this fact in concrete terms. On p. 77 of Hickmann's paper \( x_n \) is defined as

\[
x_n = - p_1 x_1 - p_2 x_2 + d_1 p_2 - d_2 p_1 + s_1 \frac{p_1^2}{p_2} + s_2 \frac{p_2^2}{p_1}
\]

and

\[
y_x = x \cdot p_2 \left[ s_1 \left( \frac{p_1}{p_2} \right)^2 + s_2 \frac{p_2^2}{p_1^2} \right] - n.
\]

Thus \( y_x = x_n \) requires the following polynomial of 3rd degree to vanish identically in the \( p_1 \)s

\[
[ s_1 (1-k) ] p_1^3 - d_2 p_1^2 p_2 - d_1 p_1 p_2^2 + [ s_2 (1-k) ] p_2^3 - \lambda p_1 p_2 \equiv 0.
\]
Sufficient conditions for this to happen are

a) \[ K = I, \ d_2 = d_1 = 1, \ M = 0 \quad \text{or} \]
b) \[ S_1 = d_1 - d_2 = S_2 = M = 0 \]

But this is ruled out by the nature of the economic problem. Especially would it mean that because of \( M = 0 \), the minimum requirements of a monetary theory could never be satisfied. Some other aspects of the system presented by Hickmann is of interest: suppose \( c \) given by (3.13) \( \mathbf{S}_1 \)

\[ \sqrt{S_1} = z \]

so that Say's law is not implied by the given structure. We may state then that there exists some \( k \), such that \( c \cdot k = z \), which transform Hickmann's system with the matrix

\[ \begin{pmatrix} -S_1 & d_1 \\ d_2 & -S_2 \end{pmatrix} = \begin{pmatrix} -S_1 & d_1 \\ -c \cdot S_1 & -c \cdot d_1 \end{pmatrix} \]

where \( c = \frac{d_2}{S_1} = \frac{S_2}{d_1} \)

and linear combination \( x = \mathbf{c} x_1 + x_2 \equiv 0 \) and \( x \not\in \mathbf{I} \) into a system with a matrix

\[ \begin{pmatrix} -S_1 & d_1 \\ d_2 & -S_2 \end{pmatrix} = \begin{pmatrix} -S_1 & d_1 \\ -c_1 \cdot S_1 & -c_1 \cdot d_1 \end{pmatrix} \]

and linear combination \( \bar{x} \equiv c \cdot k \cdot \begin{pmatrix} S_1 \ 1 \\ k \end{pmatrix} + x_2 \equiv 0 \). And because of \( c \cdot k = z \)

we have \( \bar{x} \equiv \mathbf{x}_n \equiv z \cdot x_1^* + x_2 \equiv p_1 x_1^* + p_2 x_2 \equiv 0 \) \( \text{where } x_1^* \equiv \frac{1}{k} \cdot x_1 \)

The matrices show that the solutions are invariant with respect to \( k \).

Thus we observe that Hickmann's system is equivalent to a system implying Say's law.

In Hickmann's paper there is to be found an even much more direct statement as to his view on Say's law in a monetary system. On p. 18 we read: "(Say's law) occurs...in a full monetary economy in which goods are sold only for the purchase of other goods." Thus he holds explicitly that it is analytically sound to propose simultaneously \( x_n \equiv 0 \) and \( M_x = 0 \), where \( M_x \) is an equation involving \( M \neq 0 \) and \( |p| < \infty \).

On p. 15-16 Hickmann attempted vainly to dispose of the fallacy criterion provided by the monetary identity. He starts with the monetary constraint

\[ (1) \quad M = K \cdot D_n(p) \]

where \( K \) and \( M \) are constants and \( D_n = \sum_{i=1}^{n-1} p_1 S_i \). Then he states that \( D_n \) is bdl while \( M \) and \( K \) are bdo in \( p' \). This eliminates a relation of the form

\[ (2) \quad M = K \cdot D_n(p) \]

And he states: "...this fact does not invalidate the Cambridge equation,
since it is a constraint and not an identity in the p's." Everybody will agree to this. To postulate the Cambridge equation means excluding \( M_n = 0 \), but then Hickmann cannot hold simultaneously \( X_n \neq 0 \). But the monetary identity is not at all concerned—as Hickmann thinks—whether the postulates chosen prevent \( M_n \neq 0 \). The problem is whether \( \lambda X_n \) holds or not, irrespective if \( X_n = 0 \) or \( X_n \neq 0 \). That is a different question than Hickmann's.

24) The formulation on p. 8 in Patinkin III is very definite: "...after the functions of the classical systems are modified, so as to be free of Invalidity I, the resulting system of classical excess demand functions must be either inconsistent or indeterminate."

25) op. cit., section 11. The presentation here differs insofar as a \( \lambda X \) function—possibly of Cambridge form—is postulated.

26) op. cit., p. 11.

27) There is another—but equivalent way—of visualizing the fallacy: System (3.19)—(3.20) contains \((n-1)\) variables and \(n\) equations. This fact—as shown in Patinkin III—is sufficient to imply functional dependence. Whenever the system is supposed to describe the complete universe of transactions 'Alras' law is the expression of this dependence. Utilizing this and eliminating \( X_{n} \), we are left with (3.19) a real system possibly consistent. If on the other hand an \( X_{1} \) is eliminated, we have a system possibly consistent and determinate. Such a dependence of the results from the elimination process is nonsensical but it reflects what the assumptions of \( X_{1} \) and \( \lambda X \) properties plus 'Alras' law produce. And if we give up 'Alras' law, other inconsistencies crop up. Whatever way we turn it, the system simply makes no sense, and interestingly this is not registered by invalidity I nor II.

28) This proof is exactly analogous to the reasoning applied in the last section and in Patinkin II, p. 11.

29) The derivation of a nh bond-function from a nh money-supply function and the existence of bond-transactions proceeds in the following way:

Denote by \( B^B_0 \) and \( B^P_0 \) the given stocks of bonds in the P and the B-sector respectively and by \( B^B_d \) and \( B^P_d \) the desired quantities (stocks) in the two sectors. Then let us divide all the units of the P and the B sector into two groups for each sector, such that \( P^P \) contains all P-units which for a given price-vector have a positive net demand (the same holds for \( B \)) while \( P^B \) (or \( B^P \)) contains all units with a positive net supply. In this way

\[
B^P_{n-1} = B^P_d - B^P
\]
\[
B^B_{n-1} = B^B_d - B^B
\]
\[
S^P_{n-1} = B^P - B^P_d
\]
\[
S^B_{n-1} = B^B - B^B_d
\]

Further, by utilizing relations (2.23) and (2.24) we observe that the net-injection is related to the bonds-transaction \( B^P_{n-1} \) and \( S^B_{n-1} \) between
the F and the B sector, so that
\[ S^B = B^B = L^B = S^P = p_{n-1} (B^B_d - B_d) \]
when
\[ B^B_d = B^B_d + B^B_d \quad e^B = e^B + e^B \]

This implies a relation \( B^B_d = F [L^B_S(p, \alpha)] \) when \( \frac{d}{dp} B^B_d = \frac{1}{p_{n-1}} \frac{d}{dp} L^B_S(p, \alpha) > 0 \)

By hypothesis \( B^B_d \) must be nh in prices. Together with the assumed working properties of the money creating mechanism this implies nh of its components. Suppose \( B^B_d \) and \( B^B_d \) to be hdi — by taking this hypothesis over from the Lange system—then \( B^B_{n-1} \) and \( S^P_{n-1} \) will be nh, \( B^B_{n-1} \) and \( S^P_{n-1} \) in general nh as well, while \( B_d \) will be strictly nh.

30) Provided \( p_d \) \((i=1...n-1) \neq 0 \) there is no reason why not any function may be selected to be expressed in terms of the others.

31) These divergent economic propositions do not imply different solution-vectors. The functional dependence postulated in the form of "alras" law will take care of this.

32) "We may state that all the systems with one function nh, which exclude "alras" law by their postulates imply of course inconsistent economic propositions. Take Hickmann's system: if he accepts the monetary identity, he is forced to admit \( M = 0 \), but in his monetary constraint he supposes \( M 
eq 0 \). Or if he denies a monetary identity, he implies that an individual's cash-balance might increase by more than by his net-inflow, which as an economic proposition makes no sense. The system developed on p. 22 in Patinkin II corresponds to system (3.23), with the difference (possibly) that he presented a \( L^2 \) system. In Patinkin II we find no analytical objection to this system and in Patinkin III there is an implied criticism by resorting to micro-analysis. Patinkin argues further (p. 12-13) that the (implicit) money function of the system cannot be hdi in p and the stock of money as the Cambridge function or as in a system derived from micro-analysis. This is correct and shows the inconsistency of this "modified Lange system" with one built up by a micro-analytical argument. But we must note that the latter procedure implies as well that the demand function for money (not only \( L^2 \) be hdi in p and \( M \)—which is not consistent with the Cambridge function. A direct macro-analytical argument may strengthen the case against a system which is not consistent with micro-analysis.

33) Case a has been presented on p. 5-6 and \( L^2 \); case b on p. 16-17; case c on p. 18-19; case d on p. 22 of Patinkin II.

34) It may be necessary to state that the system (5.1)-(5.3) presented on p. 15 of Patinkin III is essentially different from the one discussed above. It is able to account for a definite non-zero stock of money and finite p's without creating an analytical impasse, because the complementing property is not involved.
35) Patinkin I, p. 150.

36) Patinkin III contains an interesting argument about the relation between homogeneity in all quantities of a function and homogeneity in a proper subset. He shows that both hold only, if the function is h of the same degree in the complementary subset as well. But this implies in our case, either assets to be zero, or the derivative of market functions with respect to assets to be zero. Both statements are equivalent to market functions without assets as parameters.

37) Patinkin proved explicitly that the consistency of such a system requires zero money-stock. What consistency requires is that the real values of money-stocks be zero, which may be accomplished for non-zero nominal stocks by infinite money prices.

C. Phipps attacked in his contribution to Econometrica mainly some aspects of Patinkin I. But it seems to me that Phipps confused two essentially different phenomena: namely, the "own-price" of a commodity chosen as numeraire and the exchange-price of the numeraire commodity. The first is by necessity always unity, the price of anything in terms of itself cannot be anything else. There is no economic significance in this. The second measures the quantity of other commodities given up or acquired for a unit of the numeraire commodity. The second may be zero, finite or infinite while the first is univariantly unity.

Phipps states first that any commodity with zero utility or zero marginal utility will have a zero exchange price. Then he states explicitly the converse. These propositions are then applied to check Patinkin's analysis. Patinkin started with the postulate that money has no utility, from which followed his conclusion that money will have a zero exchange price. According to Phipps this contradicts some other assumption of Patinkin's, namely \( P_0 = 1 \). But this is simply the "own-price of money." So there is contrary to Phipps' contention no contradiction whatsoever in Patinkin's argument and postulates. But I have to add that there were in Phipps' other paper a number of suggestive statements. Especially one, referring to the Casselian system and observing that the functions of the system are incomplete, because an important variable or parameter is neglected, seems to indicate essentially the same idea as formulated in this section.

38) Harshak "The Rationale of the Demand for Money and of Money Illusion."

39) I had a highly stimulating discussion on this point with Donald Fort.

40) I am indebted to C. Hildreth for important suggestions as to the analytical argument.

41) In one of Donald Fort's discussion papers there was an able discussion of the economic significance of the timing pattern of the flow of payments through a cash-balance. "e may conceive the k as a simple characterization of such timing patterns.

42) Donald Fort's arguments were essentially organized around the problem of the timing patterns of the in- and outflow of money in and out of cash-balances. If we wish to acknowledge these facts, but treat it by a short-
cut—without incorporating it as an additional variable we proceed as above within a static framework. But already within static theory it is well possible to do much better and to have velocity as a variable emanating from the solution of the whole system. But I think that Fort misses the essential conception of this paper if he considers the postulate formulated in footnote 10) as relevant for the issue on hand. This postulate has been added to remind the reader of the type and range of analysis developed in this paper. But the main issue is dichotomization. And my contention, like Patinkin's, is, that any theory of a monetary system satisfying the minimum requirements will not be able to present a dichotomized pricing process. Fort's arguments do not contain anything which challenges his statement—they seem more of a complementary nature, in the sense that we have to expand the above results by a truly dynamic analysis: Such an extension would scarcely change our main contention.

43) In this respect I deviate definitely from O. Becker, who in essence upheld Hickmann.

44) It is not sufficient. For instance, we may define macro-system with all of its functions nh in p and postulate then "alras' law, so that

\[ x_i(p_1 \cdots p_{n-1}) = 0 \quad i = 1, \ldots, n-1 \]

But from the point of view of economic analysis this procedure is thoroughly unsatisfactory. By resorting to micro-analysis we can do better than just stating nh in p. An important economic meaning will be attached to the resulting nh functions. Thus we may state as a construction principle of monetary theories, that any macro-system ought to be consistent with micro-analysis. Such a procedure will eliminate all types of inconsistencies mentioned. While artificial and arbitrary assumptions will be avoided a maximum of economic meaning can be gained in this way.