Supplemental Material
to
Inference in a Stationary/Nonstationary Autoregressive Time-Varying Parameter Model

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Section A provides the critical values $c_\psi(\alpha)$, the α^{th} Quantile of J_ψ, for constructing CI's and MUE's for $\rho(\tau)$. Section B provides asymptotic theory. Section C is concerned with the simulation results. Section D extends the methods of the paper for TVP-AR(1) models to TVP-AR(p) models for $p > 1$. Section E provides additional empirical results.

A Critical Values $c_\psi(\alpha)$

Table SM.1 provides the critical values $c_\psi(\alpha)$ for $\alpha = .025$, .05, .5, .95, and .975 and for ψ between 0 and 500. Given these critical values, one can compute equal-tailed two-sided CI's and MUE's for $\rho(\tau)$ based on (3.9) and (3.11), respectively.

Table SM.1: Values of Relevant Quantiles of J_ψ for Use with 90% and 95% Equal-Tailed Two-Sided CI's and MUE's

Values of $c_\psi(\alpha)$, the α^{th} Quantile of J_ψ, for Use with 90% and 95% Equal-Tailed Two-Sided CI's and MUE's													
ψ	0	0.2	0.4	0.6	0.8	1	1.4	1.8	2.2	2.6	3	3.4	3.8
$c_\psi(.025)$	-3.12	-3.09	-3.05	-3.03	-2.99	-2.98	-2.93	-2.89	-2.85	-2.82	-2.79	-2.77	-2.74
$c_\psi(.05)$	-2.86	-2.83	-2.79	-2.76	-2.72	-2.70	-2.65	-2.61	-2.57	-2.53	-2.51	-2.48	-2.46
$c_\psi(.5)$	-1.57	-1.51	-1.47	-1.42	-1.37	-1.34	-1.26	-1.20	-1.14	-1.08	-1.03	-1.00	-0.96
$c_\psi(.95)$	-0.09	-0.02	0.03	0.08	0.13	0.17	0.24	0.31	0.37	0.42	0.48	0.53	0.56
$c_\psi(.975)$	0.23	0.30	0.36	0.40	0.45	0.49	0.55	0.63	0.69	0.74	0.79	0.84	0.87
ψ	4.2	4.6	5	6	7	8	9	10	11	12	13	14	15
$c_\psi(.025)$	-2.72	-2.70	-2.68	-2.65	-2.60	-2.58	-2.56	-2.54	-2.51	-2.50	-2.48	-2.46	-2.45
$c_\psi(.05)$	-2.44	-2.41	-2.39	-2.35	-2.31	-2.28	-2.26	-2.23	-2.21	-2.19	-2.18	-2.15	-2.14
$c_\psi(.5)$	-0.92	-0.90	-0.86	-0.81	-0.75	-0.71	-0.68	-0.65	-0.62	-0.59	-0.58	-0.55	-0.54
$c_\psi(.95)$	0.60	0.64	0.68	0.75	0.82	0.86	0.91	0.95	0.98	1.02	1.04	1.05	1.08
$c_\psi(.975)$	0.91	0.94	0.99	1.05	1.12	1.17	1.22	1.25	1.29	1.32	1.34	1.37	1.39
ψ	20	25	30	40	50	60	70	80	90	100	200	300	500
$c_\psi(.025)$	-2.39	-2.35	-2.32	-2.28	-2.25	-2.23	-2.20	-2.19	-2.17	-2.17	-2.11	-2.08	-2.05
$c_\psi(.05)$	-2.08	-2.05	-2.02	-1.96	-1.94	-1.91	-1.89	-1.88	-1.86	-1.85	-1.80	-1.76	-1.74
$c_\psi(.5)$	-0.47	-0.42	-0.39	-0.33	-0.30	-0.27	-0.25	-0.23	-0.23	-0.21	-0.15	-0.12	-0.09
$c_\psi(.95)$	1.15	1.20	1.24	1.30	1.33	1.37	1.39	1.40	1.41	1.43	1.49	1.52	1.55
$c_\psi(.975)$	1.46	1.51	1.56	1.61	1.65	1.67	1.71	1.72	1.72	1.74	1.80	1.83	1.87

B Theory

B.1 Weaker Assumptions on h for the Case where ρ_n, μ_n, and σ_n^2 Are Asymptotically Locally Constant

The asymptotic results in Section 7 rely on Assumption 2, which requires that the bandwidth h is small enough that $nh^5 \to 0$ as $n \to \infty$. This assumption is suitable when the functions ρ_n, μ_n, and σ_n^2 are asymptotically non-constant in a neighborhood of the time point of interest τ. However, this condition can be relaxed if the functions ρ_n, μ_n, and σ_n^2 are constant or asymptotically constant in a neighborhood of τ. In this section, we state an alternative to Assumption 2 that imposes weaker conditions on h that depend on how close the functions ρ_n, μ_n, and σ_n^2 are to being asymptotically constant in a neighborhood of τ, but still allows us to establish the results of Theorems 7.2 and 7.3. In the case of locally constant functions, the condition on h is just $h = o(\ln^{-2}(n))$.

The functions $\rho_n(\cdot)$ and $\mu_n(\cdot)$ depend on $\kappa_n(\cdot)$ and $\eta_n(\cdot)$ by the definition of the parameter space Λ_n following (7.1).

Definition. Let ℓ_n be the supremum of the Lipschitz constants of $\kappa_n(\cdot)$, $\eta_n(\cdot)$, and $\sigma_n^2(\cdot)$ and the absolute value of the second derivative of $\kappa_n(\cdot)$ over the interval I_{τ,ε_2} that contains τ.

In this definition, I_{τ,ε_2} is as defined in the parameter space Λ_n in Section 7.1.

As defined, ℓ_n is a measure of the non-constancy of the functions $\rho_n(\cdot)$, $\mu_n(\cdot)$, and $\sigma_n^2(\cdot)$. When $\ell_n \to 0$ as $n \to \infty$, these functions become closer to being constant functions in a neighborhood of τ as the sample size increases.

Assumption 2*. (i) $n\ell_n^2 h^5 \to 0$ *and* (ii) $h = o(\ln^{-2}(n))$.

Assumption 2*(i) allows h to converge to zero slower than the condition $nh^5 \to 0$ when $\ell_n \to 0$. If $n\ell_n^2 \to 0$, then h can converge to zero as slowly as $o(\ln^{-2}(n))$.

The following result shows that Assumption 2 can be replaced by the weaker condition Assumption 2* and Theorems 7.2 and 7.3 still hold.

Theorem B.1. *Theorems 7.2 and 7.3 hold with Assumption 2 replaced by Assumption 2*.*

The proof of Theorem B.1 is given in Section B.15 below.

B.2 Proof of Theorem 7.1

The proof of Theorem 7.1 uses the following lemma (which is also used elsewhere below).

By definition, $\rho_n(s) = 1 - \kappa_n(s)/b_n$, see (7.8). When $b_n \to \infty$, for $s \in I_{\tau,\varepsilon_2}$, define $\kappa_n^*(s)$ by

$$\rho_n(s) = 1 - \kappa_n(s)/b_n = \exp\{-\kappa_n^*(s)/b_n\}. \tag{B.2.1}$$

The function $\kappa_n^*(s)$ on I_{τ,ε_2} has the following property when $b_n \to \infty$.

Lemma B.1. *If $b_n \to \infty$ and $\kappa_n(s)$ satisfies part* (ii) *of the parameter space Λ_n defined following* (7.1), *then*

$$\sup_{s \in I_{\tau,\varepsilon_2}} \left| \frac{\kappa_n^*(s)}{\kappa_n(s)} - 1 \right| \to 0.$$

The proof of Lemma B.1 follows that of Theorem 7.1.

Proof of Theorem 7.1. Let

$$CP_n(\lambda) := P_\lambda(\rho(\tau) \in CI_{n,\tau}), \tag{B.2.2}$$

where $P_\lambda(\cdot)$ denotes probability under $\lambda \in \Lambda_n$. The results of Theorem 7.1 hold by Theorem 2.1 of Andrews, Cheng, and Guggenberger (2020) (ACG) provided Assumptions A1 and S of ACG hold with CP in Assumption S equal to $1-\alpha$. In applying Theorem 2.1 of ACG, we let the parameter space Λ in that paper depend on n, as it does in the present paper, which does not cause any complications for the results of that paper. Sufficient conditions for these assumptions are Assumptions B and S of ACG by Theorem 2.2 of ACG. Assumptions B and S of ACG combine to require: For any subsequence $\{p_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ and any sequence $\{\lambda_{p_n} \in \Lambda_{p_n}\}_{n\geq 1}$

$$\text{for which } h_{p_n}^*(\lambda_{p_n}) \to h^* \in H^*, \text{ we have } CP_{p_n}(\lambda_{p_n}) \to 1-\alpha, \tag{B.2.3}$$

where $\{h_n^*(\lambda)\}_{n\geq 1}$ is a suitably chosen sequence of functions.[1] In the present case, we take

$$\begin{aligned} h_n^*(\lambda) &:= \left(h_{n,1}^*(\lambda), h_{n,2}^*(\lambda), h_{n,3}^*(\lambda), h_{n,4}^*(\lambda)\right)', \text{ where} \\ h_{n,1}^*(\lambda) &:= nh(1-\rho(\tau)) = nh\left(1 - \left(1 - \frac{\kappa(\tau)}{b}\right)\right) = \frac{nh}{b}\kappa(\tau), \\ h_{n,2}^*(\lambda) &:= \frac{nh}{b}, \\ h_{n,3}^*(\lambda) &:= b/\left(nh^{1/2}\right), \text{ and} \\ h_{n,4}^*(\lambda) &:= \left(\kappa, \mu, \sigma^2\right), \end{aligned} \tag{B.2.4}$$

[1]The asterisks on $h_n^*(\lambda_{p_n}), h^*$, and H^* do not appear in ACG. They are added here to avoid confusion with the smoothing parameter h used in this paper.

where (κ, μ, σ^2) are viewed as functions on I_{τ,ε_2}, rather than on $[0,1]$. As required by ACG, the functions (κ, μ, σ^2) in $h_{n,4}^*(\lambda)$ lie in a compact metric space $\mathscr{T}$ (under the sup norm) by the definition of Λ_n. We can write the smoothing parameter h as $h = h_n$. This implies that for the subsequence $\{p_n\}_{n\geq 1}$, nh becomes $p_n h_{p_n}$.

The condition $h_{p_n}^*(\lambda_{p_n}) \to h^* \in H^*$ in (B.2.3) implies (i) $\left(\kappa_{p_n}, \mu_{p_n}, \sigma_{p_n}^2\right) \to (\kappa_0, \mu_0, \sigma_0^2) \in \mathscr{T}$ under the sup norm, which implies Assumption 3, (ii) $p_n h_{p_n}/b_{p_n} \to r_0 \in [0,\infty]$, which is imposed in the subsequence versions of Theorems 7.2 and 7.3 with $r_0 \in [0,\infty)$ and $r_0 = \infty$, respectively, (iii) $b_{p_n}/\left(p_n h_{p_n}^{1/2}\right) \to w_0 \in [0,\infty]$, which implies Assumption 4 and is imposed in the subsequence versions of Lemma 7.2(b) and Theorem 7.2 when $r_0 = 0$, and (iv) $p_n h_{p_n}(1-\rho_{p_n}(\tau)) = \frac{p_n h_{p_n}}{b_{p_n}} \kappa_{p_n}(\tau) \to r_0\kappa_0(\tau) \in [0,\infty]$, which holds by (i) and (ii). In the present case, $h^* = (r_0\kappa_0(\tau), r_0, w_0, (\kappa_0, \mu_0, \sigma_0^2))'$ and $H^* = [0,\infty] \times [0,\infty] \times [0,\infty] \times \mathscr{T}$.

By Theorem 7.2, Assumptions 1, 2, and 3, and $p_n h_{p_n}/b_{p_n} \to r_0 \in [0,\infty)$, which are implied by $h_{p_n}^*(\lambda_{p_n}) \to h^* \in H^*$ with $r_0 \in [0,\infty)$, we have: $T_{p_n}(\rho_{p_n}(\tau)) \to_d J_\psi$ with $\psi = r_0\kappa_0(\tau)$ for J_ψ defined in (3.8). By Theorem 7.3, Assumptions 1, 2, and 3, and $p_n h_{p_n}/b_{p_n} \to r_0 = \infty$, which are implied by $h_{p_n}^*(\lambda_{p_n}) \to h^* \in H^*$ with $r_0 = \infty$, we have $T_{p_n}(\rho_{p_n}(\tau)) \to_d J_\psi$ for $\psi = \infty$ and $J_\infty \sim N(0,1)$.

For $\psi \in [0,\infty]$, the quantiles $c_\psi(\alpha/2)$ and $c_\psi(1-\alpha/2)$ of the distribution of J_ψ, which appear in the definition of $CI_{n,\tau}$, are continuous at all $\psi \in [0,\infty]$. The proof of this is given in the proof of Lemma A.7 of ACG with $Z \sim N(0,1)$ replaced by $Z = 0$ (which simplifies the proof because some terms that need to be shown to be $o_p(1)$ are immediately 0 and the case $\psi = 0$ is trivial when $Z = 0$). In consequence, under $\{\lambda_{p_n} \in \Lambda_n\}_{n\geq 1}$,

$$\begin{aligned} &\psi_{p_n h_{p_n}, \rho_{p_n}(\tau)} \to r_0\kappa_0(\tau) \text{ implies that} \\ &c_{\psi_{p_n h_{p_n}, \rho_{p_n}(\tau)}}(\alpha/2) \to c_{r_0\kappa_0(\tau)}(\alpha/2) \text{ and } c_{\psi_{p_n h_{p_n} \rho_{p_n}(\tau)}}(1-\alpha/2) \to c_{r_0\kappa_0(\tau)}(1-\alpha/2). \end{aligned} \tag{B.2.5}$$

Now, we show that the convergence in the first line of (B.2.5) holds. For notational simplicity, we replace p_n by n in the proof of this convergence. We consider three cases. Case 1: $r_0 \in [0,\infty)$. Case 2: (i) $r_0 = \infty$ and (ii) $\rho_n(\tau) > 0$ for n sufficiently large. Case 3: (i) $r_0 = \infty$ and (ii) $\rho_n(\tau) \leq 0$ infinitely often as $n \to \infty$. In case 1, $nh/b_n \to r_0 < \infty$, and so, $b_n \to \infty$, Lemma B.1 applies, and $\rho_n(\tau) > 0$ for n sufficiently large. Thus, we have

$$\begin{aligned} \psi_{p_n h_{p_n}, \rho_{p_n}(\tau)} &= -nh\ln(\rho_n(\tau)) = -nh\ln(\exp\{-\kappa_n^*(\tau)/b_n\}) \\ &= nh\kappa_n(\tau)(1+o(1))/b_n \to r_0\kappa_0(\tau), \end{aligned} \tag{B.2.6}$$

where the second equality holds by (B.2.1), the third equality holds by Lemma B.1, and the convergence holds by $h_n^*(\lambda_n) \to h^* \in H^*$. Thus, case 1 is proved.

For case 2, by implication (iv) listed above, we have $d_n := nh\left(1-\rho_n\left(\tau\right)\right) \to r_0\kappa_0\left(\tau\right) = \infty$. We have $\rho_n\left(\tau\right) = 1 - d_n/nh$ and $\psi_{p_n h_{p_n},\rho_{p_n}(\tau)} = -nh\ln\left(1-d_n/nh\right)$ for n large by the definition of $\psi_{p_n h_{p_n},\rho_{p_n}(\tau)}$ and the assumption that $\rho_n\left(\tau\right) > 0$ for n sufficiently large by condition (ii) of case 2. For all $d \in (0,\infty)$, we have

$$\liminf_{n\to\infty} \psi_{p_n h_{p_n},\rho_{p_n}(\tau)} = \liminf_{n\to\infty}\left[-nh\ln\left(1-d_n/nh\right)\right] \geq \liminf_{n\to\infty}\left[-nh\ln\left(1-d/nh\right)\right] \qquad \text{(B.2.7)}$$

because the right-hand side quantity is increasing in d and $d_n \to \infty$. By a mean value expansion around 1, $\ln\left(1-d/nh\right) = (1/d_{n*})(-d/nh)$, where $d_{n*} \in [1-d/nh, 1]$ and $d_{n*} \to 1$. Hence,

$$\liminf_{n\to\infty}\left[-nh\ln\left(1-d/nh\right)\right] \to d. \qquad \text{(B.2.8)}$$

Since this holds for all $d \in (0,\infty)$, using (B.2.7), we get $\liminf_{n\to\infty}\psi_{p_n h_{p_n},\rho_{p_n}(\tau)} = \infty = r_0\kappa_0\left(\tau\right)$, as desired and case 2 is proved.

For case 3, for the subsequence of indices for which $\rho_n\left(\tau\right) \leq 0$, we have $\psi_{p_n h_{p_n},\rho_{p_n}(\tau)} = \infty$ for all indices, and so, its limit equals $\infty = r_0\kappa_0\left(\tau\right)$, as desired. For the subsequence of indices for which $\rho_n\left(\tau\right) > 0$, the argument used to prove case 2 applies and the limit is $\infty = r_0\kappa_0\left(\tau\right)$. Thus, case 3 is proved.

Now, given the convergence in the second line of (B.2.5), we have

$$\begin{aligned}
& CP_{p_n}\left(\lambda_{p_n}\right) \\
&= P_{\lambda_{p_n}}\left(\rho_{p_n}\left(\tau\right) \in CI_{p_n,\tau}\right) \\
&= P_{\lambda_{p_n}}\left(c_{\psi_{p_n h_{p_n},\rho_{p_n}(\tau)}}\left(\alpha/2\right) \leq T_{p_n}\left(\rho_{p_n}\left(\tau\right)\right) \leq c_{\psi_{p_n h_{p_n},\rho_{p_n}(\tau)}}\left(1-\alpha/2\right)\right) \\
&\to P\left(c_{r_0\kappa_0(\tau)}\left(\alpha/2\right) \leq J_{r_0\kappa_0(\tau)} \leq c_{r_0\kappa_0(\tau)}\left(1-\alpha/2\right)\right) \\
&= 1-\alpha,
\end{aligned} \qquad \text{(B.2.9)}$$

where the second equality holds by (3.9), the convergence holds by $T_{p_n}\left(\rho_{p_n}\left(\tau\right)\right) \to_d J_\psi$ with $\psi = r_0\kappa_0\left(\tau\right)$ and (B.2.5), and the last equality holds by the definition of the quantile $c_\psi\left(\alpha\right)$ following (3.8). This verifies (B.2.3) and completes the verification of Assumptions B and S of ACG with $CP = 1-\alpha$ in Assumption S, which completes the proof. □

Proof of Lemma B.1. Because $b_n \to \infty$ and $\kappa_n\left(\cdot\right)$ is nonnegative and bounded on I_{τ,ε_2} uniformly over n by part (ii) of Λ_n,

$$\sup_{s\in I_{\tau,\varepsilon_2}}\left|\rho_n\left(s\right)-1\right| = \sup_{s\in I_{\tau,\varepsilon_2}}\frac{\kappa_n\left(s\right)}{b_n} \to 0. \qquad \text{(B.2.10)}$$

We prove by contradiction that

$$\sup_{s\in I_{\tau,\varepsilon_2}} \frac{\kappa_n^*(s)}{b_n} \to 0. \tag{B.2.11}$$

Let $\{s_n\}_{n\geq 1}$ be a sequence in I_{τ,ε_2} such that $\sup_{s\in I_{\tau,\varepsilon_2}} \kappa_n^*(s)/b_n - \kappa_n^*(s_n)/b_n \to 0$. Suppose the claim does not hold. Then, there exists a subsequence $\{n_k\}_{k\geq 1}$ of $\{n\}_{n\geq 1}$ and a constant $\delta > 0$ such that $\sup_{s\in I_{\tau,\varepsilon_2}} \kappa_{n_k}^*(s_{n_k})/b_{n_k} \geq \delta\ \forall k \geq 1$, and hence, $\kappa_{n_k}^*(s_{n_k})/b_{n_k} \geq \delta/2$ for all k large. In consequence,

$$\rho_{n_k}(s_{n_k}) := \exp\{-\kappa_{n_k}^*(s_{n_k})/b_{n_k}\} \leq \exp\{-\delta/2\} < 1 \tag{B.2.12}$$

for all k large, where the equality holds by the definition of $\kappa_n^*(s)$ in (B.2.1), which contradicts (B.2.10) and establishes (B.2.11).

For each $s \in I_{\tau,\varepsilon_2}$, a mean value expansion gives

$$\exp\{-\kappa_n^*(s)/b_n\} = 1 - \exp\{-\kappa_n^{**}(s)/b_n\}\frac{\kappa_n^*(s)}{b_n}, \tag{B.2.13}$$

where $\kappa_n^{**}(s)$ lies between $\kappa_n^*(s)$ and 0. The latter and (B.2.11) give: $\sup_{s\in I_{\tau,\varepsilon_2}} \kappa_n^{**}(s)/b_n = o(1)$.

We have

$$\begin{aligned} 0 = \rho_n(s) - \rho_n(s) &= \exp\{-\kappa_n^*(s)/b_n\} - (1 - \kappa_n(s)/b_n) \\ &= -\exp\{-\kappa_n^{**}(s)/b_n\}\frac{\kappa_n^*(s)}{b_n} + \frac{\kappa_n(s)}{b_n}, \end{aligned} \tag{B.2.14}$$

where the third equality uses (B.2.13). Equation (B.2.14) gives

$$\begin{aligned} &\frac{\kappa_n^*(s)}{\kappa_n(s)} = \exp\{\kappa_n^{**}(s)/b_n\}, \text{ and so,} \\ &\sup_{s\in I_{\tau,\varepsilon_2}} \left|\frac{\kappa_n^*(s)}{\kappa_n(s)} - 1\right| = \sup_{s\in I_{\tau,\varepsilon_2}} |\exp\{\kappa_n^{**}(s)/b_n\} - 1| \to 0, \end{aligned} \tag{B.2.15}$$

where the convergence holds by a mean value expansion using $\sup_{s\in I_{\tau,\varepsilon_2}} \kappa_n^{**}(s)/b_n = o(1)$. $\square$

B.3 Proof of Lemma 7.1

Proof of Lemma 7.1. First, we prove part (a). By definition, $[T_1, T_2] = I_{n\tau,nh/2}$. Because $\kappa_n(\cdot)$ is Lipschitz on I_{τ,ε_2}, we have

$$\max_{t\in I_{n\tau,nh/2}} |\kappa_n(t/n) - \kappa_n(\tau)| \leq L_4 \max_{t\in I_{n\tau,nh/2}} |t/n - \tau| = O(h), \tag{B.3.1}$$

where the inequality holds for n sufficiently large such that $h/2 + 1/n \leq \varepsilon_2$ (because then $t \in I_{n\tau,nh/2}$ implies that $t/n \in I_{\tau,\varepsilon_2}$).

Next, we have

$$\begin{aligned}\max_{t\in I_{n\tau,nh/2}} |\rho_t - \rho_{n\tau}| &= \max_{t\in I_{n\tau,nh/2}} |\kappa_n(t/n) - \kappa_n(\tau)| / b_n \\ &= O(h/b_n), \end{aligned} \tag{B.3.2}$$

where the first equality holds because $\rho_t = \rho_n(t/n) = 1 - \kappa_n(t/n)/b_n$ and $\rho_{n\tau} = \rho_n(\tau) = 1 - \kappa_n(\tau)/b_n$ by (7.1) and the second equality uses (B.3.1). This establishes part (a).

Part (b) holds by (B.3.1) with $\sigma_n^2(\cdot)$ in place of $\kappa_n(\cdot)$.

Using (7.3), we have

$$\begin{aligned}\max_{t\in I_{n\tau,nh/2}} \max_{0\leq j\leq t-T_1} \left|c_{t,j} - \rho_{n\tau}^j\right| &:= \max_{t\in I_{n\tau,nh/2}} \max_{0\leq j\leq t-T_1} \left|\prod_{k=1}^{j} \rho_{t-k} - \rho_{n\tau}^j\right| \\ &\leq \max_{t\in I_{n\tau,nh/2}} \max_{0\leq j\leq t-T_1} (j+1) \max_{1\leq k\leq j} |\rho_{t-k} - \rho_{n\tau}| \\ &= O(nh \cdot h/b_n), \end{aligned} \tag{B.3.3}$$

where the first equality holds by (7.3), the inequality uses standard manipulations and $\max(|\rho_{t-k}|, |\rho_{n\tau}|) \leq 1$, and the second equality uses part (a) and $j + 1 \leq T_2 - T_1 = O(nh)$. Hence, part (c) holds.

Part (d) holds by (B.3.2) with $\eta_n(\cdot)$ in place of $\kappa_n(\cdot)$. □

B.4 Proof of Lemma 7.2

The proofs of Lemma 7.2(b) and Lemma 7.4 use the following lemma, which is an extension of Lemma 7.1(a)–(c).

Lemma B.2. *Under Assumptions* 1 *and* 3, *for a sequence* $\{\lambda_n = (\rho_n, \mu_n, \sigma_n^2, \kappa_n, b_n, F_n) \in \Lambda_n\}_{n\geq 1}$ *and a sequence of integer constants* $\{m_n\}_{n\geq 1}$ *for which* $m_n \to \infty$ *and* $m_n/n \to 0$,

(a) $\max_{t\in I_{n\tau,nh/2+2m_n}} |\rho_t - \rho_{n\tau}| = O((h + \frac{m_n}{n})/b_n)$,

(b) $\max_{t \in I_{n\tau,nh/2+2m_n}} |\sigma_t^2 - \sigma_{n\tau}^2| = O(h + \frac{m_n}{n})$, *and*

(c) $\max_{t \in \{T_0 - m_n, T_0\}} \max_{0 \le j \le m_n} |c_{t,j} - \rho_{n\tau}^j| = O(m_n(h + \frac{m_n}{n})/b_n)$.

Proof of Lemma B.2. The proof of Lemma B.2(a) is the same as that of Lemma 7.1(a) with the following changes. In (B.3.1), $I_{n\tau,nh/2}$ is replaced by $I_{n\tau,nh/2+2m_n}$ and $L_4 \max_{t \in I_{n\tau,nh/2}} |t/n - \tau| = O(h)$ is replaced by $L_4 \max_{t \in I_{n\tau,nh/2+2m_n}} |t/n - \tau| = O(h + \frac{m_n}{n})$ and the inequality in (B.3.1) holds for n sufficiently large that $h/2 + 2m_n/n + 1/n \le \varepsilon_2$, which uses the assumption that $m_n/n \to 0$ (because then $t \in I_{n\tau,nh/2+2m_n}$ implies that $t/n \in I_{\tau,\varepsilon_2}$). In (B.3.2), $I_{n\tau,nh/2}$ is replaced by $I_{n\tau,nh/2+2m_n}$ and $\max_{t \in I_{n\tau,nh/2}} \exp\{x_{nt}\} |\kappa_n^*(t/n) - \kappa_n^*(\tau)|/b_n = O(h/b_n)$ is replaced by $\max_{t \in I_{n\tau,nh/2+2m_n}} \exp\{x_{nt}\} |\kappa_n^*(t/n) - \kappa_n^*(\tau)|/b_n = O((h + \frac{m_n}{n})/b_n)$ using the revised version of (B.3.1). This establishes Lemma B.2(a).

The proof of Lemma B.2(b) is the same as that of part (a) with $|\kappa_n(t/n) - \kappa_n(\tau)|/b_n$ replaced by $|\sigma_t^2 - \sigma_{n\tau}^2|$.

The proof of Lemma B.2(c) is the same as that of Lemma 7.1(c) with $\max_{t \in I_{n\tau,nh/2}} \max_{0 \le j \le t - T_1}$ replaced by $\max_{t \in \{T_0, T_0 - m_n\}} \max_{0 \le j \le 2m_n}$, which implies that the largest value of j considered is bounded by $2m_n$, rather than nh. This implies that the rhs of (B.3.3) is changed from $O(nh \cdot h/b_n)$ to $O(m_n \cdot (h + \frac{m_n}{n})/b_n)$, which establishes Lemma B.2(c). $\square$

Proof of Lemma 7.2. First, we prove part (a). By recursive substitution of (2.1), we have

$$Y_{T_0}^* = \sum_{j=0}^{T_0 - 1} c_{T_0,j} \sigma_{T_0 - j} U_{T_0 - j} + c_{T_0,T_0} Y_0^*. \tag{B.4.1}$$

By Markov's inequality, we only need to show $EY_{T_0}^{*2}/n = O(1)$, which is true because

$$\begin{aligned} EY_{T_0}^{*2}/n &= n^{-1} E \left(\sum_{j=0}^{T_0 - 1} c_{T_0,j} \sigma_{T_0 - j} U_{T_0 - j} + c_{T_0,T_0} Y_0^* \right)^2 \\ &\le 2n^{-1} E \left(\sum_{j=0}^{T_0 - 1} c_{T_0,j} \sigma_{T_0 - j} U_{T_0 - j} \right)^2 + 2n^{-1} E \left(c_{T_0,T_0} Y_0^* \right)^2 \\ &= 2 \sum_{j=0}^{T_0 - 1} c_{T_0,j}^2 \sigma_{T_0 - j}^2 / n + 2c_{T_0,T_0}^2 EY_0^{*2}/n \\ &\le 2C_{3,U} \left(T_0/n \right) + O(1) = O\left(1\right), \end{aligned} \tag{B.4.2}$$

where the first inequality holds by $(a+b)^2 \le 2a^2 + 2b^2$, the second equality uses the fact that $\{U_t\}_{t=1}^n$ is a martingale difference sequence and $EU_t^2 = 1$ by the definition of the parameter space Λ_n, and the last inequality holds by $\max_{t \in [1,n]} \sigma_t^2 \le C_{3,U}$, $\max_{j \in [0,T_0]} |c_{T_0,j}| \le 1$, $T_0 = \lfloor n\tau \rfloor - \lfloor nh/2 \rfloor - 1 = O(n)$, and part (v) of Λ_n.

In consequence, by Markov's inequality, we have

$$Y^*_{T_0} = O_p\left(n^{1/2}\right), \tag{B.4.3}$$

which proves part (a).

Next, we prove part (b). By part (a), $Y^*_{T_0} = O_p(n^{1/2})$. Hence, if $w_0 = \infty$ (in Assumption 4), then

$$\frac{(nh)^{1/2}}{b_n} Y^*_{T_0} = \frac{nh^{1/2}}{b_n} O_p(1) = o_p(1) \tag{B.4.4}$$

and the result of part (b) of the lemma is proved.

Hence, to prove part (b), it remains to consider the case where $w_0 < \infty$. For notational simplicity, we suppose $\sigma_0(\tau) = 1$. By recursive substitution, as in (B.4.1), for a sequence of integer constants $\{m_n\}_{n\geq 1}$ for which $m_n \to \infty$ and $m_n/n \to 0$, we have

$$Y^*_{T_0} = \sum_{j=0}^{m_n-1} c_{T_0,j}\sigma_{T_0-j}U_{T_0-j} + c_{T_0,m_n}Y^*_{T_0-m_n}. \tag{B.4.5}$$

Similarly to (7.12), we bound $|\rho_t|$ for $t \in [T_0 - m_n, T_0]$ by $\overline{\rho}_n := \max\{\exp\{-\kappa_0(\tau)/(2b_n)\}, -1 + \varepsilon_1\}$. As in (7.12), it suffices to consider the case where $\rho_t \geq 0$ for all $t \in [0,1]$. We have

$$\begin{aligned}\max_{t\in[T_0-m_n,T_0]} |\rho_t| &\leq \max_{t\in[T_0-m_n,T_0]} |\rho_t - \rho_{n\tau}| + |\rho_{n\tau} - \exp\{-\kappa_0(\tau)/b_n\}| + \exp\{-\kappa_0(\tau)/b_n\} \\ &\leq O\left((h + \frac{m_n}{n})/b_n\right) + o(1)/b_n + \exp\{-\kappa_0(\tau)/b_n\} \leq \overline{\rho}_n, \end{aligned} \tag{B.4.6}$$

where $b_n \geq \varepsilon_3 > 0$, the second inequality uses Lemma B.2(a), (7.8), Assumption 3, and a mean value expansion, and the last inequality holds using $\kappa_0(\tau) > 0$ and the fact that, when $b_n \to \infty$, $\overline{\rho}_n - \exp\{-\kappa_0(\tau)/b_n\} \geq K/b_n$ for some constant $K > 0$. Hence, for $j = 0, ..., m_n$,

$$c_{T_0,j} \leq \overline{\rho}_n^j. \tag{B.4.7}$$

We have

$$\begin{aligned} EY^{*2}_{T_0} &= \sum_{j=0}^{m_n-1} c^2_{T_0,j}\sigma^2_{T_0-j} + c^2_{T_0,m_n}EY^{*2}_{T_0-m_n} \leq O(1)\sum_{j=0}^{m_n-1}\overline{\rho}_n^{2j} + \overline{\rho}_n^{2m_n}O(n) \\ &\leq O(1)\frac{1}{1-\overline{\rho}_n^2} + \omega^{m_n/b_n}O(n) = O(b_n) + \omega^{m_n/b_n}O(n), \text{ where} \\ \omega &:= \exp\{-\kappa_0(\tau)\}, \end{aligned} \tag{B.4.8}$$

the first equality uses (B.4.5) and the martingale difference property of $\{U_t\}_{t\geq 1}$, the first inequality uses (B.4.7), $\max_{t\leq n}\sigma_t^2 \leq C_{3,U} < \infty$ (by part (i) of Λ_n), and $EY^{*2}_{T_0-m_n} = O(n)$ (which holds by (B.4.2) with $T_0 - m_n$ in place of T_0), the second inequality holds by a bound on the geometric sum and uses the definition of ω, and the last equality uses (7.16).

We are considering the case where $\frac{b_n}{nh^{1/2}} \to w_0 < \infty$. We take $m_n = b_n c \ln(n)$ for some finite positive constant $c \in (0, -1/\ln(\omega))$. We have $m_n \to \infty$, because b_n is bounded away from zero by condition (ii) of Λ_n and c is positive. We have

$$\frac{m_n}{n} = \frac{b_n}{nh^{1/2}} ch^{1/2}\ln(n) \to 0 \text{ provided } h^{1/2}\ln(n) = o(1), \tag{B.4.9}$$

because $\frac{b_n}{nh^{1/2}} \to w_0 < \infty$ in the case that we are considering. By Assumption 2, the condition $h^{1/2}\ln(n) = o(1)$ holds. Thus, m_n satisfies the required conditions that $m_n \to \infty$ and $\frac{m_n}{n} \to 0$.

Next, we have

$$\omega^{m_n/b_n} n \to 0 \text{ iff } \frac{m_n}{b_n}\ln(\omega) + \ln(n) = (c\ln(\omega) + 1)\ln(n) \to -\infty \tag{B.4.10}$$

and the latter holds because $c\ln(\omega) + 1 < 0$ by the definition of c.

By (B.4.8) and Markov's inequality, $Y^{*2}_{T_0} = O_p(b_n) + O_p(\omega^{m_n/b_n} n)$. Using this, we have

$$\frac{nh}{b_n^2} Y^{*2}_{T_0} = O_p(\frac{nh}{b_n}) + \frac{nh}{b_n^2} O_p(\omega^{m_n/b_n} n) = o_p(1), \tag{B.4.11}$$

where the second equality uses $\frac{nh}{b_n} \to r_0 = 0$ in the present case, $\frac{nh}{b_n^2} = \frac{nh}{b_n}\frac{1}{b_n} \to 0$, and $\omega^{m_n/b_n} n \to 0$ by (B.4.10).

Equation (B.4.11) establishes the desired results for the case $w_0 < \infty$, which completes the proof of part (b).

Next, we prove part (c), which considers the case $nh/b_n \to r_0 = \infty$. Let $\{m_n\}_{n\geq 1}$ be an arbitrary sequence of positive integers for which $m_n \to \infty$ and $m_n/n \to 0$. Then, by (B.4.8), $EY^{*2}_{T_0} = O(b_n) + \omega^{m_n/b_n} O(n)$. Now, take m_n as defined above, i.e., $m_n = b_n c \ln(n)$ for $c \in (0, -1/\ln(\omega))$. As above, $m_n \to \infty$. In addition,

$$\frac{m_n}{n} = \frac{b_n}{nh} ch\ln(n) \to 0 \text{ provided } h\ln(n) = O(1), \tag{B.4.12}$$

where the convergence holds because $\frac{nh}{b_n} \to r_0 = \infty$, which holds by assumption, iff $\frac{b_n}{nh} \to 0$. The condition $h\ln(n) = O(1)$ holds by Assumption 2. Now, $\omega^{m_n/b_n} O(n) = o(1)$ by the argument in (B.4.10) above. Combining this and (B.4.8) gives

$$EY^{*2}_{T_0} = O(b_n). \tag{B.4.13}$$

This and Markov's inequality proves part (c) of the lemma. $\square$

B.5 Proof of Lemma 7.3

When $b_n \to \infty$, Lemma B.1 and $\kappa_n(\cdot) \to \kappa_0(\cdot)$ imply that $\kappa_n^*(\cdot) \to \kappa_0(\cdot)$ uniformly over I_{τ,ε_2} and $\kappa_n^*(\cdot)$ is Lipschitz with Lipschitz constant less than $2L_4$ because $\kappa_n(\cdot)$ is Lipschitz with Lipschitz constant L_4. In the following proofs, for notational simplicity, we let L_4 be the Lipschitz constant $2L_4$ for $\kappa_n^*(\cdot)$ since the factor 2 does not affect the asymptotic results. Now, we prove the local-to-unity asymptotic results with $\rho_n(\cdot)$ expressed in terms of $\kappa_n^*(\cdot)$, rather than $\kappa_n(\cdot)$.

Proof of Lemma 7.3. We suppress n from $Y^0_{n,t(s)}$ and $U_{n,t}$ in the proof. Denote $E(X|\mathscr{G}_i)$ by E_iX. By the definition of the parameter space Λ_n, we have $\{U_t\}_{t=1}^n$ is a martingale difference sequence.

We adopt the following notational convention. When the lower index of a sum exceeds the upper index, the sum is defined to equal zero. In particular, $\sum_{j=T_1+\lfloor nhs\rfloor+1}^{T_1+\lfloor nhs\rfloor} \kappa_n^*(j/n) = 0$. Then, by the definitions of Y_t^0 in (7.5), κ_n^* in (B.2.1), and $t(s)$ in (7.9), we have

$$
\begin{aligned}
&(nh)^{-1/2}\, Y^0_{t(s)} \\
&= (nh)^{-1/2} \sum_{k=T_1}^{T_1+\lfloor nhs\rfloor} \exp\left\{-\frac{1}{b_n}\sum_{j=k+1}^{T_1+\lfloor nhs\rfloor} \kappa_n^*\left(\frac{j}{n}\right)\right\}\sigma_n\left(\frac{k}{n}\right)U_k \\
&= (nh)^{-1/2} \sum_{k=T_1}^{T_1+\lfloor nhs\rfloor} \exp\left\{-\frac{1}{b_n}\left(\sum_{j=T_1}^{T_1+\lfloor nhs\rfloor} \kappa_n^*\left(\frac{j}{n}\right) - \sum_{j=T_1}^{k}\kappa_n^*\left(\frac{j}{n}\right)\right)\right\}\sigma_n\left(\frac{k}{n}\right)U_k \\
&= \exp\left\{-\frac{1}{b_n}\sum_{j=T_1}^{T_1+\lfloor nhs\rfloor}\kappa_n^*\left(\frac{j}{n}\right)\right\} \\
&\quad\times\left[(nh)^{-1/2}\sum_{k=T_1}^{T_1+\lfloor nhs\rfloor}\exp\left\{\frac{1}{b_n}\sum_{j=T_1}^{k}\kappa_n^*\left(\frac{j}{n}\right)\right\}\sigma_n\left(\frac{k}{n}\right)U_k\right] \\
&=: A_{1s}\times A_{2s}.
\end{aligned}
\tag{B.5.1}
$$

Because $\kappa_n^*(\cdot)$ is Lipschitz on I_{τ,ε_2}, we have

$$
\max_{t\in I_{n\tau,nh/2}} |\kappa_n^*(t/n) - \kappa_n^*(\tau)| \le L_4 \max_{t\in I_{n\tau,nh/2}} |t/n-\tau| = O(h), \tag{B.5.2}
$$

where the inequality holds for n sufficiently large such that $h/2+1/n \le \varepsilon_2$. Equation (B.5.2) implies that

$$\max_{s\in[0,1]}\left|\left[\frac{1}{\lfloor nhs\rfloor+1}\sum_{j=T_1}^{T_1+\lfloor nhs\rfloor}\kappa_n^*\left(\frac{j}{n}\right)\right]-\kappa_n^*\left(\tau\right)\right|\leq\max_{j\in[T_1,T_1+\lfloor nh\rfloor]}\left|\kappa_n^*\left(\frac{j}{n}\right)-\kappa_n^*\left(\tau\right)\right|=O\left(h\right) \tag{B.5.3}$$

because the range of the summation is $[T_1, T_1+\lfloor nhs\rfloor] \subset [T_1, T_1+\lfloor nh\rfloor] \subset I_{n\tau,nh/2}$ for $s\in[0,1]$ by construction. Therefore,

$$\begin{aligned}A_{1s} &= \exp\left\{-\frac{\lfloor nhs\rfloor+1}{b_n}\left[\frac{1}{\lfloor nhs\rfloor+1}\sum_{j=T_1}^{T_1+\lfloor nhs\rfloor}\kappa_n^*\left(\frac{j}{n}\right)\right]\right\}\\ &= \exp\left\{-\frac{\lfloor nhs+1\rfloor}{nh}\frac{nh}{b_n}\left(\kappa_n^*\left(\tau\right)+O\left(h\right)\right)\right\}\\ &\Rightarrow \exp\left\{-sr_0\kappa_0\left(\tau\right)\right\},\end{aligned} \tag{B.5.4}$$

where the convergence holds by Lemma B.1, Assumptions 1 and 3, $nh/b_n \to r_0$ as $n\to\infty$, and the continuous mapping theorem (CMT).

To derive the limit distribution of

$$A_{2s} = (nh)^{-1/2}\sum_{k=T_1}^{T_1+\lfloor nhs\rfloor}\exp\left\{\frac{1}{b_n}\sum_{j=T_1}^{k}\kappa_n^*\left(\frac{j}{n}\right)\right\}\sigma_n\left(\frac{k}{n}\right)U_k, \tag{B.5.5}$$

we use Theorem 2.1 of Hansen (1992). First, we present a few definitions. For any random arrays $\{D_{n,k}, W_{n,k} : T_1 \leq k \leq T_2; n\geq 1\}$, we transform the arrays into random elements on $[0,1]$ by defining

$$D_n\left(u\right) := D_{n,T_1+\lfloor nhu\rfloor} \text{ and } W_n\left(u\right) := W_{n,T_1+\lfloor nhu\rfloor}. \tag{B.5.6}$$

for $u\in[0,1]$. Define the differences $\delta_{n,k} := W_{n,k} - W_{n,k-1}$. Then, we define the stochastic integral

$$\int_0^s D_n\left(u\right)dW_n\left(u\right) := \sum_{k=T_1}^{T_1+\lfloor nhs\rfloor} D_{n,k}\delta_{n,k+1} \tag{B.5.7}$$

for $s\in[0,1]$.

We let

$$\begin{aligned}D_{n,k}^a &:= \exp\left\{\frac{k-T_1+1}{b_n}\left(\frac{1}{k-T_1+1}\sum_{j=T_1}^{k}\kappa_n^*\left(\frac{j}{n}\right)-\kappa_n^*\left(\tau\right)\right)\right\}\\ &\quad\times\exp\left\{\frac{k-T_1+1}{b_n}\kappa_n^*\left(\tau\right)\right\}\sigma_n\left(\tau\right),\end{aligned} \tag{B.5.8}$$

$$D^b_{n,k} := \exp\left\{\frac{k-T_1+1}{b_n}\left(\frac{1}{k-T_1+1}\sum_{j=T_1}^{k}\kappa^*_n\left(\frac{j}{n}\right)-\kappa^*_n(\tau)\right)\right\}$$
$$\times \exp\left\{\frac{k-T_1+1}{b_n}\kappa^*_n(\tau)\right\}\left(\sigma_n\left(\frac{k}{n}\right)-\sigma_n(\tau)\right), \text{ and} \tag{B.5.9}$$
$$W_{n,k} := (nh)^{-1/2}\sum_{j=T_1}^{k-1}U_j \tag{B.5.10}$$

for $T_1 \le k \le T_2$. Next, transform these quantities into random elements on $[0,1]$ by defining

$$D^a_n(u) := \exp\left\{\left(u+\frac{1}{nh}\right)\left(\frac{nh}{b_n}\right)\left(\frac{1}{1+nhu}\sum_{j=T_1}^{T_1+\lfloor nhu\rfloor}\left[\kappa^*_n\left(\frac{j}{n}\right)-\kappa^*_n(\tau)\right]\right)\right\}$$
$$\times \exp\left\{\left(u+\frac{1}{nh}\right)\left(\frac{nh}{b_n}\right)\kappa^*_n(\tau)\right\}\sigma_n(\tau), \tag{B.5.11}$$
$$D^b_n(u) := \exp\left\{\left(u+\frac{1}{nh}\right)\left(\frac{nh}{b_n}\right)\left(\frac{1}{1+nhu}\sum_{j=T_1}^{T_1+\lfloor nhu\rfloor}\left[\kappa^*_n\left(\frac{j}{n}\right)-\kappa^*_n(\tau)\right]\right)\right\}$$
$$\times \exp\left\{\left(u+\frac{1}{nh}\right)\left(\frac{nh}{b_n}\right)\kappa^*_n(\tau)\right\}\left(\sigma_n\left(\frac{T_1+\lfloor nhu\rfloor}{n}\right)-\sigma_n(\tau)\right), \text{ and} \tag{B.5.12}$$
$$W_n(u) := (nh)^{-1/2}\sum_{j=T_1}^{T_1+\lfloor nhu\rfloor-1}U_j \tag{B.5.13}$$

for $u \in [0,1]$.

Then, we have

$$\begin{aligned}
A_{2s} &= (nh)^{-1/2}\sum_{k=T_1}^{T_1+\lfloor nhs\rfloor}\exp\left\{\frac{1}{b_n}\sum_{j=T_1}^{k}\kappa^*_n\left(\frac{j}{n}\right)\right\}\sigma_n\left(\frac{k}{n}\right)U_k \\
&= (nh)^{-1/2}\sum_{k=T_1}^{T_1+\lfloor nhs\rfloor}\exp\left\{\frac{k-T_1+1}{b_n}\left(\frac{1}{k-T_1+1}\sum_{j=T_1}^{k}\kappa^*_n\left(\frac{j}{n}\right)-\kappa^*_n(\tau)\right)\right\} \\
&\quad \times \exp\left\{\frac{k-T_1+1}{b_n}\kappa^*_n(\tau)\right\}\sigma_n(\tau)U_k \\
&\quad + (nh)^{-1/2}\sum_{k=T_1}^{T_1+\lfloor nhs\rfloor}\exp\left\{\frac{k-T_1+1}{b_n}\left(\frac{1}{k-T_1+1}\sum_{j=T_1}^{k}\kappa^*_n\left(\frac{j}{n}\right)-\kappa^*_n(\tau)\right)\right\} \\
&\quad \times \exp\left\{\frac{k-T_1+1}{b_n}\kappa^*_n(\tau)\right\}\left(\sigma_n\left(\frac{k}{n}\right)-\sigma_n(\tau)\right)U_k \\
&= \sum_{k=T_1}^{T_1+\lfloor nhs\rfloor}D^a_{n,k}\left(W_{n,k+1}-W_{n,k}\right)+\sum_{k=T_1}^{T_1+\lfloor nhs\rfloor}D^b_{n,k}\left(W_{n,k+1}-W_{n,k}\right) \\
&= \int_0^s D^a_n(u)\,dW_n(u)+\int_0^s D^b_n(u)\,dW_n(u)
\end{aligned}$$

$$=: A_{2as} + A_{2bs}. \tag{B.5.14}$$

We obtain

$$\max_{u\in[0,1]} \left| \frac{1}{1+nhu} \sum_{j=T_1}^{T_1+\lfloor nhu \rfloor} \left[\kappa_n\left(\frac{j}{n}\right) - \kappa_n(\tau) \right] \right| \leq O(h) \to 0, \tag{B.5.15}$$

by (B.5.2) with $\kappa_n(\cdot)$ in place of $\kappa_n^*(\cdot)$, $[T_1, T_1 + \lfloor nhu \rfloor] \subset [T_1, T_1 + \lfloor nh \rfloor] \subset I_{n\tau,nh/2}$, and Assumption 1. In addition, Lemma B.1 and Assumption 3 imply that

$$\sup_{s\in I_{\tau,\varepsilon_2}} |\kappa_n^*(s) - \kappa_0(s)| \to 0. \tag{B.5.16}$$

Combining these two results, we have

$$\max_{u\in[0,1]} \left| \frac{1}{1+nhu} \sum_{j=T_1}^{T_1+\lfloor nhu \rfloor} \left[\kappa_n^*\left(\frac{j}{n}\right) - \kappa_n^*(\tau) \right] \right| \leq O(h) \to 0. \tag{B.5.17}$$

Then, we have

$$D_n^a(u) \Rightarrow D^a(u) := \exp\{ur_0\kappa_0(\tau)\}\sigma_0(\tau) \tag{B.5.18}$$

by (B.5.16), (B.5.17), and the CMT. We also have

$$W_n(\cdot) \Rightarrow B(\cdot), \tag{B.5.19}$$

where $B(\cdot)$ is a standard Brownian motion on $[0,1]$, by Theorem 2.3 of McLeish (1974).

Therefore,

$$A_{2as} = \int_0^s D_n^a(u)\, dW_n(u) \Rightarrow \int_0^s \exp\{ur_0\kappa_0(\tau)\}\sigma_0(\tau)\, dB(u) \tag{B.5.20}$$

by (B.5.18), (B.5.19), the definition of the parameter space Λ_n, and Theorem 2.1 of Hansen (1992), with $D_n^a(\cdot)$ in (B.5.11) and $D^a(\cdot)$ in (B.5.18) in the roles of $U_n(\cdot)$ and $U(\cdot)$, respectively, and $W_n(\cdot)$ in (B.5.13) and $B(\cdot)$ in the roles of $Y_n(\cdot)$ and $Y(\cdot)$, respectively, in Theorem 2.1 of Hansen (1992).

For A_{2bs}, because $\sigma_n^2(\cdot)$ is a bounded Lipschitz function on $[0,1]$ and $I_{\tau,h/2} \subset [0,1]$, we have

$$\max_{t\in I_{n\tau,nh/2}} |\sigma_n(t/n) - \sigma_n(\tau)| \leq C_\sigma \max_{t\in I_{n\tau,nh/2}} |t/n - \tau| = O(h), \tag{B.5.21}$$

where $C_\sigma := L_3/(2C_{3,L})$. Equation (B.5.21) implies that

$$\max_{u\in[0,1]}\left|\sigma_n\left(\frac{T_1+\lfloor nhu\rfloor}{n}\right)-\sigma_n(\tau)\right|\leq O(h)\to 0 \tag{B.5.22}$$

by $[T_1, T_1+\lfloor nhu\rfloor]\subset[T_1,T_1+\lfloor nh\rfloor]\subset I_{n\tau,nh/2}$ for $u\in[0,1]$, and Assumption 1. Then,

$$D^b_{n,\tau}(u)\Rightarrow 0 \tag{B.5.23}$$

by (B.5.17), (B.5.22), Assumption 3, $nh/b_n\to r_0\in[0,\infty)$ as $n\to\infty$, and the CMT. Therefore,

$$A_{2bs}=\int_0^s D^b_n(u)\,dW_n(u)\Rightarrow 0 \tag{B.5.24}$$

by the convergence result (B.5.23) and Theorem 2.1 of Hansen (1992), with $D^b_n(\cdot)$ in (B.5.12) and the zero function on $[0,1]$ in the roles of $U_n(\cdot)$ and $U(\cdot)$, respectively, and $W_n(\cdot)$ in (B.5.13) and $B(\cdot)$ in the roles of $Y_n(\cdot)$ and $Y(\cdot)$, respectively, in Theorem 2.1 of Hansen (1992).

Combining (B.5.20) and (B.5.24), we obtain

$$A_{2s}\Rightarrow\int_0^s \exp\left(ur_0\kappa_0(\tau)\right)\sigma_0(\tau)\,dB(u), \tag{B.5.25}$$

where $B(u)$ is standard Brownian motion.

Therefore, by (B.5.1), (B.5.4), and (B.5.25), we have

$$(nh)^{-1/2}\,Y^0_{n,t(s)}/\sigma_0(\tau)=A_{1s}+A_{2s}\Rightarrow\int_0^s \exp\left\{-(s-u)\,r_0\kappa_0(\tau)\right\}dB(u) \tag{B.5.26}$$

using the CMT and the assumption that $nh/b_n\to r_0$ as $n\to\infty$.

The subsequence version of Lemma 7.3, see Remark 7.2, which has $\{p_n\}_{n\geq1}$ in place of $\{n\}_{n\geq1}$, is proved by replacing n by p_n and $h=h_n$ by h_{p_n} throughout the proof above. □

B.6 Proof of Lemma 7.4

Proof of Lemma 7.4. By recursive substitution, for a sequence of integers $\{m_n\}_{n\geq1}$ such that $m_n\to\infty$, we write

$$(2\psi/nh)^{1/2}Y^*_{T_0}/\sigma_0(\tau)=D_{1n}+D_{2n},\ \text{where}$$

$$D_{1n} := (2\psi/nh)^{1/2} \sum_{j=0}^{m_n-1} c_{T_0,j}\sigma_{T_0-j}U_{T_0-j}/\sigma_0(\tau) \text{ and}$$
$$D_{2n} := (2\psi/nh)^{1/2} c_{T_0,m_n} Y^*_{T_0-m_n}/\sigma_0(\tau). \quad \text{(B.6.1)}$$

We show that $D_{1n} \to_d Z_1 \sim N(0,1)$. We choose m_n such that $D_{2n} = o_p(1)$. This requires that m_n is large enough that c_{T_0,m_n} is sufficiently small, but small enough that ρ_t is close to $\rho_{n\tau}$ for all $t \in [T_0 - 2m_n, T_0]$.

Define

$$m_n = b_n/h^{1/5}. \quad \text{(B.6.2)}$$

For this choice of m_n, we have

$$\begin{array}{ll} \text{(i)} & \frac{m_n}{n} = \left(\frac{b_n}{nh}\right) h^{4/5} = o(1), \\ \text{(ii)} & \frac{m_n h}{b_n} = h^{4/5} = o(h^{1/2}), \text{ and} \\ \text{(iii)} & \frac{m_n^2}{nb_n} = \left(\frac{b_n}{nh}\right) h^{3/5} = o(h^{1/2}), \end{array} \quad \text{(B.6.3)}$$

where (i) and (iii) use $\frac{b_n}{nh} \to \frac{1}{r_0} \in (0,\infty)$ and (i)–(iii) use $h = o(1)$ by Assumption 1. Given (B.6.3), Lemma B.2 applies and the error on the rhs of its part (c) is $O(m_n(h + \frac{m_n}{n})/b_n) = o(h^{1/2})$. Thus,

$$c_{T_0,m_n} = \rho^{m_n}_{n\tau} + o(h^{1/2}) \text{ and } c_{T_0-m_n,m_n} = \rho^{m_n}_{n\tau} + o(h^{1/2}). \quad \text{(B.6.4)}$$

In addition, we have

$$\begin{aligned} \rho^{m_n}_{n\tau} &= (1-\kappa_n(\tau)/b_n)^{m_n} = \exp\{-\kappa^*_n(\tau)m_n/b_n\} = \exp\{-\kappa^*_n(\tau)h^{-1/5}\} \\ &= \exp\{-\kappa_0(\tau)\}^{h^{-1/5}\kappa^*_n(\tau)/\kappa_0(\tau)} = (\omega^{\gamma_n}\gamma_n^{5/2})\gamma_n^{-5/2} = o(h^{1/2}), \text{ where} \\ \omega &:= \exp\{-\kappa_0(\tau)\}, \ \gamma_n := h^{-1/5}\kappa^*_n(\tau)/\kappa_0(\tau), \end{aligned} \quad \text{(B.6.5)}$$

the second equality uses the definition of $\kappa^*_n(\cdot)$ in (B.2.1), the third equality uses (B.6.2), the fifth equality uses the definitions of ω and γ_n, and the last equality on the second line holds because $\gamma_n \to \infty$ (using $\kappa^*_n(\tau)/\kappa_0(\tau) \to 1$ by Lemma B.1), $\omega \in (0,1)$ (since $\kappa(\tau) \geq \varepsilon_4 > 0$ by part (ii) of Λ_n), $\omega^x x^{5/2} = o(1)$ as $x = \gamma_n \to \infty$ (since $\ln(\omega^x x^{5/2}) = x\ln\omega + (5/2)\ln x \to -\infty$ as $x \to \infty$), and $\gamma_n^{-5/2} = h^{1/2}(\kappa^*_n(\tau)/\kappa_0(\tau))^{-5/2} = O(h^{1/2})$.

Equations (B.6.4) and (B.6.5) combine to give

$$c_{T_0,m_n} = o(h^{1/2}) \text{ and } c_{T_0-m_n,m_n} = o(h^{1/2}). \quad \text{(B.6.6)}$$

Next, by recursive substitution, we write the multiplicand $Y^*_{T_0-m_n}$ in D_{2n} as the sum of two quantities, as in (B.6.1), as follows:

$$
\begin{aligned}
Y^*_{T_0-m_n} &:= \sum_{j=0}^{T_0-m_n-1} c_{T_0-m_n,j}\sigma_{T_0-m_n-j}U_{T_0-m_n-j} + c_{T_0-m_n,T_0-m_n}Y_0^* \text{ and} \\
D_{2n} &= D_{21n} + D_{22n}, \text{ where} \\
D_{21n} &:= (2\psi/nh)^{1/2}c_{T_0,m_n}\sum_{j=0}^{T_0-m_n-1} c_{T_0-m_n,j}\sigma_{T_0-m_n-j}U_{T_0-m_n-j}/\sigma_0(\tau) \text{ and} \\
D_{22n} &:= (2\psi/nh)^{1/2}c_{T_0,m_n}c_{T_0-m_n,T_0-m_n}Y_0^*/\sigma_0(\tau). \qquad \text{(B.6.7)}
\end{aligned}
$$

By part (v) of Λ_n, $E_{F_n}(Y_0^*)^2 \leq C_5 n$. Hence, by Markov's inequality, $Y_0^* = O_p(n^{1/2})$ and

$$
D_{22n} = h^{-1/2}c_{T_0,m_n}c_{T_0-m_n,T_0-m_n}O_p(1)/\sigma_0(\tau) = o_p(h^{1/2}) = o_p(1), \qquad \text{(B.6.8)}
$$

where the second last equality holds by (B.6.6) and $\sigma_0(\tau) > 0$ (because $\sigma_0(\tau)$ is bounded below by $C_{3,L}$ by part (i) of Λ_n and $C_{3,L} > 0$ by assumption).

To show that $D_{2n} = o_p(1)$, it remains to show that $D_{21n} = o_p(1)$. By Markov's inequality, it suffices to show that $E_{F_n}D^2_{21n} \to 0$. Since $\{U_t : t = 1, ..., n\}$ is a stationary martingale difference sequence by part (iv) of Λ_n, its elements are uncorrelated. Thus, we have

$$
\begin{aligned}
ED^2_{21n} &= (2\psi/nh)c^2_{T_0,m_n}\sum_{j=0}^{T_0-m_n-1} c^2_{T_0-m_n,j}\sigma^2_{T_0-m_n-j}EU^2_{T_0-m_n-j}/\sigma_0(\tau) \\
&= (2\psi/nh)o(h)\sum_{j=0}^{T_0-m_n-1}\sigma^2_{T_0-m_n-j}/\sigma_0(\tau) = o(1), \qquad \text{(B.6.9)}
\end{aligned}
$$

where the second equality holds by the first result in (B.6.6), $c^2_{T_0-m_n,j} \leq 1$ (since $|\rho_t| \leq 1$ for all $t \leq n$ by part (i) of Λ_n), and $EU_t^2 = 1$ for all $t \leq n$ (by part (iv) of Λ_n) and the third equality holds because $\sigma^2_{T_0-m_n-j}$ is bounded by $C_{3,U} < \infty$ (by part (i) of Λ_n), $\sigma_0(\tau) > 0$ (as noted above), and $T_0 - m_n \leq n$. This completes the proof that $D_{21n} = o_p(1)$ and $D_{2n} = o_p(1)$.

Next, we consider D_{1n}. By change of variables with $i = T_0 - j$, we have

$$
D_{1n} = (2\psi/nh)^{1/2}\sum_{i=T_0-m_n+1}^{T_0} c_{T_0,T_0-i}\sigma_i U_i/\sigma_0(\tau), \qquad \text{(B.6.10)}
$$

where $\{U_t : t = 0, ..., n\}$ is a stationary martingale difference sequence under F_n. We apply the CLT in Hall and Heyde (1980) with $X_{ni} = (2\psi/nh)^{1/2}c_{T_0,T_0-i}\sigma_i U_i/\sigma_0(\tau)$, with the number of summands being $m_n - 1$, rather than n, and with the σ-fields $\mathcal{F}_{ni}$ being the σ-fields $\mathcal{G}_i$ in part (iv) of Λ_n. We need to verify a Lindeberg condition and a conditional variance condition. To verify the former, for any $\varepsilon, \delta > 0$, we have

$$
\begin{aligned}
&P\left(\sum_{i=T_0-m_n+1}^{T_0} E(X_{n,i}^2 1(|X_{n,i}| > \varepsilon)|\mathcal{F}_{n,i-1}) > \delta\right) \\
&\leq \delta^{-1} \sum_{i=T_0-m_n+1}^{T_0} EX_{n,i}^2 1(X_{n,i}^2 > \varepsilon^2) \\
&= \delta^{-1}(2\psi/nh) \sum_{i=T_0-m_n+1}^{T_0} c_{T_0,T_0-i}^2(\sigma_i^2/\sigma_0^2(\tau))EU_i^2 1((2\psi/nh)c_{T_0,T_0-i}^2(\sigma_i^2/\sigma_0^2(\tau))U_i^2 > \varepsilon^2) \\
&\leq \delta^{-1}\left[(2\psi/nh)\sum_{i=T_0-m_n+1}^{T_0} c_{T_0,T_0-i}^2\right](C_{3,U}/\sigma_0^2(\tau))EU_1^2 1(2\psi(C_{3,U}/\sigma_0^2(\tau))U_1^2 > nh\varepsilon^2) \\
&= O(1)EU_1^2 1(2\psi(C_{3,U}/\sigma_0^2(\tau))U_1^2 > nh\varepsilon^2) \\
&= o(1), \qquad \text{(B.6.11)}
\end{aligned}
$$

where the first inequality holds by Markov's inequality, the first equality holds by the definition of X_{ni}, and the second inequality holds because $\sigma_i^2 \leq C_{3,U}$ by part (i) of Λ_n, $c_{T_0,T_0-i}^2 \leq 1$ (since $|\rho_t| \leq 1$), and $\{U_i\}$ are identically distributed by part (iv) of Λ_n. The second last equality in (B.6.11) holds because

$$(2\psi/nh)\sum_{i=T_0-m_n+1}^{T_0} c_{T_0,T_0-i}^2 = 1 + o(1), \qquad \text{(B.6.12)}$$

as shown below. The last equality in (B.6.11) holds because, for $\xi := \varepsilon^2/(2\psi C_{3,U}/\sigma_0^2(\tau)) > 0$,

$$EU_1^2 1(U_1^2 > nh\xi) = EU_1^2 1([U_1^2/(nh\xi)] > 1) \leq EU_1^4(nh\xi)^{-1} \to 0 \qquad \text{(B.6.13)}$$

using $EU_1^4 \leq M < \infty$ by part (iv) of Λ_n and $nh \to \infty$ by Assumption 1.

To show (B.6.12), we have

$$(nh)^{-1}\sum_{i=T_0-m_n+1}^{T_0}(c_{T_0,T_0-i}^2 - \rho_{n\tau}^{2(T_0-i)}) = (nh)^{-1}m_n o(h^{1/2}) = o(1), \qquad \text{(B.6.14)}$$

where the first equality uses $|c_{T_0,T_0-i}^2 - \rho_{n\tau}^{2(T_0-i)}| = |c_{T_0,T_0-i} - \rho_{n\tau}^{T_0-i}| \cdot |c_{T_0,T_0-i} + \rho_{n\tau}^{T_0-i}| \leq 2|c_{T_0,T_0-i} - \rho_{n\tau}^{T_0-i}|$ (since $|\rho_{n\tau}^{T_0-i}|, |c_{T_0,T_0-i}| \leq 1$) and Lemma B.2(c) with an error $O(m_n(h + \frac{m_n}{n})/b_n)$ that is shown above to be $o(h^{1/2})$ and the last equality holds because $m_n = b_n/h^{1/5}$ by (B.6.2), and so, $(nh)^{-1}m_n h^{1/2} = b_n/(nh^{7/10}) = (b_n/nh)h^{3/10} = (1/r_0 + o(1))h^{3/10} = o(1)$ since $r_0 > 0$.

Next, we show that

$$(2\psi/nh)\sum_{i=T_0-m_n+1}^{T_0}\rho_{n\tau}^{2(T_0-i)}=1+o(1). \tag{B.6.15}$$

This holds because (i) $\sum_{i=T_0-m_n+1}^{T_0}\rho_{n\tau}^{2(T_0-i)}=\sum_{j=0}^{m_n-1}\rho_{n\tau}^{2j}=(1-\rho_{n\tau}^{2(m_n+1)})/(1-\rho_{n\tau}^2)$ using a change of variables, (ii) $\rho_{n\tau}^{2(m_n+1)}=\exp\{-2\kappa_n^*(\tau)(m_n+1)/b_n\}\to 0$, for $\kappa_n^*(\cdot)$ defined in (B.2.1), using $m_n/b_n=1/h^{1/5}\to\infty$ and $\kappa_n^*(\tau)\to\kappa_0(\tau)>0$ (by Lemma B.1, Assumption 3, and part (ii) of Λ_n, which guarantees that $\kappa_0(\tau)>0$), and

$$\text{(iii) } nh(1-\rho_{n\tau}^2)=nh(1-\rho_{n\tau})(1+\rho_{n\tau})=(1+\rho_{n\tau})nh\kappa_n(\tau)/b_n\to 2\kappa_0(\tau)r_0=2\psi. \tag{B.6.16}$$

Equations (B.6.14) and (B.6.15) combine to establish (B.6.12). This completes the verification of the Lindeberg condition in (B.6.11).

Now, we prove the conditional variance condition. We have

$$\begin{aligned}\sum_{i=T_0-m_n+1}^{T_0}E(X_{n,i}^2|\mathcal{F}_{n,i-1})-1 &= (2\psi/nh)\sum_{i=T_0-m_n+1}^{T_0}c_{T_0,T_0-i}^2\sigma_i^2/\sigma_0^2(\tau)-1\\ &= (2\psi/nh)\sum_{i=T_0-m_n+1}^{T_0}\rho_{n\tau}^{2(T_0-i)}\sigma_i^2/\sigma_0^2(\tau)-1+o(1)\\ &= (2\psi/nh)\sum_{i=T_0-m_n+1}^{T_0}\rho_{n\tau}^{2(T_0-i)}-1+o(1)\\ &= o(1),\end{aligned} \tag{B.6.17}$$

where the first equality uses $E(U_i^2|\mathcal{G}_{i-1})=1$ a.s. by part (iv) of Λ_n and the second equality holds by the same argument as used to show (B.6.14) since $\sigma_i^2/\sigma_0^2(\tau)$ is uniformly bounded. The third equality in (B.6.17) holds because

$$\begin{aligned}&(2\psi/nh)\sum_{i=T_0-m_n+1}^{T_0}\rho_{n\tau}^{2(T_0-i)}|\sigma_i^2/\sigma_0^2(\tau)-1|\\ &\le (2\psi/nh)\sum_{i=T_0-m_n+1}^{T_0}\rho_{n\tau}^{2(T_0-i)}\cdot\max_{j\in[T_0-m_n,T_0]}|\sigma_j^2-\sigma_0^2(\tau)|/\sigma_0^2(\tau)\\ &= (1+o(1))\max_{j\in[T_0-m_n,T_0]}|\sigma_j^2-\sigma_0^2(\tau)|/\sigma_0^2(\tau)\\ &= o(1),\end{aligned} \tag{B.6.18}$$

where the second last equality in (B.6.18) holds by (B.6.15) and the last equality in (B.6.18) holds by Lemma B.2(b), $O(h+\frac{m_n}{n})=o(1)$ (since $h\to 0$ and $m_n/n\to 0$), and $\sigma_{n\tau}^2=\sigma_n^2(\tau)\to$

$\sigma_0^2(\tau)$ (by Assumption 3). Hence, the conditional variance condition in (B.6.17) holds. By the CLT of Hall and Heyde (1980), we have

$$D_{1n} \to_d Z_1 \sim N(0,1), \tag{B.6.19}$$

as desired.

By (B.5.19), $W_n(\cdot) \Rightarrow B(\cdot)$ and by (B.6.19), $D_{1n} \to_d Z_1$. These results can be shown to hold jointly because $W_n(\cdot)$ and D_{1n} depend on the same random variables $\{U_t\}_{t\leq n}$. By (B.5.13), $W_n(u)$ is a linear function of $\{U_t : t \geq T_1\}$ for all $u \in [0,1]$. By (B.6.10), D_{1n} is a linear function of $\{U_t : t < T_1\}$. Since $\{U_t\}_{t\leq n}$ is a martingale difference sequence, these properties imply that $Cov(W_n(u), D_{1n}) = 0$ for all $u \in [0,1]$. In consequence, $W_n(u)$ and D_{1n} are asymptotically independent, i.e., $B(\cdot)$ and Z_1 are independent. □

B.7 Proof of Lemma 7.5

In this section, for notational simplicity in the proof, we assume that $\sigma_0^2(\tau) = 1$. If this is not assumed, numerous quantities in the proof needed to be rescaled by $1/\sigma_0(\tau)$.

Proof of Lemma 7.5. First, we prove part (a). We have

$$\begin{aligned}(nh)^{-1/2}Y_{t(s)} &= (nh)^{-1/2}(\mu_{t(s)} + Y^0_{t(s)} + c_{t(s),t(s)-T_0}Y^*_{T_0}) \\ &= o(1) + (nh)^{-1/2}Y^0_{t(s)} + (2\psi)^{-1/2}c_{t(s),t(s)-T_0}(2\psi/nh)^{1/2}Y^*_{T_0} \\ &\Rightarrow I_\psi(s) + (2\psi)^{-1/2}\exp\{-\psi s\}Z_1 =: I^*_\psi(s),\end{aligned} \tag{B.7.1}$$

where the first equality holds by (2.1) and (7.4), the second equality holds by part (i) of Λ_n and Assumption 1, the convergence holds by Lemma 7.3, Lemma 7.4, which uses the assumption that $r_0 \in (0,\infty)$, and $c_{t(s),t(s)-T_0} \Rightarrow \exp\{-\psi s\}$, which we now establish.

By Lemma 7.1(c),

$$\begin{aligned}c_{t(s),t(s)-T_0} &= \rho_{n\tau}^{t(s)-T_0} + O(nh^2/b_n) = \exp\{-\kappa_n^*(\tau)(t(s)-T_0)/b_n\} + o(1) \\ &= \exp\{-\kappa_n^*(\tau)(\lfloor nhs\rfloor + 1)/b_n\} + o(1) \to \exp\{-\kappa_0(\tau)r_0 s\} = \exp\{-\psi s\},\end{aligned} \tag{B.7.2}$$

where the $O(nh^2/b_n)$ term holds uniformly over $s \in [0,1]$, $\kappa_n^*(\tau)$ is defined in (B.2.1), the second equality uses $nh/b_n \to r_0 < \infty$ and $h \to 0$ by Assumption 1, the third equality holds by the definition of $t(s)$ in (7.9) and $T_0 = T_1 - 1$, the convergence uses Lemma B.1, Assumption 3, and $nh/b_n \to r_0$, and the final equality uses the definition of $\psi = r_0\kappa_0(\tau)$. This completes the proof of part (a).

The proofs of parts (b) and (c) use the technique developed in Phillips (1987). For example, for part (b), we have

$$(nh)^{-3/2} \sum_{t=T_1}^{T_2} Y_{t-1} = \int_0^1 \left[(nh)^{-1/2} Y_{t(s)} \right] ds \to_d \int_0^1 I_\psi^* (s) \, ds, \tag{B.7.3}$$

where the convergence holds by the CMT and part (a) and $\psi = r_0 \kappa_0 (\tau)$.

For part (c), we have

$$(nh)^{-2} \sum_{t=T_1}^{T_2} Y_{t-1}^2 = \int_0^1 \left[(nh)^{-1/2} Y_{t(s)} \right]^2 ds \to_d \int_0^1 I_\psi^{*2} (s) \, ds, \tag{B.7.4}$$

where the convergence holds by the CMT and part (a).

To prove part (d), define $W_n(s) := (nh)^{-1/2} \sum_{t=T_1}^{T_1 + \lfloor nhs \rfloor - 1} U_t$ for $s \in [0, 1]$ and define the stochastic integral

$$\int_0^1 \left[\sigma_{t(s)} \right] dW_n(s) := (nh)^{-1/2} \sum_{t=T_1}^{T_2} U_t \sigma_t. \tag{B.7.5}$$

By Assumption 3, Lemma 7.1(b), and the triangle inequality, we have

$$\max_{s \in [0,1]} \left| \sigma_{t(s)}^2 - 1 \right| \to_p 0, \tag{B.7.6}$$

which implies

$$\max_{s \in [0,1]} \left| \sigma_{t(s)} - 1 \right| \to_p 0 \tag{B.7.7}$$

since σ_t is nonnegative. By the functional central limit theorem for martingale difference sequences, we have

$$W_n(s) \Rightarrow B(s). \tag{B.7.8}$$

Therefore, we use Theorem 2.1 of Hansen (1992) and obtain

$$(nh)^{-1/2} \sum_{t=T_1}^{T_2} U_t \sigma_t = \int_0^1 \left[\sigma_{t(s)} \right] dW_n(s) \to_d \int_0^1 dB(s). \tag{B.7.9}$$

To prove part (e), we define the stochastic integral

$$\int_0^1 \left[(nh)^{-1/2} Y_{t(s)} \sigma_{t(s)} \right] dW_n(s) := (nh)^{-1} \sum_{t=T_1}^{T_2} Y_{t-1} U_t \sigma_t. \tag{B.7.10}$$

By part (a), we have

$$(nh)^{-1/2} Y_{t(s)} \Rightarrow I_\psi^* (s). \tag{B.7.11}$$

We also know that $\max_{s\in[0,1]}\left|\sigma_{t(s)}-1\right| \to_p 0$ by (B.7.7). Thus, by the CMT we have

$$(nh)^{-1/2} Y_{t(s)}\sigma_{t(s)} = (nh)^{-1/2} Y_{t(s)} + (nh)^{-1/2} Y_{t(s)}\left(\sigma_{t(s)}-1\right) \Rightarrow I_{\psi}^{*}(s). \tag{B.7.12}$$

Equations (B.7.8) and (B.7.12) hold jointly due to the common underlying martingale difference process $\{U_t\}_{t=T_1,\ldots,T_2}$. Therefore, we have

$$(nh)^{-1}\sum_{t=T_1}^{T_2} Y_{t-1}U_t\sigma_t = \int_0^1 \left[(nh)^{-1/2} Y_{t(s)}\sigma_{t(s)}\right] dW_n(s) \to_d \int_0^1 I_{\psi}^{*}(s)\, dB(s), \tag{B.7.13}$$

where the convergence holds by Theorem 2.1 of Hansen (1992), with $(nh)^{-1/2} Y_{t(\cdot)}\sigma_{t(\cdot)}$ and $I_{\psi}^{*}(\cdot)$ in the roles of $U_n(\cdot)$ and $U(\cdot)$, respectively, and $W_n(\cdot)$ and $B(\cdot)$ in the roles of $Y_n(\cdot)$ and $Y(\cdot)$, respectively, in Theorem 2.1 of Hansen (1992).

The proof of part (f) is analogous to that of part (c), and thus, is omitted.

Part (g) holds by the same argument as given above for parts (a)–(c) and (e), but with $Y_{t(s)} = \mu_{t(s)} + Y_{t(s)}^{0} + c_{t(s),t(s)-T_0} Y_{T_0}^{*}$ replaced by $\mu_{t(s)} + Y_{t(s)}^{0}$ in part (a), which simplifies the proof because the initial condition $Y_{T_0}^{*}$ does not appear.

The subsequence version of Lemma 7.5, see Remark 7.2, which has $\{p_n\}_{n\geq 1}$ in place of $\{n\}_{n\geq 1}$, is proved by replacing n by p_n and $h = h_n$ by h_{p_n} throughout the proof above. □

B.8 Proof of Theorem 7.2

In this section, for notational simplicity in the proof, we assume that $\sigma_0^2(\tau) = 1$.

Proof of Theorem 7.2. First, we prove Theorem 7.2 for the case where $r_0 \in (0,\infty)$. To start, we show that the denominator of (7.10) divided by $\sigma_0(\tau)$ converges in distribution to $\int_0^1 I_{D,\psi}^{*2}(s)\, ds$, where $\psi = r_0\kappa_0(\tau)$.

$$\begin{aligned}
&(nh)^{-2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)^2 \\
&= (nh)^{-2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}\right)^2 - (nh)^{-1}\left(\overline{Y}_{nh,-1}\right)^2 \\
&= (nh)^{-2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}\right)^2 - \left((nh)^{-3/2}\sum_{t=T_1}^{T_2} Y_{t-1}\right)^2 \\
&\to_d \int_0^1 I_{\psi}^{*2}(s)\, ds - \left(\int_0^1 I_{\psi}^{*}(s)\, ds\right)^2 = \int_0^1 I_{D,\psi}^{*2}(s)\, ds,
\end{aligned} \tag{B.8.1}$$

where the convergence holds by Lemma 7.5(b) and (c) and the CMT.

Next, we show the numerator of (7.10) converges in distribution to $\int_0^1 I_{D,\psi}^*(s)\,dB(s)$. For any $t = T_1, ..., T_2$, we have

$$\begin{aligned} Y_t - \rho_{0,n} Y_{t-1} &= (\mu_t - \rho_{0,n}\mu_{t-1}) + \left(\rho_t Y_{t-1}^* + \sigma_t U_t - \rho_{0,n} Y_{t-1}^*\right) \\ &= \sigma_t U_t + (\mu_t - \rho_{0,n}\mu_{t-1}) + (\rho_t - \rho_{0,n})\, Y_{t-1}^0 + (\rho_t - \rho_{0,n})\, c_{t-1,t-1-T_0} Y_{T_0}^*, \end{aligned} \quad \text{(B.8.2)}$$

where the first equality holds by (2.1) and the last equality holds by (7.5). Substituting (B.8.2) into the numerator of (7.10) gives

$$\begin{aligned} &(nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right)(Y_t - \rho_{0,n} Y_{t-1}) \\ &= (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right) \sigma_t U_t \\ &\quad + (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right)(\mu_t - \rho_{0,n}\mu_{t-1}) \\ &\quad + (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right)(\rho_t - \rho_{0,n})\, Y_{t-1}^0 \\ &\quad + (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right)(\rho_t - \rho_{0,n})\, c_{t-1,t-1-T_0} Y_{T_0}^* \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned} \quad \text{(B.8.3)}$$

For A_1, we have

$$\begin{aligned} A_1 &= (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right) \sigma_t U_t \\ &= (nh)^{-1} \sum_{t=T_1}^{T_2} Y_{t-1}\sigma_t U_t - \left[(nh)^{-1/2}\, \overline{Y}_{nh,-1}\right] (nh)^{-1/2} \sum_{t=T_1}^{T_2} \sigma_t U_t \\ &\to_d \int_0^1 I_\psi^*(s)\, dB(s) - \int_0^1 I_\psi^*(s)\, ds \int_0^1 dB(s) = \int_0^1 I_{D,\psi}^*(s)\, dB(s), \end{aligned} \quad \text{(B.8.4)}$$

where the convergence holds by Lemma 7.5(b), (d), and (e).

For A_2, we have

$$\begin{aligned} |A_2|^2 &= \left| (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right)(\mu_t - \rho_{0,n}\mu_{t-1}) \right|^2 \\ &\leq \left[(nh)^{-2} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right)^2 \right] \left[\sum_{t=T_1}^{T_2} (\mu_t - \rho_{0,n}\mu_{t-1})^2 \right] \end{aligned}$$

$$
\begin{aligned}
&\leq \left[(nh)^{-2} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1} \right)^2 \right] \\
&\quad \times \left[2 \sum_{t=T_1}^{T_2} \left(\mu_t - \mu_{t-1} \right)^2 + 2 \sum_{t=T_1}^{T_2} \left(\mu_{t-1} - \rho_{0,n} \mu_{t-1} \right)^2 \right] \\
&\leq \left[(nh)^{-2} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1} \right)^2 \right] \\
&\quad \times 2 \left[nh \max_{t \in [T_1, T_2]} \left(\mu_t - \mu_{t-1} \right)^2 + nh \left(1 - \rho_{0,n} \right)^2 \max_{t \in [T_1, T_2]} \mu_{t-1}^2 \right] \\
&= O_p(1) \left[nhO\left(n^{-2} \right) + nhO\left((nh)^{-2} \right) \right] = o_p(1),
\end{aligned} \tag{B.8.5}
$$

where the first inequality holds by the Cauchy-Schwarz (CS) inequality, the second inequality uses the fact that $(a+b)^2 \leq 2\left(a^2+b^2\right)$, and the second last equality holds by (B.8.1), $\max_{t \in [T_1, T_2]} |\mu_t - \mu_{t-1}| \leq L_2/n$ by the Lipschitz condition on $\mu(\cdot)$, $|1-\rho_{0,n}| = O\left((nh)^{-1}\right)$, and $\max_{t \in [T_1, T_2]} \mu_t^2$ is $O(1)$.

For A_3, we have

$$
\begin{aligned}
|A_3|^2 &= \left| (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1} \right) \left(\rho_t - \rho_{0,n} \right) Y_{t-1}^0 \right|^2 \\
&\leq \left[(nh)^{-2} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1} \right)^2 \right] \left[\sum_{t=T_1}^{T_2} \left(\rho_t - \rho_{0,n} \right)^2 \left(Y_{t-1}^0 \right)^2 \right] \\
&\leq \left[(nh)^{-2} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1} \right)^2 \right] \left[(nh)^2 \max_{t \in [T_1, T_2]} \left(\rho_t - \rho_{0,n} \right)^2 \right] \\
&\quad \times \left[(nh)^{-2} \sum_{t=T_1}^{T_2} \left(Y_{t-1}^0 \right)^2 \right] \\
&= O_p(1)\, O\left((nh)^2 n^{-2} \right) O_p(1) = o_p(1),
\end{aligned} \tag{B.8.6}
$$

where the first inequality holds by the CS inequality and the second last equality holds by (B.8.1), Lemma 7.1(a), and Lemma 7.5(f). Note that Lemma 7.1(a) implies $\max_{t \in [T_1, T_2]} \left(\rho_t - \rho_{0,n} \right)^2 = O\left(n^{-2}\right)$ because under H_0 and when $nh/b_n = O(1)$,

$$
\max_{t \in [T_1, T_2]} \left(\rho_t - \rho_{0,n} \right)^2 = \left(\max_{t \in [T_1, T_2]} \left| \rho_t - \rho_{n\tau} \right| \right)^2 = \left(O\left(h/b_n \right) \left(b_n/nh \right) nh/b_n \right)^2 = O\left(n^{-2} \right). \tag{B.8.7}
$$

For A_4, we have

$$\begin{aligned}|A_4|^2 &= \left|(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)(\rho_t-\rho_{0,n})\,c_{t-1,t-1-T_0}Y_{T_0}^*\right|^2\\ &\le \left[(nh)^{-2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)^2\right]\left[\sum_{t=T_1}^{T_2}(\rho_t-\rho_{0,n})^2\,c_{t-1,t-1-T_0}^2Y_{T_0}^{*2}\right]\\ &\le \left[(nh)^{-2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)^2\right]\left[nh\max_{t\in[T_1,T_2]}(\rho_t-\rho_{0,n})^2\right]Y_{T_0}^{*2}\\ &= O_p(1)\,O\left(nh\cdot n^{-2}\right)O_p(n)=o_p(1),\end{aligned}\tag{B.8.8}$$

where the first inequality holds by the CS inequality, the second inequality uses $\max_{t\in[T_0,T_2]}c_{t,t-T_0}^2\le 1$, and the second last equality holds by (B.4.3), (B.8.1), and (B.8.7).

Therefore, we obtain

$$(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)(Y_t-\rho_{0,n}Y_{t-1})=A_1+o_p(1)\to_d\int_0^1 I_{D,\psi}^*(s)\,dB(s)\tag{B.8.9}$$

from (B.8.4), (B.8.5), (B.8.6), and (B.8.8).

Combining (B.8.1) and (B.8.9), we have

$$nh\left(\widehat{\rho}_{n\tau}-\rho_{0,n}\right)\to_d\left(\int_0^1 I_{D,\psi}^{*2}(s)\,ds\right)^{-1}\int_0^1 I_{D,\psi}^*(s)\,dB(s)\tag{B.8.10}$$

by the CMT.

Next, for the t-statistic $T_n(\rho_{0,n})$, we have

$$T_n(\rho_{0,n})=\frac{(nh)^{1/2}\left(\widehat{\rho}_{n\tau}-\rho_{0,n}\right)}{\widehat{s}_{n\tau}}=\frac{nh\left(\widehat{\rho}_{n\tau}-\rho\right)}{\left(nh\widehat{s}_{n\tau}^2\right)^{1/2}}.\tag{B.8.11}$$

Thus, by (3.4), (3.5), (B.8.1), (B.8.10), and the CMT, we only need to show

$$\widehat{\sigma}_{n\tau}^2:=(nh)^{-1}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{Y}_{nh}-\widehat{\rho}_{n\tau}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2\to_p 1.\tag{B.8.12}$$

First, we replace $\widehat{\rho}_{n\tau}$ with $\rho_{0,n}$ in (B.8.12):

$$\widehat{\sigma}_{n\tau}^2=(nh)^{-1}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{Y}_{nh}-\widehat{\rho}_{n\tau}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2$$

$$
\begin{aligned}
&= (nh)^{-1} \sum_{t=T_1}^{T_2} \left[Y_t - \overline{Y}_{nh} - \rho_{0,n} \left(Y_{t-1} - \overline{Y}_{nh,-1} \right) \right]^2 \\
&\quad + (nh)^{-1} \sum_{t=T_1}^{T_2} \left[(\rho_{0,n} - \widehat{\rho}_{n\tau}) \left(Y_{t-1} - \overline{Y}_{nh,-1} \right) \right]^2 \\
&\quad + 2 (nh)^{-1} \sum_{t=T_1}^{T_2} \left[Y_t - \overline{Y}_{nh} - \rho_{0,n} \left(Y_{t-1} - \overline{Y}_{nh,-1} \right) \right] \left[(\rho_{0,n} - \widehat{\rho}_{n\tau}) \left(Y_{t-1} - \overline{Y}_{nh,-1} \right) \right] \\
&=: A_5 + A_6 + A_7. \qquad \text{(B.8.13)}
\end{aligned}
$$

We show that $A_5 \to_p 1$ and $A_6 \to_p 0$, which together imply $A_7 \to_p 0$ by the CS inequality.

For A_5, we have

$$
\begin{aligned}
&(nh)^{-1} \sum_{t=T_1}^{T_2} \left[Y_t - \overline{Y}_{nh} - \rho_{0,n} \left(Y_{t-1} - \overline{Y}_{nh,-1} \right) \right]^2 \\
&= (nh)^{-1} \sum_{t=T_1}^{T_2} (Y_t - \rho_{0,n} Y_{t-1})^2 + \left(\overline{Y}_{nh} - \rho_{0,n} \overline{Y}_{nh,-1} \right)^2 \\
&\quad + 2 (nh)^{-1} \sum_{t=T_1}^{T_2} \left(\overline{Y}_{nh} - \rho_{0,n} \overline{Y}_{nh,-1} \right) (Y_t - \rho_{0,n} Y_{t-1}) \\
&=: A_{51} + A_{52} + A_{53}. \qquad \text{(B.8.14)}
\end{aligned}
$$

First, we show that A_{51} converges in probability to 1. By (B.8.2), we have

$$
\begin{aligned}
A_{51} &= (nh)^{-1} \sum_{t=T_1}^{T_2} \left[\begin{array}{c} \sigma_t U_t + (\mu_t - \rho_{0,n} \mu_{t-1}) + (\rho_t - \rho_{0,n}) Y_{t-1}^0 \\ + (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* \end{array} \right]^2 \\
&= (nh)^{-1} \sum_{t=T_1}^{T_2} (\sigma_t U_t)^2 \\
&\quad + (nh)^{-1} \sum_{t=T_1}^{T_2} \left[(\mu_t - \rho_{0,n} \mu_{t-1}) + (\rho_t - \rho_{0,n}) Y_{t-1}^0 + (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* \right]^2 \\
&\quad + 2 (nh)^{-1} \sum_{t=T_1}^{T_2} \sigma_t U_t \left[(\mu_t - \rho_{0,n} \mu_{t-1}) + (\rho_t - \rho_{0,n}) Y_{t-1}^0 + (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* \right] \\
&=: A_{511} + A_{512} + A_{513}. \qquad \text{(B.8.15)}
\end{aligned}
$$

For A_{511}, we have

$$
(nh)^{-1} \sum_{t=T_1}^{T_2} (\sigma_t U_t)^2 = (nh)^{-1} \sum_{t=T_1}^{T_2} U_t^2 + (nh)^{-1} \sum_{t=T_1}^{T_2} \left(\sigma_t^2 - 1 \right) U_t^2. \qquad \text{(B.8.16)}
$$

By the weak law of large numbers, $(nh)^{-1}\sum_{t=T_1}^{T_2} U_t^2 \to_p 1$ as $n \to \infty$. We also have

$$\left|(nh)^{-1}\sum_{t=T_1}^{T_2}\left(\sigma_t^2-1\right)U_t^2\right| \leq \max_{t\in[T_1,T_2]}\left|\sigma_t^2-1\right|(nh)^{-1}\sum_{t=T_1}^{T_2}U_t^2 = o_p(1), \tag{B.8.17}$$

where the equality holds by (B.7.7). Therefore,

$$A_{511} = (nh)^{-1}\sum_{t=T_1}^{T_2}(\sigma_t U_t)^2 \to_p 1. \tag{B.8.18}$$

Next, we have $\sum_{t=T_1}^{T_2}(\mu_t - \rho_{0,n}\mu_{t-1})^2 = o(1)$, $\sum_{t=T_1}^{T_2}(\rho_t-\rho_{0,n})^2\left(Y_{t-1}^0\right)^2 = o_p(1)$, and $\sum_{t=T_1}^{T_2}(\rho_t-\rho_{0,n})^2 c_{t-1,t-1-T_0}^2 Y_{T_0}^{*2} = o_p(1)$, by (B.8.5), (B.8.6), and (B.8.8), respectively. These results and the CS inequality yield $A_{512} \to_p 0$. Finally, using the CS inequality again gives $A_{513} \to_p 0$. Therefore, by the CMT we have $A_{51} \to_p 1$.

For A_{52}, by (B.8.2) we have

$$\begin{aligned}(A_{52})^{1/2} &= \left|(nh)^{-1}\sum_{t=T_1}^{T_2}(Y_t - \rho_{0,n}Y_{t-1})\right| \\ &= \left|(nh)^{-1}\sum_{t=T_1}^{T_2}\left[\begin{array}{c}\sigma_t U_t + (\mu_t-\rho_{0,n}\mu_{t-1}) + (\rho_t-\rho_{0,n})Y_{t-1}^0 \\ +(\rho_t-\rho_{0,n})c_{t-1,t-1-T_0}Y_{T_0}^*\end{array}\right]\right| \to_p 0,\end{aligned} \tag{B.8.19}$$

where the convergence holds by Lemma 7.5(d), the results stated after (B.8.18), and the CS inequality.

The convergence of $A_{53} \to_p 0$ follows from $A_{51} \to_p 1$, $A_{52} \to_p 0$, and the CS inequality. Combining the results gives $A_5 \to_p 1$.

For A_6, we have

$$\begin{aligned}A_6 &= (nh)^{-1}\sum_{t=T_1}^{T_2}\left[(\rho_{0,n}-\widehat{\rho}_{n\tau})\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2 \\ &= (\rho_{0,n}-\widehat{\rho}_{n\tau})^2 \times (nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)^2 \\ &= O_p\left((nh)^{-2}\right)\times O_p(nh) = o_p(1),\end{aligned} \tag{B.8.20}$$

where the third equality holds by (B.8.1) and (B.8.10).

In conclusion, we have proved (B.8.12), which leads to

$$T_n(\rho_{0,n}) \to_d \left(\int_0^1 I_{D,\psi}^{*2}(s)\,ds\right)^{-1/2}\int_0^1 I_{D,\psi}^*(s)\,dB(s) \tag{B.8.21}$$

for the case where $r_0 \in (0, \infty)$.

Now, we prove the results of Theorem 7.2 for the case where $r_0 = 0$. The idea is to use the same proof as given above for $r_0 > 0$, except with Y_{t-1} $(= \mu_{t-1} + Y^0_{t-1} + c_{t-1,t-1-T_0} Y^*_{T_0})$ split into the two pieces $\mu_{t-1} + Y^0_{t-1}$ and $c_{t-1,t-1-T_0} Y^*_{T_0}$. To deal with the first component $\mu_{t-1} + Y^0_{t-1}$, we use the argument given above for $r_0 > 0$ using the results of Lemma 7.5(g), which holds even when $r_0 = 0$. Then, we show that the second component $c_{t-1,t-1-T_0} Y^*_{T_0}$ has a negligible asymptotic effect because $c_{t-1,t-1-T_0}$ is quite close to the constant 1 since $r_0 = 0$. The reason it has a negligible asymptotic effect is that the LS regression includes a constant term, and hence, $c_{t-1,t-1-T_0} Y^*_{T_0}$ only enters the LS estimator $\widehat{\rho}_{n\tau}$ through $(c_{t-1,t-1-T_0} - (nh)^{-1} \sum_{s=T_1}^{T_2} c_{s-1,s-1-T_0}) Y^*_{T_0}$, which is close to $(1-1)Y^*_{T_0}$. Combining this with Lemma 7.2(b), we show that its impact is asymptotically negligible.

We have

$$\begin{aligned}
\max_{t \in [T_1, T_2]} |1 - c_{t-1,t-1-T_0}| &\leq \max_{t \in [T_1, T_2]} |1 - \rho_{n\tau}^{t-1-T_0}| + O(nh/b_n) \\
&= 1 - \rho_{n\tau}^{nh} + O(nh/b_n) \\
&= 1 - \exp\{-\kappa_n^*(\tau) nh/b_n\} + O(nh/b_n) \\
&= 1 - (1 - \kappa_n^*(\tau)(nh/b_n) \exp\{\zeta_n\}) + O(nh/b_n) \\
&= O(nh/b_n), \qquad \text{(B.8.22)}
\end{aligned}$$

where the inequality holds by Lemma 7.1(c) and $O(nh^2/b_n) = O(nh/b_n)$, the first equality uses $T_2 - 1 - T_0 = 2\lfloor nh/2 \rfloor$, which we denote by nh for simplicity, the second equality holds by the definition of $\kappa_n^*(\tau)$ given in (B.2.1), the third equality holds by a mean value expansion with ζ_n lying between 0 and $-\kappa_n^*(\tau) nh/b_n$, and hence, $|\zeta_n| \leq \kappa_n^*(\tau) nh/b_n = O(nh/b_n) = o(1)$ using $\kappa_n^*(\tau) = O(1)$ (by $\kappa_n(\tau) \leq C_4 < \infty$ by part (ii) of Λ_n and Lemma B.1) and $nh/b_n \to r_0 = 0$, and the last equality holds by $\kappa_n^*(\tau) = O(1)$ and $\zeta_n \to 0$.

Equation (B.8.22) yields

$$\max_{t \in [T_1, T_2]} |c_{t-1,t-1-T_0} - \overline{c}_{nh}| = O(nh/b_n), \text{ where } \overline{c}_{nh} := (nh)^{-1} \sum_{s=T_1}^{T_2} c_{s-1,s-1-T_0}. \qquad \text{(B.8.23)}$$

Now, consider the denominator of the normalized LS estimator given in (7.10) and (B.8.1). We have

$$\begin{aligned}
&(nh)^{-2} \sum_{t=T_1}^{T_2} (Y_{t-1} - \overline{Y}_{nh,-1})^2 \\
= \; &(nh)^{-2} \sum_{t=T_1}^{T_2} \left(\mu_{t-1} + Y^0_{t-1} - (\overline{\mu}_{nh,-1} + \overline{Y}^0_{nh,-1}) + (c_{t-1,t-1-T_0} - \overline{c}_{nh}) Y^*_{T_0}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= (nh)^{-2}\sum_{t=T_1}^{T_2}\left(\mu_{t-1}+Y_{t-1}^0-(\overline{\mu}_{nh,-1}+\overline{Y}^0_{nh,-1})\right)^2+(nh)^{-1}\max_{t\in[T_1,T_2]}\left(c_{t-1,t-1-T_0}-\overline{c}_{nh}\right)^2Y_{T_0}^{*2}\\
&\quad +2O_p(1)(nh)^{-1/2}\max_{t\in[T_1,T_2]}|c_{t-1,t-1-T_0}-\overline{c}_{nh}|\cdot|Y_{T_0}^*|\\
&= (nh)^{-2}\sum_{t=T_1}^{T_2}\left(\mu_{t-1}+Y_{t-1}^0-(\overline{\mu}_{nh,-1}+\overline{Y}^0_{nh,-1})\right)^2\\
&\quad +(nh)^{-1}O((nh/b_n)^2)o_p\left(b_n^2/nh\right)+O_p(1)(nh)^{-1/2}O(nh/b_n)o_p\left(b_n/(nh)^{1/2}\right)\\
&\to_d \int_0^1 I_{D,\psi}^{*2}(s)ds, \qquad \text{(B.8.24)}
\end{aligned}
$$

where $\overline{\mu}_{nh,-1}=(nh)^{-1}\sum_{t=T_1}^{T_2}\mu_{t-1}$, $\overline{Y}^0_{nh,-1}=(nh)^{-1}\sum_{t=T_1}^{T_2}Y^0_{t-1}$, the first equality uses (2.1) and (7.5), the second equality uses $(nh)^{-3/2}\sum_{t=T_1}^{T_2}\left|\mu_{t-1}+Y_{t-1}^0-(\overline{\mu}_{nh,-1}+\overline{Y}^0_{nh,-1})\right|=O_p(1)$ by Lemma 7.5(g)(b) (which refers to Lemma 7.5(b) adjusted according to Lemma 7.5(g), i.e., with Y_{t-1} replaced by $\mu_{t-1}+Y^0_{t-1}$) and with absolute values added to $\mu_{t-1}+Y^0_{t-1}$ (which does not affect the argument), the third equality holds using (B.8.23) and Lemma 7.2(b), and the convergence holds because $(nh)^{-1}O((nh/b_n)^2)o_p(b_n^2/nh)=o_p(1)$, $O_p(1)(nh)^{-1/2}O(nh/b_n)\times o_p\left(b_n/(nh)^{1/2}\right)=o_p(1)$, and the first summand on the lhs converges in distribution to $\int_0^1 I_{D,\psi}^{*2}(s)ds$ by the same argument as in (B.8.1) but with $\mu_{t-1}+Y^0_{t-1}$ in place of Y_{t-1} and using Lemma 7.5(g), which uses $\mu_{t-1}+Y^0_{t-1}$ and applies when $r_0=0$.

Next, for the numerator of the normalized LS estimator $\widehat{\rho}_{n\tau}$ given in (7.10) and (B.8.3), we decompose $Y_{t-1}-\overline{Y}_{nh,-1}$ into $\mu_{t-1}+Y^0_{t-1}-(\overline{\mu}_{nh,-1}+\overline{Y}^0_{nh,-1})$ and $(c_{t-1,t-1-T_0}-\overline{c}_{nh})Y^*_{T_0}$ in each of the summands $A_1,...,A_4$ in (B.8.3). Thus, we write

$$
\begin{aligned}
A_1 &= (nh)^{-1}\sum_{t=T_1}^{T_2}(Y_{t-1}-\overline{Y}_{nh,-1})\sigma_tU_t=A_{11}+A_{12},\ \text{where}\\
A_{11} &:= (nh)^{-1}\sum_{t=T_1}^{T_2}(\mu_{t-1}+Y^0_{t-1})\sigma_tU_t-(nh)^{-1/2}(\overline{\mu}_{nh,-1}+\overline{Y}^0_{nh,-1}))(nh)^{-1/2}\sum_{t=T_1}^{T_2}\sigma_tU_t\ \text{and}\\
A_{12} &:= (nh)^{-1}\sum_{t=T_1}^{T_2}(c_{t-1,t-1-T_0}-\overline{c}_{nh})\sigma_tU_tY^*_{T_0}. \qquad \text{(B.8.25)}
\end{aligned}
$$

We have

$$
A_{11}\to_d \int_0^1 I_\psi^*(s)dB(s)-\int_0^1 I^*_\psi(s)ds\int_0^1 dB(s)=\int_0^1 I^*_{D,\psi}(s)dB(s), \qquad \text{(B.8.26)}
$$

where the convergence holds by Lemma 7.5(g)(b) (which refers to Lemma 7.5(b) adjusted according to Lemma 7.5(g)), Lemma 7.5(g)(e), and Lemma 7.5(d).

For A_{12}, we have

$$Var\left((nh)^{-1}\sum_{t=T_1}^{T_2}(c_{t-1,t-1-T_0}-\overline{c}_{nh})\sigma_t U_t\right)=(nh)^{-2}\sum_{t=T_1}^{T_2}(c_{t-1,t-1-T_0}-\overline{c}_{nh})^2\sigma_t^2$$
$$=O((nh)^{-1})\max_{t\in[T_1,T_2]}(c_{t-1,t-1-T_0}-\overline{c}_{nh})^2=O((nh)^{-1})O((nh/b_n)^2)=O(nh/b_n^2), \tag{B.8.27}$$

where the third equality uses (B.8.23). Hence,

$$A_{12}=O_p\left(\frac{(nh)^{1/2}}{b_n}\right)Y_{T_0}^*=O_p\left(\frac{(nh)^{1/2}}{b_n}\frac{b_n}{nh^{1/2}}\right)=O_p\left(\frac{1}{n^{1/2}}\right)=o_p(1), \tag{B.8.28}$$

where the second equality uses Lemma 7.2(b). Combining (B.8.25)–(B.8.28) gives: $A_1\to_d \int_0^1 I_{D,\psi}^*(s)dB(s)$ in the $r_0=0$ case, just as in the $r_0>0$ case considered in (B.8.4).

When $r_0=0$, we have $A_2=o_p(1)$ and $A_3=o_p(1)$ (where A_2 and A_3 are defined in (B.8.3)) by the same arguments as in (B.8.5) and (B.8.6) using $(nh)^{-2}\sum_{t=T_1}^{T_2}(Y_{t-1}-\overline{Y}_{nh,-1})^2=O_p(1)$ by (B.8.24).

When $r_0=0$, for A_4 (defined in (B.8.3)), we have

$$|A_4|^2\le\left[(nh)^{-2}\sum_{t=T_1}^{T_2}(Y_{t-1}-\overline{Y}_{nh,-1})^2\right]\left[nh\max_{t\in[T_1,T_2]}(\rho_t-\rho_{0,n})^2\right]Y_{T_0}^{*2}$$
$$=O_p(1)nhO(n^{-2})O_p(n)=O_p(h)=o_p(1), \tag{B.8.29}$$

where the inequality holds by the first three lines of (B.8.8) and the first equality uses (B.8.7), (B.8.24), and Lemma 7.2(a).

This completes the proof of (B.8.9) concerning the numerator of the normalized LS estimator in the $r_0=0$ case. Combined with the result for the denominator in (B.8.24), this establishes the result of (B.8.10) for the normalized LS estimator in the $r_0=0$ case.

For the t-statistic, as in (B.8.11) and (B.8.12), it remains to show that $\widehat{\sigma}_{n\tau}^2\to_p 1$. For $r_0=0$, this holds by the same argument as given in (B.8.13)–(B.8.20) for the $r_0>0$ with the only change needed being that $(nh)^{-1}\sum_{t=T_1}^{T_2}(Y_{t-1}-\overline{Y}_{nh,-1})^2=O_p(nh)$ in the third equality of (B.8.20) by (B.8.24) when $r_0=0$, rather than by (B.8.1).

The subsequence versions of Lemma 7.3, Lemma 7.5, and Theorem 7.2, see Remark 7.2, which have $\{p_n\}_{n\ge1}$ in place of $\{n\}_{n\ge1}$, are proved by replacing n by p_n and $h=h_n$ by h_{p_n} throughout the proofs above. □

B.9 Proof of Lemma 7.6

In this section, for notational simplicity in the proof, we assume that $\sigma_0^2(\tau) = 1$.

Proof of Lemma 7.6(a). To prove part (a), we let $\sum_{j=0}^{-1} c_{t,j}\sigma_{t-j}U_{t-j} = 0$. Then, by (7.4), we have

$$\begin{aligned}
& E\left[\left(1-\rho_{0,n}^2\right)^{1/2}(nh)^{-1}\sum_{t=T_1}^{T_2} Y_{t-1}^0\right]^2 \\
&= E\left[\left(1-\rho_{0,n}^2\right)^{1/2}(nh)^{-1}\sum_{t=T_1-1}^{T_2-1}\left(\sum_{j=0}^{t-T_1} c_{t,j}\sigma_{t-j}U_{t-j}\right)\right]^2 \\
&= \left(1-\rho_{0,n}^2\right)(nh)^{-2}\sum_{t=T_1}^{T_2-1} E\left(\sum_{j=0}^{t-T_1} c_{t,j}\sigma_{t-j}U_{t-j}\right)^2 \\
&\quad + \left(1-\rho_{0,n}^2\right)(nh)^{-2}\sum_{t,s=T_1}^{T_2-1} E\left(\sum_{i=0}^{t-T_1} c_{t,i}\sigma_{t-i}U_{t-i}\right)\left(\sum_{j=0}^{s-T_1} c_{s,j}\sigma_{s-j}U_{s-j}\right)\mathbb{1}\{t\neq s\} \\
&=: A_{e1}+A_{e2}, \qquad \text{(B.9.1)}
\end{aligned}$$

where the first equality holds by (7.5). We show $|A_{e1}| = o(1)$ and $|A_{e2}| = o(1)$, which establish part (a) by Markov's inequality.

For $|A_{e1}|$, we have

$$\begin{aligned}
|A_{e1}| &= \left(1-\rho_{0,n}^2\right)(nh)^{-2}\sum_{t=T_1}^{T_2-1} E\left(\sum_{j=0}^{t-T_1} c_{t,j}\sigma_{t-j}U_{t-j}\right)^2 \\
&= \left(1-\rho_{0,n}^2\right)(nh)^{-2}\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1} EU_{t-j}^2 c_{t,j}^2\sigma_{t-j}^2 \\
&\leq \left(1-\rho_{0,n}^2\right)(nh)^{-2}\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1} \overline{\rho}_n^{2j}\sigma_{t-j}^2 \\
&\leq \max_{t\in[T_1,T_2]}\sigma_t^2\left(1-\rho_{0,n}^2\right)(nh)^{-2}\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{\infty}\overline{\rho}_n^{2j} \\
&= O(1)\,O\left(b_n^{-1}\right)(nh)^{-2}\,O(nhb_n) = o(1), \qquad \text{(B.9.2)}
\end{aligned}$$

where the second equality uses the fact that $\{U_t\}_{t=1}^n$ is a martingale difference sequence, the first inequality holds by (7.13), and the second last equality holds by (B.7.7) and (7.16).

For $|A_{e2}|$, we have

$$\begin{aligned}
|A_{e2}| &= \left| \begin{array}{c} \left(1-\rho_{0,n}^2\right)(nh)^{-2} \sum_{t,s=T_1}^{T_2-1} E\left(\sum_{i=0}^{t-T_1} c_{t,i}\sigma_{t-i}U_{t-i}\right) \\ \times \left(\sum_{j=0}^{s-T_1} c_{s,j}\sigma_{s-j}U_{s-j}\right) \mathbb{1}\{t \neq s\} \end{array} \right| \\
&= 2\left|\left(1-\rho_{0,n}^2\right)(nh)^{-2} \sum_{t=T_1+1}^{T_2-1} \sum_{s=T_1}^{t-1} E \sum_{i=0}^{t-T_1} c_{t,i}\sigma_{t-i}U_{t-i} \sum_{j=0}^{s-T_1} c_{s,j}\sigma_{s-j}U_{s-j}\right| \\
&= 2\left|\left(1-\rho_{0,n}^2\right)(nh)^{-2} \sum_{t=T_1+1}^{T_2-1} \sum_{s=T_1}^{t-1} \sum_{j=0}^{s-T_1} EU_{s-j}^2 c_{t,j+(t-s)} c_{s,j} \sigma_{s-j}^2\right| \\
&\leq 2 \max_{t\in[T_1,T_2]} \sigma_t^2 \left(1-\rho_{0,n}^2\right)(nh)^{-2} \left| \sum_{t=T_1+1}^{T_2-1} \sum_{s=T_1}^{t-1} \overline{\rho}_n^{t-s} \sum_{j=0}^{s-T_1} \overline{\rho}_n^{2j} \right| \\
&\leq O(1)\, O\left(b_n^{-1}\right)(nh)^{-2} O\left(nhb_n^2\right) = O\left(b_n/nh\right) = o(1), \qquad \text{(B.9.3)}
\end{aligned}$$

where the first equality uses the fact that t and s are symmetric, the last inequality holds by (B.7.7), (7.16), and (7.19), and the last equality holds by $b_n = o(nh)$.

Therefore, by Markov's inequality, we have

$$\left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1} \sum_{t=T_1}^{T_2} Y_{t-1}^0 \rightarrow_p 0. \qquad \text{(B.9.4)}$$

□

Proof of Lemma 7.6(b). To prove part (b), we have

$$\begin{aligned}
& E\left[\left(1-\rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1}^0\right)^2\right] \\
&= \left(1-\rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1-1}^{T_2-1} E\left(\sum_{j=0}^{t-T_1} c_{t,j}\sigma_{t-j}U_{t-j}\right)^2 \\
&= \left(1-\rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1}^{T_2-1} \sum_{j=0}^{t-T_1} c_{t,j}^2 \sigma_{t-j}^2 \\
&= \left(1-\rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1}^{T_2-1} \sum_{j=0}^{t-T_1} \rho_{0,n}^{2j} + \left(1-\rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1}^{T_2-1} \sum_{j=0}^{t-T_1} \left(c_{t,j}^2 - \rho_{0,n}^{2j}\right)\sigma_{t-j}^2 \\
&\quad + \left(1-\rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1}^{T_2-1} \sum_{j=0}^{t-T_1} \rho_{0,n}^{2j} \left(\sigma_{t-j}^2 - \sigma_0^2(\tau)\right) \\
&=: A_{b1} + A_{b2} + A_{b3}. \qquad \text{(B.9.5)}
\end{aligned}$$

We show $A_{b1} = 1 + o(1)$, $A_{b2} = o(1)$, and $A_{b3} = o(1)$.

For A_{b1}, we have

$$A_{b1} = \left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1}\rho_{0,n}^{2j} = \left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2-1}\left(1-\rho_{0,n}^{2(t-T_1+1)}\right)\left(1-\rho_{0,n}^2\right)^{-1}$$
$$= 1-(nh)^{-1}\sum_{k=1}^{\lfloor nh\rfloor}\rho_{0,n}^{2k} = 1+O\left(b_n/nh\right) = 1+o\left(1\right), \tag{B.9.6}$$

where the third equality uses the change of coordinates $k = t - T_1 + 1$ and the second last equality uses (7.16).

For A_{b2}, we have

$$\begin{aligned}
|A_{b2}| &= \left(1-\rho_{0,n}^2\right)(nh)^{-1}\left|\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1}\left(c_{t,j}^2-\rho_{0,n}^{2j}\right)\sigma_{t-j}^2\right| \\
&\le \max_{t\in[T_1,T_2]}\sigma_t^2\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1}\left|c_{t,j}^2-\rho_{0,n}^{2j}\right| \\
&\le O\left(1\right)\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1}\left|c_{t,j}-\rho_{0,n}^{j}\right| \\
&\le O\left(1\right)O\left(b_n^{-1}\right)(nh)^{-1}\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1} j\bar{\rho}_n^{j-1}L_1 h/b_n = O\left(h\right) = o\left(1\right),
\end{aligned} \tag{B.9.7}$$

where the second inequality holds by (B.7.6), the last inequality uses (7.14), and the second last equality holds by (7.17).

For A_{b3}, we have

$$\begin{aligned}
|A_{b3}| &= \left(1-\rho_{0,n}^2\right)(nh)^{-1}\left|\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1}\rho_{0,n}^{2j}\left(\sigma_{t-j}^2-\sigma_0^2\left(\tau\right)\right)\right| \\
&\le \max_{t\in[T_1,T_2]}\left|\sigma_t^2-\sigma_0^2\left(\tau\right)\right|\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2-1}\sum_{j=0}^{t-T_1}\rho_{0,n}^{2j} \\
&= \ o\left(1\right)O\left(b_n^{-1}\right)(nh)^{-1}O\left(nhb_n\right) = o\left(1\right),
\end{aligned} \tag{B.9.8}$$

where the second last equality uses (B.7.6) and (7.16).

Combining (B.9.6), (B.9.7), and (B.9.8), we obtain

$$E\left[\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2\right] = 1+o\left(1\right). \tag{B.9.9}$$

Next, we show $E\left[\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2\right]^2 = 1+o(1)$ by observing

$$\begin{aligned}
&E\left[\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2\right]^2 \\
&= \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} E\left[\sum_{t_1,t_2=T_1}^{T_2}\left(Y_{t_1-1}^0\right)^2\left(Y_{t_2-1}^0\right)^2\right] \\
&= \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} E\left[\sum_{t_1,t_2=T_1-1}^{T_2-1}\left(Y_{t_1}^0\right)^2\left(Y_{t_2}^0\right)^2\right] \\
&= \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} E\left[\begin{array}{c}\left(\sum_{i=0}^{t_1-T_1} c_{t_1,i}\sigma_{t_1-i}U_{t_1-i}\right)^2 \\ \times\left(\sum_{j=0}^{t_2-T_1} c_{t_2,j}\sigma_{t_2-j}U_{t_2-j}\right)^2\end{array}\right] \\
&= \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1}\sum_{i_1,i_2=0}^{t_1-T_1}\sum_{j_1,j_2=0}^{t_2-T_1}\left[\begin{array}{c} c_{t_1,i_1}c_{t_1,i_2}c_{t_2,j_1}c_{t_2,j_2}\sigma_{t_1-i_1}\sigma_{t_1-i_2}\sigma_{t_2-j_1}\sigma_{t_2-j_2} \\ \times E\left(U_{t_1-i_1}U_{t_1-i_2}U_{t_2-j_1}U_{t_2-j_2}\right)\end{array}\right],
\end{aligned} \tag{B.9.10}$$

where the third equality holds by $Y_{T_0}^0 = 0$ and (7.5). All expectations in the last line are zero unless (i) all the indices on the four innovation terms coincide or (ii) there are two groups of two indices that each coincide or (iii) three larger indices coincide.

In case (i), we must have $i_1 = i_2 = i$, $j_1 = j_2 = j$ and $t_1 - i_1 = t_2 - j_1$, which implies

$$\begin{aligned}
&c_{t_1,i_1}c_{t_1,i_2}c_{t_2,j_1}c_{t_2,j_2}EU_{t_1-i_1}U_{t_1-i_2}U_{t_2-j_1}U_{t_2-j_2}\sigma_{t_1-i_1}\sigma_{t_1-i_2}\sigma_{t_2-j_1}\sigma_{t_2-j_2} \\
&= c_{t_1,i}^2 c_{t_2,j}^2 EU_{t-i}^4 \mathbb{1}\{t_1 - i = t_2 - j\}\sigma_{t_1-i}^4.
\end{aligned} \tag{B.9.11}$$

Substituting (B.9.11) into the right-hand side of (B.9.10), we have

$$\begin{aligned}
&\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2}\sum_{t_1,t_2=T_1}^{T_2-1}\sum_{i=0}^{t_1-T_1}\sum_{j=0}^{t_2-T_1} c_{t_1,i}^2 c_{t_2,j}^2 EU_{t-i}^4\mathbb{1}\{t_1-i=t_2-j\}\sigma_{t_1-i}^4 \\
&\leq \left(1-\rho_{0,n}^2\right)^2 \max_{t\in[T_1,T_2]}\sigma_t^4 M (nh)^{-2}\sum_{t_1,t_2=T_1}^{T_2-1}\sum_{i=0}^{t_1-T_1} c_{t_1,i}^2\sum_{j=0}^{t_2-T_1} c_{t_2,j}^2\mathbb{1}\{t_1-i=t_2-j\} \\
&\leq O(1)\left(1-\rho_{0,n}^2\right)^2\left\{2(nh)^{-2}\sum_{t_1>t_2=T_1}^{T_2-1}\sum_{i=t_1-t_2}^{t_1-T_1}\overline{\rho}_n^{4i-2(t_1-t_2)} + (nh)^{-2}\sum_{t=T_1}^{T_2-1}\sum_{i=0}^{t-T_1}\overline{\rho}_n^{4i}\right\} \\
&= O(1)\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2}\sum_{t_1>t_2=T_1}^{T_2-1}\overline{\rho}_n^{2(t_1-t_2)}\sum_{k=0}^{t_2-T_1}\overline{\rho}_n^{4k} + O(1)\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2}\sum_{t=T_1}^{T_2-1}\sum_{i=0}^{t-T_1}\overline{\rho}_n^{4i} \\
&\leq O(1)\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2}\sum_{t_1=T_1+1}^{T_2-1}\sum_{l=1}^{t_1-T_1}\overline{\rho}_n^{2l}\sum_{k=0}^{\infty}\overline{\rho}_n^{4k} + O(1)\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2}\sum_{t=T_1}^{T_2-1}\sum_{i=0}^{\infty}\overline{\rho}_n^{4i}
\end{aligned}$$

$$= O\left((nh)^{-1}\right) = o(1), \tag{B.9.12}$$

where the first inequality holds by $EU_t^4 < M$, the second inequality uses (B.7.7) and (7.13), the first equality uses the change of coordinates $k = i-(t_1 - t_2)$, and the second last equality holds by $\sum_{k=0}^{\infty} \overline{\rho}_n^{4k} = (1-\overline{\rho}_n^4)^{-1} = O(b_n)$ and (7.16).

In case (ii), we must have (ii1) ($i_1 = i_2 = i$ and $j_1 = j_2 = j$ and $t_1 - i \neq t_2 - j$) or (ii2) ($t_1 - i_1 = t_2 - j_1$ and $t_1 - i_2 = t_2 - j_2$ and $i_1 \neq i_2$) or (ii3) ($t_1 - i_1 = t_2 - j_2$ and $t_1 - i_2 = t_2 - j_1$ and $i_1 \neq i_2$).

In case (ii1), we have

$$\begin{aligned} & c_{t_1,i_1} c_{t_1,i_2} c_{t_2,j_1} c_{t_2,j_2} E U_{t_1-i_1} U_{t_1-i_2} U_{t_2-j_1} U_{t_2-j_2} \sigma_{t_1-i_1} \sigma_{t_1-i_2} \sigma_{t_2-j_1} \sigma_{t_2-j_2} \\ &= c_{t_1,i}^2 c_{t_2,j}^2 E U_{t_1-i}^2 U_{t_2-j}^2 \mathbb{1}\{t_1 - i \neq t_2 - j\} \sigma_{t_1-i}^2 \sigma_{t_2-j}^2. \end{aligned} \tag{B.9.13}$$

Substituting (B.9.13) into the right-hand side of (B.9.10), we have

$$\begin{aligned} & \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} c_{t_1,i}^2 c_{t_2,j}^2 \sigma_{t_1-i}^2 \sigma_{t_2-j}^2 E\left(U_{t_1-i}^2 U_{t_2-j}^2\right) \mathbb{1}\{t_1 - i \neq t_2 - j\} \\ =& \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} c_{t_1,i}^2 c_{t_2,j}^2 \sigma_{t_1-i}^2 \sigma_{t_2-j}^2 \mathbb{1}\{t_1 - i \neq t_2 - j\} \\ =& \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} c_{t_1,i}^2 c_{t_2,j}^2 \sigma_{t_1-i}^2 \sigma_{t_2-j}^2 \\ & - \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} c_{t_1,i}^2 c_{t_2,j}^2 \sigma_{t_1-i}^2 \sigma_{t_2-j}^2 \mathbb{1}\{t_1 - i = t_2 - j\} \\ =& \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} \rho_{0,n}^{2(i+j)} \\ & + \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} \left(c_{t_1,i}^2 c_{t_2,j}^2 - \rho_{0,n}^{2(i+j)}\right) \\ & + \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} c_{t_1,i}^2 c_{t_2,j}^2 \left(\sigma_{t_1-i}^2 \sigma_{t_2-j}^2 - \sigma_0^4(\tau)\right) - o(1) \\ =:& A_{b4} + A_{b5} + A_{b6} + o(1), \end{aligned} \tag{B.9.14}$$

where the first equality holds by $E\left[U_t^2 | \mathscr{G}_{t-1}\right] = 1$ a.s. and the third equality holds by (B.9.12).

Because $A_{b4} = A_{b1}^2$, we obtain $A_{b4} = 1 + o(1)$ from (B.9.6). Thus, for case (ii1), we only need to show $A_{b5} = o(1)$ and $A_{b6} = o(1)$.

For A_{b5}, we have

$$
\begin{aligned}
|A_{b5}| &\leq \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} \left[c_{t_2,j}^2 \left|c_{t_1,i}^2-\rho_{0,n}^{2i}\right| + \rho_{0,n}^{2i} \left|c_{t_2,j}^2-\rho_{0,n}^{2j}\right|\right] \\
&\leq \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} \left(\overline{\rho}_n^{2j} i \overline{\rho}_n^{i-1} + \overline{\rho}_n^{2i} j \overline{\rho}_n^{j-1}\right) 2L_1 h/b_n \\
&= \left(1-\rho_{0,n}^2\right)^2 (4L_1 h/b_n)(nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} i\overline{\rho}_n^{i-1} \sum_{j=0}^{t_2-T_1} \overline{\rho}_n^{2j} \\
&= O\left(b_n^{-2}(h/b_n)(nh)^{-2}(nh)^2 b_n^2 b_n\right) = O(h) = o(1), \qquad \text{(B.9.15)}
\end{aligned}
$$

where the first inequality uses the triangle inequality, the second inequality holds by (7.13) and (7.14), the first equality uses the fact that t_1 and t_2 are symmetric, and the second equality holds by (7.16) and (7.17).

For A_{b6}, we have

$$
\begin{aligned}
|A_{b6}| &\leq \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} c_{t_1,i}^2 c_{t_2,j}^2 \left|\sigma_{t_1-i}^2 \sigma_{t_2-j}^2 - \sigma_0^4(\tau)\right| \\
&\leq O\left(b_n^{-2}\right) \max_{t_1,t_2\in[T_1,T_2]} \left|\sigma_{t_1}^2\sigma_{t_2}^2 - \sigma_0^4(\tau)\right| (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} \overline{\rho}_n^{2(i+j)} = O(h), \qquad \text{(B.9.16)}
\end{aligned}
$$

where the second inequality uses (7.13) and the equality holds by

$$
\sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i=0}^{t_1-T_1} \sum_{j=0}^{t_2-T_1} \overline{\rho}_n^{2(i+j)} \leq (nh)^2 \left(1-\overline{\rho}_n^2\right)^{-2} = O\left((nhb_n)^2\right) \qquad \text{(B.9.17)}
$$

and

$$
\begin{aligned}
&\max_{t,s\in[T_1,T_2]} \left|\sigma_t^2\sigma_s^2 - \sigma_0^4(\tau)\right| \\
&\leq \max_{s\in[T_1,T_2]} \left|\sigma_s^2-\sigma_0^2(\tau)\right| \max_{t\in[T_1,T_2]} \sigma_t^2 + \max_{t\in[T_1,T_2]} \left|\sigma_t^2 - \sigma_0^2(\tau)\right| = O(h). \qquad \text{(B.9.18)}
\end{aligned}
$$

This completes case (ii1).

Since case (ii2) and (ii3) are symmetric, we only prove the result for case (ii2) and show it is $o(1)$. Observe that when $t_1-i_1=t_2-j_1$, $t_1-i_2=t_2-j_2$, and $i_1\neq i_2$,

$$
\begin{aligned}
&c_{t_1,i_1}c_{t_1,i_2}c_{t_2,j_1}c_{t_2,j_2} E U_{t_1-i_1}U_{t_1-i_2}U_{t_2-j_1}U_{t_2-j_2}\sigma_{t_1-i_1}\sigma_{t_1-i_2}\sigma_{t_2-j_1}\sigma_{t_2-j_2} \\
&= c_{t_1,i_1}c_{t_1,i_2}c_{t_2,i_1-(t_1-t_2)}c_{t_2,i_2-(t_1-t_2)} E U_{t_1-i_1}^2 U_{t_1-i_2}^2 \mathbb{1}\{i_1\neq i_2\}\sigma_{t_1-i_1}^2\sigma_{t_1-i_2}^2. \qquad \text{(B.9.19)}
\end{aligned}
$$

Substituting (B.9.19) into the right-hand side of (B.9.10), we have

$$
\begin{aligned}
&\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i_1,i_2=0}^{t_1-T_1} \sum_{j_1,j_2=0}^{t_2-T_1} \begin{bmatrix} c_{t_1,i_1}c_{t_1,i_2}c_{t_2,i_1-(t_1-t_2)}c_{t_2,i_2-(t_1-t_2)}\mathbb{1}\left\{i_1\neq i_2\right\} \\ \times EU_{t_1-i_1}^2 U_{t_1-i_2}^2 \sigma_{t_1-i_1}^2 \sigma_{t_1-i_2}^2 \end{bmatrix} \\
&\leq 2\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \sum_{i_1,i_2=t_1-t_2}^{t_1-T_1} c_{t_1,i_1}c_{t_1,i_2}c_{t_2,t_2-(t_1-i_1)}c_{t_2,t_2-(t_1-i_2)}\sigma_{t_1-i_1}^2\sigma_{t_1-i_2}^2 \\
&\quad + \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t=T_1}^{T_2-1} \sum_{i_1,i_2=0}^{t-T_1} c_{t,i_1}^2 c_{t,i_2}^2 \sigma_{t-i_1}^2 \sigma_{t-i_2}^2 \\
&\leq \left(\max_{t\in[T_1,T_2]}\sigma_t\right)^4 2\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \sum_{i_1,i_2=t_1-t_2}^{t_1-T_1} \overline{\rho}_n^{i_1+i_2+t_2-(t_1-i_1)+t_2-(t_1-i_2)} \\
&\quad + \left(\max_{t\in[T_1,T_2]}\sigma_t\right)^4 \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t=T_1}^{T_2-1} \sum_{i_1,i_2=0}^{\infty} \overline{\rho}_n^{2(i_1+i_2)} \\
&= O(1)\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \overline{\rho}_n^{2(t_2-t_1)} \sum_{i_1,i_2=t_1-t_2}^{t_1-T_1} \overline{\rho}_n^{2(i_1+i_2)} + O\left((nh)^{-1}\right) \\
&= O(1)\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \overline{\rho}_n^{2(t_1-t_2)} \sum_{l_1,l_2=0}^{t_2-T_1} \overline{\rho}_n^{2(l_1+l_2)} + O\left((nh)^{-1}\right) \\
&\leq O(1)\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \overline{\rho}_n^{2(t_1-t_2)} \left(1-\overline{\rho}_n^2\right)^{-2} + O\left((nh)^{-1}\right) \\
&= O(1)\,O\left(b_n^{-2}\right)(nh)^{-2}\,O(nhb_n)\,O\left(b_n^2\right) + O\left((nh)^{-1}\right) = O\left((nh/b_n)^{-1}\right) = o(1),
\end{aligned} \tag{B.9.20}
$$

where the second equality uses the changes of coordinates $l_1 = i_1 - (t_1 - t_2)$ and $l_2 = i_2 - (t_1 - t_2)$, the last inequality holds by (7.16), the second last equality holds by (7.19), and the last equality uses $b_n/nh = o(1)$. This completes case (ii2).

In case (iii) we must have (iii1) $(t_1 - i_1 = t_1 - i_2 = t_2 - j_1 > t_2 - j_2)$ or (iii2) $(t_1 - i_1 = t_1 - i_2 = t_2 - j_2 > t_2 - j_1)$ or (iii3) $(t_1 - i_1 = t_2 - j_2 = t_2 - j_1 > t_1 - i_2)$ or (iii4) $(t_1 - i_2 = t_2 - j_2 = t_2 - j_1 > t_1 - i_1)$.

Now, we prove the desired result for case (iii1). Note that in this case, it must be true that $i_1 = i_2 = i$ and $j_2 > j_1 = i - (t_1 - t_2)$, which implies

$$
\begin{aligned}
&c_{t_1,i_1}c_{t_1,i_2}c_{t_2,j_1}c_{t_2,j_2}EU_{t_1-i_1}U_{t_1-i_2}U_{t_2-j_1}U_{t_2-j_2}\sigma_{t_1-i_1}\sigma_{t_1-i_2}\sigma_{t_2-j_1}\sigma_{t_2-j_2} \\
&= c_{t_1,i}^2 c_{t_2,i-(t_1-t_2)}c_{t_2,j}EU_{t_1-i}^3 U_{t_2-j}\mathbb{1}\left\{j > i-(t_1-t_2)\right\}\sigma_{t_1-i}^3\sigma_{t_2-j}.
\end{aligned} \tag{B.9.21}
$$

Substituting (B.9.21) into the right-hand side of (B.9.10), we have

$$\left|\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1,t_2=T_1}^{T_2-1} \sum_{i_1,i_2=0}^{t_1-T_1} \sum_{j_1,j_2=0}^{t_2-T_1} \left[\begin{array}{c} c_{t_1,i}^2 c_{t_2,i-(t_1-t_2)} c_{t_2,j} EU_{t_1-i}^3 U_{t_2-j} \\ \times \mathbb{1}\left\{j > i-(t_1-t_2)\right\} \sigma_{t_1-i}^3 \sigma_{t_2-j} \end{array} \right]\right|$$

$$\leq \left(\max_{t\in[T_1,T_2]} \sigma_t\right)^4 2\left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \sum_{i=t_1-t_2}^{t_1-T_1} \sum_{j=i-(t_1-t_2)+1}^{t_2-T_1} \overline{\rho}_n^{3i+j-(t_1-t_2)} \left|EU_{t_1-i}^3 U_{t_2-j}\right|$$

$$+\left(\max_{t\in[T_1,T_2]} \sigma_t\right)^4 \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t=T_1}^{T_2-1} \sum_{i=0}^{t-T_1} \sum_{j=i+1}^{t-T_1} \overline{\rho}_n^{3i+j} \left|EU_{t-i}^3 U_{t-j}\right|$$

$$\leq O\left(b_n^{-2}\right)\left[(nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \sum_{i=t_1-t_2}^{t_1-T_1} \sum_{j=i-(t_1-t_2)+1}^{t_2-T_1} \overline{\rho}_n^{3i+j-(t_1-t_2)} + (nh)^{-2} \sum_{t=T_1}^{T_2-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \overline{\rho}_n^{3i+j}\right]$$

$$= O\left(b_n^{-2}\right)\left[(nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \sum_{i=t_1-t_2}^{t_1-T_1} \overline{\rho}_n^{4i-2(t_1-t_2)} \sum_{l=1}^{t_1-T_1-i} \overline{\rho}_n^{l} + O\left(b_n^2/nh\right)\right]$$

$$= O\left(b_n^{-2}\right)\left[(nh)^{-2} \sum_{t_1>t_2=T_1}^{T_2-1} \overline{\rho}_n^{2(t_1-t_2)} O\left(b_n^2\right) + O\left(b_n^2/nh\right)\right] = O\left(b_n/nh\right) = o\left(1\right), \qquad \text{(B.9.22)}$$

where the second inequality holds by HÃPlder's inequality and part (iv) of Λ_n: $|EU_t^3 U_{t-j}| \leq E\left|U_t^3 U_{t-j}\right| \leq \left(EU_t^4\right)^{3/4}\left(EU_{t-j}^4\right)^{1/4} < M$ for $j > 0$, and (B.7.7), the first equality holds by the change of coordinates $l = j - i + (t_1 - t_2) - 1$ and (7.15), the second equality uses the change of variables $k = i - (t_1 - t_2)$, (7.15), and (7.16), and the second last equality holds by (7.19).

The proofs for cases (iii2)-(iii4) are analogous to case (iii1) and thus are omitted.

Combining cases (i)–(iii), we have

$$E\left[\left(1-\rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1}^0\right)^2\right]^2 = 1 + o\left(1\right). \qquad \text{(B.9.23)}$$

Note that by Markov's inequality, for any random variable X_n

$$P\left(\left|X_n - 1\right| > \varepsilon\right) \leq \frac{E\left(X_n - 1\right)^2}{\varepsilon^2} = \frac{EX_n^2 - 2EX_n + 1}{\varepsilon^2}. \qquad \text{(B.9.24)}$$

Let $X_n := \left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2$. Then, substituting (B.9.9) and (B.9.23) into (B.9.24), we have

$$\left(1-\rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1}^0\right)^2 \to_p 1. \qquad \text{(B.9.25)}$$

□

Proof of Lemma 7.6(c). To prove part (c), by a central limit theorem for a triangular array of martingale differences as in Corollary 3.1 in Hall and Heyde (1980), it is sufficient to establish the Lindeberg condition (i) $\sum_{t=T_1}^{T_2} E\left[\zeta_t^2 \mathbb{1}\{|\zeta_t| > \delta\} | \mathscr{G}_{t-1}\right] \to_p 0$ for any $\delta > 0$ and (ii) $\sum_{t=T_1}^{T_2} E\left(\zeta_t^2 | \mathscr{G}_{t-1}\right) \to_p 1$, where $\zeta_t := (nh)^{-1/2}\left(1-\rho_{0,n}^2\right)^{1/2} Y_{t-1}^0 \sigma_t U_t$ for $t = T_1, ..., T_2$. To prove (i), by Markov's inequality, it is enough to show that $\sum_{t=T_1}^{T_2} E\left[\zeta_t^2 \mathbb{1}\{|\zeta_t| > \delta\}\right] \to 0$ for any $\delta > 0$. By $\sum_{t=T_1}^{T_2} E\left[\zeta_t^2 \mathbb{1}\{|\zeta_t| > \delta\}\right] \leq \sum_{t=T_1}^{T_2} E\left[\zeta_t^4\right]/\delta^2$, it is then sufficient to show $\sum_{t=T_1}^{T_2} E\left[\zeta_t^4\right] = o(1)$, which is true because

$$
\begin{aligned}
\sum_{t=T_1}^{T_2} E\left[\zeta_t^4\right] &= \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t=T_1}^{T_2} E\left(Y_{t-1}^0 U_t \sigma_t\right)^4 \\
&= \left(1-\rho_{0,n}^2\right)^2 (nh)^{-2} \sum_{t=T_1}^{T_2} E\left[\left(Y_{t-1}^0\right)^4 E\left(U_t^4 \middle| \mathscr{G}_{t-1}\right)\right]\sigma_t^4 \\
&\leq O\left(b_n^{-2}\right) M \max_{t\in[T_1,T_2]} \sigma_t^4 (nh)^{-2} \sum_{t=T_1}^{T_2-1} E\left(Y_t^0\right)^4 \\
&= O\left(b_n^{-2}\right)(nh)^{-2} \sum_{t=T_1}^{T_2-1} E\left[\sum_{j=0}^{t-T_1} c_{t,j}\sigma_{t-j}U_{t-j}\right]^4 \\
&\leq O\left(b_n^{-2}\right)\left(\max_{t\in[T_1,T_2]} \sigma_t\right)^4 (nh)^{-2} \sum_{t=T_1}^{T_2-1} \sum_{j_1,j_2,j_3,j_4=0}^{t-T_1} \left| \begin{array}{c} c_{t,j_1}c_{t,j_2}c_{t,j_3}c_{t,j_4} \\ \times E\left(U_{t-j_1}U_{t-j_2}U_{t-j_3}U_{t-j_4}\right) \end{array} \right| \\
&\leq O\left(b_n^{-2}\right)(nh)^{-2} \sum_{t=T_1}^{T_2-1}\left[\sum_{j=0}^{t-T_1} \overline{\rho}_n^{4j} M + 3\sum_{i,j=0}^{t-T_1} \overline{\rho}_n^{2(i+j)} \mathbb{1}\{i\neq j\} + 4\sum_{i=0}^{t-T_1}\sum_{j=i+1}^{t-T_1} \overline{\rho}_n^{3i+j} M^{1/2}\right] \\
&= O\left(b_n^{-2}\right)(nh)^{-2}\, nh\left(O(b_n) + O\left(b_n^2\right) + O\left(b_n^2\right)\right) = o(1), \qquad \text{(B.9.26)}
\end{aligned}
$$

where the second equality uses the law of iterated expectations, the first inequality uses $E\left[U_t^4 | \mathscr{G}_{t-1}\right] < M$ a.s. and (7.16), the third equality uses (B.7.7), and the last inequality holds by dividing the sum into three cases of (i) all four indices on the four innovation terms coincide, (ii) two pairs of two indices coincide, and (iii) three larger indices coincide and (7.13).

To prove (ii), by part (b) we have $\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2 = 1 + o_p(1)$. Thus,

$$
\begin{aligned}
\sum_{t=T_1}^{T_2} E\left(\zeta_t^2 \middle| \mathscr{G}_{t-1}\right) &= \left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2 E\left[U_t^2 \middle| \mathscr{G}_{t-1}\right]\sigma_t^2 \\
&= \left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2 \\
&\quad + \left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2\left(\sigma_t^2 - \sigma_0^2(\tau)\right)
\end{aligned}
$$

$$=1+o_p(1)+o_p(1)\to_p 1, \tag{B.9.27}$$

where the second equality holds by $E\left[U_t^2\middle|\mathscr{G}_{t-1}\right]=1$ a.s. and the last equality holds by part (b) and

$$\begin{aligned}
&\left|\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2\left(\sigma_t^2-\sigma_0^2(\tau)\right)\right| \\
&\leq \max_{t\in[T_1,T_2]}\left|\sigma_t^2-\sigma_0^2(\tau)\right|\left|\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2\right| \\
&= \ o(1)\,O_p(1)=o_p(1).
\end{aligned} \tag{B.9.28}$$

□

B.10 Proof of Lemma 7.7

In this section, for notational simplicity in the proof, we assume that $\sigma_0^2(\tau)=1$.

Proof of Lemma 7.7(a). First, we prove part (a). We have

$$\begin{aligned}
&\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)^2 \\
=&\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left[Y_{t-1}^0+\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)+c_{t-1,t-1-T_0}Y_{T_0}^*\right]^2 \\
=&\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2 \\
&+\left(1-\rho_{0,n}^2\right) \\
&\times(nh)^{-1}\sum_{t=T_1}^{T_2}\left[\begin{array}{c}\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)^2+c_{t-1,t-1-T_0}^2Y_{T_0}^{*2}+2\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)Y_{t-1}^0 \\ +2\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)c_{t-1,t-1-T_0}Y_{T_0}^*+2Y_{t-1}^0c_{t-1,t-1-T_0}Y_{T_0}^*\end{array}\right] \\
=:&\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2+\sum_{i=1}^{5}A_{ai}.
\end{aligned} \tag{B.10.1}$$

By Lemma 7.6(b), $\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2\to_p 1$. Hence, we need to show $\sum_{i=1}^5 A_{ai}$ in (B.10.1) converges in probability to 0. By the CS inequality, we only need to show $A_{a1}\to 0$ and $A_{a2}\to_p 0$.

For A_{a1}, we have

$$A_{a1} = \left(1 - \rho_{0,n}^2\right)(nh)^{-1} \sum_{t=T_1}^{T_2} \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right)^2 \leq O\left(b_n^{-1}\right) \max_{t,s \in [T_0, T_2]} \left(\mu_t - \mu_s\right)^2 = o(1), \tag{B.10.2}$$

where the last equality holds by Lemma 7.1(d) and the triangle inequality.

For A_{a2}, we have

$$\begin{aligned} A_{a2} &= (1 - \rho_{0,n}^2)(nh)^{-1} \sum_{t=T_1}^{T_2} c_{t-1,t-1-T_0}^2 Y_{T_0}^{*2} = O(b_n^{-1})(nh)^{-1} \sum_{t=T_1}^{T_2} \overline{\rho}_n^{2(t-1-T_0)} O_p(b_n) \\ &= O(b_n^{-1})(nh)^{-1} O(b_n) O_p(b_n) = O_p(b_n/nh) = o_p(1), \end{aligned} \tag{B.10.3}$$

where the second equality uses $1 - \rho_{0,n} = -\kappa_n(\tau)/b_n$, $\kappa_n(\tau) = O(1)$ (by part (ii) of Λ_n), (7.13), and Lemma 7.2(c), which applies because the lemma assumes that $nh/b_n \to r_0 = \infty$, the third equality uses (7.16), and the last equality holds because $nh/b_n \to r_0 = \infty$. This completes the proof of part (a). $\square$

Proof of Lemma 7.7(b). To prove part (b), it suffices to show that

$$A_b := (1 - \rho_{0,n}^2)^{1/2} (\overline{Y}_{nh,-1} - \overline{\mu}_{nh,-1}) = o_p(1). \tag{B.10.4}$$

We have

$$\begin{aligned} A_b &= A_{b1} + A_{b2}, \text{ where} \qquad\qquad (B.10.5) \\ A_{b1} &:= (1 - \rho_{0,n}^2)^{1/2} (nh)^{-1} \sum_{t=T_1}^{T_2} Y_{t-1}^0 \text{ and } A_{b2} := (1 - \rho_{0,n}^2)^{1/2} (nh)^{-1} \sum_{t=T_1}^{T_2} c_{t-1,t-1-T_0} Y_{T_0}^*. \end{aligned}$$

By Lemma 7.6(a), $A_{b1} = o_p(1)$. In addition, we have

$$|A_{b2}| \leq (1 - \rho_{0,n}^2)^{1/2} (nh)^{-1} \sum_{t=T_1}^{T_2} \overline{\rho}_n^{t-1-T_0} |Y_{T_0}^*| \leq O(b_n^{-1/2})(nh)^{-1} O(b_n) O_p(b_n^{1/2}) = o_p(1), \tag{B.10.6}$$

where the first inequality uses (7.13), the second inequality uses $\sum_{t=T_1}^{T_2} \overline{\rho}_n^{t-1-T_0} \leq (1 - \overline{\rho}_n)^{-1} = O(b_n)$ (by (7.15)) and Lemma 7.2(c), which applies because $nh/b_n \to r_0 = \infty$, which is an assumption of the lemma, and the equality holds because $b_n/nh \to 0$. Hence, $A_b = o_p(1)$ and part (b) is established. $\square$

B.11 Proof of Lemma 7.8

In this section, for notational simplicity in the proof, we assume that $\sigma_0^2(\tau) = 1$.

Proof of Lemma 7.8(a). To prove part (a), we express

$$Y_{t-1} - \overline{\mu}_{nh,-1} = Y_{t-1}^0 + \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) + c_{t-1,t-1-T_0} Y_{T_0}^* \tag{B.11.1}$$

and

$$\begin{aligned} & Y_t - \overline{\mu}_{nh} - \rho_{0,n}\left(Y_{t-1} - \overline{\mu}_{nh,-1}\right) \\ = {} & \sigma_t U_t + (\rho_t - \rho_{0,n}) Y_{t-1}^0 + (\mu_t - \overline{\mu}_{nh}) - \rho_{0,n}\left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) + (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^*. \end{aligned} \tag{B.11.2}$$

Substituting (B.11.1) and (B.11.2) into the left-hand side of part (a), we have

$$\begin{aligned} & \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{\mu}_{nh,-1}\right)\left[Y_t - \overline{\mu}_{nh} - \rho_{0,n}\left(Y_{t-1} - \overline{\mu}_{nh,-1}\right)\right] \\ = {} & \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} Y_{t-1}^0 \sigma_t U_t \\ & + \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \\ & \times \sum_{t=T_1}^{T_2} \left[\begin{array}{c} (\rho_t - \rho_{0,n})\left(Y_{t-1}^0\right)^2 + Y_{t-1}^0 (\mu_t - \overline{\mu}_{nh}) - Y_{t-1}^0 \rho_{0,n}\left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) \\ + Y_{t-1}^0 (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* + \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right)\sigma_t U_t \\ + \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right)(\rho_t - \rho_{0,n}) Y_{t-1}^0 + \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right)(\mu_t - \overline{\mu}_{nh}) \\ -\rho_{0,n}\left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right)^2 + \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right)(\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* \\ + c_{t-1,t-1-T_0} Y_{T_0}^* \sigma_t U_t + c_{t-1,t-1-T_0} Y_{T_0}^* (\rho_t - \rho_{0,n}) Y_{t-1}^0 \\ + c_{t-1,t-1-T_0} Y_{T_0}^* (\mu_t - \overline{\mu}_{nh}) - c_{t-1,t-1-T_0} Y_{T_0}^* \rho_{0,n}\left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) \\ + c_{t-1,t-1-T_0}^2 (\rho_t - \rho_{0,n}) Y_{T_0}^{*2} \end{array}\right] \\ =: {} & \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} Y_{t-1}^0 \sigma_t U_t + \sum_{i=1}^{14} A_{ci}, \end{aligned} \tag{B.11.3}$$

where A_{ci} is the i^{th} term in the second last line of (B.11.3). For example,

$$A_{c1} := \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} (\rho_t - \rho_{0,n})\left(Y_{t-1}^0\right)^2 \tag{B.11.4}$$

and

$$A_{c2} := \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} Y_{t-1}^0 \left(\mu_t - \overline{\mu}_{nh}\right). \tag{B.11.5}$$

By Lemma 7.6(c), we have

$$\left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} Y_{t-1}^0 \sigma_t U_t \to_d N(0,1). \tag{B.11.6}$$

Therefore, we need to show $\sum_{i=1}^{14} A_{ci}$ converges in probability to 0. We examine each of its components one by one.

First, we consider A_{c1}. Let Y_t^c be the constant parameter version of Y_t^0 based on $\rho_{0,n}$ and $\sigma_{n\tau}$ and with T_2-T_1 lagged innovations (which does not depend on t), rather than $t-T_1$ lags. That is,

$$Y_t^c := \sum_{j=0}^{T_2-T_1} \rho_{0,n}^j \sigma_{n\tau} U_{t-j} \text{ for } t \in [T_1, T_2]. \tag{B.11.7}$$

Here, the superscript c stands for "constant parameter."

We decompose A_{c1} into three terms:

$$\begin{aligned}
A_{c1} &= A_{c1,1} + A_{c1,2} + A_{c1,3}, \text{ where} \\
A_{c1,1} &:= (1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2} \sum_{t=T_1}^{T_2} (\rho_t - \rho_{0,n}) E(Y_{t-1}^c)^2, \\
A_{c1,2} &:= (1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2} \sum_{t=T_1}^{T_2} (\rho_t - \rho_{0,n}) \left((Y_{t-1}^0)^2 - (Y_{t-1}^c)^2\right), \text{ and} \\
A_{c1,3} &:= (1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2} \sum_{t=T_1}^{T_2} (\rho_t - \rho_{0,n}) \left((Y_{t-1}^c)^2 - E(Y_{t-1}^c)^2\right).
\end{aligned} \tag{B.11.8}$$

We show that $A_{c1,b} = o_p(1)$ for $b = 1, 2, 3$, provided Assumption 2 holds, which yields $A_{c1} = o_p(1)$.

Now, we consider $A_{c1,1}$. We have

$$E(Y_{t-1}^c)^2 = \sigma_{n\tau}^2 \sum_{j=0}^{T_2-T_1} \sum_{k=0}^{T_2-T_1} \rho_{0,n}^j \rho_{0,n}^k E U_{t-j-1} U_{t-k-1} = \sigma_{n\tau}^2 \sum_{j=0}^{T_2-T_1} \rho_{0,n}^{2j}, \tag{B.11.9}$$

which does not depend on t, and hence, can be taken out of the sum over t in the definition of $A_{c1,1}$.

By definition,

$$\rho_t - \rho_{0,n} := \rho_{0,n}(t/n) - \rho_{0,n}(\tau) = \kappa_{0,n}(\tau)/b_n - \kappa_{0,n}(t/n)/b_n \tag{B.11.10}$$

using part (ii) in the definition of Λ_n. Using the condition in Λ_n that $\kappa(\cdot)$ is a twice continuously differentiable function , by a two-term Taylor expansion of $\kappa_{0,n}(t/n)$ around τ, we obtain

$$\kappa_{0,n}(t/n) - \kappa_{0,n}(\tau) = \kappa'_{0,n}(\tau)(t/n-\tau) + \kappa''_{0,n}(\widetilde{\tau}_{n,t})(t/n-\tau)^2, \tag{B.11.11}$$

where $\widetilde{\tau}_{n,t}$ lies between t/n and τ, and hence, lies between T_1/n and T_2/n for $t \in [T_1, T_2]$.

Let $\alpha_n := \lfloor nh/2 \rfloor$. Using $T_1 := \lfloor n\tau \rfloor - \lfloor nh/2 \rfloor$ and $T_2 := \lfloor n\tau \rfloor + \lfloor nh/2 \rfloor$ by (3.1), we have

$$\sum_{t=T_1}^{T_2} (t-n\tau) = \sum_{t=\lfloor n\tau \rfloor+1}^{\lfloor n\tau \rfloor+\alpha_n} (t-n\tau) + \sum_{t=\lfloor n\tau \rfloor-\alpha_n}^{\lfloor n\tau \rfloor-1} (t-n\tau) + \lfloor n\tau \rfloor - n\tau. \tag{B.11.12}$$

In addition, we have

$$\begin{aligned} \sum_{t=\lfloor n\tau \rfloor+1}^{\lfloor n\tau \rfloor+\alpha_n} (t-n\tau) &= \sum_{s=1}^{\alpha_n} (s+\lfloor n\tau \rfloor - n\tau) \text{ and} \\ \sum_{t=\lfloor n\tau \rfloor-\alpha_n}^{\lfloor n\tau \rfloor-1} (t-n\tau) &= \sum_{s=1}^{\alpha_n} (-s+\lfloor n\tau \rfloor - n\tau), \end{aligned} \tag{B.11.13}$$

where the first line uses a change of variables with $s = t - \lfloor n\tau \rfloor$ and the second line uses a change of variables with $s = -t + \lfloor n\tau \rfloor$. Combining (B.11.12) and (B.11.13) gives

$$\sum_{t=T_1}^{T_2} (t/n-\tau) = 2n^{-1} \sum_{s=1}^{\alpha_n} (\lfloor n\tau \rfloor - n\tau) + n^{-1}(\lfloor n\tau \rfloor - n\tau) = O(n^{-1}\alpha_n) = O(h). \tag{B.11.14}$$

Using (B.11.8)–(B.11.11) and (B.11.14), we get

$$\begin{aligned} |A_{c1,1}| &= \Bigg| E(Y_{t-1}^c)^2 b_n^{-1}(1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2}\kappa'_{0,n}(\tau)\sum_{t=T_1}^{T_2}(t/n-\tau) \\ &\quad + E(Y_{t-1}^c)^2 b_n^{-1}(1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\kappa''_{0,n}(\widetilde{\tau}_{n,t})(t/n-\tau)^2 \Bigg| \\ &\le CE(Y_{t-1}^c)^2 b_n^{-1}(1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2}\left(O(h) + \sum_{t=T_1}^{T_2}(t/n-\tau)^2\right) \\ &= O(b_n^{-1}(1-\rho_{0,n}^2)^{-1/2}(nh)^{-1/2}\left(h + nh\cdot h^2\right)) \\ &= O(b_n^{-1}b_n^{1/2}\left((h/n)^{1/2} + (nh^5)^{1/2}\right)) = O((h/n)^{1/2} + (nh^5)^{1/2}) = o(1), \end{aligned} \tag{B.11.15}$$

for some finite constant C for which $\left|\kappa'_{0,n}(\tau)\right| \le C$ and $\sup_{t,n}\left|\kappa''_{0,n}(\widetilde{\tau}_{n,t})\right| \le C$, where the inequality uses these inequalities and (B.11.14), the second equality uses $E(Y_{t-1}^c)^2 = O((1-\rho_{0,n}^2)^{-1})$ by (B.11.9) and $|t/n-\tau| \le h$ because $|t-n\tau| \le nh/2$ for $t \in [T_1, T_2]$, and the third equality uses $(1-\rho_{0,n}^2)^{-1/2} = O(b_n^{1/2})$ (by (7.16) with $\rho_{0,n}$ in place of $\overline{\rho}_n$), the fourth equality holds because $b_n^{-1/2} = O(1)$, and the last equality holds by $h \to 0$ (by Assumption 1) and $nh^5 \to 0$ (by Assumption 2).

Next, we consider $A_{c1,2}$. We have

$$
\begin{aligned}
|A_{c1,2}| &\leq (1-\rho_{0,n}^2)^{1/2} \max_{t\in[T_1,T_2]} |\rho_t - \rho_{0,n}|(nh)^{-1/2} \sum_{t=T_1}^{T_2} |Y_{t-1}^0 + Y_{t-1}^c| \cdot |Y_{t-1}^0 - Y_{t-1}^c| \\
&= O(b_n^{-1/2})O(h/b_n)(nh)^{-1/2} \sum_{t=T_1}^{T_2} |Y_{t-1}^0 + Y_{t-1}^c| \cdot |Y_{t-1}^0 - Y_{t-1}^c|, \qquad \text{(B.11.16)}
\end{aligned}
$$

where the equality uses Lemma 7.1(a).

We have

$$
Y_{t-1}^0 - Y_{t-1}^c = \sum_{j=0}^{t-1-T_1} (c_{t-1,j}\sigma_{t-j-1} - \rho_{0,n}^j \sigma_{n\tau})U_{t-j-1} - \sum_{j=t-T_1}^{T_2-T_1} \rho_{0,n}^j \sigma_{n\tau} U_{t-j-1}. \qquad \text{(B.11.17)}
$$

Combining (B.11.16) and (B.11.17) gives

$$
\begin{aligned}
E|A_{c1,2}| \leq\ & (b_n^{-1/2})O(h/b_n)(nh)^{-1/2} \sum_{t=T_1}^{T_2} 2\max\{(E(Y_{t-1}^0)^2)^{1/2}, (E(Y_{t-1}^c)^2)^{1/2}\} \\
& \times 2\max\left\{ \left(E\left(\sum_{j=0}^{t-1-T_1} (c_{t-1,j}\sigma_{t-j-1} - \rho_{0,n}^j \sigma_{n\tau})U_{t-j-1} \right)^2 \right)^{1/2}, \right. \\
& \left. \left(E\left(\sum_{j=t-T_1}^{T_2-T_1} \rho_{0,n}^j \sigma_{n\tau} U_{t-j-1} \right)^2 \right)^{1/2} \right\} \qquad \text{(B.11.18)}
\end{aligned}
$$

using the triangle and CS inequalities.

We have $(E(Y_{t-1}^c)^2)^{1/2} = O(b_n^{1/2})$ uniformly over $t \in [T_1, T_2]$ by the discussion following (B.11.15). In addition,

$$
\begin{aligned}
E(Y_{t-1}^0)^2 &= E\left(\sum_{j=0}^{t-1-T_1} c_{t-1,j}\sigma_{t-j-1}U_{t-j-1} \right)^2 = \sum_{j=0}^{t-1-T_1} c_{t-1,j}^2 \sigma_{t-1-j}^2 \\
&\leq C_{3,U} \sum_{j=0}^{\infty} \overline{\rho}_n^2 = C_{3,U}(1-\overline{\rho}_n^2)^{-1} = O(b_n) \qquad \text{(B.11.19)}
\end{aligned}
$$

uniformly over $t \in [T_1, T_2]$, where the inequality uses (7.13) and the bound $C_{3,U}$ on $\sigma^2(\cdot)$ in part (i) of Λ_n and the last equality uses (7.16). So, $(E(Y_{t-1}^0)^2)^{1/2} = O(b_n^{1/2})$.

Next, we have

$$|c_{t-1,j}\sigma_{t-j-1}-\rho_{0,n}^{j}\sigma_{n\tau}| \leq \sigma_{t-j-1}|c_{t-1,j}-\rho_{0,n}^{j}|+|\rho_{0,n}|^{j}|\sigma_{t-j-1}-\sigma_{n\tau}| \leq j\overline{\rho}_{n}^{j-1}O(h/b_{n})+\overline{\rho}_{n}^{j}O(h), \tag{B.11.20}$$

where the second inequality uses (7.14), a uniform bound on σ_{t-j-1} across t, j, and $\max_{t\in[T_1,T_2]}|\sigma_{t-j-1}-\sigma_{n\tau}| = O(h)$ by Lemma 7.1(b) and $\sigma_{t-j-1}-\sigma_{n\tau} = (\sigma_{t-j-1}^{2}-\sigma_{n\tau}^{2})/(\sigma_{t-j-1}+\sigma_{n\tau})$. In consequence, we obtain

$$\begin{aligned} E\left(\sum_{j=0}^{t-1-T_1}(c_{t-1,j}\sigma_{t-j-1}-\rho_{0,n}^{j}\sigma_{n\tau})U_{t-j-1}\right)^{2} &= \sum_{j=0}^{t-1-T_1}(c_{t-1,j}\sigma_{t-j-1}-\rho_{0,n}^{j}\sigma_{n\tau})^{2}EU_{t-j-1}^{2} \\ = O\left(\sum_{j=0}^{\infty}j^{2}\overline{\rho}_{n}^{2j-2}h^{2}/b_{n}^{2}\right)+O\left(\sum_{j=0}^{\infty}\overline{\rho}_{n}^{2j}h^{2}\right) &= O(b_{n}^{3}h^{2}/b_{n}^{2})+O(b_{n}h^{2}) = O(b_{n}h^{2}), \end{aligned} \tag{B.11.21}$$

where the second equality uses (B.11.20) and $EU_t^2 = 1$ for all t, and the third equality uses (7.16) and (7.18).

In addition,

$$\begin{aligned} E\left(\sum_{j=t-T_1}^{T_2-T_1}\rho_{0,n}^{j}\sigma_{n\tau}U_{t-j-1}\right)^{2} &= \sum_{j=t-T_1}^{T_2-T_1}\rho_{0,n}^{2j}\sigma_{n\tau}^{2}EU_{t-j-1}^{2} \leq \sigma_{n\tau}^{2}\rho_{0,n}^{2(t-T_1)}\sum_{j=0}^{\infty}\rho_{0,n}^{2j} \\ &= \sigma_{n\tau}^{2}\rho_{0,n}^{2(t-T_1)}(1-\rho_{0,n}^{2})^{-1} = \rho_{0,n}^{2(t-T_1)}O(b_n). \end{aligned} \tag{B.11.22}$$

And so,

$$\begin{aligned} \sum_{t=T_1}^{T_2}\left(E\left(\sum_{j=t-T_1}^{T_2-T_1}\rho_{0,n}^{j}\sigma_{n\tau}U_{t-j-1}\right)^{2}\right)^{1/2} &\leq \sum_{t=T_1}^{T_2}|\rho_{0,n}|^{t-T_1}O(b_n^{1/2}) \\ \leq O(b_n^{1/2})\sum_{t=0}^{\infty}|\rho_{0,n}|^{t} = O(b_n^{1/2})(1-\overline{\rho}_n)^{-1} &= O(b_n^{1/2})O(b_n) = O(b_n^{3/2}). \end{aligned} \tag{B.11.23}$$

Combining (B.11.18), (B.11.19), (B.11.21), and (B.11.23), gives

$$\begin{aligned} E|A_{c1,2}| \leq\ & O(b_n^{-1/2})O(h/b_n)(nh)^{-1/2}nhO(b_n^{1/2})O(b_n^{1/2}h) \\ & +O(b_n^{-1/2})O(h/b_n)(nh)^{-1/2}O(b_n^{1/2})O(b_n^{3/2}) \\ =\ & O(n^{1/2}h^{5/2}b_n^{-1/2})+O(n^{-1/2}h^{1/2}b_n^{1/2}) = O((nh^5)^{1/2})+O((b_n/nh)^{1/2}h) = o(1), \end{aligned} \tag{B.11.24}$$

where the last equality uses $nh^5 \to 0$ by Assumption 2, $h \to 0$ by Assumption 1, and $b_n/nh \to 0$ in the "stationary" case. By Markov's inequality, this gives $A_{c1,2} = o_p(1)$.

Now, we consider $A_{c1,3}$. Below we reuse calculations in (B.9.10)–(B.9.22), which bound the term

$$ES_n^2 := E\left[(1-\rho_{0,n}^2)(nh)^{-1}\sum_{t=T_1}^{T_2}(Y_{t-1}^0)^2\right]^2 = (1-\rho_{0,n}^2)^2(nh)^{-2}E\left[\sum_{t=T_1}^{T_2}(Y_{t-1}^0)^2\right]^2, \text{ whereas}$$

$$EA_{c1,3}^2 = (1-\rho_{0,n}^2)(nh)^{-1}E\left[\sum_{t=T_1}^{T_2}(\rho_t-\rho_{0,n})\left((Y_{t-1}^c)^2 - E(Y_{t-1}^c)^2\right)\right]^2. \qquad \text{(B.11.25)}$$

The terms ES_n^2 and $EA_{c1,3}^2$ are similar, but differ as follows. The quantities $(1-\rho_{0,n}^2)^2 = O(b_n^{-2})$ and $(nh)^{-2}$ appear in ES_n^2, whereas $(1-\rho_{0,n}^2) = O(b_n^{-1})$ and $(nh)^{-1}$ appear in $EA_{c1,3}^2$. The quantity $\max_{t\in[T_1,T_2]}|\rho_t-\rho_{0,n}|^2 = O(h^2/b_n^2)$ (by Lemma 7.1(a)) appears in the bound we obtain on $EA_{c1,3}^2$, but does not appear in the bound on ES_n^2. The difference between Y_{t-1}^0, which appears in S_n, and Y_{t-1}^c, which appears in $EA_{c1,3}^2$, is not important because the same bounds can be employed with either one. The term S_n is based on summands $(Y_{t-1}^0)^2$, which do not have mean 0, whereas $A_{c1,3}$ is based on mean zero summands $(Y_{t-1}^c)^2 - E(Y_{t-1}^c)^2$, which is an important difference.

Analogously to (B.9.10), we can write

$$\begin{aligned}
&E\left[\sum_{t=T_1}^{T_2}(Y_{t-1}^c)^2 - E(Y_{t-1}^c)^2\right]^2\\
&= E\sum_{t_1,t_2=T_1-1}^{T_2-1}\left(\left(\sum_{i=0}^{t_1-T_1}\rho_{0,n}^i\sigma_{n\tau}U_{t_1-i}\right)^2 - E(Y_{t_1}^c)^2\right)\left(\left(\sum_{j=0}^{t_2-T_1}\rho_{0,n}^j\sigma_{n\tau}U_{t_2-j}\right)^2 - E(Y_{t_2}^c)^2\right)\\
&= \sigma_{n\tau}^4 E\sum_{t_1,t_2=T_1-1}^{T_2-1}\left(\sum_{i_1,i_2=0}^{t_1-T_1}\rho_{0,n}^{i_1}\rho_{0,n}^{i_2}(U_{t_1-i_1}U_{t_1-i_2} - EU_{t_1-i_1}U_{t_1-i_2})\right)\\
&\quad\times\left(\sum_{j_1,j_2=0}^{t_2-T_1}\rho_{0,n}^{j_1}\rho_{0,n}^{j_2}(U_{t_2-j_1}U_{t_2-j_2} - EU_{t_2-j_1}U_{t_2-j_2})\right)\\
&= \sigma_{n\tau}^4\sum_{t_1,t_2=T_1-1}^{T_2-1}\sum_{i_1,i_2=0}^{t_1-T_1}\sum_{j_1,j_2=0}^{t_2-T_1}\rho_{0,n}^{i_1}\rho_{0,n}^{i_2}\rho_{0,n}^{j_1}\rho_{0,n}^{j_2}\\
&\quad\times E(U_{t_1-i_1}U_{t_1-i_2} - EU_{t_1-i_1}U_{t_1-i_2})(U_{t_2-j_1}U_{t_2-j_2} - EU_{t_2-j_1}U_{t_2-j_2})\\
&= \sigma_{n\tau}^4\sum_{t_1,t_2=T_1-1}^{T_2-1}\sum_{i_1,i_2=0}^{t_1-T_1}\sum_{j_1,j_2=0}^{t_2-T_1}\rho_{0,n}^{i_1}\rho_{0,n}^{i_2}\rho_{0,n}^{j_1}\rho_{0,n}^{j_2}\\
&\quad\times(EU_{t_1-i_1}U_{t_1-i_2}U_{t_2-j_1}U_{t_2-j_2} - EU_{t_1-i_1}U_{t_1-i_2}\cdot EU_{t_2-j_1}U_{t_2-j_2}).
\end{aligned} \qquad \text{(B.11.26)}$$

The term $EA_{c1,3}^2$ equals the rhs of (B.11.26) multiplied by $(1-\rho_{0,n}^2)(nh)^{-1}$ and with $(\rho_{t_1+1} - \rho_{0,n})(\rho_{t_2+1}-\rho_{0,n})$ inserted after the three summation signs.

As discussed following (B.9.10), the expectations $EU_{t_1-i_1}U_{t_1-i_2}U_{t_2-j_1}U_{t_2-j_2}$ in the last line of (B.9.10) are all zero except when the indices fall in cases (i), (ii), or (iii), which are defined there. Furthermore, case (ii) is subdivided into cases (ii1), (ii2), and (ii3) just above (B.9.13). The difference between the expectations on the rhs of (B.11.26) and that of (B.9.10) is that the former has $\eta_{t_1t_2i_1i_2j_1j_2} := EU_{t_1-i_1}U_{t_1-i_2} \cdot EU_{t_2-j_1}U_{t_2-j_2}$ subtracted off, whereas the latter does not. The quantity $\eta_{t_1t_2i_1i_2j_1j_2}$ is non-zero iff $i_1 = i_2$ and $j_1 = j_2$. The case $i_1 = i_2$ and $j_1 = j_2$ with $t_1 - i_1 = t_2 - j_1$ is case (i). The case $i_1 = i_2$ and $j_1 = j_2$ with $t_1 - i_1 \neq t_2 - j_1$ is case (ii1). Hence, the expectations on the rhs of (B.11.26) are all zero except when the indices fall in cases (i), (ii), and (iii), just as in (B.9.10). In addition, in case (ii1), the expectations on the rhs of (B.11.26) are zero, because $EU_{t_1-i_1}U_{t_1-i_2}U_{t_2-j_1}U_{t_2-j_2} = EU^2_{t_1-i_1}U^2_{t_2-j_1} = EU^2_{t_1-i_1}EU^2_{t_2-j_1} = \eta_{t_1t_2i_1i_2j_1j_2}$, where the second equality holds because $t_1 - i_1 \neq t_2 - j_1$ in case (ii1). We conclude that the expectations on the rhs of (B.11.26) are non-zero only in cases (i), (ii2), (ii3), and (iii).

Now, we use the calculations in (B.9.11)–(B.9.22) to bound the terms in (B.11.26) when the indices fall in cases (i), (ii2), (ii3), and (iii).

For case (i), using (B.9.11) and (B.9.12), we have: the sum over the indices in case (i) on the rhs of (B.11.26) is

$$O(1)\sum_{t_1=T_1-1}^{T_2-1}\sum_{\ell=1}^{t_1-T_1}\overline{\rho}_n^{2\ell}\sum_{k=0}^{\infty}\overline{\rho}_n^{4k} + O(1)\sum_{t_1=T_1}^{T_2-1}\sum_{i=0}^{\infty}\overline{\rho}_n^{4i} = O(nhb_n^2), \tag{B.11.27}$$

where the last equality uses (7.16) and $\sum_{k=0}^{\infty}\overline{\rho}_n^{4k} = O(b_n)$. As noted above, $EA^2_{c1,3}$ equals the rhs of (B.11.26) multiplied by $(1-\rho^2_{0,n})(nh)^{-1} = O(b_n^{-1})(nh)^{-1}$ and with $(\rho_{t_1+1}-\rho_{0,n})(\rho_{t_2+1}-\rho_{0,n})$ inserted after the three summands, where $\max_{t\in[T_1,T_2]}|\rho_t-\rho_{0,n}|^2 = O(h^2/b_n^2)$ by Lemma 7.1(a). In consequence, a bound on the sum of the terms in $EA^2_{c1,3}$ that correspond to indices in case (i) is

$$O(b_n^{-1})(nh)^{-1}O(h^2/b_n^2)O(nhb_n^2) = O(b_n^{-1}h^2) = o(1). \tag{B.11.28}$$

For case (ii2), using (B.9.19) and (B.9.20), we have: the sum over the indices in case (ii2) on the rhs of (B.11.26) is

$$O(1)\sum_{t_1>t_2=T_1}^{T_2-1}\overline{\rho}_n^{2(t_1-t_2)}(1-\overline{\rho}^2)^{-2} + O(1)\sum_{t=T_1}^{T_2-1}\sum_{i_1,i_2=0}^{\infty}\overline{\rho}_n^{2(i_1-i_2)} = O(nhb_n^3) + O(nhb_n^2) = O(nhb_n^3), \tag{B.11.29}$$

where the first equality uses (7.16). Since the bound in (B.11.29) is larger than that in (B.11.27) by the factor b_n, (B.11.28) implies that the sum of the terms in $EA^2_{c1,3}$ that correspond to indices in case (ii2) is $O(h^2) = o(1)$. Cases (ii2) and (ii3) are symmetric. So, the same result holds for case (ii3).

For case (iii), using (B.9.21) and (B.9.22), we have: the sum over the indices in case (iii) on the rhs of (B.11.26) is

$$\sum_{t_1>t_2=T_1}^{T_2-1} \overline{\rho}_n^{2(t_1-t_2)} O(b_n^2) + O(nhb_n^2) = O(nhb_n^3) + O(nhb_n^2) = O(nhb_n^3). \tag{B.11.30}$$

Since the bounds in (B.11.29) and (B.11.30) are the same, the sum of the terms in $EA_{c1,3}^2$ that correspond to indices in case (iii) is $O(h^2) = o(1)$.

To conclude, we have $EA_{c1,3}^2 = o(1)$. Hence, by Markov's inequality, $A_{c1,3} = o_p(1)$. This concludes the proof that

$$A_{c1} = o_p(1). \tag{B.11.31}$$

For A_{c2}, we have

$$\begin{aligned}
EA_{c2}^2 &= E\left[\left(1-\rho_{0,n}^2\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(\mu_t-\overline{\mu}_{nh}\right)Y_{t-1}^0\right]^2 \\
&= \left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t,s=T_1}^{T_2}\left(\mu_t-\overline{\mu}_{nh}\right)\left(\mu_s-\overline{\mu}_{nh}\right)E\left(Y_{t-1}^0Y_{s-1}^0\right) \\
&= O\left(b_n^{-1}\right)(nh)^{-1}\sum_{t,s=T_1}^{T_2}\left(\mu_t-\overline{\mu}_{nh}\right)\left(\mu_s-\overline{\mu}_{nh}\right)\sum_{i=0}^{t-T_1}\sum_{j=0}^{s-T_1} c_{t-1,i}c_{s-1,j}\sigma_{t-1-i}\sigma_{s-1-j} \\
&\quad \times E\left(U_{t-1-i}U_{s-1-j}\right) \\
&\le O\left(b_n^{-1}\right)(nh)^{-1}\left(\max_{t,s\in[T_1,T_2]}|\mu_t-\mu_s|\right)^2\left(2\sum_{t>s=T_1}^{T_2}\sum_{i=t-s}^{t-T_1}\overline{\rho}_n^{2i-(t-s)}+\sum_{t=T_1}^{T_2}\sum_{i=0}^{t-T_1}\overline{\rho}_n^{2i}\right) \\
&= O\left(b_n^{-1}\right)(nh)^{-1}O\left(h^2/b_n^2\right)\left(O\left(nhb_n^2\right)+O\left(nhb_n\right)\right) = O\left(h^2/b_n\right) = o\left(1\right),
\end{aligned} \tag{B.11.32}$$

where the inequality holds by dividing the case into $t = s$ and $t \neq s$ and (7.13) and the fourth equality uses the change of coordinates $l = i - (t - s)$ and $k = t - s$, (7.16), (7.19), Lemma 7.1(d) and the triangle inequality. Then, we have $A_{c2} \to_p 0$ by Markov's inequality.

The proof of $A_{c3} \to_p 0$ is the same as that of $A_{c2} \to_p 0$ and is omitted.

For A_{c4}, we have

$$\begin{aligned}
EA_{c4}^2 &= E\left[\left(1-\rho_{0,n}^2\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(\rho_t-\rho_{0,n}\right)c_{t-1,t-1-T_0}Y_{t-1}^0Y_{T_0}^*\right]^2 \\
&= O\left(b_n^{-1}\right)(nh)^{-1}\sum_{t,s=T_1}^{T_2}\left(\rho_t-\rho_{0,n}\right)\left(\rho_s-\rho_{0,n}\right)c_{t-1,t-1-T_0}c_{s-1,s-1-T_0}EY_{t-1}^0Y_{s-1}^0Y_{T_0}^{*2}
\end{aligned}$$

$$\leq O\left(b_n^{-1}\right)(nh)^{-1}\sum_{t,s=T_1+1}^{T_2}\left[\begin{array}{c}\left(\rho_t-\rho_{0,n}\right)\left(\rho_s-\rho_{0,n}\right)c_{t-1,t-1-T_0}c_{s-1,s-1-T_0}\\ \times\sum_{i=0}^{t-1-T_1}\sum_{j=0}^{s-1-T_1}c_{t-1,i}c_{s-1,j}EU_{t-1-i}U_{s-1-j}Y_{T_0}^{*2}\end{array}\right]C_{3,U}$$
$$=O\left(b_n^{-1}\right)(nh)^{-1}2\sum_{t>s=T_1+1}^{T_2}\left[\begin{array}{c}\left(\rho_t-\rho_{0,n}\right)\left(\rho_s-\rho_{0,n}\right)c_{t-1,t-1-T_0}c_{s-1,s-1-T_0}\\ \times\sum_{i=t-s}^{t-1-T_1}c_{t-1,i}c_{s-1,i-(t-s)}EU_{t-1-i}^2Y_{T_0}^{*2}\end{array}\right]$$
$$+O\left(b_n^{-1}\right)(nh)^{-1}\sum_{t=T_1+1}^{T_2}\left[\left(\rho_t-\rho_{0,n}\right)^2c_{t-1,t-1-T_0}^2\sum_{i=0}^{t-1-T_1}c_{t-1,i}^2EU_{t-1-i}^2Y_{T_0}^{*2}\right]$$
$$\leq O\left(b_n^{-1}\right)(nh)^{-1}O\left(h^2/b_n^2\right)\sum_{t>s=T_1+1}^{T_2}\left[\overline{\rho}_n^{t-1-T_0+s-1-T_0}\sum_{i=t-s}^{t-1-T_1}\overline{\rho}_n^{2i-(t-s)}\right]O(n)$$
$$+O\left(b_n^{-1}\right)(nh)^{-1}O\left(h^2/b_n^2\right)\sum_{t=T_1+1}^{T_2}\left[\overline{\rho}_n^{2(t-1-T_0)}\sum_{i=0}^{t-1-T_1}\overline{\rho}_n^{2i}\right]O(n)$$
$$=O\left(b_n^{-1}\right)(nh)^{-1}O\left(h^2/b_n^2\right)O\left(b_n^3\right)O(n)+O\left(b_n^{-1}\right)(nh)^{-1}O\left(h^2/b_n^2\right)O\left(b_n^2\right)O(n)$$
$$=O(h)=o(1), \tag{B.11.33}$$

where the first inequality holds by $\max_{t\in[T_1,T_2]}\sigma_t^2\leq C_{3,U}$, the second inequality holds by Lemma 7.1(a), (B.4.2), (7.13), and $E\left[U_t^2|\mathscr{G}_{t-1}\right]=1$ a.s., the third equality uses the law of iterated expectations (LIE) and part (iv) of Λ_n, and the fourth equality holds by the change of variables $l=2i-(t-s)$ and (7.16). Then, by Markov's inequality, we have $A_{c4}\rightarrow_p 0$.

For A_{c5}, we have

$$EA_{c5}^2=E\left[\left(1-\rho_{0,n}^2\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)\sigma_tU_t\right]^2$$
$$=\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)^2\sigma_t^2EU_t^2$$
$$\leq O\left(b_n^{-1}\right)(nh)^{-1}nhO\left(h^2/b_n^2\right)C_{3,U}=o(1), \tag{B.11.34}$$

where the inequality holds by

$$\max_{t\in[T_1,T_2]}\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)^2\leq\max_{t,s\in[T_0,T_2]}\left(\mu_t-\mu_s\right)^2=O\left(h^2/b_n^2\right) \tag{B.11.35}$$

using Lemma 7.1(d) and the triangle inequality. By Markov's inequality, we obtain $A_{c5}\rightarrow_p 0$.

The proof of

$$A_{c6}:=\left(1-\rho_{0,n}^2\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)\left(\rho_t-\rho_{0,n}\right)Y_{t-1}^0\rightarrow_p 0 \tag{B.11.36}$$

is quite similar to that of $A_{c2}\rightarrow 0$ given in (B.11.32), and hence, is omitted.

The proofs of

$$|A_{c7}| := \left| \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) \left(\mu_t - \overline{\mu}_{nh}\right) \right| \to 0 \tag{B.11.37}$$

and

$$|A_{c8}| := \left| \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} \rho_{0,n} \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right)^2 \right| \to 0 \tag{B.11.38}$$

are identical, thus we only prove the result for A_{c8}. An application of the triangle inequality gives

$$|A_{c8}| \leq \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} O(nh) \max_{t,s\in[T_0,T_2]} (\mu_t - \mu_s)^2 = O\left(\left(nh^5/b_n^5\right)^{1/2}\right) = o(1), \tag{B.11.39}$$

where the second last equality holds by Lemma 7.1(d) and the triangle inequality and the last equality holds by Assumption 2.

For A_{c9}, we have

$$\begin{aligned} |A_{c9}| &= \left| \left(1-\rho_{0,n}^2\right)^{1/2} (nh)^{-1/2} \sum_{t=T_1}^{T_2} \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* \right| \\ &\leq O\left(b_n^{-1/2}\right) (nh)^{-1/2} \max_{t,s\in[T_0,T_2]} |\mu_t - \mu_s| \max_{t\in[T_1,T_2]} |\rho_t - \rho_{0,n}| \sum_{t=T_1}^{T_2} \overline{\rho}_n^{t-1-T_0} \left|Y_{T_0}^*\right| \\ &= O\left(b_n^{-1/2}\right) (nh)^{-1/2} O(h) O(h/b_n) O(b_n) O_p\left(n^{1/2}\right) = O_p\left(h^{3/2} b_n^{-1/2}\right), \end{aligned} \tag{B.11.40}$$

where the second last equality holds by Lemma 7.1(a) and (d), (B.4.3), and (7.15).

For A_{c10}, we have

$$\begin{aligned} EA_{c10}^2 &= E\left[(1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2} \sum_{t=T_1}^{T_2} c_{t-1,t-1-T_0} Y_{T_0}^* \sigma_t U_t\right]^2 \\ &= (1-\rho_{0,n}^2)(nh)^{-1} \sum_{t=T_1}^{T_2} c_{t-1,t-1-T_0}^2 \sigma_t^2 E U_t^2 Y_{T_0}^{*2} \\ &\leq (1-\rho_{0,n}^2)(nh)^{-1} \sum_{t=T_1}^{T_2} \overline{\rho}_n^{2(t-1-T_0)} \sigma_t^2 E Y_{T_0}^{*2} \\ &\leq O(b_n^{-1})(nh)^{-1} O(b_n) O(b_n) = O(b_n/nh) = o(1), \end{aligned} \tag{B.11.41}$$

where the second equality uses the martingale difference properties of $\{U_t\}_{t\leq n}$, the first inequality uses (7.13) and $E(U_t^2|Y_{T_0}^{*2}) = 1$ a.s. by part (iv) of Λ_n, the second inequality uses Lemma 7.2(c), which applies because $nh/b_n \to r_0 = \infty$ is an assumption of the lemma, and

(7.15), and the last equality holds because $b_n/nh \to 0$. Equation (B.11.41) and Markov's inequality give $A_{c10} = o_p(1)$.

For A_{c11}, we have: $A_{c11} = A_{c4} = o_p(1)$ by (B.11.33).

For A_{c12}, we have

$$\begin{aligned}|A_{c12}| &= \left|(1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2} c_{t-1,t-1-T_0}(\mu_{t-1}-\overline{\mu}_{nh,-1})Y_{T_0}^*\right| \\ &\le (1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\overline{\rho}_n^{t-1-T_0}\max_{t\in[T_1,T_2]}|\mu_{t-1}-\overline{\mu}_{nh,-1}|\cdot|Y_{T_0}^*| \\ &= O(b_n^{-1/2})(nh)^{-1/2}O(b_n)O(h/b_n)O_p(b_n^{1/2}) = O_p((nh)^{-1/2}h) = o_p(1),\end{aligned} \tag{B.11.42}$$

where the inequality uses (7.13), the second equality uses (7.15), Lemma 7.1(d), and Lemma 7.2(c).

We have $|A_{c13}| = |\rho_{0,n}A_{c12}| = o_p(1)$ (since $A_{c12} = o_p(1)$ and $|\rho_{0,n}| \le 1$).

For A_{c14}, we have

$$\begin{aligned}|A_{c14}| &= \left|(1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2} c_{t-1,t-1-T_0}^2(\rho_t-\rho_{0,n})Y_{T_0}^{*2}\right| \\ &\le (1-\rho_{0,n}^2)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\overline{\rho}_n^{2(t-1-T_0)}\max_{t\in[T_1,T_2]}|\rho_t-\rho_{0,n}|Y_{T_0}^{*2} \\ &= O(b_n^{-1/2})(nh)^{-1/2}O(b_n)O(h/b_n)O_p(b_n) \\ &= O_p((b_n/nh)^{1/2}h) = o_p(1),\end{aligned} \tag{B.11.43}$$

where the inequality uses (7.13), the second equality uses (7.15), Lemma 7.1(a), and Lemma 7.2(c).

By (B.11.31)–(B.11.43), we obtain

$$\sum_{i=1}^{14} A_{ci} \to_p 0. \tag{B.11.44}$$

Combining (B.11.3), (B.11.6), and (B.11.44), we have

$$\left(1-\rho_{0,n}^2\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\left[\ Y_t-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\ \right]\to_d N\left(0,1\right), \tag{B.11.45}$$

as desired. □

Proof of Lemma 7.8(b). To prove part (b), by Lemma 7.7(b), we have

$$\left(1-\rho_{0,n}^2\right)^{1/2}\left(\overline{Y}_{nh,-1}-\overline{\mu}_{nh,-1}\right)=o_p\left(1\right). \tag{B.11.46}$$

Thus, we only need to show

$$(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\right]=O_p\left(1\right). \tag{B.11.47}$$

This is done by examining each component of (B.11.47) and showing it is $O_p(1)$. Specifically, by (B.11.2), we expand

$$\begin{aligned}&(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\right]\\ =&\,(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left[\begin{array}{c}\sigma_tU_t+\left(\rho_t-\rho_{0,n}\right)Y_{t-1}^0+\left(\mu_t-\overline{\mu}_{nh}\right)\\ -\rho_{0,n}\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)+\left(\rho_t-\rho_{0,n}\right)c_{t-1,t-1-T_0}Y_{T_0}^*\end{array}\right]=:\sum_{i=1}^{5}A_{di}.\end{aligned} \tag{B.11.48}$$

We observe that $A_{d1}=O_p(1)$ by the central limit theorem for martingale difference sequences. The proof of $A_{d2}=o_p(1)$ is almost the same as that of A_{c2} except that (i) A_{d2} does not have the $\left(1-\rho_{0,n}^2\right)^{1/2}$ multiplicand, which is $O\left(b_n^{-1/2}\right)$ and (ii) A_{d2} has $\rho_t-\rho_{0,n}$ in place of $\mu_t-\overline{\mu}_{nh}$, both of which have the same order of $O(h/b_n)$ of maximum intertemporal difference on $[T_1,T_2]$ by Lemma 7.1(a) and (d). Thus, following an argument identical to that in (B.11.32), we have

$$A_{d2}=O\left(b_n^{1/2}\right)O_p\left(hb_n^{-1/2}\right)=O_p\left(h\right)=o_p\left(1\right). \tag{B.11.49}$$

By the definitions of A_{d3} and A_{d4}, we get $A_{d3}=A_{d4}=0$. Finally, we derive $A_{d5}=o_p(1)$ by observing

$$\begin{aligned}\left|A_{d5}\right|&\le\max_{t\in[T_1,T_2]}\left|\rho_t-\rho_{0,n}\right|(nh)^{-1/2}\sum_{t=T_1}^{T_2}\overline{\rho}_n^{t-T_1}\left|Y_{T_0}^*\right|\\&=O\left(h/b_n\right)(nh)^{-1/2}O\left(b_n\right)O_p\left(n^{1/2}\right)=O_p\left(h^{1/2}\right)=o_p\left(1\right),\end{aligned} \tag{B.11.50}$$

where the first equality holds by Lemma 7.1(a), (B.4.3), and (7.15).

Thus, $\sum_{i=1}^{5}A_{di}=o_p(1)$, (B.11.47) holds, and the proof is complete. □

B.12 Proof of Theorem 7.3

In this section, for notational simplicity in the proof, we assume that $\sigma_0^2(\tau)=1$.

Proof of Theorem 7.3. Recall from (7.20) that

$$\begin{aligned}&\left(1-\rho_{0,n}^{2}\right)^{-1/2}(nh)^{1/2}\left(\widehat{\rho}_{n\tau}-\rho_{0,n}\right)\\=&\frac{\left(1-\rho_{0,n}^{2}\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\left(Y_t-\rho_{0,n}Y_{t-1}\right)}{\left(1-\rho_{0,n}^{2}\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)^{2}}.\end{aligned} \tag{B.12.1}$$

We analyze the denominator and numerator of (B.12.1) separately.

First, for the denominator of (B.12.1), we have

$$\begin{aligned}&\left(1-\rho_{0,n}^{2}\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)^{2}\\=&\left(1-\rho_{0,n}^{2}\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)^{2}-\left(1-\rho_{0,n}^{2}\right)\left(\overline{Y}_{nh,-1}-\overline{\mu}_{nh,-1}\right)^{2}\\=:&\ A_{f1}+A_{f2}.\end{aligned} \tag{B.12.2}$$

By Lemma 7.7(a) and (b), we have $A_{f1}\to_p 1$ and $A_{f2}\to_p 0$, respectively. Therefore, we have

$$\left(1-\rho_{0,n}^{2}\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)^{2}\to_p 1. \tag{B.12.3}$$

Next, for the numerator of (B.12.1), we have

$$\begin{aligned}&\left(1-\rho_{0,n}^{2}\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\left(Y_t-\rho_{0,n}Y_{t-1}\right)\\=&\left(1-\rho_{0,n}^{2}\right)^{1/2}(nh)^{-1/2}\\&\times\sum_{t=T_1}^{T_2}\left[\ Y_{t-1}-\overline{\mu}_{nh,-1}-\left(\overline{Y}_{nh,-1}-\overline{\mu}_{nh,-1}\right)\ \right]\left[\ Y_t-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\ \right]\\=&\left(1-\rho_{0,n}^{2}\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\left[Y_t-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\right]\\&+\left(1-\rho_{0,n}^{2}\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(\overline{Y}_{nh,-1}-\overline{\mu}_{nh,-1}\right)\left[Y_t-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\right]\\=:&A_{f3}+A_{f4}.\end{aligned} \tag{B.12.4}$$

By Lemma 7.8(a) and (b), we have $A_{f3}\to_d N(0,1)$ and $A_{f4}\to_p 0$, respectively. Therefore, we obtain

$$\left(1-\rho_{0,n}^2\right)^{1/2}(nh)^{-1/2}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\left(Y_t-\rho_{0,n}Y_{t-1}\right)\to_d N(0,1). \tag{B.12.5}$$

Combining (B.12.3) and (B.12.5), we have

$$\left(1-\rho_{0,n}^2\right)^{-1/2}(nh)^{1/2}\left(\widehat{\rho}_{n\tau}-\rho_{0,n}\right)\to_d N(0,1). \tag{B.12.6}$$

For the t-statistic $T_n(\rho_{0,n})$, by (3.4), (3.5), (B.12.6), and the CMT, we only need to show

$$\left(1-\rho_{0,n}^2\right)^{-1}\widehat{s}_{n\tau}^2=\frac{(nh)^{-1}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{Y}_{nh}-\widehat{\rho}_{n\tau}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2}{\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)^2}\to_p 1. \tag{B.12.7}$$

Equation (B.12.3) shows that the denominator converges in probability to one. For the numerator of (B.12.7), we have

$$\begin{aligned}
&(nh)^{-1}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{Y}_{nh}-\widehat{\rho}_{n\tau}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2\\
=&(nh)^{-1}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{Y}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2\\
&+(nh)^{-1}\sum_{t=T_1}^{T_2}\left[\left(\widehat{\rho}_{n\tau}-\rho_{0,n}\right)\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2\\
&+2(nh)^{-1}\sum_{t=T_1}^{T_2}\left\{\begin{array}{c}\left(\widehat{\rho}_{n\tau}-\rho_{0,n}\right)\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\\ \times\left[Y_t-\overline{Y}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]\end{array}\right\}\\
=:&A_{f5}+A_{f6}+A_{f7}.
\end{aligned} \tag{B.12.8}$$

We show $A_{f5}\to_p 1$ and $A_{f6}\to_p 0$, which imply $A_{f7}\to_p 0$ by the CS inequality.

For A_{f5}, we have

$$\begin{aligned}
&(nh)^{-1}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{Y}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2\\
=&(nh)^{-1}\sum_{t=T_1}^{T_2}\left[Y_t-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\right]^2\\
&+\left[\overline{Y}_{nh}-\overline{\mu}_{nh}-\rho_{0,n}\left(\overline{Y}_{nh,-1}-\overline{\mu}_{nh,-1}\right)\right]^2\\
&+2(nh)^{-1}\sum_{t=T_1}^{T_2}\left\{\begin{array}{c}\left[Y_t-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\right]\\ \times\left[\overline{Y}_{nh}-\overline{\mu}_{nh}-\rho_{0,n}\left(Y_{t-1}-\overline{\mu}_{nh,-1}\right)\right]\end{array}\right\}
\end{aligned}$$

$$=:A_{f51} + A_{f52} + A_{f53}. \tag{B.12.9}$$

We show $A_{f51} \to_p 1$ and $A_{f52} \to_p 0$, which imply $A_{f53} \to_p 0$. Recall that by (B.11.2), we have

$$\begin{aligned}
A_{f51} =& (nh)^{-1} \sum_{t=T_1}^{T_2} \begin{bmatrix} \sigma_t U_t + (\rho_t - \rho_{0,n}) Y_{t-1}^0 + (\mu_t - \overline{\mu}_{nh}) \\ -\rho_{0,n} \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) + (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* \end{bmatrix}^2 \\
=& (nh)^{-1} \sum_{t=T_1}^{T_2} (\sigma_t U_t)^2 \\
&+ (nh)^{-1} \sum_{t=T_1}^{T_2} \begin{bmatrix} (\rho_t - \rho_{0,n}) Y_{t-1}^0 + (\mu_t - \overline{\mu}_{nh}) \\ -\rho_{0,n} \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) + (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* \end{bmatrix}^2 \\
&+ 2 (nh)^{-1} \sum_{t=T_1}^{T_2} \sigma_t U_t \begin{bmatrix} (\rho_t - \rho_{0,n}) Y_{t-1}^0 + (\mu_t - \overline{\mu}_{nh}) \\ -\rho_{0,n} \left(\mu_{t-1} - \overline{\mu}_{nh,-1}\right) + (\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^* \end{bmatrix} \\
=:& A_{f511} + A_{f512} + A_{f513}.
\end{aligned} \tag{B.12.10}$$

By (B.8.18), we have $A_{f511} \to_p 1$. For A_{f512}, we have

$$\begin{aligned}
&(nh)^{-1} \sum_{t=T_1}^{T_2} \left[(\rho_t - \rho_{0,n}) Y_{t-1}^0\right]^2 \\
\leq& \max_{t \in [T_1, T_2]} (\rho_t - \rho_{0,n})^2 \left(1 - \rho_{0,n}^2\right)^{-1} \left(1 - \rho_{0,n}^2\right) (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1}^0\right)^2 \\
=& O\left((h/b_n)^2\right) O(b_n) O_p(1) = o_p(1),
\end{aligned} \tag{B.12.11}$$

where the first equality holds by Lemmas 7.1(a) and 7.6(b), and the fact that $\left(1 - \rho_{0,n}^2\right)^{-1} = O(b_n)$. Additionally, by Lemmas 7.1(a), 7.1(d), and 7.2(a), and (7.16), we have

$$(nh)^{-1} \sum_{t=T_1}^{T_2} (\mu_t - \overline{\mu}_{nh})^2 \to_p 0, \tag{B.12.12}$$

and

$$\begin{aligned}
&(nh)^{-1} \sum_{t=T_1}^{T_2} \left[(\rho_t - \rho_{0,n}) c_{t-1,t-1-T_0} Y_{T_0}^*\right]^2 \\
\leq& (nh)^{-1} \max_{t \in [T_1, T_2]} (\rho_t - \rho_{0,n})^2 \sum_{t=T_1}^{T_2} \overline{\rho}_n^{2(t-T_1)} O_p(n) = O_p\left((nh)^{-1} (h/b_n)^2 n b_n\right) = o_p(1).
\end{aligned} \tag{B.12.13}$$

Similarly to (B.12.12), we have

$$(nh)^{-1}\sum_{t=T_1}^{T_2}\left[\rho_{0,n}\left(\mu_{t-1}-\overline{\mu}_{nh,-1}\right)\right]^2\to_p 0 \tag{B.12.14}$$

since $|\rho_{0,n}|\leq 1$. By (B.12.11)–(B.12.14) and the CS inequality, we have $A_{f512}\to_p 0$. Then, by $A_{f511}\to_p 1$, $A_{f512}\to_p 0$, and the CS inequality, we have $A_{f513}\to_p 0$. Thus, $A_{f51}\to_p 1$.

Next, for A_{f52} we have

$$\begin{aligned}&\overline{Y}_{nh}-\overline{\mu}_{nh}-\rho_{0,n}\left(\overline{Y}_{nh,-1}-\overline{\mu}_{nh,-1}\right)\\&=(nh)^{-1}\sum_{t=T_1}^{T_2}\left[\sigma_t U_t+(\rho_t-\rho_{0,n})\,Y_{t-1}^0+(\rho_t-\rho_{0,n})\,c_{t-1,t-1-T_0}Y_{T_0}^*\right]\\&=:A_{f521}+A_{f522}+A_{f523}.\end{aligned} \tag{B.12.15}$$

By the weak law of large numbers, $A_{f521}\to_p 0$. For A_{f522}, we have

$$\begin{aligned}A_{f522}^2&=\left\{(nh)^{-1}\sum_{t=T_1}^{T_2}\left[(\rho_t-\rho_{0,n})\,Y_{t-1}^0\right]\right\}^2\\&\leq\max_{t\in[T_1,T_2]}(\rho_t-\rho_{0,n})^2\left(1-\rho_{0,n}^2\right)^{-1}\left(1-\rho_{0,n}^2\right)(nh)^{-1}\sum_{t=T_1}^{T_2}\left(Y_{t-1}^0\right)^2\\&=O\left(h^2/b_n^2\right)O\left(b_n\right)O_p\left(1\right)=o_p\left(1\right),\end{aligned} \tag{B.12.16}$$

where the inequality holds by the CS inequality and the second last equality holds by Lemmas 7.1(a) and 7.6(b).

For A_{f523}, we have

$$\begin{aligned}|A_{f523}|&=\left|(nh)^{-1}\sum_{t=T_1}^{T_2}\left[(\rho_t-\rho_{0,n})\,c_{t-1,t-1-T_0}Y_{T_0}^*\right]\right|\\&\leq(nh)^{-1}O\left(h/b_n\right)O\left(b_n\right)O_p\left(n^{1/2}\right)=o_p\left(1\right),\end{aligned} \tag{B.12.17}$$

where the inequality holds by Lemmas 7.1(a) and 7.2(a), (7.13), and (7.15). Combining the results, we have $A_{f5}\to_p 1$.

For A_{f6}, we use (B.12.6) and obtain

$$(nh)^{-1}\sum_{t=T_1}^{T_2}\left[(\widehat{\rho}_{n\tau}-\rho_{0,n})\left(Y_{t-1}-\overline{Y}_{nh,-1}\right)\right]^2$$

$$= \left[\left(1 - \rho_{0,n}^2\right)^{-1} nh \left(\widehat{\rho}_{n\tau} - \rho_{0,n}\right)^2 \right] \left[\left(1 - \rho_{0,n}^2\right) (nh)^{-1} \sum_{t=T_1}^{T_2} \left(Y_{t-1} - \overline{Y}_{nh,-1}\right)^2 \right] / (nh)$$
$$= O_p(1) O_p(1) / (nh) = o_p(1), \quad \text{(B.12.18)}$$

where the second equality holds by (B.12.3) and (B.12.6).

Therefore, we have shown that the quantity in (B.12.8) equals $1 + o_p(1)$. This and (B.12.3) establish (B.12.7) and the proof of the result for the t-statistic is complete.

The subsequence versions of Lemmas 7.6–7.8 and Theorem 7.3, see Remark 7.3, which have $\{p_n\}_{n\geq 1}$ in place of $\{n\}_{n\geq 1}$, are proved by replacing n by p_n and $h = h_n$ by h_{p_n} throughout the proofs above. □

B.13 Proof of Lemma 7.9

Proof of Lemma 7.9. By the definition of $\widehat{h}_{opt}$, $L_n(\widehat{h}_{opt})/L_n(\widehat{h}) \leq 1 \; \forall n \geq 1$. So, the result of the lemma follows from $L_n(\widehat{h}_{opt})/L_n(\widehat{h}) \geq 1 + o_p(1)$. We have

$$\begin{aligned}
\frac{L_n(\widehat{h}_{opt})}{L_n(\widehat{h})} &= \frac{L_n(\widehat{h}_{opt}) - 2C_n(\widehat{h}_{opt})}{L_n(\widehat{h})} + 2\frac{C_n(\widehat{h}_{opt})}{L_n(\widehat{h})} \\
&\geq \frac{L_n(\widehat{h}) - 2C_n(\widehat{h})}{L_n(\widehat{h})} + 2\frac{C_n(\widehat{h}_{opt})}{L_n(\widehat{h})} \\
&= 1 - 2\frac{C_n(\widehat{h})}{L_n(\widehat{h})} + 2\frac{C_n(\widehat{h}_{opt})}{L_n(\widehat{h})} \\
&= 1 + o_p(1), \qquad \text{(B.13.1)}
\end{aligned}$$

where the inequality holds because $\widehat{h}$ minimizes $L_n(h) - 2C_n(h)$ over $\mathcal{H}_n$ and the last equality holds using Assumption 5 and $|\frac{C_n(\widehat{h}_{opt})}{L_n(\widehat{h})}| \leq |\frac{C_n(\widehat{h}_{opt})}{L_n(\widehat{h}_{opt})}| = o_p(1)$ since $\widehat{h}_{opt}$ minimizes $L_n(h)$ and using Assumption 5 again. □

B.14 Proof of Lemma 7.10

Proof of Lemma 7.10. We have: $EC_{2n}(h) = 0$ because (i) $E(U_t|\mathcal{G}_{t-1}) = 0$ a.s. by the definition of Λ_n which applies by Assumption 6(a) and (ii) Y_{t-1} and $\widehat{\rho}_{t-1}(h)$ are functions of $(U_{t-1}, ..., U_1, Y_0^*)$ provided $t > nh$ and these variables are in $\mathcal{G}_{t-1}$ by the definition of Λ_n. Let $n_* := n - nh$. Next, we have

$$Var\left(C_{2n}(h)\right) = E\left(n_*^{-1} \sum_{t=nh_{\max}+1}^{n} \sigma_t U_t(\widehat{\mu}_{t-1}(h) - \mu_t + Y_{t-1}(\widehat{\rho}_{t-1}(h) - \rho_t))\right)^2$$

$$
\begin{aligned}
&= n_*^{-2} \sum_{t=nh_{\max}+1}^{n} \sigma_t^2 E U_t^2 (\widehat{\mu}_{t-1}(h) - \mu_t + Y_{t-1}(\widehat{\rho}_{t-1}(h) - \rho_t))^2 \\
&= n_*^{-2} \sum_{t=nh_{\max}+1}^{n} \sigma_t^2 E (\widehat{\mu}_{t-1}(h) - \mu_t + Y_{t-1}^2(\widehat{\rho}_{t-1}(h) - \rho_t))^2 \\
&\leq C_{3U} \cdot n_*^{-1} E L_{2n}(h), \qquad \text{(B.14.1)}
\end{aligned}
$$

where the first equality holds because $EC_{2n}(h) = 0$, the second equality holds because, if $t > s$ (and $t > nh_{\max}$), $EU_t(\widehat{\mu}_{t-1}(h) - \mu_t + Y_{t-1}(\widehat{\rho}_{t-1}(h) - \rho_t))U_s(\widehat{\mu}_{s-1}(h) - \mu_s + Y_{s-1}(\widehat{\rho}_{s-1}(h) - \rho_s)) = 0$ by (i) since $(\widehat{\mu}_{t-1}(h), Y_{t-1}, \widehat{\rho}_{t-1}(h), U_s, \widehat{\mu}_{s-1}(h), Y_{s-1}, \widehat{\rho}_{s-1}(h))$ are functions of $(U_{t-1}, ..., U_1, Y_0^*)$ and these variables are in $\mathcal{G}_{t-1}$, and analogously if $t < s$, the third equality holds because $E(U_i^2|\mathcal{G}_{t-1}) = 1$ a.s. by the definition of Λ_n, and the inequality holds by the bound C_{3U} on the variance function $\sigma^2(\cdot)$ by the definition of Λ_n, which implies that $\sigma_t^2 \leq C_{3U}$ and the definition of $L_{2n}(h)$.[2]

For any positive constant K,

$$
\begin{aligned}
P\left(\sup_{h\in\mathcal{H}_n}\left|\frac{C_{2n}(h)}{\xi_n^{1/2} Var^{1/2}(C_{2n}(h))}\right| > K\right) &\leq \sum_{h\in\mathcal{H}_n} P\left(\left|\frac{C_{2n}(h)}{\xi_n^{1/2} Var^{1/2}(C_{2n}(h))}\right| > K\right) \\
&\leq \sum_{h\in\mathcal{H}_n} \frac{EC_{2n}(h)^2}{\xi_n Var(C_{2n}(h))K^2} = \frac{1}{K^2} \qquad \text{(B.14.2)}
\end{aligned}
$$

for all $n \geq 1$, where the second inequality holds by Markov's inequality. In consequence,

$$
O_p(1) = \sup_{h\in\mathcal{H}_n}\left|\frac{C_{2n}(h)}{\xi_n^{1/2} Var^{1/2}(C_{2n}(h))}\right| \geq \sup_{h\in\mathcal{H}_n}\left|\frac{n_*^{1/2} C_{2n}(h)}{C_{3U}^{1/2}\xi_n^{1/2}(EL_{2n}(h))^{1/2}}\right|, \qquad \text{(B.14.3)}
$$

where the equality holds by (B.14.2) and the inequality holds by (B.14.3).We have

$$
\begin{aligned}
\sup_{h\in\mathcal{H}_n}\left|\frac{C_{2n}(h)}{L_{2n}(h)}\right| &= \sup_{h\in\mathcal{H}_n}\left|\frac{n_*^{1/2} C_{2n}(h)}{C_{3U}^{1/2}\xi_n^{1/2}(EL_{2n}(h))^{1/2}} \frac{C_{3U}^{1/2}\xi_n^{1/2}(EL_{2n}(h))^{1/2}}{n_*^{1/2} L_{2n}(h)}\right| \\
&= O_p(1) \cdot \sup_{h\in\mathcal{H}_n} \frac{\xi_n^{1/2}(EL_{2n}(h))^{1/2}}{n_*^{1/2} L_{2n}(h)}. \qquad \text{(B.14.4)}
\end{aligned}
$$

[2] One could consider $EC_{2n}(h)^m$ for some even number $m > 2$, rather than the variance of $C_{2n}(h)$. With i.i.d. observations $\{Y_i\}_{i\leq n}$, this would yield a bound that decreases to zero faster as a function of n_* than n_*^{-1}, which appears in (B.14.1) for the case of $m = 2$. However, in the present model, a faster rate is not obtained for $m > 2$ because the summands in the m-fold sum are zero only when the largest index of $U_a U_b \cdots U_t$ is unique, not when any index is unique, as occurs with i.i.d. summands.

The right-hand side expression is $o_p(1)$ iff

$$\inf_{h\in\mathcal{H}_n} \frac{n_*^{1/2} L_{2n}(h)}{\xi_n^{1/2}(R_{2n}(h))^{1/2}} \to_p \infty \text{ iff } \inf_{h\in\mathcal{H}_n} \frac{n_*^{1/2}(R_{2n}(h))^{1/2}}{\xi_n^{1/2}} \to \infty \text{ iff } \frac{n \inf_{h\in\mathcal{H}_n} R_{2n}(h)}{\xi_n} \to \infty, \tag{B.14.5}$$

where the first "iff" uses $R_{2n}(h) = EL_{2n}(h)$, the second "iff" uses Assumption 6(c), the third "iff" uses Assumption 6(d), and the last condition holds by Assumption 6(e). In consequence, $\sup_{h\in\mathcal{H}_n} \left|\frac{C_{2n}(h)}{L_{2n}(h)}\right| = o_p(1)$. This, combined with Assumption 6(b) and $L_{2n}(h) \geq 0\ \forall h$, verifies Assumption 5. □

B.15 Proof of Theorem B.1

To show that Theorem 7.3 holds with Assumption 2 replaced by Assumption 2*, the proof of Theorem B.1 uses the following extension of Lemma 7.1 that improves its bounds in the case where $\ell_n \to 0$ as $n \to \infty$.

Lemma B.3. *Under Assumptions 1 and 3,*

(a) $\max_{t\in[T_1,T_2]} |\rho_t - \rho_{n\tau}| = O(\ell_n h/b_n)$,

(b) $\max_{t\in[T_1,T_2]} |\sigma_t^2 - \sigma_{n\tau}^2| = O(\ell_n h)$,

(c) $\max_{t\in[T_1,T_2]} |c_{t,j} - \rho_{n\tau}^j| = O(\ell_n n h^2/b_n)$, *and*

(d) $\max_{t\in[T_1,T_2]} |\mu_t - \mu_{n\tau}| = O(\ell_n h/b_n)$.

Proof of Lemma B.3. Part (a) holds by the proof of Lemma 7.1(a) by replacing the Lipschitz bound L_4 in (B.3.1) by ℓ_n, which implies that the rhs bound in (B.3.1) becomes $O(\ell_n h)$. In turn, the rhs bound in (B.3.2) becomes $O(\ell_n h/b_n)$. Parts (b)-(d) then hold by the same argument as in the proof of Lemma 7.1 with the additional term ℓ_n appearing in each of the error bounds. □

Proof of Theorem B.1. First, we show that Theorem 7.2 holds with Assumption 2 replaced by Assumption 2*. Assumption 2 enters the proof of Theorem 7.2 only through its application of Lemma 7.2(b), which relies on Assumption 2. In turn, Assumption 2 enters the proof of Lemma 7.2(b) only through its use in (B.4.10) to show that $h^{1/2}\ln(n) = o(1)$. Since the latter holds under Assumption 2*(ii), this completes the proof for Theorem 7.2 under Assumption 2*.

Next, we show that Theorem 7.3 holds with Assumption 2 replaced by Assumption 2*. Assumption 2 enters the proof of Theorem 7.3 only through its application of Lemmas

7.7 and 7.8(a), which both use Assumption 2. Assumption 2 enters the proof of Lemma 7.7 only through its application of Lemma 7.2(c), which relies on Assumption 2. In turn, Assumption 2 enters the proof of Lemma 7.2(c) only through its use in (B.4.12) to show that $h \ln(n) = o(1)$. The latter holds under Assumption 2*(ii).

Assumption 2 is used in the proof of Lemma 7.8(a) in equations (B.11.15), (B.11.24), and (B.11.39) and because the proof applies Lemma 7.2(c) (which we have just shown to hold under Assumption 2*(ii)). To verify (B.11.15) using Assumption 2* in place of Assumption 2, we bound $|\kappa''_{0,n}(\widetilde{\tau}_{n,t})|$ by ℓ_n in the fourth last line of (B.11.15) (which is why ℓ_n is defined to bound the absolute value of second derivative of $\kappa_n(\cdot) = \kappa_{0,n}(\cdot)$ over the interval I_{τ,ε_2}). This leads to $\ell_n \sum_{t=T_1}^{T_2}(t/n - \tau)^2$ appearing in place of $\sum_{t=T_1}^{T_2}(t/n - \tau)^2$ in the third last line of (B.11.15), which in turn leads to $\ell_n nh \cdot h^2$ appearing in place of $nh \cdot h^2$ in the second last line of (B.11.15). The latter leads to $(n\ell_n^2 h^5)^{1/2}$ appearing in place of $(nh^5)^{1/2}$ in two places in the last line of (B.11.15). Since $(n\ell_n^2 h^5)^{1/2} = o(1)$ by Assumption 2*(i), (B.11.15) holds under Assumption 2*.

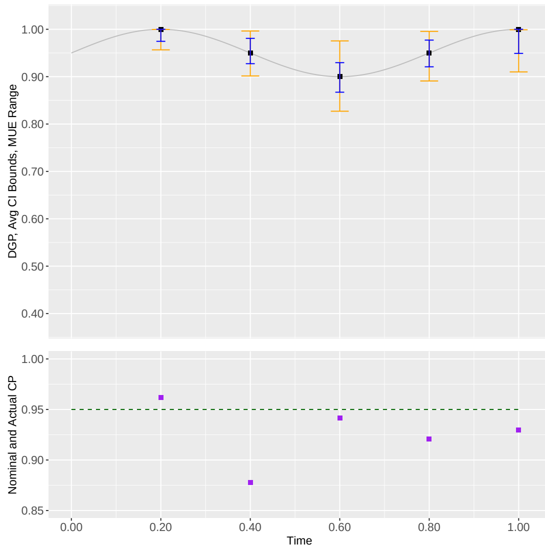
To verify (B.11.24) under Assumption 2*, we employ Lemma B.3(b) and (c) to yield the rhs of (B.11.20) to be $j\overline{\rho}_n^{j-1}O(\ell_n h/b_n) + \overline{\rho}_n^j O(\ell_n h)$ rather than $j\overline{\rho}_n^{j-1}O(h/b_n) + \overline{\rho}_n^j O(h)$ (which is why ℓ_n is defined to bound the Lipschitz constants for $\kappa_n(\cdot)$ and $\sigma_n^2(\cdot)$ over the interval I_{τ,ε_2}). In consequence, each of the terms on the last line of (B.11.21) gets multiplied by ℓ_n^2, and so, the rhs of (B.11.21) becomes $O(\ell_n^2 b_n h^2)$. In turn, this causes $O(\ell_n b_n^{1/2} h)$ to appear in place of $O(b_n^{1/2} h)$ at the end of the first line of (B.11.24). And this causes $O(n^{1/2}\ell_n h^{5/2} b_n^{-1/2})$ and $O((n\ell_n^2 h^5)^{1/2})$ to appear in place of $O(n^{1/2} h^{5/2} b_n^{-1/2})$ and $O((nh^5)^{1/2})$, respectively, in the last line of (B.11.24). Since $O((n\ell_n^2 h^5)^{1/2}) = o(1)$ under Assumption 2*(i), (B.11.24) holds under Assumption 2*.

To verify (B.11.39) under Assumption 2*, we employ Lemma B.3(d) to yield the bound $O((n\ell_n^2 h^5/b_n^5)^{1/2})$ rather than the bound $O((nh^5/b_n^5)^{1/2})$ in (B.11.39). Since $O((n\ell_n^2 h^5/b_n^5)^{1/2}) = o(1)$ under Assumption 2*(i) (which is why ℓ_n is defined to bound the Lipschitz constant for $\eta_n(\cdot)$ over the interval I_{τ,ε_2}), (B.11.39) holds under Assumption 2*. This completes the proof. □

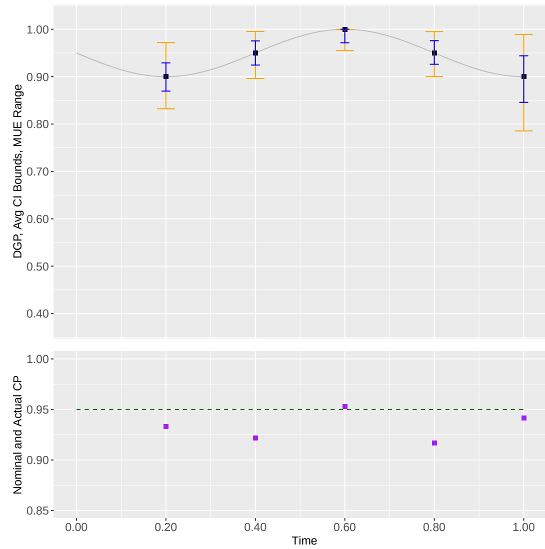
C Additional Simulation Results

For a discussion of the results given in Figures SM.1–SM.4 below, see Section 5 of the paper.

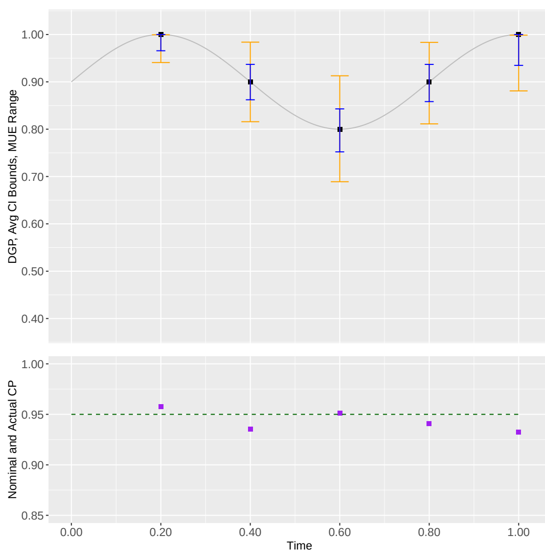
(a) sin 1.00-0.90-1.00, time-varying μ and σ



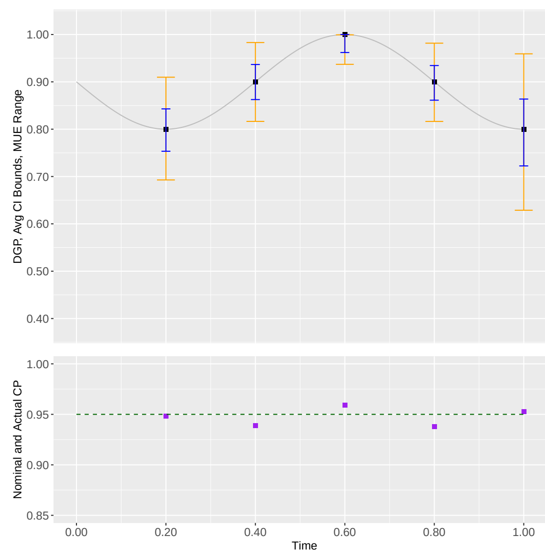
(b) sin 0.90-1.00-1.00, time-varying μ and σ



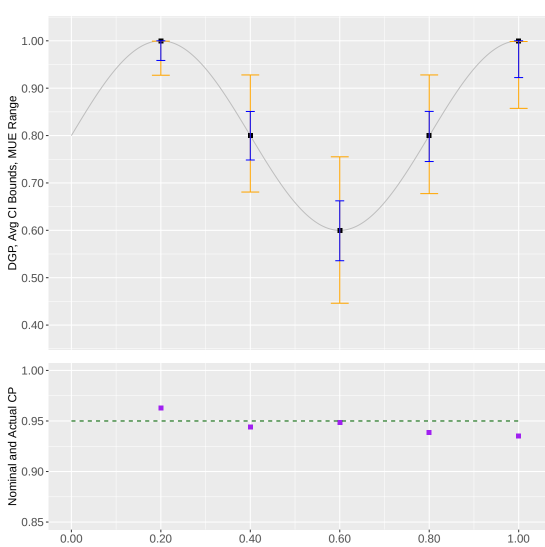
(c) sin 1.00-0.80-1.00, time-varying μ and σ



(d) sin 0.80-1.00-0.80, time-varying μ and σ



(e) sin 1.00-0.60-1.00, time-varying μ and σ



(f) sin 0.60-1.00-0.60, time-varying μ and σ

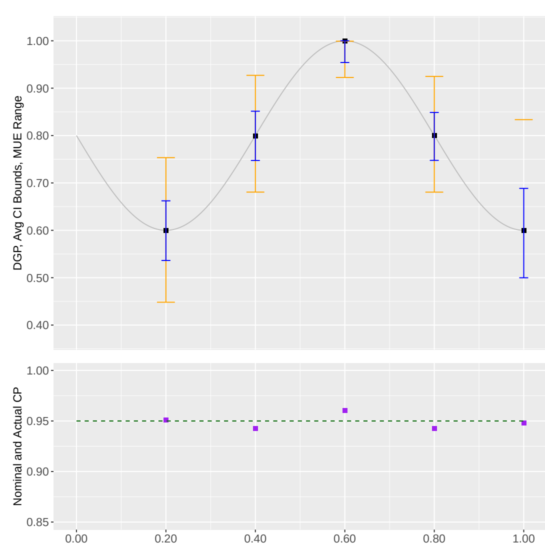
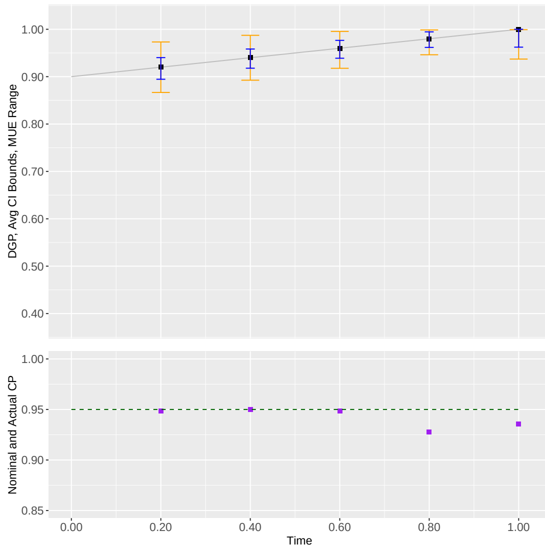
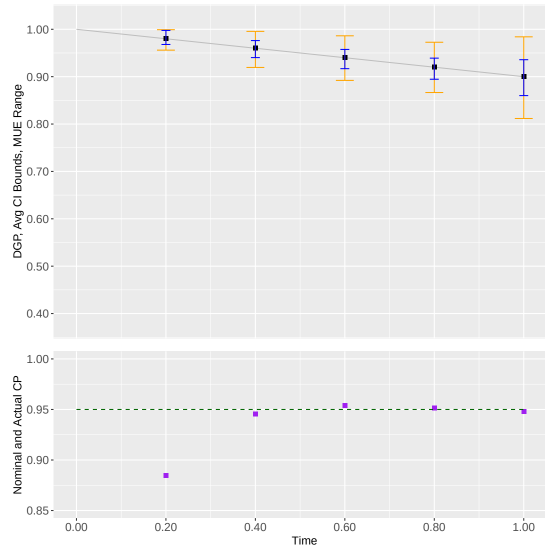


Figure SM.1: CP's and AL's of CI's for $\rho(\tau)$ and MAD's of the MUE of $\rho(\tau)$

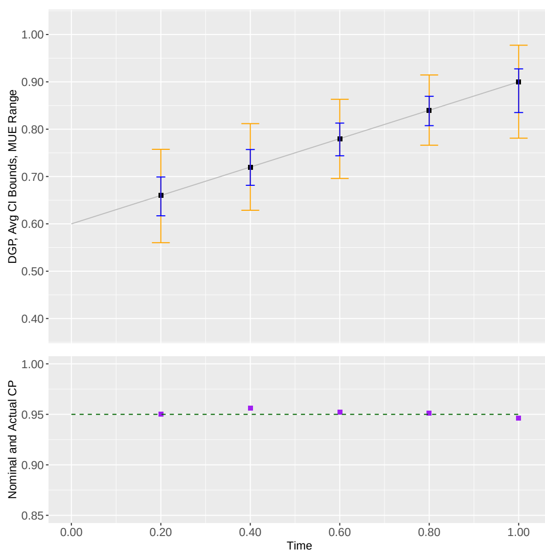
(a) linear 0.90-1.00, time-varying μ and σ



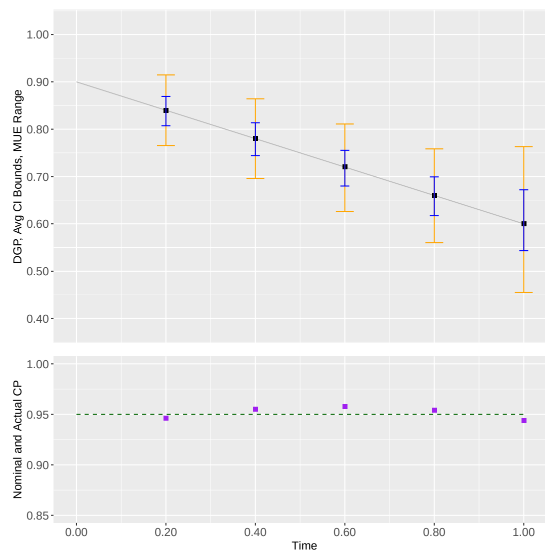
(b) linear 1.00-0.90, time-varying μ and σ



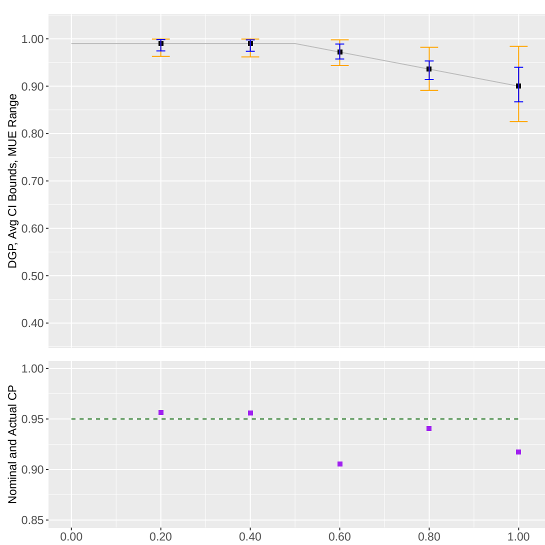
(c) linear 0.60-0.90, time-varying μ and σ



(d) linear 0.90-0.60, time-varying μ and σ



(e) flat-lin 0.99-0.90, constant μ and σ



(f) flat-lin 0.99-0.90, time-varying μ and σ

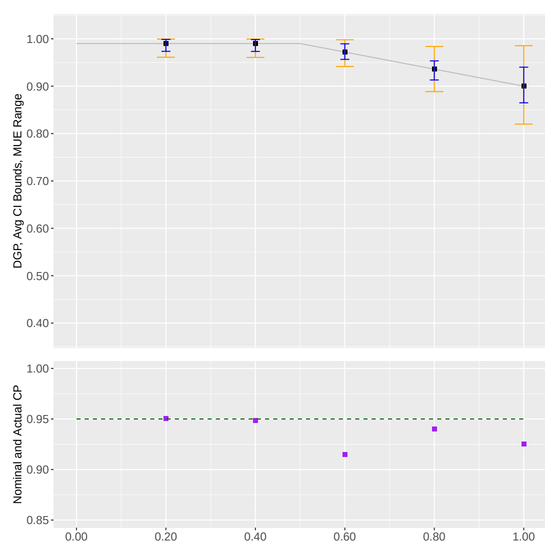
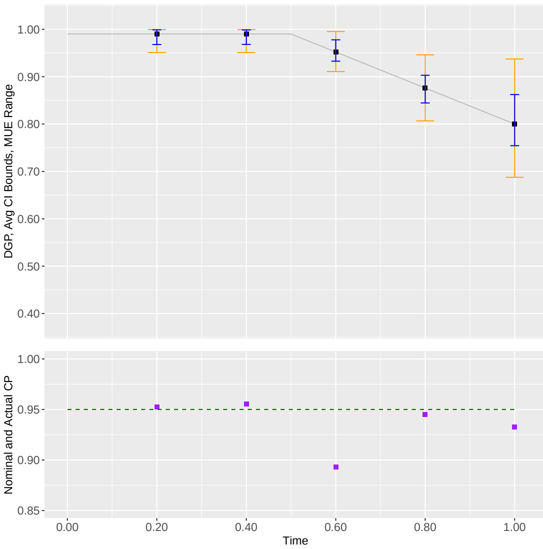
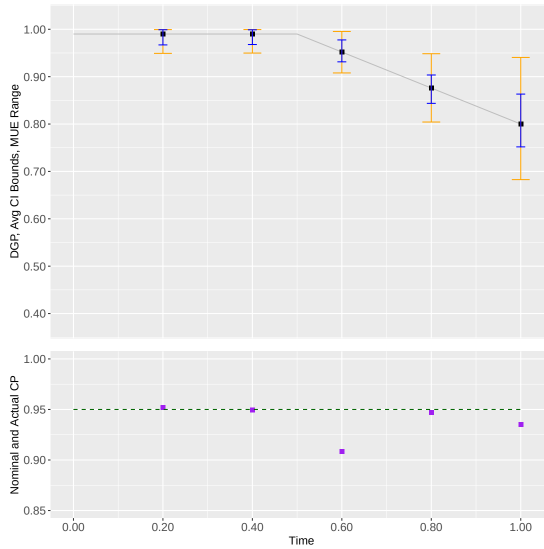


Figure SM.2: CP's and AL's of CI's for $\rho(\tau)$ and MAD's of the MUE of $\rho(\tau)$

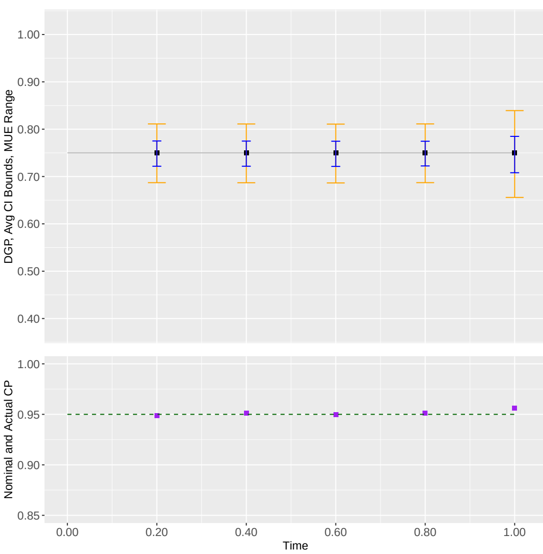
(a) flat-lin 0.99-0.80, constant μ and σ



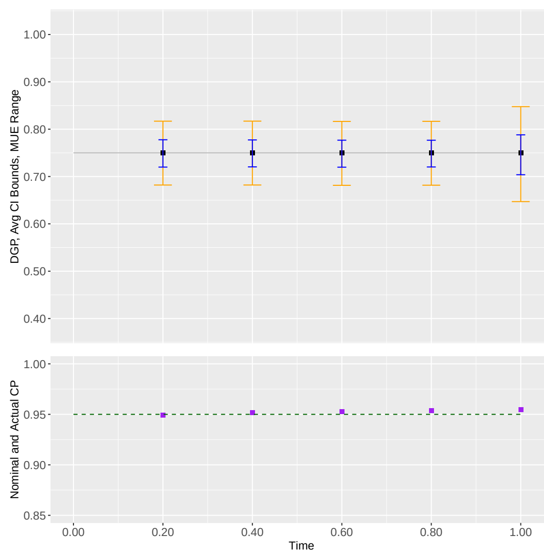
(b) flat-lin 0.99-0.80, time-varying μ and σ



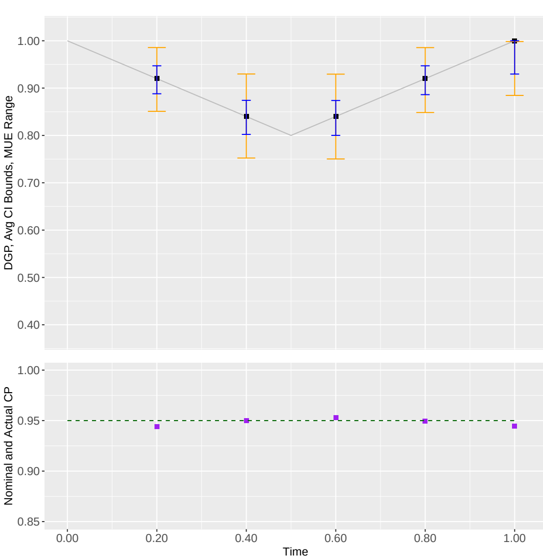
(c) flat 0.75, constant μ and σ



(d) flat 0.75, time-varying μ and σ



(e) kinked 1.00-0.80-1.00, constant μ and σ



(f) kinked 1.00-0.80-1.00, time-varying μ and σ

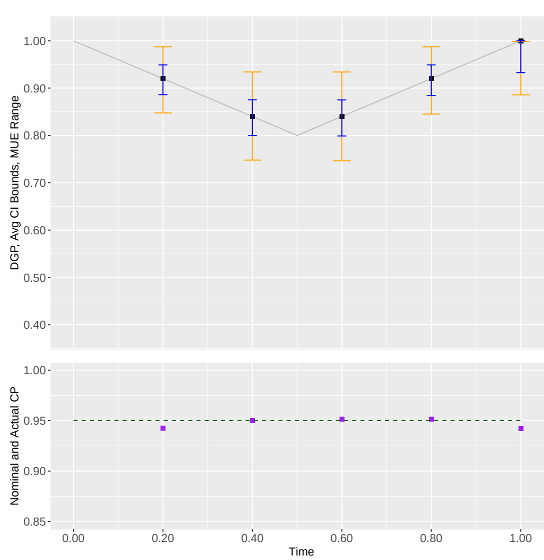
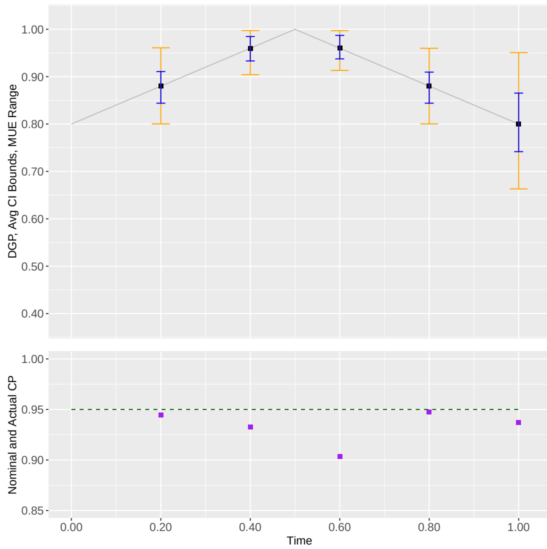
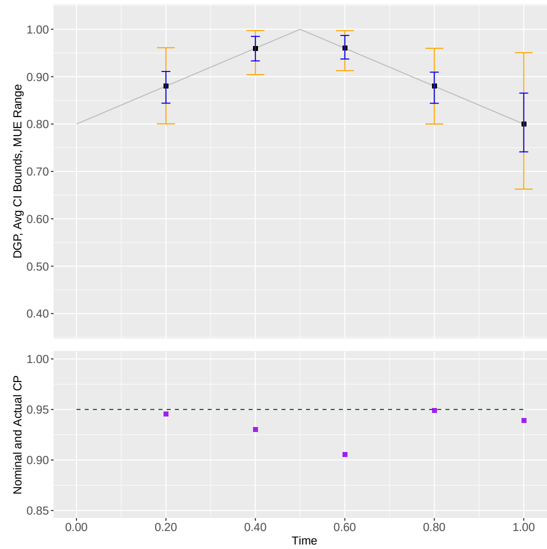


Figure SM.3: CP's and AL's of CI's for $\rho(\tau)$ and MAD's of the MUE of $\rho(\tau)$

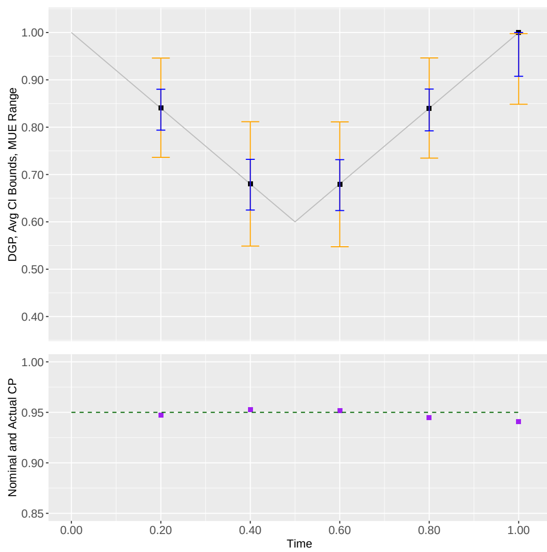
(a) kinked 0.80-1.00-0.80, constant μ and σ



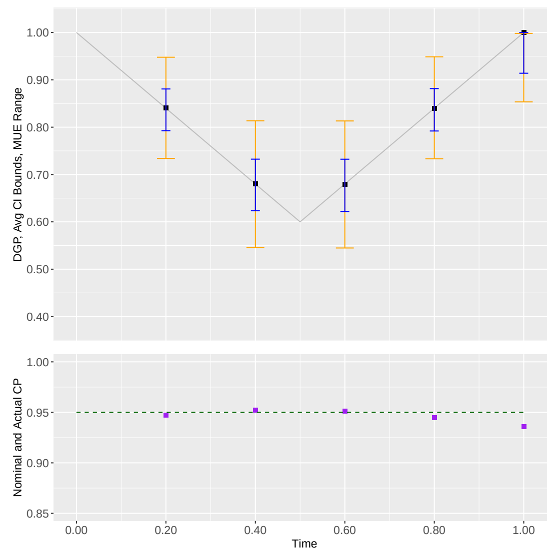
(b) kinked 0.80-1.00-0.80, time-varying μ and σ



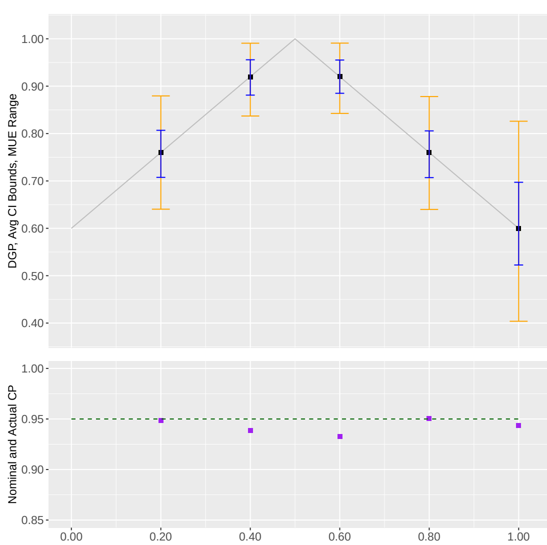
(c) kinked 1.00-0.60-1.00, constant μ and σ



(d) kinked 1.00-0.60-1.00, time-varying μ and σ



(e) kinked 0.60-1.00-0.60, constant μ and σ



(f) kinked 0.60-1.00-0.60, time-varying μ and σ

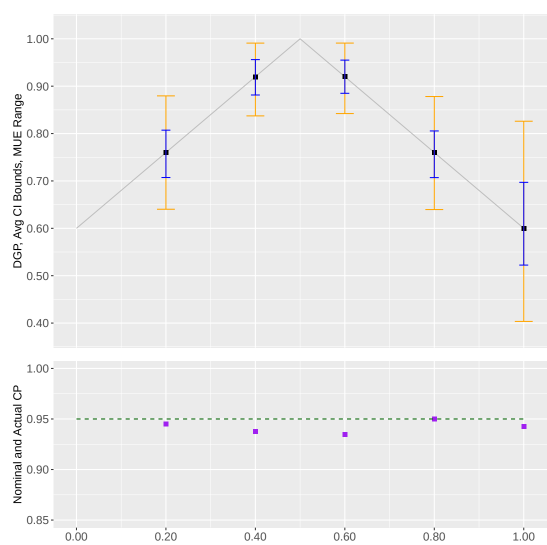


Figure SM.4: CP's and AL's of CI's for $\rho(\tau)$ and MAD's of the MUE of $\rho(\tau)$

D Extension to TVP-AR(p) Models

Here we discuss how the methods introduced in the paper for a TVP-AR(1) model can be extended to a TVP-AR(p) model. We combine the approach discussed above with a method for constant parameter AR(p) models that is similar to, but somewhat different from, methods that have been considered in the literature to date. It is similar to Hansen's (1999) grid bootstrap, also see Mikusheva (2007, Sec. 7), but uses asymptotic critical values, rather than bootstrap critical values, which eases computation considerably in the time-varying case because the tabulated quantiles for the AR(1) case can be utilized (with an adjustment of the ψ_{nh} value that is employed). Asymptotic results for $p > 1$ are beyond the scope of this paper and are not provided here.

Consider the following TVP-AR(p) model written in augmented Dickey-Fuller (ADF) form:

$$\begin{aligned} &Y_t = \mu_t + Y_t^* \text{ and} \\ &Y_t^* = \rho_t Y_{t-1}^* + \sum_{j=1}^{p-1} \beta_{jt} \Delta Y_{t-j}^* + \sigma_t U_t, \text{ for } t = 1, ..., n, \end{aligned} \tag{D.1}$$

where $\Delta Y_{t-j}^* = Y_{t-j}^* - Y_{t-j-1}^*$ for $j = 1, ..., p-1$. Here, μ_t, ρ_t, σ_t, and U_t are as in Section 2. The coefficients β_{jt} are possibly time varying and satisfy analogous properties to those of μ_t. The parameter ρ_t is the sum of the p AR coefficients. It is the parameter of interest because it is a suitable measure of the persistence of the time series, see Andrews and Chen (1994, Sec. 2.2) for a discussion. As in Section 2, $\rho_t := \rho(t/n)$ and, for $\tau \in (0, 1)$, we consider estimation and inference concerning $\rho(\tau)$.

To construct a CI for $\rho(\tau)$ in the AR(p) model, we proceed as follows. First, consider the regression of Y_t on a constant, Y_{t-1}, $\Delta Y_{t-1}, ..., \Delta Y_{t-p+1}$ for $t = T_1, ..., T_2$, where $\Delta Y_s := Y_s - Y_{s-1}$. For arbitrary $\rho_0 \in (-1, 1]$, let $T_n(\rho_0, p)$ be the t-statistic for testing the null hypothesis that the coefficient on the regressor Y_{t-1} in this regression equals ρ_0. Second, compute $\widehat{\beta}(\rho_0) \in R^{p-1}$ from the regression of $Y_t - \rho_0 Y_{t-1}$ on a constant, $\Delta Y_{t-1}, ..., \Delta Y_{t-p+1}$ for $t = T_1, ..., T_2$. Third, one computes

$$\psi_{nh,\rho_0}^p := \frac{-nh \ln(\rho_0)}{\widehat{\lambda}(\rho_0)} \text{ for } \rho_0 > 0 \text{ and } \psi_{nh,\rho_0}^p := \infty \text{ for } \rho_0 \leq 0, \text{ where } \widehat{\lambda}(\rho_0) := 1 - \sum_{j=1}^{p-1} \widehat{\beta}_j(\rho_0). \tag{D.2}$$

A nominal $1 - \alpha$ equal-tailed two-sided CI for $\rho(\tau)$ is given by the formula in (3.10) with

$T_n(\rho_0, p)$ in place of $T_n(\rho_0)$ and with ψ^p_{nh,ρ_0} in place of ψ_{nh,ρ_0} in the critical values. A median-unbiased interval estimator $\widetilde{\rho}_{n\tau}$ of $\rho(\tau)$ is defined as in Section 3.3 with the same changes. For the motivation behind the definition of ψ^p_{nh} above, see Hansen (1995, 1999).

Note that the only computational difference between the above CI's for ρ in the TVP-AR(1) and TVP-AR(p) models is that the latter requires the computation of $\widehat{\beta}(\rho_0)$ for a fine grid of ρ_0 values and each value τ of interest. In contrast, if one replaces the critical values $c_{\psi^p_{nh}}(\alpha/2)$ and $c_{\psi^p_{nh}}(1-\alpha/2)$ by the $\alpha/2$ and $1-\alpha/2$ quantiles of a bootstrap test statistic, e.g., as in Hansen's (1999) grid bootstrap, then one needs to simulate these quantiles for a fine grid of ρ_0 values for each τ of interest, which is computationally quite expensive for reasonable choices of the number of simulation repetitions.

Some empirical applications with $p = 6$ and 12 are reported in Section E below.

E Additional Empirical Results

In this section, we present results for some additional time series in the IFS dataset and some in the FRED dataset. Some of these series require a TVP-AR(p) model for $p > 1$.

As noted in the Introduction, and described in Section D of the Supplemental Material, the methods introduced above for the TVP-AR(1) model can be extended to TVP-AR(p) models with $p > 1$. In the TVP-AR(p) model, the parameter ρ_t is the sum of the autoregressive coefficients at time t, or equivalently, the coefficient at time t on the lagged Y_t value in the augmented Dickey-Fuller representation of the model. This coefficient is a suitable measure of the persistence of the time series at time t, e.g., see Andrews and Chen (1994, Sec. 2.2).

For each time series, we estimate a TVP-AR(p) model with $p = 1, 6, 12$ and examine the degree of autocorrelation of the corresponding residuals by computing Ljung-Box tests with six lags of the residuals. For each time series, we present the results from the TVP-AR(p) model with the smallest value of p for which the null hypothesis of no autocorrelation is not rejected at the 5% level. When $p \in \{6, 12\}$, the MUE's and CI's are for the time-varying autoregressive parameter corresponding to the lagged dependent variable in ADF form. We group the results based on the selected p in the figures.

First, we consider additional time series from the IFS dataset, which include real exchange rate series for Norway, Canada, and Japan, interest rate series for Australia, Canada, and the US, and inflation series for Switzerland. The definition of real exchange rates and inflation are the same as described in Section 6.1 and 6.2, respectively. For the interest rate series, we use the monthly interbank interest rate, which is a key monetary tool for central banks to achieve their policy goals. More details about the length, time period, and frequency of the additional IFS series can be found in Tables SM.2–SM.4.

Figure SM.5 presents the MUE's and 90% CI's of $\rho(t)$ for the additional real exchange rate series. We fit a TVP-AR(1) model for Norway and TVP-AR(6) model for Canada and Japan based on the Ljung-Box tests results. Across all three countries, the MUE's are close to one with reasonably tight 90% CI's. The selected $n\widehat{h}_{us}$ values are quite large, consistent with the parameter estimates that show little time variation. The results echo the empirical findings of high real exchange rate persistence for the developed countries presented in Section 6.2.

Figure SM.6 shows that the MUE's of $\rho(t)$ for the interest rate series from a TVP-AR(6) model are close to one with a moderate degree of variation over time. Most notable are the estimates of $\rho(t)$ for Canada and the US during the period around 2012 when the MUE's drop to as low as .7. The 90% CI's are fairly tight. In comparison, the constant parameter MUE's from an AR(6) model are uniformly one for the three series.

Figure SM.9(a)–(b) summarizes the results for estimating a TVP-AR(12) model for the Switzerland inflation series. The MUE's of $\rho(t)$ are quite volatile over time, ranging between -.6 and 1 in Figure SM.9(a). This is different from the constant parameter estimate which is close to .9, highlighting the importance of allowing for possible time variation in the autoregressive parameters in these models.

Second, we consider the FRED series. We have a total of eight time series for the US, including the 10 year bond yield, average wages for the manufacturing sector, industrial production, real GDP per capita, real GNP, real GNP per capita, S&P 500 index, and the unemployment rate. We provide details about the length, time period, and frequency of the FRED series in Tables SM.3–SM.4.

Figures SM.7–SM.8 show the results from estimating a TVP-AR(6) model for the US FRED series for which the null hypothesis of no autocorrelation is not rejected at the 5% level. In Figure SM.7(e) and (f), there are some variation in the MUE's of $\rho(t)$ for the US unemployment rate series, however the magnitude is small. For all other series in Figures SM.7–SM.8, the MUE's of $\rho(t)$ are uniformly one or very close to one over time and almost the same as constant parameter estimates. All of the 90% CI's are tight with a length smaller than .02. The selected $n\widehat{h}_{us}$ values are large, in line with the parameter estimates that show little time variation. Hence, the methods proposed in the paper deliver a constant parameter unit root, or near unit root, model in circumstances in which such a model is appropriate.

Figure SM.9(c)–(f) provides the results for fitting a TVP-AR(12) model to the time series on the S&P 500 index and the US industrial production. For the S&P 500 index series, the

MUE's of $\rho(t)$ are one and similar to the constant parameter estimates. The results are consistent with predictions from a random walk hypothesis for the US stock markets. For the US industrial production series, the MUE's of $\rho(t)$ in Figure SM.9(e) are close to 1 for most of the time except before 1925 and after 2010. It may be caused by a boundary effect.

Table SM.2: Autocorrelation Test Results for Residuals from Estimating TVP-AR(1) Model

	From	To	Frequency	n	$n\widehat{h}_{us}$	Ljung-Box Test p-Value
US Inflation	1/2/1955	1/10/2022	monthly	813	125	.27
US Inflation	1/2/1955	1/10/2022	monthly	813	188	.28
Canada Inflation	1/2/1955	1/10/2022	monthly	813	125	.38
Canada Inflation	1/2/1955	1/10/2022	monthly	813	188	.22
Germany Inflation	1/2/1955	1/10/2022	monthly	813	125	.76
Germany Inflation	1/2/1955	1/10/2022	monthly	813	188	.81
UK Real Exchange Rate	1/1/1957	1/8/2022	monthly	788	823	.52
UK Real Exchange Rate	1/1/1957	1/8/2022	monthly	788	1,234	.51
Sweden Real Exchange Rate	1/1/1957	1/8/2022	monthly	788	823	.28
Sweden Real Exchange Rate	1/1/1957	1/8/2022	monthly	788	1,234	.28
Switzerland Real Exchange Rate	1/1/1957	1/8/2022	monthly	788	393	.56
Switzerland Real Exchange Rate	1/1/1957	1/8/2022	monthly	788	590	.54
Norway Real Exchange Rate	1/1/1957	1/8/2022	monthly	788	823	.37
Norway Real Exchange Rate	1/1/1957	1/8/2022	monthly	788	1,234	.38

Table SM.3: Autocorrelation Test Results for Residuals from Estimating TVP-AR(6) Model

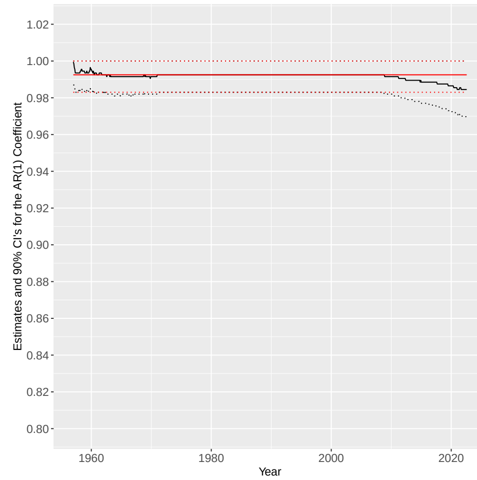
	From	To	Frequency	n	$n\widehat{h}_{us}$	Ljung-Box Test p-Value
Canada Real Exchange Rate	1/1/1957	1/8/2022	Monthly	788	823	1.00
Canada Real Exchange Rate	1/1/1957	1/8/2022	Monthly	788	1,234	1.00
Japan Real Exchange Rate	1/1/1957	1/8/2022	Monthly	788	823	.81
Japan Real Exchange Rate	1/1/1957	1/8/2022	Monthly	788	1,234	.84
Australia Interest Rate	1/5/1976	1/4/2017	Monthly	492	160	.97
Australia Interest Rate	1/5/1976	1/4/2017	Monthly	492	240	.98
Canada Interest Rate	1/5/1976	1/4/2017	Monthly	492	95	.95
Canada Interest Rate	1/5/1976	1/4/2017	Monthly	492	143	.97
US Interest Rate	1/5/1976	1/4/2017	Monthly	492	103	.98
US Interest Rate	1/5/1976	1/4/2017	Monthly	492	155	.97
US 10yr Bond Yield	2/1/1962	20/1/2023	Daily	15248	16,006	1.00
US 10yr Bond Yield	2/1/1962	20/1/2023	Daily	15248	24,009	1.00
US Average Wages Manufacturing	1/1/1939	1/12/2022	Monthly	1008	152	.12
US Average Wages Manufacturing	1/1/1939	1/12/2022	Monthly	1008	228	.27
US Unemployment Rate	1/1/1948	1/12/2022	Monthly	900	900	.98
US Unemployment Rate	1/1/1948	1/12/2022	Monthly	900	1,350	.99
US Real GDP Per Capita	1/1/1947	1/7/2022	Quarterly	303	317	.97
US Real GDP Per Capita	1/1/1947	1/7/2022	Quarterly	303	476	.98
US Real GNP	1/1/1947	1/7/2022	Quarterly	303	317	1.00
US Real GNP	1/1/1947	1/7/2022	Quarterly	303	476	.48
US Real GNP Per Capita	1/1/1947	1/7/2022	Quarterly	303	317	.96
US Real GNP Per Capita	1/1/1947	1/7/2022	Quarterly	303	476	.99

Table SM.4: Autocorrelation Test Results for Residuals from Estimating TVP-AR(12) Model

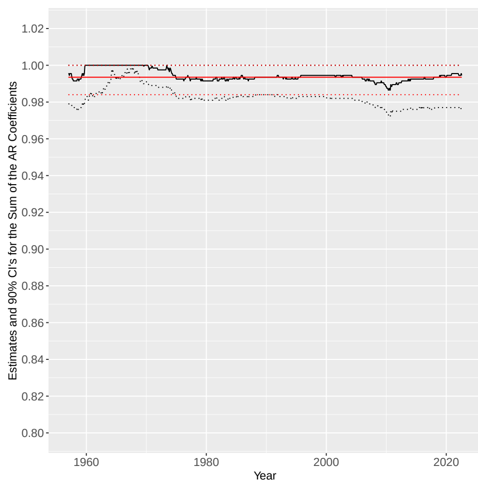
	From	To	Frequency	n	nh	Ljung-Box Test p-Value
Switzerland Inflation	1/2/1955	1/10/2022	Monthly	813	125	.96
Switzerland Inflation	1/2/1955	1/10/2022	Monthly	813	188	.63
S&P 500 Index	24/1/2013	23/1/2023	Daily	2517	2,518	1.00
S&P 500 Index	24/1/2013	23/1/2023	Daily	2517	3,777	1.00
US Industrial Production	1/1/1919	1/12/2022	Monthly	1248	368	1.00
US Industrial Production	1/1/1919	1/12/2022	Monthly	1248	551	.97

(a) Norway Real Exchange Rate, $n\widehat{h}_{us} = 823$

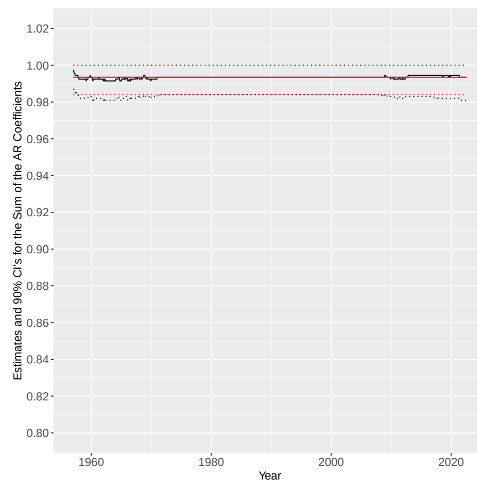
(b) Norway Real Exchange Rate, $1.5n\widehat{h}_{us} = 1{,}234$



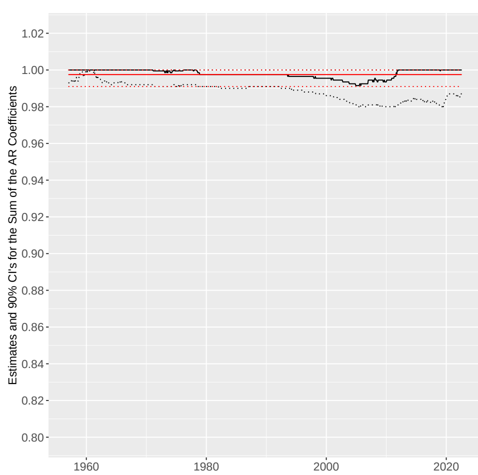
(c) Canada Real Exchange Rate, $n\widehat{h}_{us} = 823$



(d) Canada Real Exchange Rate, $1.5n\widehat{h}_{us} = 1{,}234$



(e) Japan Real Exchange Rate, $n\widehat{h}_{us} = 823$



(f) Japan Real Exchange Rate, $1.5n\widehat{h}_{us} = 1{,}234$

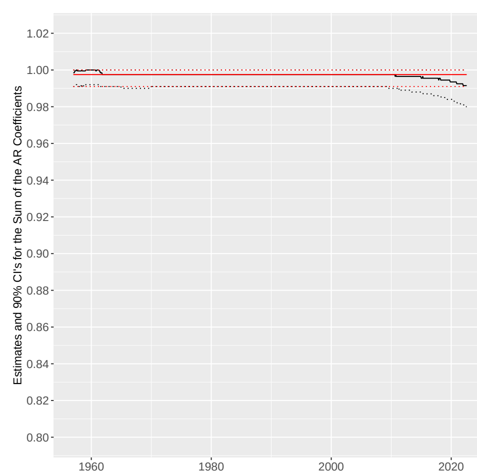
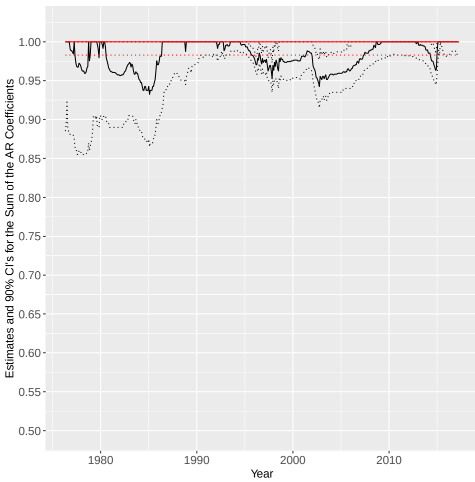
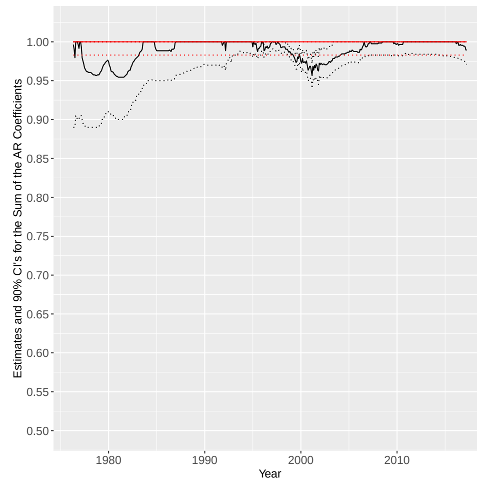


Figure SM.5: Estimates and 90% CI's for the Sum of the AR Coefficients in TVP-AR Models: Norway Real Exchange Rate (TVP-AR(1)), Canada and Japan Real Exchange Rate (TVP-AR(6))

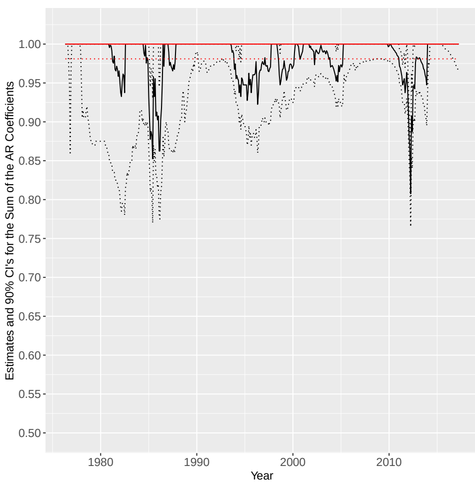
(a) Australia Interest Rate, $n\widehat{h}_{us} = 160$



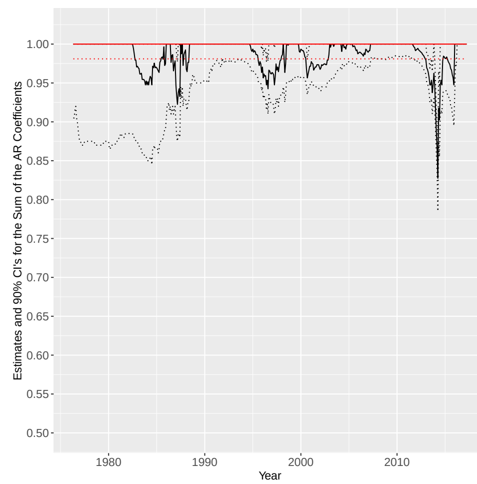
(b) Australia Interest Rate, $1.5n\widehat{h}_{us} = 240$



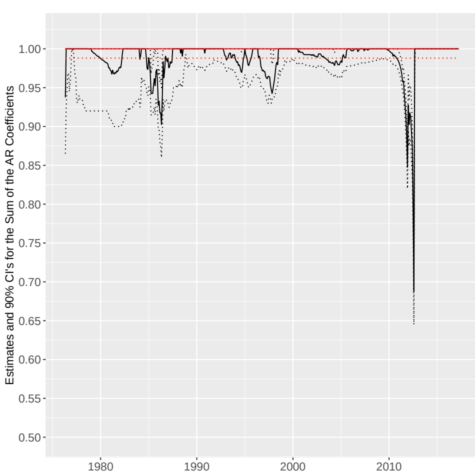
(c) Canada Interest Rate, $n\widehat{h}_{us} = 95$



(d) Canada Interest Rate, $1.5n\widehat{h}_{us} = 143$



(e) US Interest Rate, $n\widehat{h}_{us} = 103$



(f) US Interest Rate, $1.5n\widehat{h}_{us} = 155$

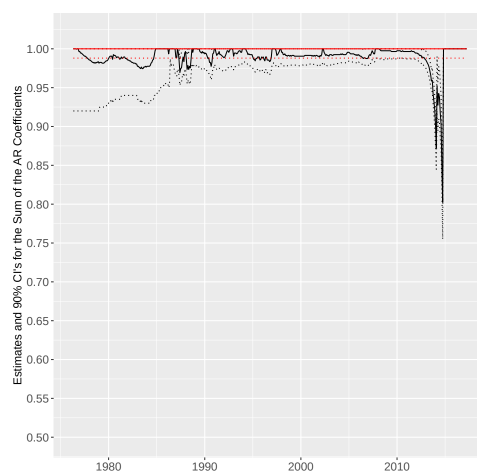
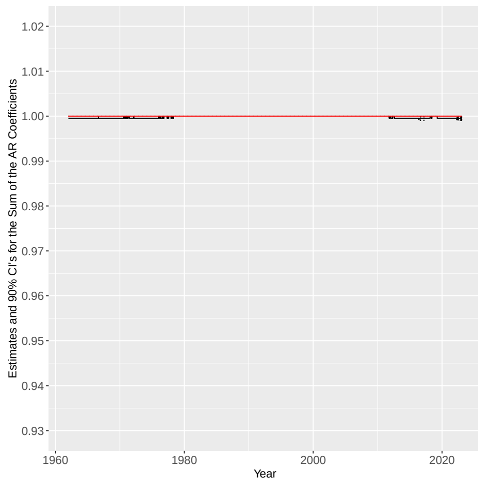
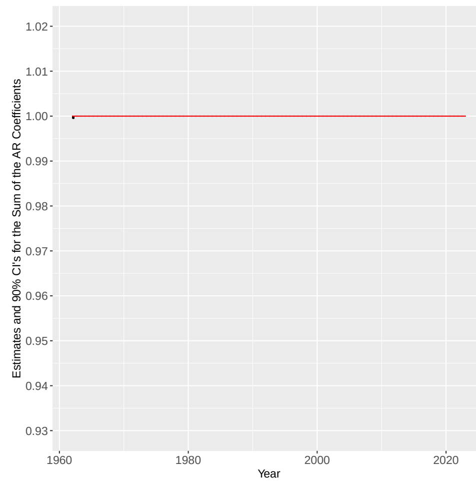


Figure SM.6: Estimates and 90% CI's for the Sum of the AR Coefficients in TVP-AR(6) Models: Australia, Canada, and the US Interest Rate

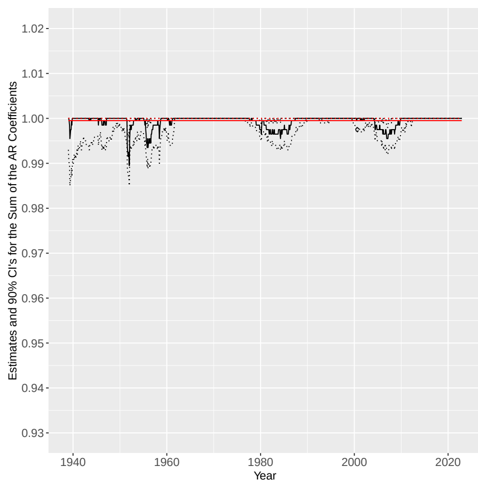
(a) US 10yr Bond Yield, $\widehat{nh}_{us} = 16{,}006$



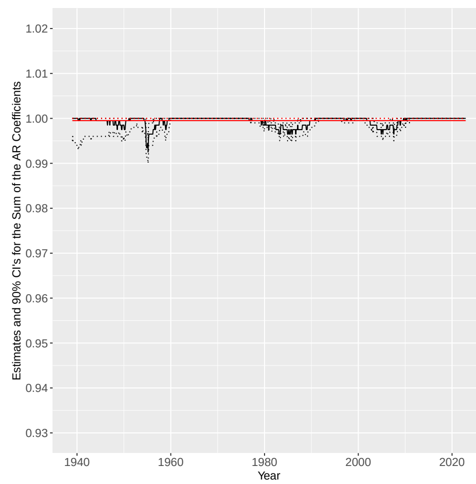
(b) US 10yr Bond Yield, $1.5\widehat{nh}_{us} = 24{,}009$



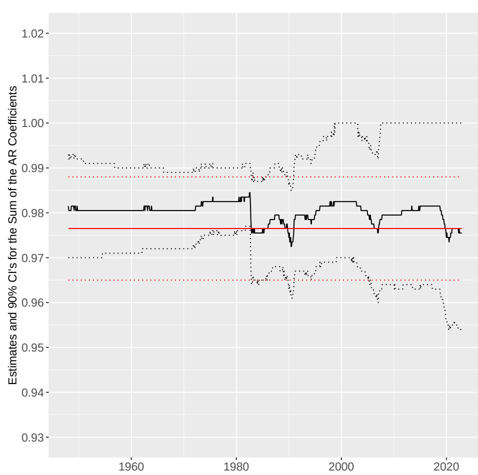
(c) US Avg Wages Manufacturing, $\widehat{nh}_{us} = 152$



(d) US Avg Wages Manufacturing, $1.5\widehat{nh}_{us} = 228$



(e) US Unemployment Rate, $\widehat{nh}_{us} = 900$



(f) US Unemployment Rate, $1.5\widehat{nh}_{us} = 1{,}350$

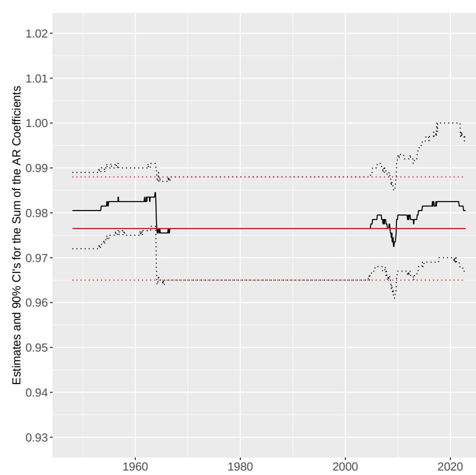
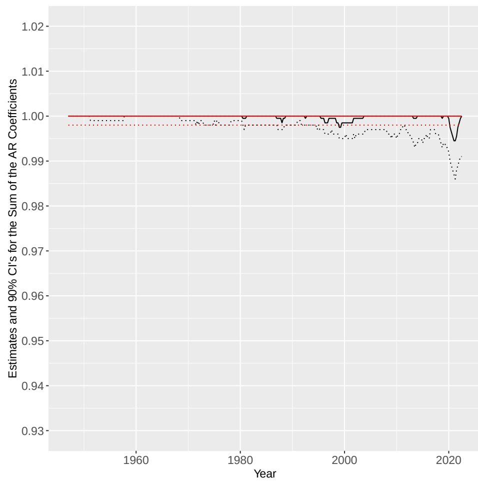
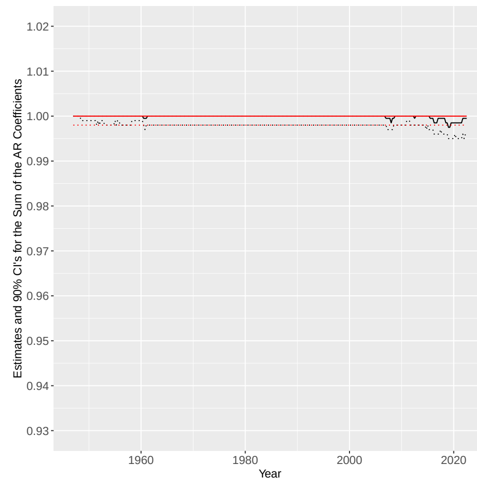


Figure SM.7: Estimates and 90% CI's for the Sum of the AR Coefficients in TVP-AR(6) Models: US 10yr Bond Yield, US Average Wages Manufacturing, and US Unemployment Rate

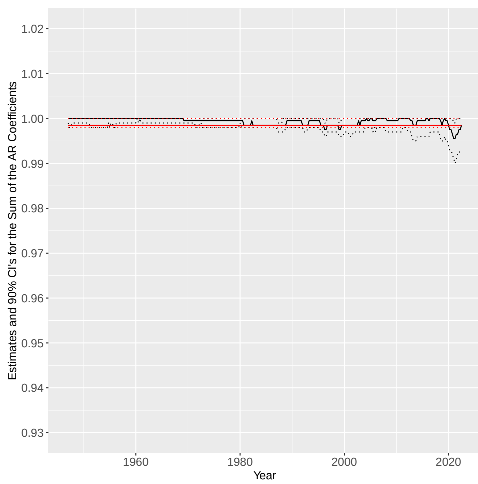
(a) US Real GDP Per Capita, $n\widehat{h}_{us} = 317$



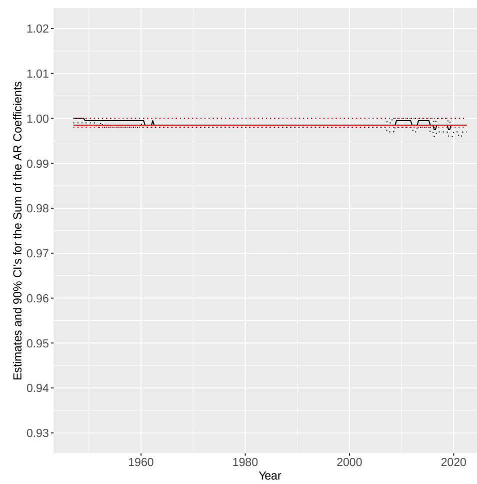
(b) US Real GDP Per Capita, $1.5n\widehat{h}_{us} = 476$



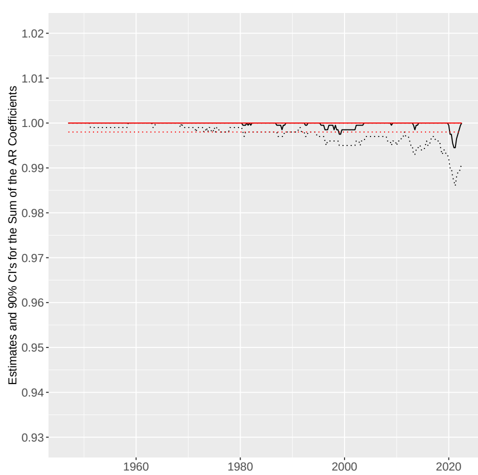
(c) US Real GNP, $n\widehat{h}_{us} = 317$



(d) US Real GNP, $1.5n\widehat{h}_{us} = 476$



(e) US Real GNP Per Capita, $n\widehat{h}_{us} = 317$



(f) US Real GNP Per Capita, $1.5n\widehat{h}_{us} = 476$

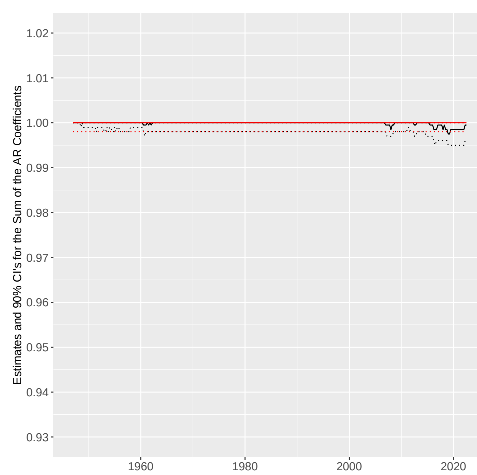
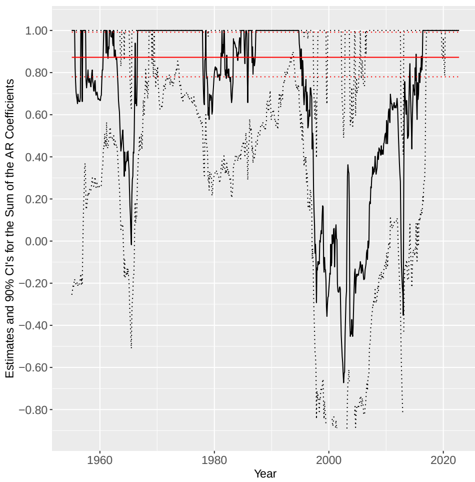
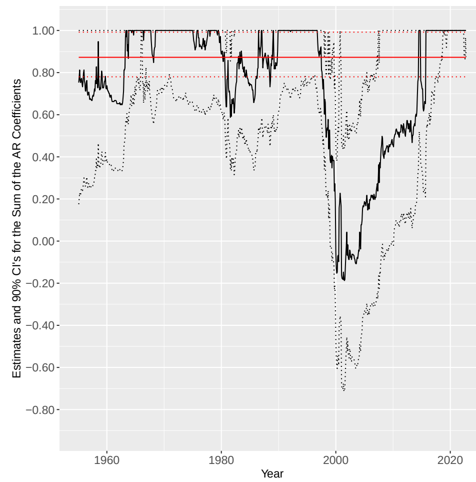


Figure SM.8: Estimates and 90% CI's for the Sum of the AR Coefficients in TVP-AR(6) Models: US Real GDP Per Capita, US Real GNP, and US Real GNP Per Capita

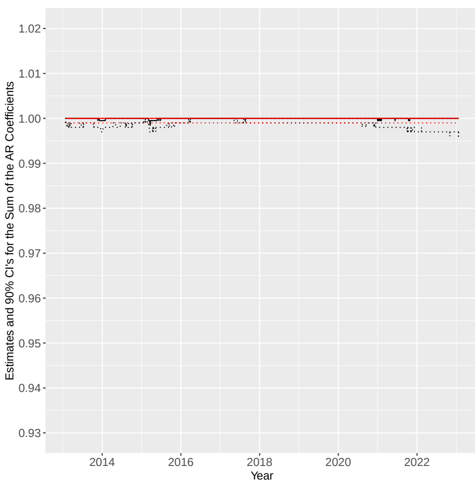
(a) Switzerland Inflation, $n\widehat{h}_{us} = 125$



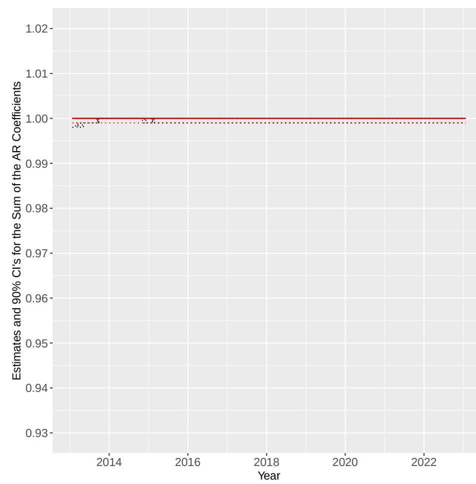
(b) Switzerland Inflation, $1.5n\widehat{h}_{us} = 188$



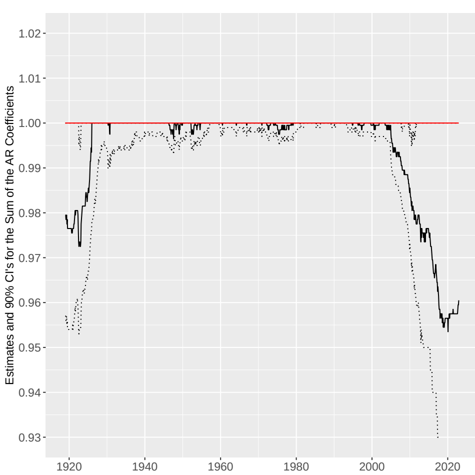
(c) S&P 500 Index, $n\widehat{h}_{us} = 2{,}518$



(d) S&P 500 Index, $1.5n\widehat{h}_{us} = 3{,}777$



(e) US Industrial Production, $n\widehat{h}_{us} = 368$



(f) US Industrial Production, $1.5n\widehat{h}_{us} = 551$

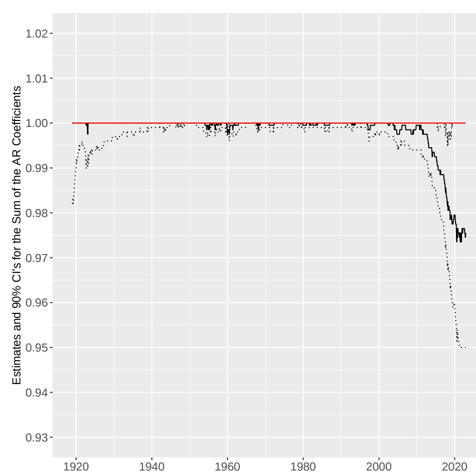


Figure SM.9: Estimates and 90% CI's for the Sum of the AR Coefficients in TVP-AR(12) Models: Switzerland Inflation, S&P 500 Index, and US Industrial Production

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