The Stock Market in the Overlapping
Generations Model with Production

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1. Introduction

This paper addresses the following question: does the stock market influence the process of capital accumulation? If exchanging ownership of firms on a stock market is equivalent to exchanging the ownership of their capital on a capital goods market, then introducing a stock market will not affect the predictions of the real models of capital accumulation—the Ramsey model (1928) if agents are infinitely lived, or the Diamond model (1965) if agents are finitely lived. The assumption that ownership of firms is transferred through the stock market rather than the capital goods market can lead to a different outcome only if there is a friction which makes the stock market into a financial entity distinct from the real capital goods market.

The friction that we study in this paper is the firm specificity of capital which makes it costly, if not impossible, to detach part of the tangible or intangible capital of a firm to sell it on a (second-hand) market for capital goods. To quote Tobin (1998, p. 147) “The various physical assets of a business enterprise are often designed, installed and used in complex combinations specific to the technology. It is costly or impossible to detach and move individual assets or to apply them to alternative purposes.” We take this observation to the theoretical limit by assuming that capital, once installed, is a sunk cost: it cannot be transformed back into a consumption good or used by other firms.

Under this assumption, when capital is durable, firms must be long lived and if transferred, must be kept intact in their entirety. If, as we shall assume, economic agents are short lived, then there is a need for a market which makes such transfers possible, and this is one of the important roles of the stock market: each firm becomes a separate legal entity which issues equity shares to its future income stream, and ownership of firms can be transferred in perfectly divisible amounts across an indefinite succession of finite-lived shareholders, while retaining in perpetuity the full physical and organizational entity of the firm.¹

We are thus led to study the role of the stock market as an instrument for transferring firms in the setting of the standard Diamond model to which we add the friction that capital once installed in a firm cannot subsequently be sold (i.e. has a zero price) on the market for current output. We do not assume any frictions on “new” investment: thus the financial value of a firm cannot exceed its replacement cost, for the young agents could always recreate the capital of the firm out

¹Blackstone (1765) in his Commentaries on the Laws of England, (Book I, Chapter XVIII), referred to “perpetual succession” as the “very end of incorporation: for there can not be a succession forever without an incorporation”. He explained “it has been found necessary when it is for the advantage of the public to have any particular rights kept on foot and continued, to constitute artificial persons, who may maintain a perpetual succession, and enjoy a kind of legal immortality. These artificial persons are called... corporations.”
of current output if it were less expensive to do so, and the firm would not sell at its current equity price. However, and this is the important point, the assumption that previously installed capital is a sunk cost, permits the equity price to be less than the replacement cost without creating arbitrage opportunities.

While old agents have no choice but to sell their firms on the equity market, young agents have several choices. They can decide whether to invest in equity or bonds and whether or not to invest new capital in their firms. Absence of arbitrage requires that the rate of return on each of these “investments” is the same. We show that if the equity price of each firm is its replacement cost less a lump sum discount, then all rates of return are equalized and investment is positive, provided that the discount grows at the rate of interest and does not become too large—in a sense made precise in Section 2.

The valuation of equity in this model is akin to a two-part tariff, with a marginal price of one for the installed capital and a negative fixed part given by the discount. This “two-part tariff” valuation formula for the equity of firms leads to an interesting new mechanism by which the stock market influences investment, especially for the class of economies regarded by many economists (see e.g. Abel et al (1989)) as empirically the most relevant, namely those characterized by underaccumulation. For in such economies the savings of the young are scarce and, in the standard Diamond model, do not suffice to lead the economy to the Golden Rule. However when there is a discount on the equity prices of firms, this discount — no matter how small — frees some of the scarce savings of the young and enables them to be used to purchase new investment rather than paying for previously installed capital. Although the investment behavior of firms in our model is the same as in Diamond’s model, it is “as if” there were more savings in the economy (thanks to the discount), so that the equality “savings = investment” occurs at a lower interest rate than in the Diamond equilibrium. As a result there is more investment, and hence more output, wages and savings in the next period, and this virtuous cycle fuels a sufficient increase in investment to lead the economy to the Golden Rule rather than to the Diamond steady state. Since the Golden Rule is the efficient steady state, the dynamic analysis reveals a new benefit derived from the stock market, as an instrument for the transfer of ownership of firms between generations, which is separate from its liquidity role mentioned above, and its risk-sharing role which typically takes preeminence in models of financial economics such as CAPM, but which is inevitably absent in the simple deterministic framework of this model.

The idea that frictions, or adjustment costs, may importantly influence the process of capital accumulation has a long tradition in economics (Lucas (1967), Gould (1968), Uzawa (1969),
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Kydland-Prescott (1982)) and some authors have derived from such adjustment costs the existence of a Tobin’s $q$ different from 1 (Hayashi (1982), Basu (1987), Abel (1999)). These papers study the effects of adjustment costs in installing new capital, and typically assume that these costs are convex. Our approach is different in that it focuses on the adjustment costs that would need to be incurred if previously installed capital were to be put to an alternative use, rather than on the cost of installing new capital. As a consequence in our model, Tobin’s $q$ is always less than or equal to one.

Since the equilibrium of the model yields both the market values of firms and their endogenously determined dividend streams, it is natural to ask whether the market values coincide with the discounted sums of their future dividends, or if “rational bubbles” on equity can occur in equilibrium. Since the traditional Diamond model does not have an explicit stock market, the Tirole (1985) and Weil (1987) analysis of bubbles in the Diamond model focused on money, modeled as an asset with a zero exogenous dividend stream. They showed that sustainable bubbles can only exist in the case of overaccumulation, albeit with no real effects on the long-run dynamics of the equilibrium.

Our results are somewhat different, for the following reason. Whether or not bubbles occur in equilibrium depends on the asymptotic behavior of interest rates. Since the presence of a discount on the market values of firms tends to lower interest rates, it makes the existence of bubbles more likely. In fact we show that bubbles are the rule rather than the exception, even in economies with underaccumulation: the only trajectories on which firms’ market values coincide with the discounted sums of their dividends are those of an economy with underaccumulation on which the discount is zero at all dates—the Diamond trajectories. Since these trajectories are saddlepoint stable, they are exceptional in the dynamics of our model. On every other path, despite the fact that firms are “cheap” when compared to the replacement cost of their capital, the market value of a firm exceeds the fundamental value of its dividends. The fact that equity contracts permit firms to be transferred from one generation to the next give them a transmission value over and above the value of their dividends. To the extent that sunk costs can affect the valuation of firms, this model suggests that the fundamentalist view of standard finance theory should perhaps be broadened to take this additional source of value into account.

The paper is organized as follows. Section 2 describes the model and introduces the concept of a stock market equilibrium. The asymptotic properties of such an equilibrium are studied in Section 3. The comparison between firms’ market values and the discounted sums of their dividends is the subject of Section 4.
2. The Stock Market Model

Consider a standard OLG model with production: at each date $t$, $N_t$ young agents are born who live for two periods, $t$ and $t + 1$ and each of these agents is endowed with 1 unit of labor when young, having no initial resources when old. Agents of all generations are identical, with the same endowment (1 unit of labor when young) and the same preferences, represented by a utility function $u(c^s_0, c^s_1)$ over consumption streams $c^t = (c^0_t, c^1_t)$, where $c^s_t, s = 0, 1$, represents the consumption at date $t + s$ of an agent born at date $t$. The population is assumed to grow at the exogenous rate $n_t (n \geq 0)$, i.e. $N_{t+1} = (1 + n)N_t$.

On the production side, there is a collection of $J$ firms ($j = 1, \ldots, J$), each firm producing at each date $t$ an all-purpose good — which we will call the output—from capital and labor, with the time-invariant technology $Y^j_t = F(K^j_t, L^j_t)$ where the function $F$ is the same for all firms and is smooth, concave, strictly increasing and homogeneous of degree 1. The output of firms can be used either directly for consumption or to create new capital, where it takes one unit of the good to produce one unit of new capital for any firm. Capital in each firm is durable and depreciates at the rate $\beta$ ($0 < \beta < 1$), and needs to be installed one period before it is used: thus the capital $K^j_t$ used by firm $j$ is the capital that it has carried over from date $t - 1$.

The model that we introduce differs from that of Diamond (1965) by the assumption that capital once installed in a firm cannot be “unbolted” and transformed back into the homogeneous current output, without incurring significant adjustment costs — which for simplicity we take to be infinite. Thus once capital has been installed in a firm, it cannot be used for consumption, nor can it be used for new investment (i.e. additional capital) by any other firm: in short, it is sunk in the firm. As indicated in the introduction this assumption is designed to capture the fact that many resources invested in a firm have to be adapted in a way which is firm specific to make the whole production process function smoothly and efficiently. Since the precise way in which the resources have been adapted typically makes them inappropriate for use by other firms, such installed capital has limited value on a resale market. For example, software written specifically for a firm and incorporating its specific needs may be very expensive — it consumes a great deal of labor not used for producing the consumption good — but has essentially no resale value. Even those capital goods which have a resale value, for example plant and machinery, usually have a low value on the used capital market relative to their replacement cost, since significant “adjustment costs” have to be incurred to adapt them for use by other firms. To capture this phenomenon in a simple way, we study the theoretical limit in which the installed capital of a firm is completely
firms to be infinitely lived. Invested capital has no value if the firm is liquidated, and has value only if the firm retains its identity as an income generating unit in the economy. The natural market structure which permits short-lived agents to transfer ownership of long-lived firms from one generation to the next is an equity market for ownership shares of firms. Thus to have a market structure consistent with the firm specificity of capital, we assume that each firm is a corporation with an infinite life whose ownership shares are transmitted from one generation to the next through the stock market. Let $q_t^j$ denote the price of a share of the the $j$th firm’s equity at date $t$.

At each date $t$, in addition to the stock market, there are three other markets: a market for current output, a labor market, and a bond market. Since this is a real (as opposed to a monetary) model, the price of a unit of current output is normalized to be 1. Let $w_t$ denote the wage rate at date $t$ on the labor market on which the (homogeneous) services of labor supplied by the young generation are sold to the firms. The bond market provides firms with a source of external funds for financing investment which is an alternative to issuing new equity shares, and gives young agents a way of borrowing and lending. Let $r_{t+1}$ denote the interest rate on a loan from date $t$ to date $t+1$ and let $(1,(q_t^j)_{j=1}^J,w_t,r_{t+1})$ denote the vector of prices on these four markets at date $t$ ($t = 0, 1, \ldots$).

**Agent’s maximum problem.** The representative young agent born at date $t$ purchases a portfolio of securities

$$(z_t, \theta_t^1, \ldots, \theta_t^J)$$

consisting of an amount $z_t$ of bonds and a number $\theta_t^j$ of shares of firm $j$ (for $j = 1, \ldots, J$) so as to maximize lifetime utility $u(c_0^t, c_1^t)$ subject to the budget constraints

$$c_0^t = w_t - z_t - \sum_{j=1}^J \theta_t^j q_t^j$$

$$c_1^t = z_t (1 + r_{t+1}) + \sum_{j=1}^J \theta_t^j (d_{t+1}^j + q_{t+1}^j)$$

(1)

where $d_t^j$ denotes the dividend per share paid by firm $j$ at date $t$. The agent takes the prices $(1,(q_t^j)_{j=1}^J,w_t,r_{t+1})$ as given, and correctly anticipates the next-period dividends and prices of the firms $(d_{t+1}^j,q_{t+1}^j)_{j=1}^J$. The maximum problem of the agent has a solution if and only if the no-arbitrage condition between the stock and the bond market

$$q_t^j = \frac{1}{1 + r_{t+1}} (d_{t+1}^j + q_{t+1}^j), \quad j = 1, \ldots, J$$

(2)
holds for the equity price of each firm. Since by (2) the rate of return on the bond and each of
the equity contracts is the same, the agent is indifferent between investing in any firm or investing
in the bond market: all that matters is the total sum invested in the capital markets, namely the
agent’s total savings $s_t$. When (2) holds the budget equations (1) can be written as

$$
c_0 = u_t - s_t
$$
$$
c_1 = s_t(1 + r_{t+1})
$$

(3)

where

$$
s_t = z_t + \sum_{j=1}^J \theta_{t,j} q_j
$$

(4)

The maximizing behavior of the agent is summarized by the savings function $s : \mathbb{R}_t^2 \rightarrow \mathbb{R}$ defined by

$$
s(r_{t+1}, u_t) = u_t - c_0(r_{t+1}, u_t)
$$

where $(c_0(r, w), c_1(r, w))$ is the solution of the problem of maximizing $u(c_0, c_1)$ subject to the budget
equations (3), or equivalently the solution of the problem

$$
\max_{(c_0, c_1) \in \mathbb{R}_t^2} \left\{ u(c_0, c_1) \left| c_0 \frac{c_1}{1 + r} = w \right. \right\}
$$

Assumption C. The utility function $u(c_0, c_1)$ is such that the induced savings function $s(r, w)$ satisfies

(a) $0 < s'_w(r, w) < 1$, \hspace{1em} \forall (r, w) \gg 0

(b) $s'_w(r, w) \geq 0$, \hspace{1em} \forall (r, w) \gg 0

(a) is the assumption that consumption in each period is a normal good, while (b) implies that
when the interest rate increases, the substitution effect dominates the income effect, so that savings
increase.

Corporation’s decision problem. At the beginning of date $t$, a firm inherits the capital stock $K_t^j$ from date $t - 1$. Two real decisions have to be made by the corporation: how much labor $(L_t^j)$
to hire and how much to invest in new capital $(I_t^j)$. To operate the firm during period $t$, the firm
hires labor at the current wage $w_t$; current profit $F(K_t^j, L_t^j) - w_t L_t^j$ is maximized when

$$
F'_L(K_t^j, L_t^j) = w_t, \hspace{1em} t \geq 0
$$

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Operating the firm depreciates the capital stock to \((1 - \beta)K_t^j\): investment in the current period will augment this stock to \((1 - \beta)K_t^j + I_t^j\), and this is the stock which will be carried over into period \(t + 1\) when it will be put into operation with an amount of labor \(L_{t+1}^j\). The date \(t\) investment is chosen to maximize the net present value of the investment

\[
-I_t^j + \frac{1}{1 + r_{t+1}} \left[ F\left((1 - \beta)K_t^j + I_t^j, L_{t+1}^j\right) - w_{t+1}L_{t+1}^j + \hat{q}_{t+1}^j \left((1 - \beta)^2K_t^j + (1 - \beta)I_t^j\right)\right] \quad (6)
\]

anticipating next period’s labor decision \((L_{t+1}^j)\) and the market price \(\hat{q}_{t+1}^j\) that the current owners will receive next period from the sale of their shares. If \(I_t^j > 0\), then the first-order condition for maximizing (6) is given by

\[
F'_K\left((1 - \beta)K_t^j + I_t^j, L_{t+1}^j\right) = 1 + r_{t+1} - (1 - \beta)\hat{q}_{t+1}^j \left((1 - \beta)^2K_t^j + (1 - \beta)I_t^j\right)
\]

Since the economy is (weakly) growing \((n \geq 0)\), we will focus on equilibrium paths on which investment is positive at all dates. If \(I_{t+1}^j > 0\) then the buyers of the shares of firm \(j\) will be willing to pay 1 unit for an additional unit of installed capital which they buy on the stock market since it reduces the need for 1 unit of investment.\(^2\) Thus on a path on which investment is positive at all dates, \(\hat{q}_{t+1}^j(\cdot)\) must satisfy

\[
\hat{q}_{t+1}^j \left((1 - \beta)^2K_t^j + (1 - \beta)I_t^j\right) = 1 \quad (7)
\]

It follows that the first-order conditions characterizing the optimal decisions of a firm at each date \(t \geq 0\) are given by (5) and

\[
F'_K\left((1 - \beta)K_t^j + I_t^j, L_{t+1}^j\right) = r_{t+1} + \beta, \quad t \geq 0 \quad (8)
\]

Since the production function is homogeneous of degree one, conditions (5) and (8) only determine an optimal capital-labor ratio for each firm at each date. Since each firm has the same production function, this ratio is the same for all firms. If we let \(k = K/L\) denote the capital-labor ratio and introduce the production function per unit of labor

\[f(k) = \left(\frac{1}{L}\right)F(K,L) = F(k,1)\]

then \(F'_L(K,L) = f\left(\frac{K}{L}\right) - \left(\frac{K}{L}\right)f'\left(\frac{K}{L}\right)\) and \(F'_K(K,L) = f'\left(\frac{K}{L}\right)\). If \((w_t, r_{t+1})_{t \geq 0}\) is a price sequence for the economy, then given \(K_t^j\), the sequence \((K_t^j, L_t^j)_{t \geq 0}\) is profit maximizing for firm \(j\) with

\(^2\)Note that if \(K_t\) is increased by 1 unit and \(I_t^j\) is decreased by \(1 - \beta\), then \(F(\cdot) + \hat{q}_{t+1}^j(\cdot)\) is unchanged, but \(-I_t^j\) decreases by \(1 - \beta\), and the same will be true at date \(t + 1\).
positive investment if \( K_{t+1}^j = (1 - \beta)K_t^j + I_t^j \) with \( I_t^j > 0 \), and the sequence \((k_t)_{t \geq 0}\) of capital-labor ratios, with \( k_t = \frac{K_t^j}{L_t^j} \), satisfies
\[
f(k_t) - k_t f'(k_t) = w_t, \quad t \geq 0 \tag{9}
\]
\[
f'(k_{t+1}) = r_{t+1} + \beta, \quad t \geq 0 \tag{10}
\]
Note that to be compatible with (9) and (10), the price sequence \((w_t,r_{t+1})_{t \geq 0}\) must satisfy
\[
w_{t+1} = f(k_{t+1}) - k_{t+1}(r_{t+1} + \beta), \quad t \geq 0 \tag{11}
\]
where \( k_{t+1} \) is defined by (10).

Thus as long as investment is positive, the irreversibility constraint on capital is not binding and firms’ decisions are determined by the same first-order conditions (9) and (10) as in the Diamond model. So how can the irreversibility constraint possibly affect the outcome of the model?

**Equity price.** Let us show that the effect comes from outside the firm, through the change it induces in the equity price of the firm. More precisely, we derive a pricing rule for equity which generates trajectories on which the no-arbitrage equation (2) is satisfied on the capital markets, and (9) and (10) describe profit maximization. As we shall see, the real outcome is not influenced either by the timing of the investment decision \( I_t^j \) (whether it is made before or after the sale of equity) or by the way \( I_t^j \) is financed (retained earnings, new issues, or debt): however both timing and financing influence the equity value of the firm. Let \( Q_t^j(\xi,B) \) denote the equity price of firm \( j \) when its current capital stock is \( \xi \) and its outstanding debt is \( B \). To study equity prices which are compatible with agents’ optimization and equilibrium in the presence of the irreversibility constraint, let us start with the simple scenario where investment is paid by the young after purchasing the firm from old agents on the stock market and is financed “out of their pockets”. Then in equation (6), \( \tilde{Q}_{t+1}^j \) coincides with the equity price of firm \( j \) on the stock market at date \( t + 1 \), so that

\[
\tilde{Q}_{t+1}^j(\xi) = Q_{t+1}^j(\xi,0) \quad \text{with} \quad \xi = (1 - \beta)K_{t+1}^j = (1 - \beta)^2 K_t^j + (1 - \beta)I_t^j \tag{12}
\]

The first-order conditions (5) and (8), which are equivalent to (9) and (10), were based on the argument that one unit of investment at date \( t \) increases the resale price of equity at date \( t + 1 \) by \( 1 - \beta \), i.e. that relation (7) holds. Since (9) and (10) only determine a capital-labor ratio, relation (7) must hold for all \( I_t^j > 0 \), which implies that for any amount of capital \( \xi > (1 - \beta)^2 K_t^j \), we must have \( Q_{t+1}^j(\xi,0) = 1 \). Integrating gives

\[
Q_{t+1}^j(\xi,0) = \xi - V_{t+1}^j
\]
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where $V^j_{t+1}$ is a constant of integration. Let us show that certain sequences $(V^j_t)_{t \geq 0}$ with $V^j_t > 0$, \( t \geq 0 \) (which we shall describe shortly) are compatible with optimizing behavior of firm $j$ and investors (shareholders) of firm $j$. Note first that since young agents could obtain the same capital stock $\xi$ by purchasing $\xi$ units of current output, they will not accept to buy the firm if its equity price exceeds its replacement cost $\xi$. Thus $Q^j_{t+1}(\xi, 0) \leq \xi$ which implies $V^j_{t+1} \geq 0$. On the other hand, since old agents can always abandon their capital rather than having to pay to get rid of it, $Q^j_{t+1}(\xi, 0) \geq 0$. Thus more precisely

$$Q^j_{t+1}(\xi, 0) = \max(\xi - V^j_{t+1}, 0) \quad (12)$$

Formula (12) indicates how the irreversibility constraint can affect the equity price of a firm: for while the firm can be sold on the stock market, there is no alternative (second-hand capital goods market) on which the installed capital $\xi$ can be sold. Thus its equity price can be less than its replacement cost $\xi$ without creating an arbitrage opportunity. In Diamond’s model, there is no irreversibility constraint, and installed capital can always be unbolted and converted into current output: thus $V^j_{t+1} = 0$ and $Q^j_{t+1}(\xi, 0) = \xi$, so that the equity price of a firm coincides with its replacement cost on the new capital goods market. We refer to $V^j_{t+1}$ as the discount on the equity price of firm $j$ induced by the irreversibility of capital.

Several additional restrictions on the pricing formula (12) need to be established in order to ensure that it is compatible with the optimizing behavior of investors and firms described earlier. First, the discount $V^j_{t+1}$ must not be too large, otherwise it discourages investment which may become zero; second the sequence of discounts $(V^j_t)_{t \geq 0}$ must be such that the rate of return on equity is the same as the rate of return on the bond market $(r_{t+1})$, for otherwise the no-arbitrage condition (2) does not hold and the investors’ maximum problem has no solution.

When (12) holds, the term $\bar{Q}^j_{t+1}\left((1 - \beta)^2 K^j_t + (1 - \beta) I^j_t\right)$ in the maximum problem (6) is equal to $\max\{(1 - \beta)^2 K^j_t + (1 - \beta) I^j_t - V^j_{t+1}, 0\}$. If

$$V^j_{t+1} \leq (1 - \beta)^2 K^j_t, \quad t \geq 0 \quad (13)$$

then for all $I^j_t \geq 0$, $(1 - \beta)^2 K^j_t + (1 - \beta) I^j_t - V^j_{t+1} > 0$ and the discount $V^j_{t+1}$ plays the role of an unavoidable fixed cost. The first-order conditions (5) and (8) (or (9) and (10)) then characterize the optimal solutions. If (13) is violated at some $t$, then the problem (6) is non-convex and the first-order conditions do not suffice to characterize the optimal solutions: in the Appendix we show that in this case $I^j_t = 0$ is the only optimal solution.

**Proposition 1:** If the anticipated price for equity at date $t+1$ is given by (12) and if $V^j_{t+1} >
(1 − β)^2K_j^i, then the optimal solution to the investment problem (6) at date t is I_t^j = 0.

Proof: (see Appendix).

The intuition is clear: for when the discount on equity is too large \( (V_{t+1}^j > (1 - \beta)^2K_j^i) \), when agents come to resell their equity they will only recover

\[
(1 - \beta)^2K_j^i + (1 - \beta)I_t^j - V_{t+1}^j < (1 - \beta)I_t^j
\]

which is less than the (depreciated) value of their investment, so it is better not to invest. According to (13) the capital to which a “discount” can be applied is the capital which was installed by agents who are no longer alive at date \( t+1 \): their investment has received the “normal” rate of return \( r_{t+1}, \tau \geq 0 \) at the time that it was undertaken and the part of it which is left (which is now a sunk cost) does not need to be further rewarded.

To derive the condition that must be satisfied by the discounts \( (V_t^j)_{t \geq 0} \) to ensure that the rate of return on equity is the same as on the bond market \( (r_{t+1}) \), recall that under the current financing scenario (the general case is considered below) an investor purchasing a share of firm \( j \) finances the corresponding share of investment \( I_t^j \). Thus the cost at date \( t \) is (a share of)

\[
Q_t^j + I_t^j = (1 - \beta)K_t^j + I_t^j - V_t^j = K_t^j - V_t^j,
\]

which gives the right to an equal share of the dividend

\[
D_t^j + Q_t^j = F(K_t^j, L_t^j) - w_{t+1}L_t^j = F(K_t^j)K_t^j = (r_{t+1} + \beta)K_t^j + V_t^j
\]

and the capital value \( Q_t^j \). Thus

\[
D_t^j + Q_t^j = (1 + r_{t+1})(K_t^j - V_t^j)
\]

and \( (D_t^j + Q_t^j)/(Q_t^j + I_t^j) = 1 + r_{t+1} \) is equivalent to

\[
V_{t+1}^j = (1 + r_{t+1})V_t^j, \quad t \geq 0
\]

which is the “rate of return” condition that must be satisfied by the discounts \( (V_t^j)_{t \geq 0} \).

Financial policy. The assumption that shareholders are directly called upon to finance investment “out of their pockets” is not realistic. A corporation typically finances investment either internally through retained earnings or externally by issuing new equity or debt — or a combination of these methods. When a firm with a given capital stock is sold with a debt outstanding, the equity price of the firm has to be decreased by the amount of the debt, so that the total market value — the sum of equity plus debt — reflects the production possibilities embodied in the firm. Thus if at date \( t \) a firm is sold with capital \( \xi \) and outstanding debt \( B \), then formula (12) for the equity price of the firm becomes

\[
Q_t^j(\xi, B) = \max\{\xi - B - V_t^j, 0\}
\]

Consider a sequence of real investment-labor decisions \( (K_t^j, L_t^j)_{t \geq 0} \) and an associated financial policy \( (B_t^j, \Gamma_t^j)_{t \geq 0} \) where \( B_t^j \) is the debt incurred by firm \( j \) at date \( t \) and \( \Gamma_t^j \) denotes the funds obtained
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from the new issues of equity. Then the dividend \( D^j_t \) at date \( t \) is defined by the accounting equation

\[
D^j_t + \tilde{P}_t^j = F(K^i_t, L^j_t) - w_t L^j_t - B^j_{t-1}(1 + r_t) + B^j_t + \Gamma^j_t
\]  \hspace{1cm} (16)

the retained earnings being the residual amount \( \tilde{P}_t^j + B^j_{t-1}(1 + r_t) - B^j_t - \Gamma^j_t \). If \( K^j_{t+1} - B^j_t - V^j_t > 0 \) the equity value of the firm at date \( t \) is

\[
Q^j_t = K^j_{t+1} - B^j_t - V^j_t
\]  \hspace{1cm} (17)

and its market value \( M^j_t \) is

\[
M^j_t = Q^j_t + B^j_t = K^j_{t+1} - V^j_t
\]

the firm being sold with capital \( K^j_{t+1} \). Feasibility calls for a minimal consistency between the real and financial policies of a firm. Given the real decisions, financing must not create negative dividends (limited liability), the issue of debt cannot exceed the firm’s market value and the value of new issues cannot exceed existing equity.

**Definition 1.** A sequence \((B^j_t, \Gamma^j_t)_{t \geq 0}\) is a feasible financial policy\(^3\) for the real sequence \((K^j_t, L^j_t)_{t \geq 0}\) if (i) \( D^j_t \geq 0 \) (ii) \( Q^j_t > 0 \) (iii) \( \Gamma^j_t < Q^j_t \), \( t \geq 0 \).

The following proposition shows that the analysis of the model does not depend on the choice of a particular financial policy, as long as the policy is feasible.

**Proposition 2:** Let \((w_t, r_{t+1})_{t \geq 0}\) be a sequence of prices satisfying (11), \((K^j_t, L^j_t)_{t \geq 0}\) a sequence of investment-labor decisions satisfying (9), (10) and \((V^j_t)_{t \geq 0}\) a sequence of discounts satisfying (13) and (14). If the equity price is given by (15) then

(i) The set of feasible financial policies \((B^j_t, \Gamma^j_t)_{t \geq 0}\) is nonempty.

(ii) The shareholders of the firm are indifferent between all feasible financial policies and cannot improve on the investment-labor decision.

(iii) For any feasible financial policy and for all \( t \geq 0 \), the rate of return on equity bought at date \( t \) is \( r_{t+1} \).

**Proof** (i) To ensure that dividends are non-negative, consider the financial policy which finances new investment at each date by issuing new shares i.e. \( B^j_t = 0, \Gamma^j_t = \tilde{P}_t^j, t \geq 0 \). Then the firm has

\[\text{Proof} \, (i) \text{ To ensure that dividends are non-negative, consider the financial policy which finances new investment at each date by issuing new shares i.e. } B^j_t = 0, \Gamma^j_t = \tilde{P}_t^j, t \geq 0. \text{ Then the firm has} \]

3If \( \Gamma^j_t \geq 0 \) then (iii) \( \Rightarrow \) (ii). However we do not restrict \( \Gamma^j_t \) to be non-negative. If \( \Gamma^j_t < 0 \) i.e. if firm \( j \) repurchases part of its equity then (ii) \( \Rightarrow \) (iii). To cover all cases we impose (ii) and (iii).
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no debt and is sold in period \( t \) for the equity price \( Q_t^j = (1 - \beta)K_t^j + I_t^j - V_t^j \), of which \( \tilde{Q}_t^j = Q_t^j - \Gamma_t^j \) goes to the old agents and \( \Gamma_t^j \) to the firm.\(^4\) \( \tilde{Q}_t^j = (1 - \beta)K_t^j - V_t^j > (1 - \beta)I_{t-1}^j \) by (13) and is thus positive, so that (iii) and (ii) hold. By (16) the dividends distributed by the firm at date \( t \) are \( F(K_t^j, L_t^j) - w_tL_t^j = (r_t + \beta)K_t^j > 0 \).

(ii) Let \((B_t^j, \Gamma_t^j)_{t \geq 0}\) be a feasible financial policy. Agents born at date \( t \) buy the firm for \( Q_t^j \) given by (17): next period they are paid dividends \( D_{t+1}^j \) given by (16) and receive \( \tilde{Q}_{t+1}^j = Q_{t+1}^j - \Gamma_{t+1}^j \) from the sale of the equity (the value of their diluted shares) where \( Q_t^j, Q_{t+1}^j, D_{t+1}^j \) are all positive by feasibility. Note that the present value of their investment is independent of the firm’s financial policy \((B_t^j, \Gamma_t^j)\) and by (14) is equal to

\[
-Q_t^j + \frac{1}{1 + r_{t+1}}(D_{t+1}^j + \tilde{Q}_{t+1}^j) = -(1 - \beta)K_t^j - I_t^j + \frac{1}{1 + r_{t+1}} \left[ F((1 - \beta)K_t^j + I_t^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j + (1 - \beta)((1 - \beta)K_t^j + I_t^j) \right]
\]

(18)

For the shareholders at date \( t \), \( K_t^j \) is exogenous since it is inherited from date \( t - 1 \), and (18) is maximized when (5) and (8) (or equivalently (9) and (10)) are satisfied.

(iii) Substituting \( F(K_{t+1}^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j = (r_{t+1} + \beta)K_{t+1}^j \) leads to \( D_{t+1}^j + \tilde{Q}_{t+1}^j = (1 + r_{t+1})Q_t^j \) so that the return on equity is \( r_{t+1} \).

Property (ii) of Proposition 1 is the Modigliani-Miller theorem for this economy: it asserts that the market value of a firm only depends on the real decisions and is independent of the financial policy chosen, subject to the proviso that this policy is feasible. Property (i) is important to ensure that the presence of a discount on equity \( j \) does not introduce anomalies into the model such as the need for negative dividends or negative prices.\(^5\)

\(^4\)The actual mechanism can be described as follows. After operating the firm and receiving dividends, the old agents sell their shares — assume \( \nu_t^j \) shares are outstanding — and the firm issues new shares \( \Delta \nu_t^j \) so that \( \frac{\Delta \nu_t^j}{\nu_t^j} = \frac{I_t^j}{(1 - \beta)K_t^j - V_t^j} \). Then if \( q_t^j \) is the price of a share,

\[
\frac{\Delta \nu_t^j q_t^j}{I_t^j} = \frac{\nu_t^j q_t^j}{(1 - \beta)K_t^j - V_t^j} = \frac{Q_t^j}{(1 - \beta)K_t^j + I_t^j - V_t^j} = 1
\]

so that \( \Gamma_t^j = \Delta \nu_t^j q_t^j = I_t^j \) and investment is financed by the new issues \( \Delta \nu_t^j \). At date \( t + 1 \) the number of shares outstanding is \( \nu_t^j + \Delta \nu_t^j \).

\(^5\)Since the market value of a firm at date \( t \) is equal to \( K_{t+1}^j - V_t^j \) where \( (V_t^j)_{t \geq 0} \) satisfies the “rate of return” equation (15), it is tempting to interpret the discount \(-V_t^j\) as a “negative bubble” added to the “fundamental value” \( K_{t+1}^j \) of firm \( j \). This statement is inaccurate for the following reason. In financial economics, the price of an
Equilibrium. The definition of a financial market equilibrium is usually made using the price per share of equity $q_t^j$ rather than the total equity value $Q_t^j$ of the firm. If $\nu_t^j$ denotes the number of shares outstanding at the beginning of period $t$, and $\Delta \nu_t^j$ is the number of new shares issued by firm $j$ at date $t$, then

$$q_t^j = \frac{Q_t^j}{\nu_t^j + \Delta \nu_t^j}$$

and $\nu_{t+1}^j = \nu_t^j + \Delta \nu_t^j$ is the number of shares outstanding at the beginning of period $t + 1$. A financial market equilibrium for the economy is then defined in extensive form by a sequence of prices $(w_t, r_{t+1}, q_t^j)_{t \geq 0}$ and discounts $(V_t^j)_{t \geq 0}$ on equity, portfolio decisions $(z_t, \theta_t)_{t \geq 0}$ with $\theta_t = (\theta_t^1, \ldots, \theta_t^J)$ for the sequence of representative consumers born at each date $t$, and production $(Y_t^j, L_t^j, I_t^j)_{t \geq 0}$ and financing $(B_t^j, \Delta \nu_t^j)_{t \geq 0}$ decisions for the $J$ firms such that

(i) firms maximize their market value (conditions (9) and (10)) and $(V_t^j)_{t \geq 0}$ satisfy (13) and (14)

(ii) consumers maximize their lifetime utility subject to their budget constraints (1), with $(q_t^j)_{t \geq 0}$ given by (19), $(Q_t^j)_{t \geq 0}$ defined by (17), $d_t^j = \frac{D_t^j}{v_t}$ and $D_t^j$ given by (16), $t \geq 0$

(iii) the output, labor and financial (bond, equity) markets clear at every date $t \geq 0$.

By Walras Law the output market clears once the labor market ($\sum_{j=1}^J L_t^j = N_t$) and financial markets clear. Given the indeterminacy of financial policies of both firms and consumers, market clearing on the bond and equity markets

$$N_t z_t^j = \sum_{j=1}^J B_t^j, \quad N_t \theta_t^j = \nu_t^j + \Delta \nu_t^j, \quad j = 1, \ldots, J$$

only requires that the financial markets clear in aggregate:

$$N_t s_t = \sum_{j=1}^J B_t^j + \sum_{j=1}^J Q_t^j$$

The young agents must buy the total market values of the firms (equity plus debt) the proceeds serving to reimburse the old shareholders and to finance new investment.\(^6\) Thus the condition for the financial markets to clear is

$$N_t s(r_{t+1}, w_t) = \sum_{j=1}^J (K_t^j - V_t^j)$$

asset is said to have a bubble if it does not coincide with the present value of its future stream of dividends (the asset’s fundamental value), the bubble being the difference between the price and the fundamental value. As Tirole (1985) has shown, when the price of an asset is non-negative, the bubble can only be positive. Since Proposition 1 shows that the equity prices are positive, there cannot be negative bubbles in our economy. In Section 4 we will evaluate the present value of firms’ dividends in equilibrium and find that they do not coincide with the replacement value of the capital $K_t^j$ when $V_t^j \neq 0$: thus $-V_t^j$ is not the difference between the price and the fundamental value of equity and hence is not a negative bubble.

\(^6\)More precisely, the part which has not been financed by retained earnings.
2. The Stock Market Model

The equilibrium can thus be expressed in terms of the simpler reduced-form variables consisting of the prices \( (w_i, r_{i+1})_{i \geq 0} \) and the discounts on equity \( (V_i^j)_{i \geq 0} \), the savings \((s_i)_{i \geq 0}\) of consumers and the labor-investment \((L_i^j, I_i^j)_{i \geq 0}\) decisions of firms. Note that the discounts \( (V_i^j)_{i \geq 0} \) are the only variables linked to the operation of the equity markets that need to remain in a reduced-form definition of equilibrium.

The economy starts date 0 with firms having initial capital stocks \( (K_0^i, V_0^j) \) reflecting the operation of equity markets before date 0. To reduce the analysis to the study of the aggregate economy we study only balanced growth equilibria in which firms have at all times the same relative sizes and stock market values. Consider therefore initial conditions \( (K_0^j, V_0^j) = \mu_j (K_0, V_0) \) with \( \mu_j > 0 \) and \( \sum_{j=1}^J \mu_j = 1 \). If the sequence of prices \( (w_i, r_{i+1})_{i \geq 0} \), aggregate discounts \( (V_i)_{i \geq 0} \) and labor-investment decisions \( (L_i, I_i)_{i \geq 0} \) satisfy (9), (10), (13), (14), then \( (V_t^j, L_t^j, I_t^j) = \mu_j (V_t, L_t, I_t) \) satisfy (9), (10), (13), (14) so that for each firm \( j \), \( (L_t^j, I_t^j) \) is optimal, its market value is positive and the return on its equity is \( r_{i+1} \). Thus the maximizing behavior of individual firms can be summarized as the optimal choice of aggregate capital and labor. Equilibrium on the labor market, which can be expressed by \( L_t = N_t \), is satisfied if we require the capital-labor ratio to be equal to the per-capita capital stock \( k_t = K_t/N_t \). Using lower-case letters \( (k_t, i_t, v_t) \) to denote per-capita capital, investment and discount, a balanced-growth equilibrium can be summarized in the following per-capita aggregate reduced form:

**Definition 2.** A path of savings, capital accumulation, wages and security prices \( ((s_t, i_t, k_{t+1}), (w_t, r_{i+1}, v_t),_{t \geq 0} \) with initial conditions \( (k_0, v_0) \) is an equilibrium of the stock market economy if the following conditions are satisfied for all \( t \geq 0 \)

\[
\begin{align*}
(i) & \quad f(k_t) - k_t f'(k_t) = w_t \\
(ii) & \quad f'(k_{t+1}) = \beta + r_{t+1} \\
(iii) & \quad 0 \leq (1 + n) v_{t+1} \leq (1 - \beta)^2 k_t \\
(iv) & \quad (1 + n) k_{t+1} = (1 - \beta) k_t + i_t \\
(v) & \quad (1 + n) v_{t+1} = (1 + r_{t+1}) v_t \\
(vi) & \quad s(r_{t+1}, w_t) = (1 + n) k_{t+1} - v_t
\end{align*}
\]

Conditions (i) - (iii) ensure that firms choose their production decisions in the interest of their current shareholders when the (correctly anticipated) market value of firms at each date is \( (1 + n) k_{t+1} - v_t \) (in per-capita terms), (iv) describes the evolution of capital and (v) ensures that the rate of return on the bond and equity markets is the same. (vi) summarizes the maximizing behavior of consumers and equilibrium on the financial markets at each date.

**Relation to Tobin's q theory.** The concept of a stock market equilibrium in Definition 2 is closely related to Tobin’s q theory, where the essential idea is that frictions on the capital markets can lead to a divergence between the market value of a firm (as assessed by the stock market)
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and the replacement cost of its capital (determined in the capital goods producing sector), Tobin’s
$q$ being the ratio of the market value to the replacement cost. If a firm’s capital stock could be
instantaneously and costlessly adjusted then $q$ would necessarily be 1. However when there are costs
of adjusting its capital stock—over and above the price of the capital goods used—$q$ can be either
greater or less than 1. When $q > 1$ ($q < 1$) there is an incentive for the firm to invest (disinvest).
Our framework is similar to Tobin’s in that there is a friction which permits the “financial” value
and the “real” replacement cost to differ. The friction in our model arises however not from a cost
of adjusting capital, but rather from the assumption that capital once installed is a sunk cost.

If $\gamma^j_t$ denote Tobin’s $q$ for firm $j$ at date $t$, then

$$\gamma^j_t = \frac{M^j_t}{K^j_{t+1}}$$

Since $M^j_t = K^j_{t+1} - V_t$, in our model $\gamma^j_t$ is always less than or equal to 1. Note that $\gamma^j_t$
is the ex-post result of the firm’s choice of investment: the firm’s anticipation of its next period market
value is not of the form $M^j_{t+1} = \gamma^j_{t+1} K^j_{t+2}$, but rather of the linear form $M^j_{t+1} = K^j_{t+2} - V^j_{t+1}$. As
a result our model does not exhibit the direct feedback between $\gamma^j_t > 1$ ($\gamma^j_t < 1$) and incentives
for the firm to invest (disinvest), which is characteristic of Tobin’s theory. In our equilibrium, as
in that of Diamond, current investment is only influenced by the current interest rate (equation
(ii) is the same as in a Diamond equilibrium) and is not influenced by the firm’s market value
$M^j_{t+1} = K^j_{t+2} - V^j_{t+1}$ next period.

However since security prices (expressed in per-capita form) evolve according to equations (v)
and (vi), a lower interest rate will result in a lower discount $v_{t+1}$ at date $t + 1$, and thus in a
higher ratio $\gamma^j_{t+1}$. Thus lower interest rates, higher investment and higher $q$ ratios go together, but
the causality is different from that underlying Tobin’s theory where a higher $q$ ratio (or security
price) induces more investment. The study of the dynamics of a stock market equilibrium in the
next section will show that while the discounts $V^j_t$ do not directly affect the current investment
decisions of firms, they have a decisive indirect effect by permitting more savings of the young
agents to be used for investment rather than buying the firms from the old. Thus the discounts on
the market values of the firms relative to their replacement costs, by making it cheaper to buy the
firms, frees funds which are channeled into increased savings to finance new investment: the result
is an equilibrium path with lower interest rates and higher investment.
3. Dynamics of Stock Market Equilibrium

In this section we study the long-run dynamics of a stock market equilibrium: as we shall see, the most interesting properties arise when the stock market value of each firm differs from its replacement cost. In the model outlined above, this difference between the two values of a firm was traced to an attribute of capital — namely that once installed in a firm, it ceases to be a perfect substitute for current output or current investment. The model however contains as a special case the classic Diamond model in which capital is perfectly malleable and firms can be liquidated at any time, their capital being sold on the market for current output: in this case the financial value of a firm must coincide with its replacement cost. More precisely, if \( v_0 = 0 \), equation (v) of of Definition 2 implies that \( v_t = 0 \) for all \( t \), and the equations (i), (ii), (iv) and (vi) are the equations defining a Diamond equilibrium. In particular equation (vi)

\[
(1 + n)k_{t+1} = s(r_{t+1}, w_t)
\]

is the basic “investment = savings” equation of Diamond’s model. Since the properties of a stock market equilibrium in which there is a discount on the equity prices of firms (equilibrium with \( v_0 > 0 \)) depend in an essential way on the properties of the underlying Diamond equilibrium (\( v_0 = 0 \)), we recall briefly the requisite properties of such an equilibrium.

**Diamond Equilibrium.** (i) and (ii) in Definition 2 define the wage and interest rate \((w_t, r_{t+1})\) as functions of the capital-labor ratios \((k_t, k_{t+1})\): substituting these functions into (20) gives the first-order difference equation

\[
\Phi(k_{t+1}, k_t) \equiv (1 + n)k_{t+1} - s(r_{t+1}, w_t) = 0, \quad \forall \ t \geq 0
\]

(\(E_D\))

with initial condition \( k_0 > 0 \), which defines an equilibrium path of capital accumulation of Diamond’s model. A **Diamond steady state** \( k_D \) is a solution of the equation

\[
(1 + n)k_D - s(r_D, w_D) = 0
\]

(21)

For general preferences and technology \((u, F, n)\) there can be several non-trivial steady states and the dynamics \((E_D)\) can exhibit complex behavior. We restrict attention to economies \(E(u, F, n)\) for which there is a unique positive steady state \( k_D \) and every solution of \((E_D)\) converges to \( k_D \): as noted by Galor-Ryder (1989), the standard Assumption \(C\) on preferences combined with Inada (and the usual concavity and homogeneity) conditions on \( F \) do not suffice to give this property.
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Assumption C(b) and the concavity of \( F \) imply that there exists a unique solution

\[
k_{t+1} = \phi(k_t)
\]

(\( E_D' \))

to the equation \( (E_D) \). By the implicit function theorem, \( \phi \) is differentiable. An additional assumption is needed to ensure that the graph of \( \phi \) cuts the diagonal with a positive slope at a unique \( k_D > 0 \). The following condition — which is less restrictive and simpler to verify than the one given by Ryder-Galor (1989) — is sufficient\(^7\).

**Assumption S.** Define \( S(k) = s(r(k), w(k)) \). The function \( S(k)/k \) is decreasing for all \( k > 0 \), \( \lim_{k \to 0^+} S(k)/k > 1 + n \), and \( \lim_{k \to +\infty} S(k)/k < 1 \).

The property \( S(k)/k \) decreasing is equivalent to \( \log(S(k)/k) \) decreasing and this is equivalent to the elasticity of \( S \) being less than one \( (\eta_k = \frac{dS/S}{dk/k} < 1) \): a given percentage increase in the capital stock \( k \) gives rise to a smaller percentage increase in savings \( S \). Although this assumption is a joint assumption on preferences and technology, it can be decomposed into separate assumptions on the consumption and the production sides. For example, it holds if

- \( u \) is homothetic and satisfies Assumption C
- \( f \) is such that \( w(k)/k \) is a decreasing function of \( k \) with \( \lim_{k \to 0^+} \frac{w(k)}{k} = \infty \) and \( \lim_{k \to +\infty} \frac{w(k)}{k} = 0 \)

These conditions are satisfied if both \( u \) and \( F \) are CES with elasticity of substitution greater than or equal to 1—which includes Cobb-Douglas utility and production functions.

**Proposition 3:** Under assumptions (C, S), the Diamond steady-state capital \( k_D \) is globally stable for the dynamics \( (E_D') \): for any initial condition \( k_0 \), the per-capita capital stock on an equilibrium trajectory of the Diamond economy converges to \( k_D > 0 \).

**Proof:** See Appendix.

Typically the Diamond steady state is not efficient since there is another steady state which sustains a higher permanent level of per-capita consumption. A permanent level of per-capita consumption and capital \( (c_1, c_2, k) \) is feasible if

\[
c_1 + \frac{c_2}{1 + n} = f(k) - (n + \beta)k
\]

\(^7\)A similar assumption was used by Weil (1987). For sake of completeness we prove in Appendix that assumption S implies uniqueness and global stability of the Diamond steady state.
3. Dynamics of Stock Market Equilibrium

The steady state \( k^* \) which maximizes permanent per-capita consumption \( c_1 + \frac{c_2}{1 + n} \) is called the Golden Rule steady state and is characterized by

\[
f'(k^*) = n + \beta \iff r(k^*) = n
\]

The Golden Rule \( k^* \) is determined purely by technological \((f, \beta)\) and demographic factors \((n)\): the Diamond steady state \( k_D \) defined by (21) depends in addition on agents’ preferences (savings behavior). Thus for typical economies \( k_D \neq k^* \) so that for most economies the Diamond steady state is inefficient. When \( k_D < k^* \), the interest rate \( r_D = f'(k_D) - \beta \) at the Diamond steady state exceeds the Golden Rule interest rate \( r^* = f'(k^*) - \beta = n \). The Diamond economy converges to a steady state of underaccumulation characterized by a low level of capital, low output and a high interest rate. Under assumption \( S \), \( k_D < k^* \) is equivalent to \( s(r(k^*), w(k^*)) < (1 + n)k^* \), so that an alternative definition of underaccumulation is that the savings of the consumers at the prices \((r(k^*), w(k^*))\) at the Golden Rule are not sufficient to sustain the Golden Rule capital stock. When \( k_D > k^* \), \( r_D < n \) and \( s(r(k^*), w(k^*)) > (1 + n)k^* \) so that the savings of consumers at \( k^* \) can “buy” more capital than \( k^* \): the Diamond economy converges to a steady state of overaccumulation characterized by a high level of capital, a low interest \( r_D \) and high output level \( y_D \), much of which is absorbed by the need to maintain the capital stock rather than being used for consumption.

**Stock Market Equilibrium.** In the general case where \( v_0 \neq 0 \), a stock market equilibrium can also be reduced to the solution of a difference equation describing the path of capital accumulation \((k_t)_{t \geq 0}\); from this path the other equilibrium quantities and prices can in turn be derived. As before equations (i) and (ii) of Definition 2 define the wage and interest rate. The “savings–investment” equation (equation (iv)) is now

\[
(1 + n)k_{t+1} = s(r_{t+1}, w_t) + v_t
\]

(22)

Thus when \( v_t > 0 \), the (per capita) capital stock that young agents are able to acquire for use in the subsequent period \(((1 + n)k_{t+1})\), exceeds their savings because firms are sold on the equity market with a discount relative to their replacement cost. The discount \( v_t \) in essence acts like an additional “source of funds” that enables them to finance a higher level of capital accumulation than would be warranted by their savings in a Diamond equilibrium, where firms are sold for their replacement cost.

To guarantee that there is no-arbitrage the evolution of the discount \( v_t \) must satisfy equation (v). If (22) is used to define the discount \( v_t \) as a function of \((k_{t+1}, k_t)\),

\[
v_t = v(k_{t+1}, k_t) \equiv (1 + n)k_{t+1} - s(r_{t+1}, w_t) \]

(22)
3. Dynamics of Stock Market Equilibrium

then a stock market equilibrium is a solution of the second-order difference equation in $k_t$

$$(1 + n)v(k_{t+2}, k_{t+1}) = (1 + r(k_{t+1}))v(k_{t+1}, k_t)$$  \hspace{2cm} (E_S)

with initial condition $(k_1, k_0)$ satisfying $k_0 > 0$ and $v(k_1, k_0) \geq 0$. In order that the young have the incentive to undertake positive investment in their firms at each date, the inequality (iii) of Definition 2 must be satisfied at every date, or equivalently $(1 + r(k_{t+1}))v(k_{t+1}, k_t) \leq (1 - \beta)^2 k_t$. Thus the feasible initial conditions for a stock market equilibrium with positive investment are defined by the inequality

$$0 \leq (1 + r(k_1))v(k_1, k_0) \leq (1 - \beta)^2 k_0$$  \hspace{2cm} (IC)

A steady state solution of $(E_S)$, $k_t = k$, $\forall \; t \geq 0$, must satisfy

$$(n - r(k))v(k, k) = 0$$  \hspace{2cm} (23)

and the initial condition

$$0 \leq v(k, k) \leq \frac{(1 - \beta)^2 k}{1 + n}$$  \hspace{2cm} (IC')

Under Assumptions $(\mathcal{C}, \mathcal{S})$, there are only two positive solutions of (23): the Diamond steady state given by $v(k_D, k_D) = 0$ and the Golden Rule defined by $r(k^*) = n$. $k_D$ always satisfies (IC'). However $k^*$ is not a steady state equilibrium of an economy with overaccumulation, since $v(k^*, k^*) < 0$. For an economy with underaccumulation $v(k^*, k^*) > 0$, but the characteristics $(u, F, n)$ of the economy must be such that

$$v^* = (1 + n)k^* - s(r(k^*), w(k^*)) \leq \frac{(1 - \beta)^2 k^*}{1 + n}$$

so that the right side of (IC') is satisfied, namely that the young have the incentive to undertake positive investment at $k^*$. Thus there is a limit to the extent to which the discount on equity can make up for the deficiency in savings needed to sustain the Golden Rule.

This condition can also be written as

$$s(r(k^*), w(k^*)) \geq i^* + i^*(1 - \beta)$$  \hspace{2cm} (24)

where $i^* = (n + \beta)k^*$ is the (per-capita) investment needed to sustain the Golden Rule; thus, while the savings of the young may not be sufficient to cover the combined costs of new investment and installed capital at replacement value, they must be sufficient to cover current new investment and the depreciated investment of the previous period.

**Assumption $\mathcal{S}^*$.** The characteristics of the economy $(u, F, n)$ are such that (24) is satisfied.
3. Dynamics of Stock Market Equilibrium

A stock market equilibrium can also be characterized by the pair of equations \((v_t, (22))\): this leads to the following first-order difference equation in \((k_t, v_t)\)

\[
(1 + n)k_{t+1} = s(r(k_{t+1}), w(k_t)) + v_t \\
(1 + n)v_{t+1} = (1 + r(k_{t+1}))v_t
\]

\((E^S)\)

Under Assumption \(C\), \((E^S)\) can be written as

\[
k_{t+1} = \psi(k_t, v_t) \\
v_{t+1} = \frac{1 - \beta + f'(\psi(k_t, v_t))v_t}{1 + n}
\]

\((E^S_t)\)

where \(\psi : \mathbb{R}^2 \to \mathbb{R}\) is an increasing, differentiable function. \((E^S_t)\) is a difference equation system which has been extensively studied in the OLG literature\(^8\) in the case where \(v_t \leq 0\): in the present setting we are interested in the polar case \(v_t \geq 0\). The phase diagram is determined by the curves \(\mathcal{V}\) and \(\mathcal{K}\) defined by

\[
\mathcal{V} = \left\{(k_t, v_t) \in \mathbb{R}^2_+ | v_{t+1} = v_t \right\} = \left\{(k_t, v_t) \in \mathbb{R}^2_+ | v_t = 0 \text{ or } \psi(k_t, v_t) = k^* \right\}
\]

\[
= \left\{(k_t, v_t) \in \mathbb{R}^2_+ | v_t = 0 \text{ or } v_t = (1 + n)k^* - s(r^*, w(k_t)) \right\}
\]

and

\[
\mathcal{K} = \left\{(k_t, v_t) \in \mathbb{R}^2_+ | k_{t+1} = k_t \right\} = \left\{(k_t, v_t) \in \mathbb{R}^2_+ | v_t = (1 + n)k_t - s(r(k_t), w(k_t)) \right\}
\]

Under assumptions \((C, S)\), \(\mathcal{V}\) is the union of the axis \(v = 0\) and the graph of a decreasing function, while \(\mathcal{K}\) is a U-shaped curve passing through the origin. The resulting phase diagrams for an economy with overaccumulation and underaccumulation are shown in Figure 1(a) and (b).

As Figure 1(a) suggests, the Diamond steady state \(k_D\) is globally stable for an economy with overaccumulation.

**Proposition 4:** Under assumptions \((C, S)\), if \(k^* < k_D\) then any solution \((k_t, v_t)_{t \geq 0}\) of \((E^S)\) with \(k_0 > 0, v_0 \geq 0\) converges to \((k_D, 0)\).

**Proof:** If \(v_0 = 0\), then by Proposition 3, the trajectory converges to the Diamond steady state. Suppose \(v_0 > 0\). Consider the three regions \(A, B, C\) shown in Figure 1(a). \(A\) is defined by \(k \leq \hat{k}\)

\(^8\)The mathematical equations \((E^C)\) or \((E^S)\) are the same as those of Tirole (1985). In his model \(v_t < 0\) would be a negative bubble or, as some people say, “negative money”: but negative money is excluded by the assumption of free disposal. The same equations would also be obtained if a government were to perpetually run a budget surplus, lent to the private sector each period, but such a perpetual surplus is considered very implausible (see Azariadis (1993)). The presence of a discount on security prices relative to replacement costs leads to a market-based way of generating these equilibrium equations, which does not contradict the non-negativity requirement for security prices.
3. Dynamics of Stock Market Equilibrium

Figure 1: Phase diagram for the stock market equilibrium equations (E'_S) in the cases of (a) overaccumulation and (b) underaccumulation.

where $\bar{k}$ is such that $(1+n)k^*-s(n,w(\bar{k})) = 0$: since $(1+n)k^*-s(n,w(\bar{k})) < 0$, $0 < \bar{k} < k^*$. The definitions of $B$ and $C$ are clear from Figure 1(a). Let us show that if $(k_0,v_0) \in A$ then the trajectory must leave the region $A$ in a finite number of periods and enter $B \cup C$. $v_t > 0$, $t \geq 0$ implies by induction that for $t \geq 1$, $k_t > k^D_t$ where $(k^D_t)_{t \geq 0}$ is the Diamond trajectory beginning at $(k_0,0)$: $k_{t+1} = \psi(k_t,v_t) > \psi(k_t,0) > \psi(k^D_t,0) = k^D_{t+1}$. Since $k^D_t$ converges to $k^D > \bar{k}$ the property follows. Let us show that if $(k_0,v_0) \in B \cup C$, the trajectory stays in $B \cup C$. If $(k_t,v_t) \in B$ then $k_{t+1} > k_t$, so that $(k_{t+1},v_{t+1}) \in B \cup C$. If $(k_t,v_t) \in C$, then $k_t > K_D$ so that $k_{t+1} = \psi(k_t,v_t) > \psi(k_D,0) = K_D$. Thus $(k_t,v_t) \in B \cup C$. When $(k_t,v_t) \in B \cup C$ the sequence $(v_t)_{t \geq 0}$ is a decreasing sequence which is bounded below since $v_t > 0$, $\forall t \geq 0$: thus $v_t \rightharpoonup \overline{v}$. Either $\overline{v} > 0$ or $\overline{v} = 0$. Suppose $\overline{v} > 0$ then $k_t \to k$ defined by $\overline{v} = \frac{1}{1+\beta} \left( 1 - \beta + f'(\psi(\overline{k},\overline{v}))) \right) \overline{v} \iff \psi(\overline{k},\overline{v}) = k^*$ so that $(\overline{k},\overline{v})$ lies on the curve $\mathcal{K}$. Since there is no intersection of the $\mathcal{V}_1$ curve and the $\mathcal{K}$ curve in the non-negative orthant, it follows that $\overline{v} \in \mathcal{V}_2$. Thus $\overline{v} = 0$ and $\overline{k} = K_D$. $
$
A solution of (E'_S) is an equilibrium trajectory for our economy if the inequality $(1+n)v_{t+1} < (1-\beta)^2k_t$ is satisfied at all dates. All initial conditions $(k_0,v_0) \in B \cup C$ such that $(1+n)v_1 < (1-\beta)^2 \min(k_0,k_D)$ lead to trajectories which have this property, since $(v_t)_{t \geq 0}$ is decreasing and, by the above reasoning, $k_t \geq \min(k_0,k_D)$, $t \geq 0$.

Thus in the case of overaccumulation it is easy to prove the existence of a stock market equilibrium trajectory and its convergence to the Diamond equilibrium. In this case the existence of
a discount on equity does not improve the long-run efficiency of the equilibrium. This was to be expected since in the case of overaccumulation the propensity to save of the young agents is too high when compared to the productivity of capital. The discount on equity which is akin to an increase in savings can only make things worse. In the long run however the effect vanishes, since the discount on equity increases at a slower rate than the population and tends to disappear in per-capita terms, so that the equilibrium converges to the stable Diamond steady state. A variety of methods have been proposed for absorbing the excess savings to restore convergence to the Golden Rule: social security, land as a third factor of production (McCallum (1987), Rhee (1991)) or unbacked debt (Pingle-Tesfatson (1998)): each of these methods is applicable to our model.

For an economy with underaccumulation, the savings of the young are “scarce” and the discount on the equity prices acts like an additional source of funds, permitting increased investment. The phase diagram (Figure 1(b)) suggests that the equilibrium trajectories converge to the Golden Rule steady state. Global properties are more difficult to establish for economies with underaccumulation than for those with overaccumulation. Indeed even to prove the local stability of the Golden Rule \((k^*, v^*)\), a stronger assumption is needed than that which assures the stability of the Diamond steady state under the Diamond dynamics, namely assumption \((C, S)\).

**Assumption \(\mathcal{P}\).** The production function \(f\) is such that \(kf'(k)\) is an increasing function of \(k\).

\(\mathcal{P}\) is satisfied only if capital and labor are sufficiently substitutable: it requires that the marginal product of capital \(f'(k)\) does not decrease too fast as the capital-labor ratio \(k\) increases, so that the amount of output (per capita) going as payment to capital \((kf'(k))\) decreases. This property is satisfied for CES production functions \(F(K, L)\) with elasticity of substitution greater than or equal to 1 (and hence for Cobb-Douglas production functions): thus for the class of CES functions \(\mathcal{P}\) requires no additional restrictions over those needed to satisfy assumption \(S\).

**Proposition 5:** Under assumptions \((C, S, \mathcal{P})\), if \(k^* > k_D\) then under the stock market equilibrium dynamics \(E^*_S\), the Golden Rule \((k^*, v^*)\) is locally stable and the Diamond steady state \((k_D, 0)\) is locally saddlepoint stable.

**Proof:** See Appendix

Proposition 5 implies that for all economies satisfying assumption \(S^*\), a stock market equilibrium which converges to the Golden Rule exists for all initial conditions \((k_0, v_0)\) in a neighborhood of the Golden Rule \((k^*, v^*)\). Assumption \(S^*\) in essence imposes a restriction on how far the Diamond
steady state $k_D$ is from the Golden Rule $k^*$: if $k^*$ is too much greater than $k_D$ then the funds in excess of the savings of the young needed to finance investment become too large to permit them to be covered by the discount on the equity prices. To get a feel for how the equilibrium behaves and to what extent these conditions are restrictive, let us consider a family of Cobb-Douglas economies.

**Example:** Let $E(u, F, n)$ be a Cobb-Douglas economy:

$$u(c_0, c_1) = c_1^{1-\alpha}c_0^\alpha, \quad 0 < \alpha < 1, \quad F(K, L) = AK^\gamma L^{1-\gamma}, \quad 0 < \gamma < 1$$

There are four parameters $(\alpha, \gamma, \beta, n)$ which characterize an economy: the parameter $A$ is just a scale factor which does not matter for the analysis (we choose $A = 50$). $\alpha$ gives the propensity to save of the young $(s(r, w) = \alpha w)$, $\gamma$ determines the share of capital in output, $0 < \beta < 1$ is the depreciation rate of capital and $n$ the population growth rate. Let us fix $\gamma = 0.25$ and $n = 0.35$ (which corresponds to an annual increase of population of about 1% for 30 years). The Golden Rule capital-labor ratio is $k^* = \left(\frac{A\gamma}{\beta + n}\right)^{\frac{1}{1-\gamma}}$ and there is underaccumulation if

$$(1 + n)k^* \geq A\alpha(1 - \gamma)(k^*)^\gamma \iff (1 + n) \geq \frac{\alpha(1 - \gamma)}{\gamma}(\beta + n) \iff \alpha \leq \frac{\gamma(1 + n)}{(1 - \gamma)(\beta + n)} \quad (25)$$

The Golden Rule $k^*$ satisfies condition (IC') if

$$A\alpha(1 - \gamma)(k^*)^\gamma \geq (\beta + n)k^* + \frac{1 - \beta}{1 + n}(\beta + n)k^* \iff \alpha \geq \frac{\gamma}{1 - \gamma} + \frac{(1 - \beta)^2}{1 + n} \frac{\gamma}{1 - \gamma} \quad (26)$$
4. Market Value of Firms

For the chosen parameters \( (\gamma, \eta) = (0.25, 0.35) \), (25) and (26) give the admissible values of the parameters \( (\alpha, \beta) \in (0, 1) \times (0, 1) \) for which the Golden Rule is a stock market equilibrium. Let \( \ell(\beta) = \frac{1}{1-\gamma} \left( \frac{1}{\beta + n} \right) \) denote the function in (25) defining economies with "low" savings (underaccumulation) and let \( p(\beta) = \frac{\gamma}{1-\gamma} \left( 1 + \frac{1-\beta}{1+n} \right) \) denote the function in (26) defining economies in which there is "positive" investment, then the admissible parameters \( (\alpha, \beta) \) are given by the shaded region in Figure 2(a). Thus, for example, if \( \beta = 0.4 \) (which corresponds to an annual depreciation rate of 1.7% for 30 years) then the interval of admissible \( \alpha \) values is [0.48, 0.6]. Figure 2(b) shows equilibrium trajectories when \( (\alpha, \beta) = (0.5, 0.4) \). The region lying below the dashed curve \( T \) defines the set of \( (k_t, v_t) \) pairs satisfying condition E(iii) which ensures that the young have the incentive to make positive investment in their firms. For every initial condition satisfying (IC), the trajectory satisfies E(vi) for all \( t \geq 0 \) and converges to the Golden Rule.

4. Market Value of Firms

The previous section analyzed the real equilibrium outcome of an economy in which capital once installed is a sunk cost, making it possible for the market value of firm to be less than its replacement cost. This section examines the financial side: we study the relation between the market value of a firm and its fundamental value, namely the present discounted sum of its future stream of dividends. When the market value exceeds the fundamental value, the firm acquires a secondary source of value — its transmission value. This is the extra value (over and above its fundamental or dividend value) that the market is prepared to pay for the role of equity as an instrument which permits the transfer of firms from one generation to the next.

When an infinite-lived security acquires a value in excess of its fundamental value, it is said to have a bubble and in an equilibrium model the possibility of a bubble depends on the asymptotic behavior of interest rates. Tirole (1985) showed that in the standard Diamond model, bubbles can only exist in an economy with overaccumulation. In an economy with underaccumulation equilibrium interest rates converge to the steady state interest rate \( r_D = r(k_D) \) which exceeds the rate of growth \( n \) of the population. Since a bubble grows at the rate of interest, and output grows asymptotically at the rate \( n \), a bubble would always end up exceeding the total output of the economy, which is impossible. For a stock market economy in which capital is a sunk cost, we reach a different conclusion: the reason is that the presence of a discount on equity changes the asymptotic behavior of interest rates. In an economy with underaccumulation, instead of converging to \( r_D \), the interest rate converges to \( n \), and this makes possible the existence of a bubble.
4. Market Value of Firms

From the relation

\[ Q^j_t = \frac{1}{1 + r_{t+1}} (D^j_{t+1} + \tilde{Q}^j_{t+1}) \]

shown in Proposition 2, where the financial dividend \( D^j_{t+1} \) is given by (16) and \( \tilde{Q}^j_{t+1} = Q^j_{t+1} - \Gamma^j_{t+1} \) is the equity value net of new issues, it is easy to deduce that the market value satisfies the rate of return condition

\[ M^j_t = \frac{1}{1 + r_{t+1}} (\tilde{D}^j_{t+1} + M^j_{t+1}) \]

where the dividend \( \tilde{D}^j_{t+1} \) is defined as the “real” dividend

\[ \tilde{D}^j_{t+1} = F(K^j_{t+1}, L^j_{t+1}) - w_{t+1}L^j_{t+1} - I^j_{t+1} \]

of firm \( j \) at date \( t+1 \). Since we are studying a balanced growth equilibrium, \( K^j_{t+1} = \mu_j K_{t+1}, L^j_{t+1} = \mu_j L_{t+1}, V^j_{t+1} = \mu_j V_t \), it suffices to study the relation between market value and “real” dividends at the aggregate level, so that we now drop the superscript \( j \). By successive substitutions, one obtains the classical formula

\[ M_t = \sum_{\tau=1}^{T} \frac{\tilde{D}_{t+\tau}}{(1 + r_{t+1}) \ldots (1 + r_{t+\tau})} + \frac{M_{t+T}}{(1 + r_{t+1}) \ldots (1 + r_{t+T})}, \quad T \geq 1 \]

When the limit exists, we define the fundamental value of the firms at date \( t \) by

\[ F_t = \lim_{T \to \infty} \sum_{\tau=1}^{T} \frac{\tilde{D}_{t+\tau}}{(1 + r_{t+1}) \ldots (1 + r_{t+\tau})} \]

and we call the difference between the market and fundamental value

\[ B_t = \lim_{T \to \infty} \frac{M_{t+T}}{(1 + r_{t+1}) \ldots (1 + r_{t+T})} \]

the bubble component of the value, when it is not zero. As noted earlier, since \( M_{t+T} = K_{t+T} - V_{t+T} > 0 \) in equilibrium, \( B_t \geq 0 \). Clearly the behavior of \( B_t \) depends on the asymptotic behavior of market value relative to the rate of interest.

**Proposition 6:** The market value of firms \((M_t)_{t \geq 0}\) in a stock market equilibrium has the following properties:

(i) If \((K_t, V_t)_{t \geq 0}\) is an equilibrium trajectory of an economy with underaccumulation, then

\[ (\alpha) \text{ if } V_0 = 0, \text{ then } M_t = F_t \text{ and } B_t = 0 \]

\[ (\beta) \text{ if } V_0 > 0, \text{ and the trajectory converges to the Golden Rule, then } M_t > F_t \text{ and } B_t > 0 \]
4. Market Value of Firms

(ii) If \((K_t, V_t)_{t \geq 0}\) is an equilibrium trajectory of an economy with overaccumulation, then the sum

\[
F^T_t = \sum_{\tau = 1}^{T} \frac{D_{t+\tau}}{(1 + r_{t+1}) \cdots (1 + r_{t+\tau})}
\]
diverges, \(\lim_{T \to \infty} F^T_t = -\infty\), and \(\lim_{T \to \infty} \frac{M_{t+T}}{(1 + r_{t+1}) \cdots (1 + r_{t+T})} = 0\).

\textbf{Proof:} (i)(\(\alpha\)): Since \(V_t = 0\), \(\forall t \geq 0\),

\[
\frac{M_{t+T}}{(1 + r_{t+1}) \cdots (1 + r_{t+T})} = \frac{N_t (1 + n)^{(T+1)k_{t+T+1}}}{(1 + r_{t+1}) \cdots (1 + r_{t+T})}
\]
and since the equilibrium is a Diamond equilibrium, \(k_{t+T+1} \to k_D\), \(r_{t+T} \to r_D\) with \(r_D > n\), and

\[
\lim_{T \to \infty} \frac{(1 + n)^{T+1}}{(1 + r_{t+1}) \cdots (1 + r_{t+T})} = 0.
\]

(i)(\(\beta\)): Since \(V_t > 0\), \(\forall t \geq 0\),

\[
\frac{M_{t+T}}{(1 + r_{t+1}) \cdots (1 + r_{t+T})} = \frac{N_t (1 + n)^T ((1 + n)k_{t+T+1} - v_{t+T})}{(1 + r_{t+1}) \cdots (1 + r_{t+T})}
\]
and since the equilibrium is a stock market equilibrium, \(k_{t+T+1} \to k^*,\ v_{t+T} \to v^*\) with \((1 + n)k^* - v^* = s(n, w(k^*)) > 0\) and \(A_t = \frac{N_t v_t s(n,w(k^*))}{v^*} > 0\).

(ii) If \(V_0 = 0\), then the expression for \(M_{t+T}\) is the same as in (i)(\(\alpha\)). Since \(k_{t+T+1} \to k_D\), \(r_{t+T} \to r_D\) with \(r_D < n\), it follows that \(\lim_{T \to \infty} \frac{(1 + n)^T}{(1 + r_{t+1}) \cdots (1 + r_{t+T})} = +\infty\). When \(V_0 > 0\), the expression for \(M_{t+T}\) is the same as in (i)(\(\beta\)) and since \(k_{t+T+1} \to k_D, v_{t+T+1} \to 0\), it follows that \(\lim_{T \to \infty} \frac{M_{t+T}}{(1 + r_{t+1}) \cdots (1 + r_{t+T})} = +\infty\). Since \(M_t\) is finite, it follows that \(\lim_{T \to \infty} V^T_t = -\infty\), which can also be checked directly by evaluating \(F_t\).

Thus in a stock market equilibrium, the only case where the market value of a firm coincides with the fundamental value of its dividends is in the Diamond equilibrium of an economy with underaccumulation (case (i)(\(\alpha\))). In this case the three valuations of a firm, its replacement value, its market value and its dividend value, coincide

\[
K_{t+1} = M_t = V_t
\]

In an economy with underaccumulation, when firms have sunk costs and are sold at a discount relative to their replacement costs, the effect of the discount is to lower interest rates and increase investment and capital accumulation. When the economy converges to the Golden Rule, the replacement value, market value and dividend value are all distinct

\[
K_{t+1} > M_t > V_t
\]
4. Market Value of Firms

What is crucial is that the equity market permits firms to be purchased at a cost which is less than their replacement value ($K_{t+1} > M_t$); the equity market becomes distinct from the market for current output, permitting the cheaper transfer of firms from one generation to the next. On the valuation side, unlike in the Diamond equilibrium where firms are valued for their dividends, the market now values firms in addition for their transmission value, that is as instruments permitting the transfer of firms from one generation to the next ($M_t > F_t \iff B_t > 0$).

This property can be seen in its most stark form at the Golden Rule steady state which is a stock market equilibrium with $K_t = N_t k^*, r_t = n, V_t = N_t v^*$. In each period, investment is equal to the return to capital since $i^* = (1 + n)k^* - (1 - \beta)k^* = (n + \beta)k^* = f'(k^*)k^*$. Thus the firms can finance investment by retained earnings if they give zero dividends to their shareholders. With this financial policy, the bond market is inactive and the young agents use all their savings to buy the firms’ equity, which they sell when old with a capital gain, obtaining the return $r^* = n$ on their investment. Equity is a pure bubble—it gives zero dividends—and, like money in the OLG exchange model, has value purely by virtue of its role as a store of value.

More generally, it is instructive to see how the two components $F_t$ and $B_t$ evolve on a typical path leading to the Golden Rule. $B_t$ has been evaluated in the proof of (i)(ii) and is $B_t = N_t v_t s^*/v^*$. An easy calculation shows that $F_t = N_t(1 + n)(k_{t+1} - \frac{n}{\beta}k^*)$. If a trajectory begins with a low capital-labor ratio and a small discount (i.e. $(k_0, v_0)$ small), on the initial segment of the trajectory the market value is not very different from the replacement cost of capital and is almost entirely attributable to dividends ($v_t/v^*$ small implies $B_t/N_t$ small and $F_t/N_t \simeq (1 + n)k_{t+1}$). As the economy grows, the capital-labor increases and the trajectory approaches the Golden Rule: dividends tend to disappear ($k_{t+1} \rightarrow k^*, v_t \rightarrow v^*$ imply $F_t/N_t \rightarrow 0$) and the market value is almost entirely attributable to the transmission value of the firm.

In an economy with overaccumulation the interest rate is so low that the infinite sum which defines the fundamental value diverges. This property holds for a Diamond equilibrium ($v_0 = 0$) and a stock market equilibrium ($v_0 > 0$). The fact that “real dividends” are negative\(^9\) is a manifestation of the inefficiency of the equilibrium in which the investment is too high.

The valuation of firms in a stock market equilibrium thus stands in striking contrast to the valuations predicted by many of the general equilibrium models of financial economics. Typically financial markets are studied in models with infinite-lived agents and the conditions on agents’ preferences required to establish the existence of equilibrium — namely that agents be impatient

\(^9\)Note however that, as we have shown in Section 2, there exist financial policies (e.g. financing part or all of the investment by issuing new shares) which permit firms to distribute non-negative “financial dividends” on equity.

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— rule out bubbles on securities in positive supply like equity (Tirole (1982), Magill-Quinzii (1996), Woodford-Santos (1998)). Infinite-lived agent models have moulded the thinking of many economists and the conventional wisdom of the finance literature that security prices are fundamental values.

As Tirole (1985) pointed out, the situation is quite different in the OLG framework: Tirole (1985) and Weil (1987) showed that in a Diamond model, bubbles can arise for infinite-lived securities in positive supply paying no dividends (such as money viewed purely as a store of value), but only in economies with overaccumulation. The analysis of Tirole, Weil and the ensuing literature on OLG models with production did not study the existence of bubbles on equity, since typically in the Diamond framework firms are viewed as short-lived entities which (like their owners) are recreated each period.

Our analysis has shown that if firms are infinitely lived and equity prices differ from the replacement value of capital, then bubbles on equity are the rule rather than the exception in the OLG model with production, even in economies with underaccumulation. On every equilibrium trajectory there is a bubble, except on the Diamond trajectory of an economy with underaccumulation: in this case, and only in this case, the equity price of a firm coincides with its fundamental value. Since the Diamond steady state is saddlepoint stable while the Golden Rule is stable, this trajectory is from the point of view of our model truly exceptional.

Appendix

Proof of Proposition 1: If \( V_{t+1}^j < (1 - \beta)^2 K_i^j \) then, when \( I_i^j \in [0, \bar{I}] \) with \( \bar{I} = V_{t+1}^j - (1 - \beta)^2 K_i^j \), \( \bar{Q}_{t+1}^j((1 - \beta)K_{t+1}^j) = 0 \). On the other hand if \( I_i^j \geq \bar{I} \) then \( \bar{Q}_{t+1}^j((1 - \beta)K_{t+1}^j) = (1 - \beta)K_{t+1}^j - V_{t+1}^j \). Thus the objective function (6), to be maximized with respect to \((I_i^j, L_{t+1}^j)\), that we will denote \( \pi(I_i^j, L_{t+1}^j) \), is equal to

\[
\pi(I_i^j, L_{t+1}^j) = \begin{cases} 
-I_i^j + \frac{1}{1 + r_{t+1}} \left[ F((1 - \beta)K_i^j + I_i^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j \right] & \text{if } 0 \leq I_i^j \leq \bar{I} \\
-I_i^j + \frac{1}{1 + r_{t+1}} \left[ F((1 - \beta)K_i^j + I_i^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j \right] + (1 - \beta)^2 K_i^j + (1 - \beta)I_i^j & \text{if } I_i^j \geq \bar{I} 
\end{cases}
\]

(27)

When \( I_i^j \geq \bar{I} \), substituting \( I_i^j = K_{t+1}^j - (1 - \beta)K_i^j \), this function can also be written as a function of capital and labor as

\[
\pi(I_i^j, L_{t+1}^j) = \frac{1}{1 + r_{t+1}} \left[ F(K_{t+1}^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j - (\beta + r_{t+1})K_{t+1}^j \right] + (1 - \beta)(1 + r_{t+1})K_i^j - V_{t+1}^j
\]

(28)
Appendix

Suppose that there exists a solution \((I^*_t, L^*_t)\) to maximizing (27) with \(I^*_t > 0\). Suppose first that \(I^*_t \in (0, \bar{I})\). Then \((I^*_t, L^*_t)\) must satisfy the first-order conditions of maximizing (27) in \([0, \bar{I}] \times \mathbb{R}_+\), where the constraint \(I^*_t \geq 0\) is not binding. This implies that

\[
F'_K(K^*_t, L^*_t) \geq 1 + r_{t+1}, \quad F'_L(K^*_t, L^*_t) = w_{t+1} \tag{29}
\]

with \(K^*_t = (1 - \beta)K^*_t + I^*_t\). Consider the associated capital-labor ratio \(K^*_t\) and large values of investment and labor such that \(K^*_t > (1 - \beta)K^*_t + \bar{I}, K^*_t/L^*_t = K^*_t\). Then (28) evaluated at such pairs gives

\[
\pi(I^*_t, L^*_t) = \frac{L^*_t}{1 + r_{t+1}} \left( f(K^*_t) - w_{t+1} - (\beta + r_{t+1})K^*_t \right) + \frac{1}{1 + r_{t+1}} \left( (1 - \beta)(1 + r_{t+1})K^*_t - V^*_t \right)
\]

which, when (29) holds, can be made arbitrarily large (since \(\beta < 1\)). Thus \(I^*_t \leq \bar{I}\) cannot be a solution.

Suppose \((I^*_t, L^*_t) \in (I, \infty)\). Then \((K^*_t, L^*_t)\) must be an interior solution to maximizing (28), so that the first-order conditions

\[
F'_K(K^*_t, L^*_t) = \beta + r_{t+1}, \quad F'_L(K^*_t, L^*_t) = w_{t+1} \tag{30}
\]

must hold and the value of the objective function is

\[
\pi(I^*_t, L^*_t) = \frac{1}{1 + r_{t+1}} \left( (1 - \beta)(1 + r_{t+1})K^*_t - V^*_t \right)
\]

Since \(V^*_t > (1 - \beta)^2K^*_t\), \(\pi(I^*_t, L^*_t) < \frac{1}{1 + r_{t+1}} \left( (1 - \beta)(\beta + r_{t+1})K^*_t \right)\). Let us show that the shareholders would be made better off by not investing. If they do not invest the objective function will be larger or equal to \(\pi(0, \bar{L}^*_t)\) where \(\bar{L}^*_t\) is chosen so that \(((1 - \beta)K^*_t)/\bar{L}^*_t = K^*_t\). When the FOC (30) hold, \(\pi(0, \bar{L}^*_t) = \frac{1}{1 + r_{t+1}} \left( (1 - \beta)(\beta + r_{t+1})K^*_t \right) > \pi(I^*_t, L^*_t)\). Thus the problem of maximizing (27) cannot have a solution such that \(I^*_t > 0\).

\[\square\]

**Proof of Proposition 3**: A Diamond steady state is a solution of the equation \(S(k)/k = 1 + n\) and it is clear that assumption \(S\) implies that the equation has a unique positive solution \(k_D\). To prove global stability we show (i) \(\phi\) is increasing (ii) \(\phi(k) > k\) if \(0 < k < k_D\) and (iii) \(\phi(k) < k\) if \(k > k_D\).

\[
(i) \ s(r(k_{t+1}), w(k_t)) = (1+n)k_{t+1} \iff s(r(\phi(k_t)), w(k_t)) = (1+n)\phi(k_t) \implies s' r'(k_{t+1}) \phi'(k_t) + s' w'(k_t) = (1+n)\phi'(k_t) \implies [(1+n) - s'_r f''(k_{t+1})] \phi'(k_t) = -s'_w k_t f''(k_t) \implies
\]

\[
\phi'(k_t) = \frac{-s'_w k_t f''(k_t)}{(1+n) - s'_r f''(k_{t+1})} > 0 \tag{31}
\]
Appendix

(ii) Suppose not, \( \phi(k) \leq k \); then \( r(\phi(k)) \geq r(k) \) and \( s(r(\phi(k)), w(k)) \geq s(r(k), w(k)) > (1 + n)k \geq (1 + n)\phi(k) \) where the first inequality follows from \( s'_r \geq 0 \) and the second from \( S(k)/k > \frac{S(k_D)}{k_D} = (1 + n) \) since \( k < k_D \); but this contradicts \( (E_D) \), namely \( s(r(\phi(k)), w(k)) = (1 + n)\phi(k) \).

(iii) Suppose not, \( \phi(k) \geq k \); then \( r(\phi(k)) \geq r(k) \) and \( s(r(\phi(k)), w(k)) \leq s(r(k), w(k)) < (1 + n)k \leq (1 + n)\phi(k) \), contradicting \( (E_D) \).

To complete the proof, suppose \( 0 < k_0 > k_D \) (resp. \( k_0 < k_D \)) then (i)-(iii) imply that \( k_t \) is an increasing (decreasing) sequence which is bounded above (below) by \( k_D \); thus \( k_t \to k_D \) as \( t \to \infty \). \( \square \)

**Proof of Proposition 5:** The difference equation system \( (E_D^5) \) can be written as

\[
\begin{align*}
\kappa_{t+1} &= \psi(k_t, v_t) \\
v_{t+1} &= h(k_t, v_t)
\end{align*}
\]

where \( \psi \) is defined implicitly by the equation

\[
(1 + n)\psi(k_t, v_t) - s(f'(\psi(k_t, v_t)) - \beta, w(k_t)) - v_t = 0
\]

and \( h(k_t, v_t) = g(\psi(k_t, v_t))v_t \) with \( g(x) = \frac{1 - \beta + f'(x)}{1 + n} \). Thus the linearized system associated with \( (E_D^5) \) around a steady state \((\tilde{k}, \tilde{v})\), expressed in terms of the deviation variables \((\kappa_t, \nu_t) = (k_t - \tilde{k}, v_t - \tilde{v})\) is given by

\[
\begin{bmatrix}
\kappa_{t+1} \\
\nu_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\psi'_k(\tilde{k}, \tilde{v}) & \psi'_v(\tilde{k}, \tilde{v}) \\
H'_k(\tilde{k}, \tilde{v}) & H'_v(\tilde{k}, \tilde{v})
\end{bmatrix}
\begin{bmatrix}
\kappa_t \\
\nu_t
\end{bmatrix}
\]

where

\[
\psi'_k(\tilde{k}, \tilde{v}) = \frac{-s'_w f''(\tilde{k})}{1 + n - s'_r f''(\tilde{k})}, \quad \psi'_v(\tilde{k}, \tilde{v}) = \frac{1}{1 + n - s'_r f''(\tilde{k})},
\]

\[
H'_k(\tilde{k}, \tilde{v}) = \frac{f''(\tilde{k})}{1 + n} \left( \frac{-s'_w f''(\tilde{k})}{1 + n - s'_r f''(\tilde{k})} \right) \tilde{v}, \quad H'_v(\tilde{k}, \tilde{v}) = g(\tilde{k}) + \frac{f''(\tilde{k})\tilde{v}}{(1 + n)(1 + n - s'_r f''(\tilde{k}))}
\]

Let \( \bar{M} \) denote the matrix of coefficients in \((L_S)\) evaluated at \((\tilde{k}, \tilde{v})\), and let \( p(\lambda) = \lambda^2 - tr(\bar{M})\lambda + \det(\bar{M}) = 0 \) denote the associated characteristic polynomial. To show that the Golden Rule steady state \((\tilde{k}, \tilde{v}) = (k^*, v^*)\) is locally stable we show that both roots of the characteristic polynomial lie inside the unit circle \( (|\lambda_i| < 1, i = 1, 2)\). Note that \( \det(\bar{M}^*) = \psi'_k(k^*, v^*) = \frac{-s'_w k^* f''(k^*)}{1 + n - s'_r f''(k^*)} > 0 \) by assumption \( \mathcal{C} \). Since there is underaccumulation, \( k^* > k_D \) and by assumption \( \mathcal{S}, S'(k^*) < \frac{S(k_D)}{k_D} = 1 + n \). Since \( S'(k^*) = s'_r f''(k^*) - s'_w k^* f''(k^*) \), this implies \( 0 < \det(\bar{M}^*) < 1 \). Since \( \det(\bar{M}^*) = \lambda_1 \lambda_2 \), if
both roots are complex, they lie inside the unit circle. The condition $0 < \det M^* < 1$ implies that if both roots are real they lie in the unit interval $(-1, 1)$ if and only if $p(1) > 0$ and $p(-1) > 0$. Now

$$p(1) = 1 - trM^* + \det M^* = \frac{-f''(k^*)v^*}{(1+n)(1+n-s'_r f''(k^*))} > 0$$

since $v^* > 0$, $s'_r \geq 0$, $f''(k^*) < 0$ and

$$p(-1) = 1 + trM^* + \det M^* = 2 - \frac{2s'_w k^* f''(k^*)}{1+n-s'_r f''(k^*)} + \frac{f''(k^*)v^*}{(1+n)(1+n-s'_r f''(k^*))}$$

The first two terms are positive: to show $p(-1) > 0$ it suffices to show that the third term is bounded below by -1. Since $S(k^*) > 0$, $v^* < (1+n)k^*$ and since assumption $\mathcal{P}$ implies $k^* f''(k^*) \geq -f'(k^*) = -(\beta + n)$, it follows that

$$\frac{f''(k^*)v^*}{(1+n)(1+n-s'_r f''(k^*))} > \frac{f''(k^*)k^*}{1+n-s'_r f''(k^*)} \geq \frac{-f'(k^*)}{1+n} > -(\beta + n) > -1$$

Thus both roots lie inside the unit circle and $(k^*, v^*)$ is locally stable.

At the Diamond steady state $(\bar{k}, \sigma) = (k_D, 0)$, $h'_k(k_D, 0) = 0$ so that $M_D$ is triangular and

$$P(\lambda) = (\psi'_k(k_D, 0) - \lambda)(h'_v(k_D, 0) - \lambda).$$

Since $0 < \psi'_k(k_D, 0) = -\frac{s'_w k_D f''(k_D)}{1+n-s'_r f''(k_D)} < 1$ where the latter inequality follows from $S'(k_D) < \frac{S(k_D)}{k_D} = 1+n$, and $h'_v(k_D, 0) \geq g(k_D) = \frac{1+r(k_D)}{1+n} > 1$, it follows that the Diamond steady state is locally saddlepoint stable. \hfill \Box

References


Appendix


