Supplemental Appendix to "Screening and Segmenting: A Consumer Surplus Perspective"

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A Appendix

Proof of Lemma 1. We begin by describing the profit-maximizing menu. Define

$$J_m(t) \equiv F_m^{-1}(t)(t-1).$$

We denote by t_m the largest minimizer of J_m and by vex$[J_m]$ the convexification (i.e. the pointwise largest convex function smaller than $J_m(t)$). Then, a buyer with value v buys quality q satisfying:

$$c'(q) = \frac{\partial}{\partial t}\text{vex}\left[J_m\right](F_m(v)),$$

if $F_m(v) \geq t_m$, and $q = 0$ otherwise. Note that $J_m'(F_m(v)) = \phi_m(v)$, so the expression coincides with the one in the main text when m is regular.

Consider the set of markets X_{p_m} where menu p_m is seller-optimal:

$$X_{p_m} \equiv \left\{m' \in \Delta V : \Pi(m', p_{m'}) = \Pi(m', p_m)\right\}.$$

The seller's profit $\Pi(m', p_{m'})$ is a pointwise supremum of the weak*-continuous affine maps $m' \mapsto \Pi(m', p)$, and hence convex and weak*-lower semicontinuous (lsc); since $\Pi(m', p_{m'}) \geq \Pi(m', p_m)$ and $m' \mapsto \Pi(m', p_m)$ is affine and weak*-continuous, X_{p_m} is a sublevel set of a convex weak*-lsc function, and therefore convex and weak*-closed (Theorem 7.6 and Lemma 2.42, Aliprantis and Border (2006)). Because V is a compact metric space, ΔV is itself a convex, weak*-compact, metrizable set (Theorem 15.11, Aliprantis and Border (2006)), and the closed, convex subset X_{p_m} inherits all three properties. Therefore, by Choquet's theorem (Phelps (2001)), $m \in X_{p_m}$ can be written as a segmentation supported on its extreme points.

Now, observe that $m' \in X_{p_m}$ if and only if

$$t_m = t_{m'} \text{ and } \mathrm{vex}\,[J_m]\,(t) = \mathrm{vex}\,[J_{m'}]\,(t) \text{ for all } t \geq t_m.$$

On the other hand, m is irregular if and only if $J_m(t) > \mathrm{vex}[J_m](t)$ on some interval $[t_1, t_2]$ where $F_m^{-1}(t)$ is not constant. For such m, sufficiently small perturbations of $F_m(v)$ over $v \in [F_m^{-1}(t_1), F_m^{-1}(t_2)]$ leave $\mathrm{vex}[J_m]$ unchanged (and hence remain in X_{p_m}), meaning irregular markets are not extreme points. Thus, irregular markets can be segmented into regular markets without changing the profit-maximizing menu.

The preceding reduction also allows us to restrict to markets without singular-continuous components. Let $A_m \subseteq (0,1)$ denote the union of the interiors of all maximal intervals on which F_m^{-1} is constant; these correspond exactly to the atoms of F_m. Define

$$H(t) \equiv \frac{\mathrm{vex}[J_m](t)}{t-1} \text{ for } t \in (0,1).$$

Since $\mathrm{vex}[J_m]$ is convex, H is locally absolutely continuous. Moreover, regularity implies $F_m^{-1}(t) = H(t)$ for Lebesgue-a.e. $t \in A_m^c$. Take any $t \in A_m^c$ where $H'(t)$ exists; except at the endpoint of a terminal constant interval, there exists $s > t$ such that $F_m^{-1}(s) > F_m^{-1}(t)$. Convexity and the fact that $\mathrm{vex}[J_m] \leq J_m$ implies

$$\mathrm{vex}[J_m]'(t) \leq \frac{\mathrm{vex}[J_m](s) - \mathrm{vex}[J_m](t)}{s-t} \leq \frac{J_m(s) - J_m(t)}{s-t}.$$

Thus,

$$H'(t) \geq \frac{1-s}{1-t} \cdot \frac{F_m^{-1}(s) - F_m^{-1}(t)}{s-t} > 0 \text{ for Lebesgue-a.e. } t \in A_m^c. \tag{A.1}$$

Let $B \subseteq V$ be any set with Lebesgue measure zero containing no values carrying atoms, meaning that $F_m^{-1}(t) \in B \implies t \in A_m^c$. Since H is locally absolutely continuous, H' exists a.e. on $(0,1)$, and the Serrin-Varberg theorem (Serrin and Varberg (1969)) gives

$$\lambda(\{t : H'(t) \neq 0 \text{ and } H(t) \in B\}) = 0,$$

where λ denotes the Lebesgue measure. Combining this with (A.1) implies that

$$\lambda(\{t \in A_m^c : H(t) \in B\}) = 0.$$

Finally, since $F_m^{-1}(t) = H(t)$ for Lebesgue-a.e. $t \in A_m^c$,

$$m(B) = \lambda(\{t : F_m^{-1}(t) \in B\}) \leq \lambda(\{t \in A_m^c : H(t) \in B\}) = 0,$$

meaning m has no singular-continuous component. $\square$

To make the notation more compact, we denote the survival function by:

$$D_m \equiv 1 - F_m.$$

Proof of Proposition 1. To begin, we claim that for any regular market m,

$$\int_v^{\overline{v}} D_m(t)\ dt = \begin{cases} \int_v^{\overline{v}} h_m(t)\ dF_m(t) & \text{if } \Delta_m(v) = 0; \\ \int_v^{\overline{v}} h_m(t)\ dF_m(t) + \Delta_m(v) D_m(v) & \text{if } \Delta_m(v) > 0. \end{cases} \tag{A.2}$$

In any regular market, $\Delta_m(v) = 0$ if and only if F_m is absolutely continuous at v in the local sense that the continuous non-atomic component admits a density there; by the observation in Lemma 1, regularity rules out a singular-continuous component. If F_m is absolutely continuous at v with density $f_m(v)$, then by definition $D_m(v) = f_m(v)h_m(v)$. Also, in any regular market, at the lower boundary of a gap there is always an atom. That is, at any interval (v_1, v_2) satisfying

$$(v_1, v_2) \cap \text{supp}(m) = \{\emptyset\} \text{ and } v_1, v_2 \in \text{supp}(m),$$

we must have that $a(v_1) > 0$. In any such interval,

$$\int_v^{v_2} D_m(t)\ dt = \Delta_m(v) D_m(v) = \Delta_m(v) D_m(v) + \int_v^{v_2} h_m(t)\ dF_m(t).$$

where in the second equality we are integrating zero because $dF(v)$ is zero when $v \notin \text{supp}(m)$. At the limit $v = v_1$ we obtain

$$\int_{v_1}^{v_2} D_m(t)\ dt = a_m(v_1) h_m(v_1) = \lim_{w \uparrow v_1} \int_w^{v_2} h_m(t)\ dF_m(t)$$

where we take the left limit of the lower boundary of the integral to include the atom at v_1. Thus, combining the intervals, we get (A.2).

We now apply this to the aggregate market:

$$\begin{aligned} \int_v^{\overline{v}} h^*(t)\ dF^*(t) &= \int_v^{\overline{v}} \int_{\Delta V} D_m(t)\ d\sigma(m)\ dt = \int_{\Delta V} \int_v^{\overline{v}} D_m(t)\ dt\ d\sigma(m) \\ &= \int_{\Delta V} \int_v^{\overline{v}} h_m(t)\ dF_m(t)\ d\sigma(m) + \int_{\Delta V} \mathbb{1}\{v \notin \text{supp}(m)\} \Delta_m(v) D_m(v)\ d\sigma(m), \end{aligned} \tag{A.3}$$

where the change in the order of integration uses Fubini's theorem (Theorem 11.27, Aliprantis and Border (2006)) and the second line is applying (A.2). The first term can be written as:

$$\int_{\Delta V}\int_{v}^{\overline{v}} h_m(t)\ dF_m(t)\ d\sigma(m) = \int_{v}^{\overline{v}}\int_{\Delta V} h_m(t)\ d\Sigma_t(m)\ dF^*(t) = \int_{v}^{\overline{v}} h_\sigma(t)\ dF^*(t).$$

This again uses Fubini's theorem and the disintegration/conditional measure notation embedded in the definition of Σ_t. Subtracting this from the lhs of (A.3) yields the result. □

Proof of Proposition 2. Endow ΔV and $\Delta(\Delta V)$ with the weak topology on measures, so that ΔV and $\Delta(\Delta V)$ are closed and compact. The maximization problem (23) is taken over $h \in L_+^p[\underline{v},\overline{v}]$, where $1 < p < \infty$. Boundedness guarantees Q (and u) lie in L_+^p, and our tie-breaking assumptions ensure that Q and u are upper semicontinuous (usc). The assumption that $f^*(v) > 0$ and F^* is real-analytic means that h^* is real-analytic. A crucial tool for our analysis will be the concave upper envelope of $u(v,h)$ with respect to h (i.e. the pointwise smallest concave function larger than $u(v,h)$), which we denote by $\overline{u}(v,h)$. Since $u(v,h)$ is bounded and zero when $h \geq v$, $\overline{u}(v,h)$ is constant for $h \geq \overline{h}(v)$.

The proof is divided into two main parts. Proposition A.1 proves that there exists an h which attains the supremum of (23), and characterizes the solution using a KKT system, which allows us to derive certain properties of the solution (Lemma A.1). Proposition A.2 then provides a particular construction method for a uniform segmentation implementing any h which satisfies the technical properties of Lemma A.1.

Proposition A.1 (Supremum is Attained)
The supremum of (23) *is attained. Furthermore, h solves* (23) *if and only if both* (i) $\frac{\partial}{\partial h}u(v,h(v))$ *is non-decreasing in v and constant in v when $E_h(v) > 0$, and* (ii) $u(v,h(v)) = \overline{u}(v,h(v))$ *hold at almost all v.*

Proof. For any $h(v)$ we have that

$$\widehat{h}(v) = \min\{h(v), \overline{h}(v)\}$$

generates higher local informational rents and relaxes the majorization constraint. Hence, it is without loss to make the feasible set

$$\mathcal{H} \equiv \Big\{h \in L_+^p[\underline{v},\overline{v}] \mid h \prec h^* \text{ and } 0 \leq h(v) \leq \overline{h}(v) \text{ a.e.}\Big\}.$$

We then study the relaxed problem:

$$\sup_{\mathcal{H}} \int_{\underline{v}}^{\overline{v}} \overline{u}(v,h(v))\ dF^*(v). \tag{A.4}$$

Endow this space with the weak topology. By construction, $\overline{u}(v, h)$ is concave (and hence also continuous) in h, and additionally bounded because u is bounded. By Rockafeller (1968)'s theorem on integral functionals, the rhs is concave and weakly usc over $\mathcal{H}$. Furthermore, since $\mathcal{H}$ is a closed subset of a weakly compact space, it is itself weakly compact (Theorem 6.25, Aliprantis and Border (2006)). Hence, by Weierstrass Theorem, the supremum is attained.

Next, we wish to characterize the solution of (A.4) using a KKT system. The classic Slater constraint qualification fails, due to the fact that both the box and majorization constraints have no topological interior in their corresponding function spaces. However, by applying an appropriate generalization of this constraint qualification to infinite dimensional settings (Jeyakumar and Wolkowicz (1992); Boţ et al. (2008)), we get that the KKT conditions are both necessary and sufficient.[1] Thus, for any optimal $h \prec h^*$, there exists a function $\lambda(v) \in L^q_+[\underline{v}, \overline{v}]$, where $\frac{1}{p} + \frac{1}{q} = 1$, and a finite non-negative measure μ such that (Theorem 13.18, Aliprantis and Border (2006))

$$\frac{\partial}{\partial h}\overline{u}(v, h(v)) = \mu[\underline{v}, v] - \lambda(v) \text{ and } \lambda(v)h(v) = E_h(v)d\mu(v) = 0 \text{ a.e.}$$

Here λ is the multiplier on the non-negativity constraint, and μ the multiplier on the majorization constraint (we know the upper bound never binds).

In fact, the non-negativity constraint is never binding (i.e. $\lambda(v) = 0$ a.e.). Suppose there is an interval (v_1, v_2) such that the constraint binds in this interval (that is, $\lambda(v) > 0$ and, consequently, also $h(v) = 0$ on this interval). We must clearly have that the constraint slacks in this interval, so $\mu[\underline{v}, v]$ is constant for all $v \in [v_1, v_2]$. Furthermore, if $v_1 > \underline{v}$, then the constraint must slack in $v \in [v_1 - \epsilon, v_2]$ for an ϵ small enough (which follows from the fact that $E_h(v)$ is strictly decreasing in $[v_1, v_2]$). If $v_1 = \underline{v}$, then $\mu[\underline{v}, v]$ must be equal to zero in $v \in [v_1, v_2]$, but this contradicts the first-order condition. If $v_1 > \underline{v}$, then we must have that:

$$\frac{\partial}{\partial h}\overline{u}(v_1 - \epsilon, h(v_1 - \epsilon)) = \mu[\underline{v}, v_1] > \mu[\underline{v}, v_1] - \lambda(v_1 + \epsilon),$$

where we simply used that the constraint slacks so $\mu[\underline{v}, v]$ stays constant in $[v_1 - \epsilon, v_1 + \epsilon]$ (for a small enough ϵ). However, we also have that:

$$\frac{\partial}{\partial h}\overline{u}(v_1 - \epsilon, h(v_1 - \epsilon)) \leq \frac{\partial}{\partial h}\overline{u}(v_1 - \epsilon, 0) \leq \frac{\partial}{\partial h}\overline{u}(v_1 + \epsilon, 0),$$

[1] In particular, we need a feasible point in the *quasi-relative interior* of $\mathcal{H}$, i.e. an $H \in \mathcal{H}$ such that

$$0 < h(v) < \overline{h}(v) \text{ a.e. and } \int_v^{\overline{v}} h(t)\ dF^*(t) < \int_v^{\overline{v}} h^*(t)\ dF^*(t) \text{ for all } v < \overline{v},$$

which is satisfied by truncating h^* by $\overline{h}$ and then scaling down by $0 < \alpha < 1$.

where in the first inequality we used that $\overline{u}$ is concave and in the second one we used that $\partial\overline{u}(v,0)/\partial h$ is increasing in v. We thus reach a contradiction with the fact that the first-order condition is satisfied at $v = v_1 + \epsilon$. Thus, the KKT conditions simplify to

$$\frac{\partial}{\partial h}\overline{u}(v,h(v)) = \mu[\underline{v},v] \text{ and } E_h(v)d\mu(v) = 0 \text{ a.e.} \tag{A.5}$$

Our final step is to show that (A.5) implies that there is no relaxation gap by considering the concavification in (A.4). We prove that for any h satisfying (A.5), $u(v,h(v)) = \overline{u}(v,h(v))$ a.e. An implication of (A.5) is that

$$\frac{d}{dv}\frac{\partial}{\partial h}\overline{u}(v,h(v)) = \frac{\partial^2}{\partial v \partial h}\overline{u}(v,h(v)) + \frac{dh}{dv}\cdot\frac{\partial^2}{\partial h^2}\overline{u}(v,h(v)) \geq 0 \text{ a.e.}$$

since μ is a non-negative measure. Take any v where $u(v,h(v)) < \overline{u}(v,h(v))$. The second term of the above expression is zero ($\overline{u}$ is affine). Let (h_B,h_T) be the bottom and top of the support point of $\overline{u}(v,h(v))$, respectively, so that

$$\frac{\partial\overline{u}(v,h(v))}{\partial h} = \frac{\partial u(v,h_B(v))}{\partial h} = \frac{\partial u(v,h_T(v))}{\partial h} = \frac{u(v,h_T(v)) - u(v,h_B(v))}{h_T(v) - h_B(v)}. \tag{A.6}$$

Here we are using that $\overline{u}(v,h) = u(v,h)$ for all $h < \epsilon$ (for a small enough ϵ), so the concavification differs from u only at interior hazard rates. We thus get that

$$\frac{d}{dv}\frac{\partial\overline{u}(v,h(v))}{\partial h} = \frac{1}{h_T(v) - h_B(v)}\left(\frac{\partial u(v,h_T(v))}{\partial v} - \frac{\partial u(v,h_B(v))}{\partial v}\right).$$

Here we took the derivative of the rhs of (A.6), and then used (A.6) to cancel out all the terms with derivatives of $h_T(v)$ and $h_B(v)$, so only the partial derivative with respect to v is left. Finally, using the middle equality of (A.6), we get that:

$$\frac{d}{dv}\frac{\partial\overline{u}(v,h(v))}{\partial h} = \frac{1}{h_T(v) - h_B(v)}\int_{h_B(v)}^{h_T(v)}\left(\frac{\partial^2 u(v,h)}{\partial h^2} + \frac{\partial^2 u(v,h)}{\partial h \partial v}\right)dh < 0$$

where the inequality follows from

$$\frac{\partial^2 u(v,h)}{\partial h^2} + \frac{\partial^2 u(v,h)}{\partial h \partial v} = -Q'(v-h) < 0. \tag{A.7}$$

Thus, there can be no positive measure of points where $u(v,h) < \overline{u}(v,h)$ for any h that satisfies (A.5). Since m^* is absolutely continuous, this means any solution of the relaxed problem (A.4) achieves the same value in the original problem:

$$\max_{\mathcal{H}} \int_{\underline{v}}^{\overline{v}} \overline{u}(v, h(v)) \; dF^*(v) = \max_{\mathcal{H}} \int_{\underline{v}}^{\overline{v}} u(v, h(v)) \; dF^*(v),$$

and the solution to (A.4) is also a maximizer of (23). $\square$

Before moving on to the construction of the uniform segmentation implementing the solution, we prove that the solution satisfies regularity and the constraint binds near $\overline{v}$.

Lemma A.1 (Regularity of Optimal h)
For any $h \prec h^*$ which satisfies (A.5), (i) $h'(v) < 1$ *a.e., and* (ii) *there exists a $\delta > 0$ such that $h(v) = h^*(v)$ for almost all $v \in [\overline{v} - \delta, \overline{v}]$.*

It is then without loss to assume that (i) and (ii) hold exactly everywhere, since measure 0 sets do not matter when m^* is absolutely continuous. Then, (i) both implies that $v - h(v)$ is monotone, i.e. the canonical segmentation implementing h is supported on (strictly) regular markets only, and that h has bounded total variation.

Proof. Take the total derivative of the marginal local information rent with respect to v:

$$\frac{d}{dv}\frac{\partial u(v, h(v))}{\partial h} = h'(v)\frac{\partial^2 u(v, h(v))}{\partial h^2} + \frac{\partial^2 u(v, h(v))}{\partial h \partial v} \geq 0.$$

Observe that $u(v, h(v)) = \overline{u}(v, h(v))$ implies u is concave in h at $h(v)$ (Theorem 7.23, Aliprantis and Border (2006)); (i) then follows from (A.7). For (ii), suppose by way of contradiction that there is an interval $(\overline{v} - \delta, \overline{v})$ such that $E_h(v) > 0$ on this interval (E_h is continuous even if h is not continuous). By (A.5), $\frac{\partial}{\partial h}\overline{u}(v, h(v))$ must be constant on this interval, and at most $Q(\overline{v} - \delta)$. But, $h^*(v) \to 0$, meaning $\frac{\partial}{\partial h}\overline{u}(v, h^*(v)) \to Q(\overline{v})$. This means for all v sufficiently close to $\overline{v}$, $h(v) > h^*(v)$, which means the majorization constraint is not satisfied. We thus reach a contradiction. $\square$

Proposition A.2 (Implementability of Optimal h)
There exists a uniform segmentation σ implementing the solution to (23).

Proof. Consider a continuum of markets indexed by $m \in [0, 1]$. The support is determined by a function $s : V \to [0, 1]$ such that $v \in \text{supp}(m)$ if and only if $m \leq s(v)$. The distribution is absolutely continuous in market m at all v such that $m < s(v)$ with density $f_m(v)$:

$$f_m(v) = \frac{D_m(v)}{h(v)}, \tag{A.8}$$

and there is possibly an atom of size $a_m(v)$ at v at market $m = s(v)$:

$$a_m(v) = \frac{\Delta_m(v) D_m(v)}{h(v)}. \tag{A.9}$$

Finally, $s(v)$ satisfies (21), which in this case can be written more simply as follows:

$$\int_{s(v)}^{1} \Delta_m(v) D_m(v)\ dm = E_h(v). \tag{A.10}$$

The proof proceeds in two steps. First, we show that given a solution to (A.10), the associated segmentation given by (A.8)-(A.9) aggregates to m^*. That is, for all v,

$$D^*(v) = \int_0^1 D_m(v)\ dm \iff \int_v^{\overline{v}} D^*(t)\ dt = \int_v^{\overline{v}} \int_0^1 D_m(t)\ dm\ dt. \tag{A.11}$$

Here, the equivalence simply states that two functions are the same if and only if their integrals are the same. As before, we have that for any m,

$$\int_v^{\overline{v}} D_m(t)\ dt = \int_v^{\overline{v}} h(t)\ dF_m(t) + \Delta_m(v) D_m(v).$$

The second term is only nonzero when $m > s(v)$ (or, equivalently, $v \notin \mathrm{supp}(m)$), so:

$$\int_v^{\overline{v}} \int_0^1 D_m(t)\ dm\ dt = \int_0^1 \int_v^{\overline{v}} h(t)\ dF_m(t)\ dm + \int_{s(v)}^1 \Delta_m(t) D_m(t)\ dm,$$

where we apply Fubini's theorem to exchange the integration order. Now, using (A.10) to replace the second term on the rhs, we get

$$\int_v^{\overline{v}} \int_0^1 D_m(t)\ dm\ dt = \int_0^1 \int_v^{\overline{v}} h(t)\ dF_m(t)\ dm + \int_v^{\overline{v}} D^*(t)\ dt - \int_v^{\overline{v}} h(t)\ dF^*(t).$$

Let $\widehat{v}$ be the largest value such that the lhs of (A.11) fails; note that the rhs of (A.11) still holds for $v = \widehat{v}$. Plugging this into the above equation, we get

$$\int_{\widehat{v}}^{\overline{v}} D^*(t)\ dt = \int_0^1 \int_{\widehat{v}}^{\overline{v}} h(t)\ dF_m(t)\ dm + \int_{\widehat{v}}^{\overline{v}} D^*(t)\ dt - \int_{\widehat{v}}^{\overline{v}} h(t)\ dF^*(t)$$

which contradicts that the lhs of (A.11) fails at $\widehat{v}$.

The second part of the proof is to show that (A.10) has a solution. We do this by taking a sequence $\{h_n\}$ approximating h, finding a solution s_n for each h_n, and argue that the sequence $\{s_n\}$ converges. We take h_n to be a function with the following properties:

$$h_n \text{ is real-analytic, } \epsilon_n \leq h_n(v) \leq \overline{h}(v), \text{ and (19) holds for all } v \in [\underline{v}, \overline{v} - \delta],$$

where $\epsilon_n \to 0$ and $h_n(v) = h^*(v)$ for $v \geq \overline{v} - \delta$. Since the space of real-analytic functions is dense in the space of functions with bounded total variation, we can construct a sequence $\{h_n\}$ converging pointwise a.e. to h.[2]

We construct $s(v)$ inductively. Consider v_2 such that $s(v)$ satisfies (A.10) for all $v \in [v_2, \overline{v}]$ and $s'(v_2) = 0$. We construct $s(v)$ in a new interval $[v_1, v_2]$ satisfying feasibility and such that $s'(v) = 0$. We consider two cases, and show that we move between cases only when E has a strict inflection point. Since h_n and h^* are real-analytic, E_{h_n} has a finite number of strict inflection points. Hence, the process finishes after a finite number of steps. Finally, note that $s(v) = 1$ for $v \geq \overline{v} - \delta$, so we start the induction with Case 1.

(Case 1: $E''_h(v) \geq 0$ for all v in some neighborhood $[v_2 - \epsilon, v_2]$) Define, for each m,

$$\widehat{D}_m(v) = D_m(v_2) \exp\left(\int_v^{v_2} \frac{1}{h_n(t)}\, dt \right).$$

Since h_n is strictly bounded away from 0, $\frac{1}{h_n}$ is integrable and this is well-defined. These survival functions satisfy the constant hazard rate requirement. Now, define $s_n(v)$ by the following differential equation:

$$\frac{ds_n}{dv} = \frac{1}{\widehat{D}_{s_n(v)}(v)} \cdot \frac{d^2 E_{h_n}}{dv^2}. \tag{A.12}$$

This is an ordinary differential equation where the rhs is continuous in s (recall that h_n is real-analytic), so by Carathéodory's existence theorem, a solution exists. It is clear that in this interval, s is increasing. Furthermore, (A.12) is derived from (A.10) by taking the second derivative and setting $\Delta_m(v) = 0$ (since s_n is increasing), and hence (A.10) is satisfied.

(Case 2: $E''_h(v) < 0$ for all v in some neighborhood $[v_2 - \epsilon, v_2]$) For each m, let $\widehat{v}(m)$ be the lowest inverse of $s_n(v)$ in $[v_2, \overline{v}]$:

$$\widehat{v}(m) \equiv \inf \left\{ v \in [v_2, \overline{v}] \mid s_n(v) \geq m \right\}.$$

[2] Technically, the sequence converges to h in the L^p-norm, and by the Riesz-Fisher theorem we can then extract a pointwise convergent subsequence.

For all $m \geq s_n(v_2)$, this value exists, since $s_n(\overline{v}) = 1$. We define $s_n(v)$ so that

$$\int_{s_n(v)}^{1} \big(\widehat{v}(m) - v\big) D_m(\widehat{v}(m))\ dm = E_{h_n}(v). \tag{A.13}$$

Note that this is just (A.10) under the assumption that $s_n(v)$ is decreasing in $[v, v_2]$. Let v_1 be the largest value less than v_2 such that the solution to (A.13) is decreasing in $[v_1, v_2]$. Taking the second derivative of (A.10), and evaluating at v_1 (we have $s_n'(v_1) = 0$) we get that

$$E''_{h_n}(v_1) = -s''_n(v_1)\big(\widehat{v}(s_n(v_1)) - v_1\big) D_m(\widehat{v}(s_n(v_1))) \geq 0$$

meaning that $[v_1, v_2]$ is at least as large as the gap between the strict inflection points of E_{h_n}. Thus, we move back to Case 1, and after a finite number of steps, the process ends.

Finally, we note that since $h'(v) < 1$ everywhere, when s_n is increasing,

$$\frac{ds_n}{dv} = \frac{1}{\widehat{D}_{s(v)}(v)} E''_{h_n}(v) < \frac{1}{D^*(\overline{v} - \delta)} \left(\max_v \left[2f^*(v) + \frac{df^*(v)}{dv} \overline{h}(v) \right] \right).$$

Hence, all s_n are bounded in $[0, 1]$ with uniform upper bound on $s_n'(v)$, which in turn means they have uniformly upper bounded total variation. Thus, by Helly's Selection Theorem, there is a subsequence of s_n converging pointwise. □

Together, these results establish Proposition 2. □

As a corollary of Proposition A.1, we get necessary and sufficient conditions for h^* to solve (23), extending Corollary 3 to the general setting.

Corollary A.1 (Zero Segmentation)
Zero segmentation is optimal if and only if both (i) $\frac{\partial}{\partial h} u(v, h^*(v))$ *is non-decreasing and* (ii) $u(v, h^*(v)) = \overline{u}(v, h^*(v))$ *hold a.e.*

Proof of Theorem 1. Theorem 1 consists of two parts. The first, that the value of segmentation is characterized by (25), follows from Proposition 2 after we rule out non-uniform segmentations. Any segmentation that is supported only on regular markets generates consumer surplus

$$\int_{\Delta V} \int_{\underline{v}}^{\overline{v}} u(v, h_m(v))\ d\Sigma_v(m)\ dF^*(v) \leq \int_{\underline{v}}^{\overline{v}} \int_{\Delta V} \overline{u}(v, h_m(v))\ d\Sigma_v(m)\ dF^*(v).$$

But, by Jensen's inequality,

$$\int_{\underline{v}}^{\overline{v}} \int_{\Delta V} \overline{u}(v, h_m(v)) \; d\Sigma_v(m) \; dF^*(v) \le \int_{\underline{v}}^{\overline{v}} \overline{u}(v, h_\sigma(v)) \; dF^*(v),$$

with strict inequality whenever $h_m \neq h_\sigma$ a.e. (indeed, equality in Jensen would require $h_\sigma(v)$ to lie on an affine face of $\overline{u}(v, \cdot)$, which is ruled out at an optimum by the no-relaxation-gap argument). But, by Propositions 1-2, the maximum possible value of the rhs is achievable with a uniform segmentation, so any non-uniform segmentation achieves lower consumer surplus than the optimal uniform segmentation. This proves that the optimal segmentation attains value (25).

The second part of the theorem requires us to rule out segmentations supported on irregular markets that nonetheless achieve the same value. To do this, take any irregular market, and decompose it via Lemma 1 to weakly regular markets (of which there must be a positive measure). Our first step is to show that

$$\max_{\sigma \in \mathrm{MPS}(m)} \int_{x \in \Delta V} U(x) \; d\sigma(x) \ge \sup_{h \prec h_m} \int_{\underline{v}}^{\overline{v}} \overline{u}(v, h(v)) \; dF_m(v). \tag{A.14}$$

If m is absolutely continuous, this is immediate because it follows from (25). To accommodate markets with atoms, we first prove that the rhs of (A.14) is lsc while the lhs is usc in m.

For the rhs, we assume all $h \in \mathcal{H}$ are lsc (this convention obviously does not matter when m is continuous). Since u is increasing in h, the objective function is lsc in F_m (Theorem 15.3, Aliprantis and Border (2006)). On the other hand, the set

$$\{h \in \mathcal{H} : h \prec h_m\}$$

is lower hemicontinuous in m. Lower hemicontinuity is straightforward because any $\widehat{h}(v)$ that is pointwise smaller than $h(v)$, with $h \prec h_m$, satisfies $\widehat{h} \prec h_{m'}$ for all m' close enough to m. Hence, following the maximum theorem, the value is lsc in m.

For the lhs of (A.14), note that, under our tie-breaking rules, $U(m)$ is bounded and usc (see Yang (2023)), meaning that (Theorem 15.3, Aliprantis and Border (2006))

$$\int_{x \in \Delta V} U(x) \; d\sigma(x)$$

is itself usc in σ. Furthermore, by standard results, the mapping $m \mapsto \mathrm{MPS}(m)$ is nonempty, compact-valued, and continuous. Hence, by Berge's theorem, the lhs is usc.

Now, if m is only weakly regular, there exists a positive measure subinterval $[v_1, v_2]$ (possibly with atoms) such that $\frac{\partial}{\partial h}\overline{u}(v, h_m(v))$ is strictly decreasing (if $u(v, h_m(v)) = \overline{u}(v, h_m(v))$, this follows from (A.7); else, see (A.6)). Hence, h_m does not satisfy the first-order condition, meaning it does not solve (A.4).[3] We thus have that:

$$\int_{\underline{v}}^{\overline{v}} \overline{u}(v, h_m(v))\ dF_m(v) < \sup_{h \prec h_m} \int_{\underline{v}}^{\overline{v}} \overline{u}(v, h(v))\ dF_m(v) \leq \max_{\sigma \in \mathrm{MPS}(m)} \int_{x \in \Delta V} U(x)\ d\sigma(x).$$

But, the lhs is weakly larger than the consumer surplus in m. So, there is a further segmentation of m which strictly improves consumer surplus. □

Proof of Proposition 3. We first note that for any h,

$$\frac{\partial u(v,h)}{\partial h} \geq 0 \implies \frac{\partial^2 u(v,h)}{\partial h^2} < 0. \tag{A.15}$$

This follows from the fact that

$$\frac{\partial u(v,h)}{\partial h} \geq 0 \iff \frac{1}{h} \geq \frac{Q'(v-h)}{Q(v-h)} \tag{A.16}$$

and

$$h\left(Q''(v-h) - \frac{2Q'(v-h)}{h}\right) \leq h\left(Q''(v-h) - 2\frac{(Q'(v-h))^2}{Q(v-h)}\right) \leq -h\frac{(Q'(v-h))^2}{Q(v-h)},$$

where the first inequality uses (A.16) while the second one follows from (27). We thus obtain (A.15). This implies that $u(v,h)$ is strictly quasi-concave in h. Furthermore, we have that:

$$\frac{\partial^2 u(v,h)}{\partial h \partial v} = h\left(\frac{Q'(v-h)}{h} - Q''(v-h)\right) \geq 0,$$

where the inequality follows as before from (A.16) and (27). This implies that $\overline{h}(v)$ is increasing, which, in turn, implies that $\overline{h}(v)$ and $h^*(v)$ cross at most once (where we use that h^* is decreasing for this last implication). Hence, we have that $u(v, h(v)) = \overline{u}(v, h(v))$ for $h(v)$ defined as in (28). Finally, we show that $\partial u(v, h(v))/\partial h$ is increasing in v. For all $v \leq \widehat{v}$ we have that the derivative is zero; for all $v \geq \widehat{v}$ we have that the derivative is positive.

[3] When m has atoms, the KKT system may not characterize the solution, but the FOC is still necessary; see (24) and the discussion that follows.

Finally, we have that the derivative is also increasing for all $v \geq \widehat{v}$:

$$\frac{d}{dv}\left(\frac{\partial u(v,h^*(v))}{\partial h}\right) = \left(1-\frac{dh^*(v)}{dv}\right)\left(\frac{\left(1-2\frac{dh^*(v)}{dv}\right)}{\left(1-\frac{dh^*(v)}{dv}\right)}Q'(v-h^*(v)) - h^*(v)Q''(v-h^*(v))\right)$$
$$> \left(1-\frac{dh^*(v)}{dv}\right)\left(Q'(v-h^*(v)) - h^*(v)Q''(v-h^*(v))\right) > 0,$$

where the first inequality follows from the fact that $h^*(v)$ is decreasing, so the fraction is larger than 1, while the second inequality follows from (A.16) and (27). Thus, $\partial u(v,h(v))/\partial h$ is increasing in v. Following Proposition A.1, this is the solution to (25). □

Proof of Theorem 2. The only thing left is to show that the canonical segmentation is gapless and atomless. By the proof of Proposition A.2, this is equivalent to s being increasing, or equivalently, E_h being convex. By (A.10),

$$\frac{d}{dv}E_h(v) = \big[h(v) - h^*(v)\big]f^*(v).$$

For $v \geq \widehat{v}$, $h(v) = h^*(v)$; otherwise, $h(v) < h^*(v)$ with h increasing and h^* decreasing. So, it is immediate that E_h is convex. □

Proof of Proposition 4. We use the conditions in Corollary A.1 to know when zero segmentation is optimal. First, we note that $u(v,h^*(v)) = \overline{u}(v,h^*(v))$ if and only if

$$h^*(v) \leq \overline{h}(v) = \frac{\gamma-1}{\gamma}v.$$

Since $h(v)$ is increasing in γ, the condition becomes less restrictive as γ increases. Second, we note that $\partial u(v,h^*(v))/\partial h$ is increasing in v if and only if:

$$(\gamma-1)\left(v - \frac{\partial h^*(v)}{\partial v}(2v - h^*(v))\right) + \left(\frac{\partial h^*(v)}{\partial v} - 1\right)h^*(v) \geq 0. \tag{A.17}$$

Finally, we have that $\partial h^*(v)/\partial v \leq 1$ so the second term is negative (otherwise, this is an irregular market, so zero segmentation is not optimal due to Theorem 1). Thus, if at some v, (A.17) is satisfied with equality, then (A.17) will not be satisfied also for all lower values of γ. Hence, among regular markets, (A.17) also becomes less restrictive as γ increases. □

We now provide the proofs of the results in Section 6. We consider the following problem:

$$\overline{W} \equiv \max_{\sigma \in \text{MPS}(m^*)} \int_{\Delta V} \int_{\underline{v}}^{\overline{v}} w(v, h_m(v)) \; dF_m(v) \; d\sigma(m) \tag{A.18}$$
$$\text{subject to: every } m \in \text{supp}(\sigma) \text{ is regular.}$$

This is the same as (16), but replaced $u(v,h) \to w(v,h)$. Lemma 1 will be satisfied in all the extensions we consider, so considering this formulation will indeed be justified. We assume that w is not too supermodular relative to the concavity:

$$\frac{\partial^2 w(v,h)}{\partial h^2} + \frac{\partial^2 w(v,h)}{\partial h \partial v} \leq 0. \tag{A.19}$$

This is an economically substantive assumption—it implies that higher values are associated with only moderately higher inverse hazard rates (or lower inverse hazard rates)—and is used to guarantee the solution to (23) is regular. Second, we assume that there exists ϵ such that for all $h \in (0, \epsilon)$:

$$\frac{\partial w(v,h)}{\partial h} \text{ is non-decreasing in } v \text{ and is greater than } \frac{w(v,h')}{h'} \text{ for all } h' \in [\epsilon, \infty). \tag{A.20}$$

This assumption rules out corner cases at $h = 0$ in the relaxed problem. In particular, it ensures that whenever the concavification differs from the original objective, this happens away from the boundary. This is used immediately after (A.6) in our main argument.

Theorem 3 (Generalized Upper Bound)
Under (A.19)-(A.20)*, the maximum of* (A.18) *is given by:*

$$\overline{W} = \max_{h \prec h^*} \int_{\underline{v}}^{\overline{v}} w(v, h(v)) \; dF^*(v). \tag{A.21}$$

Furthermore, every segmentation achieving this value is a uniform segmentation implementing some h which solves the rhs of (A.21)*.*

The theorem generalizes our main result to objective functions that do not consist of maximizing the local informational rents. The theorem simply identifies the conditions of u we used to prove Theorem 1, so it does not involve any new analysis. We now provide the proofs of the results in the main text.

Proof of Proposition 5. Following standard techniques, the seller's supply is:

$$Q(\phi_m(v)) = v - \frac{d\tau(v)}{dv} \frac{1 - F_m(v)}{f_m(v)}.$$

Hence, the expected consumer surplus,

$$\int_{\underline{v}}^{\overline{v}} \frac{d\tau(v)}{dv} \frac{1-F_m(v)}{f_m(v)} \left(v - \frac{d\tau(v)}{dv} \frac{1-F_m(v)}{f_m(v)} \right) dF_m(v).$$

We thus obtain that the consumer surplus is of the form (A.18) with $w(v,h)$ given by (40). Lemma 1 is clearly satisfied as we can apply the same arguments. Furthermore,

$$\frac{\partial^2 w(v,h)}{\partial h^2} + \frac{\partial^2 w(v,h)}{\partial h \partial v} = \frac{d\tau(v)}{dv} \left(1 + \frac{\frac{v d^2\tau(v)}{dv^2}}{\frac{d\tau(v)}{dv}} - 2\frac{d\tau(v)}{dv} \right) - 4h(v) \frac{d\tau(v)}{dv} \frac{d^2\tau(v)}{dv^2} < 0$$

and

$$\frac{\partial w(v,h)}{\partial h} \mid_{h=0} = v \frac{d\tau(v)}{dv}.$$

Thus, (A.19)-(A.20) are satisfied. □

Proof of Proposition 6. We can verify that for all $\lambda \geq -1$, Lemma 1 holds and $w(v,h)$ satisfies:

$$\frac{\partial^2 w(v,h)}{\partial h^2} + \frac{\partial^2 w(v,h)}{\partial h \partial v} = -(1+\lambda) Q'(v-h) \text{ and } \frac{\partial w(v,h)}{\partial h} \mid_{h=0} = Q(v).$$

Thus, (A.19)-(A.20) are satisfied. □

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