

Supplemental Material to
Limit Theory for Locally Flat Functional Coefficient Regression

By

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Online Supplement to “Limit Theory for Locally Flat Functional Coefficient Regression”*

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This Online Supplement contains three parts. Each of these addresses concerns arising from the fact that the flatness degree parameter L is typically unknown in practical work. The first part proves that the oracle estimator L_n^\dagger is consistent for L . The second part provides a key result that is used to show that the plausible practical estimator \hat{L} defined in (3.12) of the main paper is not consistent for L . The third part demonstrates that the adaptive bias estimator $\hat{B}(z)$ that is designed for inference in the absence of knowledge of L is not consistent for the true bias $h^{L^*} \mathcal{B}_L(z)$ in either the stationary or the nonstationary case. Some further complications with the adaptive approach to inference are also discussed.

1 Proof that $L_n^\dagger \rightarrow_p L$

We use the Taylor series representation $\beta(z_t) - \beta(z) = \frac{\beta^{(L)}(\tilde{z}_t)}{L!} (z_t - z)^L$ where \tilde{z}_t lies on the line segment between z_t and z and $\beta^{(L)}(z) \neq 0$ by assumption. Taking sample averages of the magnitudes of the weighted differentials $|\beta(z_t) - \beta(z)| K_{tz}$ we have

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n |\beta(z_t) - \beta(z)| K_{tz} &= \frac{1}{n} \sum_{t=1}^n \left| \frac{\beta^{(L)}(\tilde{z}_t)}{L!} \right| |z_t - z|^L K_{tz} \\ &\sim_a \int \left| \frac{\beta^{(L)}(\tilde{z}_t)}{L!} \right| |z_t - z|^L K \left(\frac{z_t - z}{h} \right) f(z_t) dz_t \\ &= h \int \left| \frac{\beta^{(L)}(z + \tilde{s}h)}{L!} \right| |s|^L h^L K(s) f(z + sh) ds \sim_a h^{L+1} f(z) \left| \frac{\beta^{(L)}(z)}{L!} \right| \int |s|^L K(s) ds, \end{aligned}$$

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with \tilde{s} on the line segment between s and 0. Further,

$$\frac{1}{n} \sum_{t=1}^n K_{tz} \sim_a \int K \left(\frac{z_t - z}{h} \right) f(z_t) dz_t = h \int K(s) f(z + sh) ds \sim_a hf(z).$$

Then

$$\begin{aligned} \log \left(\frac{1}{n} \sum_{t=1}^n |\beta(z_t) - \beta(z)| w_{tz} \right) &= \log \left(\frac{1}{n} \sum_{t=1}^n |\beta(z_t) - \beta(z)| K_{tz} \right) - \log \left(\frac{1}{n} \sum_{t=1}^n K_{tz} \right) \\ &\sim_a \log \left(h^{L+1} f(z) \left| \frac{\beta^{(L)}(z)}{L!} \right| \int |s|^L K(s) ds \right) - \log(hf(z)) \\ &= (L+1) \log(h) + \log \left(\left| \frac{\beta^{(L)}(z)}{L!} \right| \int |s|^L K(s) ds \right) - \log(h) \\ &= L \log(h) + \log \left(\left| \frac{\beta^{(L)}(z)}{L!} \right| \int |s|^L K(s) ds \right). \end{aligned}$$

It follows that as $n \rightarrow \infty$ and $h \rightarrow 0$

$$L_n^\dagger = \frac{1}{\log(h)} \log \left(\sum_{t=1}^n |\beta(z_t) - \beta(z)| w_{tz} \right) \sim_a L + \frac{1}{\log(h)} \log \left(\left| \frac{\beta^{(L)}(z)}{L!} \right| \int |s|^L K(s) ds \right) \rightarrow_p L, \quad (1.1)$$

since $\beta^{(L)}(z) \neq 0$, and thus

$$\log(h)(L_n^\dagger - L) = \log \left(\left| \frac{\beta^{(L)}(z)}{L!} \right| \right) + \log \left(\int |s|^L K(s) ds \right) + o_p(1), \quad (1.2)$$

so that the rate of convergence of L_n^\dagger is $O(\log(h))$ but with a deterministic bias function as given on the right side of (1.2). Further, since the true value $L \geq 1$ is known to be integer, it follows that $\check{L} = \max\{[L_n^\dagger], 1\} \rightarrow_p L$, giving consistency of the implied integer estimator \check{L} . ■

2 Order of magnitude of $\hat{\beta}(z + ph) - \hat{\beta}(z)$

We start with the stationary case. In view of

$$\begin{aligned} \hat{\beta}(z) &= \left(\sum_{t=1}^n x_t x_t' K_{tz} \right)^{-1} \left\{ \sum_{t=1}^n x_t x_t' \beta(z_t) K_{tz} + \sum_{t=1}^n x_t u_t K_{tz} \right\} \\ &\sim_a (nh)^{-1} \Sigma_{xx}^{-1} f^{-1}(z) \left\{ \sum_{t=1}^n x_t x_t' \beta(z_t) K_{tz} + \sum_{t=1}^n x_t u_t K_{tz} \right\}, \end{aligned} \quad (2.1)$$

we have

$$\begin{aligned}
& \hat{\beta}(z + ph) - \hat{\beta}(z) \\
& \sim_a (nh)^{-1} \Sigma_{xx}^{-1} f^{-1}(z + ph) \left(\sum_{s=1}^n x_s x'_s \beta(z_s) K\left(\frac{z_s - z - ph}{h}\right) + \sum_{s=1}^n x_s u_s K\left(\frac{z_s - z - ph}{h}\right) \right) \\
& - (nh)^{-1} \Sigma_{xx}^{-1} f^{-1}(z) \left(\sum_{s=1}^n x_s x'_s \beta(z_s) K\left(\frac{z_s - z}{h}\right) + \sqrt{nh} \sum_{s=1}^n \frac{x_s u_s}{\sqrt{nh}} K\left(\frac{z_s - z}{h}\right) \right) \\
& \sim_a \frac{1}{nh} \Sigma_{xx}^{-1} \frac{1}{f(z + ph)} \left[nh \Sigma_{xx} \int \beta(z + hp + hw) K(w) f(z + hp + hw) dw + \sqrt{nh} \sum_{s=1}^n \frac{x_s u_s}{\sqrt{nh}} K\left(\frac{z_s - z - ph}{h}\right) \right] \\
& - (nh)^{-1} \Sigma_{xx}^{-1} f^{-1}(z) \left(nh \Sigma_{xx} \int \beta(z + qh) K(q) f(z + qh) dq + \sqrt{nh} \sum_{s=1}^n \frac{x_s u_s}{\sqrt{nh}} K\left(\frac{z_s - z}{h}\right) \right) \\
& \sim_a [f(z) + O(h)]^{-1} \left[\beta(z) f(z) + h \beta(z) f^{(1)}(z) \int (p + w) K(w) dw + O(h^2) \right] \\
& - f^{-1}(z) \left[\beta(z) f(z) + h^2 \beta(z) \frac{1}{2} f^{(2)}(z) \int q^2 K(q) dq + O(h^4) \right] \\
& + \frac{1}{\sqrt{nh}} \Sigma_{xx}^{-1} [f(z) + O(h)]^{-1} \sum_{s=1}^n \frac{x_s u_s}{\sqrt{nh}} \left[K\left(\frac{z_s - z}{h} - p\right) \right] - \frac{1}{\sqrt{nh}} \Sigma_{xx}^{-1} f^{-1}(z) \sum_{s=1}^n \frac{x_s u_s}{\sqrt{nh}} K\left(\frac{z_s - z}{h}\right) \\
& \sim_a h p f^{-1}(z) \beta(z) f^{(1)}(z) + o_p(h) + O_p\left(\frac{1}{\sqrt{nh}}\right) = O_p\left(h + \frac{1}{\sqrt{nh}}\right). \tag{2.2}
\end{aligned}$$

In the nonstationary case, the proof can be carried out in similar fashion. We only outline the essentials.

$$\begin{aligned}
& \hat{\beta}(z + ph) - \hat{\beta}(z) \\
& \sim_a \left[f(z + ph) \int B_x B'_x \right]^{-1} \left(\frac{1}{nh} \sum_{s=1}^n \frac{x_s}{\sqrt{n}} \frac{x'_s}{\sqrt{n}} \beta(z_s) K\left(\frac{z_s - z - ph}{h}\right) + \frac{1}{n\sqrt{h}} \sum_{s=1}^n \frac{x_s}{\sqrt{n}} \frac{u_s}{\sqrt{nh}} K\left(\frac{z_s - z - ph}{h}\right) \right) \\
& - \left[f(z) \int B_x B'_x \right]^{-1} \left(\frac{1}{nh} \sum_{s=1}^n \frac{x_s}{\sqrt{n}} \frac{x'_s}{\sqrt{n}} \beta(z_s) K\left(\frac{z_s - z}{h}\right) + \frac{1}{n\sqrt{h}} \sum_{s=1}^n \frac{x_s}{\sqrt{n}} \frac{u_s}{\sqrt{nh}} K\left(\frac{z_s - z}{h}\right) \right) \\
& \sim_a \left[f(z + ph) \int B_x B'_x \right]^{-1} \left(\int B_x^2 \int \beta(z + ph + qh) K(q) f(z + ph + qh) dq \right) \\
& - \left[f(z) \int B_x B'_x \right]^{-1} \left(\int B_x^2 \int \beta(z + qh) K(q) f(z + qh) dq \right) \\
& + \left[f(z) \int B_x B'_x \right]^{-1} \frac{1}{n\sqrt{h}} \sum_{s=1}^n \frac{x_s}{\sqrt{n}} \frac{u_s}{\sqrt{nh}} \left[K\left(\frac{z_s - z - ph}{h}\right) - K\left(\frac{z_s - z}{h}\right) \right] \\
& \sim_a f^{-1}(z) \left[\beta(z) f(z) + h \beta(z) f^{(1)}(z) \int (p + q) K(q) dq - \beta(z) f(z) + O(h^2) \right] \\
& + \left[f(z) \int B_x B'_x \right]^{-1} \frac{1}{n\sqrt{h}} \sum_{s=1}^n \frac{x_s}{\sqrt{n}} \frac{u_s}{\sqrt{nh}} \left[K\left(\frac{z_s - z - ph}{h}\right) - K\left(\frac{z_s - z}{h}\right) \right]
\end{aligned}$$

$$\sim_a h p f^{-1}(z) \beta(z) f^{(1)}(z) + o(h) + O_p\left(\frac{1}{n\sqrt{h}}\right) = O_p\left(h + \frac{1}{n\sqrt{h}}\right). \quad (2.3)$$

■

3 Failure of the adaptive approach

We provide some details to show that the adaptive approach does not work in the case of $L > 1$. More specifically, we will show the adaptive bias estimator $\hat{B}(z)$, which is

$$\hat{B}(z) = A_n(z)^{-1} \left(\sum_{t=1}^n x_t x_t' \right) \left(\frac{1}{n} \sum_{s=1}^n [\hat{\beta}(z_s) - \hat{\beta}(z)] K \left(\frac{z_s - z}{h} \right) \right), \quad (3.1)$$

is not consistent for the true bias $h^{L^*} \mathcal{B}_L(z) = h^{L^*} \frac{\mu_{L^*}(K)}{f(z)} C_L(z)$ when $L > 1$. As before we examine the stationary and nonstationary x_t cases separately.

(i) Stationary x_t In an attempt to show $\hat{B}(z) \sim_a h^{L^*} \frac{\mu_{L^*}(K)}{f(z)} C_L(z) + o(h^{L^*})$, the critical step is to obtain the following asymptotic representation

$$\frac{1}{n} \sum_{t=1}^n [\hat{\beta}(z_t) - \hat{\beta}(z)] K \left(\frac{z_t - z}{h} \right) \sim_a \mathbb{E} \xi_{\beta t} = h^{L^*+1} \mu_{L^*}(K) C_L(z) + o(h^{L^*+1}) \quad (3.2)$$

with $C_L(z) = \frac{f(z)\beta^{(L)}(z)}{L!} \mathbf{1}_{\{L=\text{even}\}} + \left[\frac{\beta^{(L)}(z)}{L!} f^{(1)}(z) + \frac{\beta^{(L+1)}(z)}{(L+1)!} f(z) \right] \mathbf{1}_{\{L=\text{odd}\}}$, and $L^* = L \mathbf{1}_{\{L=\text{even}\}} + (L+1) \mathbf{1}_{\{L=\text{odd}\}}$. However, the asymptotic representation (3.2) does not hold, as we now show.

To begin, we write

$$\begin{aligned} & \frac{1}{n} \sum_{t=1}^n [\hat{\beta}(z_t) - \hat{\beta}(z)] K \left(\frac{z_t - z}{h} \right) \\ &= \frac{1}{n} \sum_{t=1}^n \left\{ \beta(z_t) - \beta(z) + \hat{\beta}(z_t) - \beta(z_t) - [\hat{\beta}(z) - \beta(z)] \right\} K \left(\frac{z_t - z}{h} \right) \\ &= \frac{1}{n} \sum_{t=1}^n \left\{ \beta(z_t) - \beta(z) \right\} K \left(\frac{z_t - z}{h} \right) + \frac{1}{n} \sum_{t=1}^n \left\{ \hat{\beta}(z_t) - \beta(z_t) - [\hat{\beta}(z) - \beta(z)] \right\} K \left(\frac{z_t - z}{h} \right) \\ &=: \Pi_{1n} + \Pi_{2n}. \end{aligned} \quad (3.3)$$

By Assumption 1(i) and standard manipulations, $\Pi_{1n} = \frac{1}{n} \sum_{t=1}^n \left\{ \beta(z_t) - \beta(z) \right\} K \left(\frac{z_t - z}{h} \right) \sim_a \mathbb{E} \xi_{\beta t}$.

Now consider Π_{2n} . Note that

$$\hat{\beta}(z) - \beta(z) = \left(\sum_{t=1}^n x_t x_t' K_{tz} \right)^{-1} \left\{ \sum_{t=1}^n x_t x_t' [\beta(z_t) - \beta(z)] K_{tz} + \sum_{t=1}^n x_t u_t K_{tz} \right\}$$

$$\sim_a (nh)^{-1} \Sigma_{xx}^{-1} f^{-1}(z) \left\{ \sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z)] K_{sz} + \sum_{s=1}^n x_s u_s K_{sz} \right\}. \quad (3.4)$$

Then

$$\begin{aligned} \hat{\beta}(z_t) - \hat{\beta}(z) &= \hat{\beta}(z_t) - \beta(z_t) - [\hat{\beta}(z) - \beta(z)] \\ &\sim_a (nh)^{-1} \Sigma_{xx}^{-1} \left\{ f^{-1}(z_t) \left(\sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z_t)] K_{st} + \sum_{s=1}^n x_s u_s K_{st} \right) \right. \\ &\quad \left. - f^{-1}(z) \left(\sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z)] K_{sz} + \sum_{s=1}^n x_s u_s K_{sz} \right) \right\} \\ &= (nh)^{-1} \Sigma_{xx}^{-1} \left\{ f^{-1}(z_t) \sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z_t)] K_{st} - f^{-1}(z) \sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z)] K_{sz} \right\} \\ &\quad + (nh)^{-1} \Sigma_{xx}^{-1} \left\{ f^{-1}(z_t) \sum_{s=1}^n x_s u_s K_{st} - f^{-1}(z) \sum_{s=1}^n x_s u_s K_{sz} \right\}. \end{aligned} \quad (3.5)$$

Hence

$$\begin{aligned} \Pi_{2n} &= \frac{1}{n} \sum_{t=1}^n \left\{ \hat{\beta}(z_t) - \beta(z_t) - [\hat{\beta}(z) - \beta(z)] \right\} K \left(\frac{z_t - z}{h} \right) \\ &\sim_a (nh)^{-1} \Sigma_{xx}^{-1} \frac{1}{n} \sum_{t=1}^n \left\{ f^{-1}(z_t) \sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z_t)] K_{st} - f^{-1}(z) \sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z)] K_{sz} \right\} K_{tz} \\ &\quad + (nh)^{-1} \Sigma_{xx}^{-1} \frac{1}{n} \sum_{t=1}^n \left\{ f^{-1}(z_t) \sum_{s=1}^n x_s u_s K_{st} - f^{-1}(z) \sum_{s=1}^n x_s u_s K_{sz} \right\} K_{tz} \\ &= \Sigma_{xx}^{-1} \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n} \sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z_t)] K_{st} - \Sigma_{xx}^{-1} f^{-1}(z) \frac{1}{nh} \sum_t K_{tz} \frac{1}{n} \sum_s x_s x'_s [\beta(z_s) - \beta(z)] K_{sz} \\ &\quad + h \Sigma_{xx}^{-1} \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{nh} \sum_{s=1}^n x_s u_s K_{st} - h \Sigma_{xx}^{-1} f^{-1}(z) \frac{1}{nh} \sum_{t=1}^n K_{tz} \frac{1}{nh} \sum_{s=1}^n x_s u_s K_{sz} \\ &=: \Pi_{2n1} - \Pi_{2n2} + \Pi_{2n3} - \Pi_{2n4}. \end{aligned} \quad (3.6)$$

Consider the terms Π_{2ni} , $i = 1, 2, 3, 4$ in turn. Start with Π_{2n1} and to simplify derivations in what follows assume that the z_t are independent and identically distributed. The weakly dependent case involves much longer calculations but can be handled under Assumption 1 along lines such as those used in the proof of Lemma B.2. Following a similar line of argument as that leading to (B.6) we have, using conditional expectations $\mathbb{E}_{z,t}$ given z_t ,

$$\Pi_{2n1} = \Sigma_{xx}^{-1} \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n} \sum_{s=1}^n x_s x'_s [\beta(z_s) - \beta(z_t)] K_{st}$$

$$\begin{aligned}
&= \Sigma_{xx}^{-1} \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n} \sum_{s \neq t}^n x_s x'_s [\beta(z_s) - \beta(z_t)] K_{st} \\
&\sim_a \Sigma_{xx}^{-1} \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \Sigma_{xx} \mathbb{E}_{z,t} [\beta(z_s) - \beta(z_t)] K_{st} \\
&\sim_a \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \mathbb{E}_{z,t} [\beta(z_s) - \beta(z_t)] K_{st} \\
&\sim_a \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \int_{-1}^1 [\beta(z_s) - \beta(z_t)] K \left(\frac{z_s - z_t}{h} \right) f(z_s) dz_s \\
&\sim_a \frac{1}{n} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \int_{-1}^1 [\beta(z_t + ph) - \beta(z_t)] K(p) f(z_t + ph) dp \\
&\sim_a \frac{1}{n} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \int_{-1}^1 \sum_{j=1}^{L+1} \frac{1}{j!} \beta^{(j)}(z_t) (ph)^j K(p) [f(z_t) + f^{(1)}(z_t) ph] dp \\
&\sim_a \frac{1}{n} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \sum_{j=1}^{L+1} \frac{1}{j!} \beta^{(j)}(z_t) h^j \left[\int p^j K(p) dp f(z_t) + \int p^{j+1} K(p) dp f^{(1)}(z_t) h \right] \\
&\sim_a \frac{1}{n} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \sum_{j=1}^{L+1} \frac{1}{j!} \beta^{(j)}(z_t) h^j [\mu_j f(z_t) + \mu_{j+1} f^{(1)}(z_t) h] \\
&\sim_a \int K_{tz} \sum_{j=1}^{L+1} \frac{1}{j!} \beta^{(j)}(z_t) h^j [\mu_j f(z_t) + \mu_{j+1} f^{(1)}(z_t) h] dz_t \\
&\sim_a \sum_{j=1}^{L+1} \frac{1}{j!} h^j \mu_j \int \beta^{(j)}(z_t) K_{tz} f(z_t) dz_t + \sum_{j=1}^{L+1} \frac{1}{j!} h^{j+1} \mu_{j+1} \int \beta^{(j)}(z_t) K_{tz} f^{(1)}(z_t) dz_t \\
&\sim_a \sum_{j=1}^{L+1} \frac{1}{j!} h^{j+1} \mu_j \int \beta^{(j)}(z + hu) K(u) f(z + hu) du + \sum_{j=1}^{L+1} \frac{1}{j!} h^{j+2} \mu_{j+1} \int \beta^{(j)}(z + hu) K(u) f^{(1)}(z + hu) du \\
&\sim_a \sum_{j=1}^{L+1} \frac{1}{j!} h^{j+1} \mu_j \int \sum_{k=0}^{L-j+1} \frac{1}{k!} \beta^{(j+k)}(z) (hu)^k K(u) f(z + hu) du \\
&+ \sum_{j=1}^{L+1} \frac{1}{j!} h^{j+2} \mu_{j+1} \int \sum_{k=0}^{L+1-j} \frac{1}{k!} \beta^{(j+k)}(z) (hu)^k K(u) f^{(1)}(z + hu) du \\
&\sim_a \sum_{j=1}^L \frac{1}{j!} h^{j+1} \mu_j \frac{1}{(L-j)!} \beta^{(L)}(z) \int (hu)^{L-j} K(u) f(z + hu) du \\
&+ \sum_{j=1}^{L+1} \frac{1}{j!} h^{j+1} \mu_j \frac{1}{(L+1-j)!} \beta^{(L+1)}(z) \int (hu)^{L+1-j} K(u) f(z + hu) du \\
&+ \sum_{j=1}^L \frac{1}{j!} h^{j+2} \mu_{j+1} \frac{1}{(L-j)!} \beta^{(L)}(z) \int (hu)^{L-j} K(u) f^{(1)}(z + hu) du
\end{aligned}$$

$$\begin{aligned}
&\sim_a h^{L+1} \sum_{j=1}^L \frac{1}{j!} \mu_j \frac{1}{(L-j)!} \beta^{(L)}(z) \mu_{L-j} f(z) + h^{L+2} \sum_{j=1}^L \frac{1}{j!} \mu_j \frac{1}{(L-j)!} \beta^{(L)}(z) \mu_{L+1-j} f^{(1)}(z) \\
&+ h^{L+2} \sum_{j=1}^{L+1} \frac{1}{j!} \mu_j \frac{1}{(L+1-j)!} \beta^{(L+1)}(z) \mu_{L+1-j} f(z) + h^{L+2} \sum_{j=1}^L \frac{1}{j!} \mu_{j+1} \frac{1}{(L-j)!} \beta^{(L)}(z) \mu_{L-j} f^{(1)}(z) \\
&\sim_a h^{L+1} \left\{ f(z) \beta^{(L)}(z) \sum_{j=1}^L \frac{1}{j!} \mu_j \frac{1}{(L-j)!} \mu_{L-j} \right\} \times 1_{\{L=\text{even}\}} \\
&+ h^{L+2} \left\{ \beta^{(L)}(z) f^{(1)}(z) \sum_{j=1}^L \frac{1}{j!(L-j)!} [\mu_j \mu_{L+1-j} + \mu_{j+1} \mu_{L-j}] \right. \\
&\left. + \beta^{(L+1)}(z) f(z) \sum_{j=1}^{L+1} \frac{1}{j!(L+1-j)!} \mu_j \mu_{L+1-j} \right\} \times 1_{\{L=\text{odd}\}} \\
&=: h^{L^*+1} D_L(z).
\end{aligned}$$

The second to last equation is due to the fact that when $L = \text{even}$, $\mu_j \mu_{L+1-j} = 0$ and when $L = \text{odd}$, $\mu_j \mu_{L-j} = 0$.

For the term Π_{2n2} we have

$$\begin{aligned}
\Pi_{2n2} &= \Sigma_{xx}^{-1} f^{-1}(z) \frac{1}{nh} \sum_t K_{tz} \frac{1}{n} \sum_s x_s x'_s [\beta(z_s) - \beta(z)] K_{sz} \\
&\sim_a \Sigma_{xx}^{-1} f^{-1}(z) f(z) \mathbb{E}(x_s x'_s) \mathbb{E}[\beta(z_s) - \beta(z)] K_{sz} \sim_a \mathbb{E} \xi_{\beta t}.
\end{aligned} \tag{3.7}$$

So the leading terms of Π_{1n} and Π_{2n2} are both asymptotically equivalent to $\mathbb{E} \xi_{\beta t}$. Below we will see $\Pi_{2n3} - \Pi_{2n4}$ has zero mean and does not contribute to the bias centering expression. It follows that the leading term of $\Pi_{1n} + \Pi_{2n}$ is determined by that of Π_{2n1} , which is very complicated and involves the unknown value of L . Hence, (3.2) is unlikely to hold. As a result, $\hat{B}(z)$ is not a consistent estimate of the bias $h^{L^*} \mathcal{B}_L(z)$; and, due the dependence of the limit of Π_{2n1} on unknown L , it is impossible to scale adjust $\hat{B}(z)$ to get consistent bias estimate.

Next, consider the remaining two terms of (3.6) involving the difference

$$\Pi_{2n3} - \Pi_{2n4} = h \Sigma_{xx}^{-1} \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{nh} \sum_{s=1}^n x_s u_s K_{st} - h \Sigma_{xx}^{-1} f^{-1}(z) \frac{1}{nh} \sum_{t=1}^n K_{tz} \frac{1}{nh} \sum_{s=1}^n x_s u_s K_{sz}, \tag{3.8}$$

which has zero mean and therefore does not contribute to the bias centering expression. But, as in [Phillips and Wang \(2020\)](#), we need to analyze this term's contribution to the variance of the limit distribution of the statistic. For ease of presentation and calculation, set $g = f^{-1}$,

take the scalar x_t case and let $u_t \sim_{iid} (0, \sigma_u^2)$ and z_t be *iid*, as above. Write (3.8) as

$$\Pi_{2n3} - \Pi_{2n4} = \frac{1}{n} \Sigma_{xx}^{-1} \frac{1}{\sqrt{nh}} \sum_{s=1}^n x_s u_s \left(\frac{1}{\sqrt{nh}} \sum_{t=1}^n K_{tz} [K_{st}g(z_t) - K_{sz}g(z)] \right), \quad (3.9)$$

and since $x_s u_s$ is a martingale difference, this expression has mean zero and variance

$$\begin{aligned} & \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^2 h} \mathbb{E} \left(\frac{1}{\sqrt{nh}} \sum_{t=1}^n K_{tz} [K_{st}g(z_t) - K_{sz}g(z)] \right)^2 \\ &= \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^3 h^2} \mathbb{E} \left(\sum_{t,r=1}^n K_{tz} K_{rz} [K_{st}g(z_t) - K_{sz}g(z)] [K_{sr}g(z_r) - K_{sz}g(z)] \right) \\ &= \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^2 h^2} \mathbb{E} \left(K_{tz}^2 [K_{st}g(z_t) - K_{sz}g(z)]^2 \right) + \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^3 h^2} \sum_{t \neq r} \mathbb{E} (K_{tz} K_{rz} [K_{st}g(z_t) - K_{sz}g(z)] [K_{sr}g(z_r) - K_{sz}g(z)]) \\ &=: D_1 + D_2. \end{aligned} \quad (3.10)$$

Noting that the particular case where $s = t$ in the dual summation (3.9) is of smaller order and can therefore be neglected in (3.9), we have

$$\begin{aligned} D_1 &= \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^2 h^2} \mathbb{E} \left(K_{tz}^2 [K_{st}g(z_t) - K_{sz}g(z)]^2 \right) \\ &= \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^2 h^2} \int_{-1}^1 \int_{-1}^1 K \left(\frac{z_t - z}{h} \right)^2 \left[K \left(\frac{z_s - z_t}{h} \right) g(z_t) - K \left(\frac{z_s - z}{h} \right) g(z) \right]^2 f(z_s) f(z_t) dz_s dz_t \\ &= \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^2} \int_{-1}^1 \int_{-1}^1 K(p)^2 [K(p-q)g(z+ph) - K(q)g(z)]^2 f(z+qh) f(z+ph) dpdq \\ &\sim_a \frac{\sigma_u^2 \Sigma_{xx}^{-1} g(z)^2 f(z)^2}{n^2} \int_{-1}^1 \int_{-1}^1 K(p)^2 [K(p-q) - K(q)]^2 dpdq \\ &= \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^2} \int_{-1}^1 \int_{-1}^1 K(p)^2 [K(p-q) - K(q)]^2 dpdq. \end{aligned} \quad (3.11)$$

Next, again noting that the case where $s = t$ is of smaller order and can be neglected, we have

$$\begin{aligned} D_2 &= \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{n^3 h^2} \sum_{t \neq r} \mathbb{E} (K_{tz} K_{rz} [K_{st}g(z_t) - K_{sz}g(z)] [K_{sr}g(z_r) - K_{sz}g(z)]) \\ &\sim_a \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{nh^2} \mathbb{E} (K_{tz} K_{rz} [K_{st}g(z_t) - K_{sz}g(z)] [K_{sr}g(z_r) - K_{sz}g(z)]) \\ &\sim_a \frac{\sigma_u^2 \Sigma_{xx}^{-1}}{nh^2} \int \int \int K \left(\frac{z_t - z}{h} \right) K \left(\frac{z_r - z}{h} \right) \left[K \left(\frac{z_s - z_t}{h} \right) g(z_t) - K \left(\frac{z_s - z}{h} \right) g(z) \right] \\ &\quad \times \left[K \left(\frac{z_s - z_r}{h} \right) g(z_r) - K \left(\frac{z_s - z}{h} \right) g(z) \right] f(z_s) f(z_t) f(z_r) dz_s dz_t dz_r \end{aligned}$$

$$\begin{aligned}
&= h \frac{\sigma_u^2 \sum_{xx}^{-1}}{n} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K(p) K(w) [K(p-q)g(z+ph) - K(q)g(z)] \\
&\times [K(q-w)g(z+hw) - K(q)g(z)] f(z+qh) f(z+ph) f(z+wh) dpdqdw \\
&\sim_a h \frac{\sigma_u^2 \sum_{xx}^{-1}}{n} f(z) \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K(p) K(w) [K(p-q) - K(q)] [K(q-w) - K(q)] dpdqdw.
\end{aligned} \tag{3.12}$$

It follows that $\Pi_{2n3} - \Pi_{2n4}$ has zero mean and asymptotic variance $D_1 + D_2 = O(h/n)$ in view of (3.10) - (3.12), so that $\Pi_{2n3} - \Pi_{2n4} = O_p(\sqrt{h/n})$. As a result, $A_n(z)^{-1}(\sum_t x_t x_t')(\Pi_{2n3} - \Pi_{2n4}) = O_p(\frac{1}{nh} n \sqrt{\frac{h}{n}}) = O_p(1/\sqrt{nh})$, which is of the same order with the asymptotic variance term in the stationary case since the convergence rate is \sqrt{nh} . Consequently,

$$\begin{aligned}
\hat{B}(z) &= A_n(z)^{-1} \left(\sum_t x_t x_t' \right) [\Pi_{1n} + \Pi_{2n1} - \Pi_{2n2} + \Pi_{2n3} - \Pi_{2n4}] \\
&\sim_a \frac{1}{h f(z)} (h^{L^*+1} D_L(z) + o_p(h^{L^*+1})) + A_n(z)^{-1} \left(\sum_t x_t x_t' \right) (\Pi_{2n3} - \Pi_{2n4}) \\
&\sim_a h^{L^*} f^{-1}(z) D_L(z) + o_p(h^{L^*}) + O_p(1/\sqrt{nh}).
\end{aligned} \tag{3.13}$$

This analysis reveals that $\hat{B}(z)$ is not consistent for the true bias $h^{L^*} \mathcal{B}_L(z)$. And also, it retains a random element that is $O_p(1)$ after standardization by the convergence rate \sqrt{nh} in the stationary case. So the bias term adjustment $\hat{B}(z)$ affects the limit distribution of the bias corrected estimation error $\hat{\beta}(z) - \beta(z) - \hat{B}(z)$. In effect, the adaptive bias adjustment $\hat{B}(z)$ introduces an estimation error through the presence of the element $\hat{\beta}(z_t) - \hat{\beta}(z)$. This estimation error contributes a random term $A_n(z)^{-1}(\sum_t x_t x_t')(\Pi_{2n3} - \Pi_{2n4})$ to the limit distribution that has the same order $O_p(\frac{1}{\sqrt{nh}})$ as the asymptotic variance term (the last term in (A.8) in the main paper).

In view of these difficulties, we do not consider the adaptive variance estimators $\hat{\Omega}_n(z)$ (in the stationary case) and $\hat{\Omega}_n^*(z)$ (in the nonstationary case). In short, without an adaptive consistent bias estimator $\hat{B}(z)$ to adjust estimation error $\hat{\beta}(z) - \beta(z)$ even a correctly adjusted variance matrix estimator would be unable to produce an asymptotically valid test statistic.

(ii) Nonstationary x_t The adaptive approach to constructing a test statistic does not work in this case either. First, as shown below, the adaptive bias estimator is not consistent for the same reason as in the stationary case. Moreover, the adaptive bias estimator introduces additional variation in the limit distribution and this additional variance takes a more complicated form in the nonstationary case than in the stationary case and depends on the unknown flatness parameter L and the derivative $\beta^{(L)}(z)$. Attempts to estimate these variances in an adaptive way introduces further bias terms in the adaptive variance estimator. These interactions among bias and variance effects make it difficult to formulate a successful adaptive approach to inference that is not reliant on knowledge or consistent estimation of the parameter L in the nonstationary

case. Thus, in spite of the apparent simplicity of the adaptive statistic and its formulation independent of L , the high technical complexity of its asymptotics and interactive bias and variance effects are serious challenges that are left for subsequent research on adaptation in the presence of flat function behavior.

We first show the bias estimator $\hat{B}(z)$ is not consistent. The analysis follows lines similar to those used above while allowing for limiting moment behavior of the nonstationary x_t . We have the same decomposition as in (3.3), viz.,

$$\frac{1}{n} \sum_{t=1}^n [\hat{\beta}(z_t) - \hat{\beta}(z)] K \left(\frac{z_t - z}{h} \right) = \Pi_{1n}^* + \Pi_{2n}^*. \quad (3.14)$$

As before, we have $\Pi_{1n}^* = \frac{1}{n} \sum_{t=1}^n \{\beta(z_t) - \beta(z)\} K \left(\frac{z_t - z}{h} \right) \sim_a \mathbb{E} \xi_{\beta t}$. Proceeding in the nonstationary case, we have

$$\begin{aligned} \Pi_{2n}^* &= \frac{1}{n} \sum_{t=1}^n \left\{ \hat{\beta}(z_t) - \beta(z_t) - [\hat{\beta}(z) - \beta(z)] \right\} K \left(\frac{z_t - z}{h} \right) \\ &\sim_a \frac{1}{n^2 h} \left(\int B_x B_x' \right)^{-1} \frac{1}{n} \sum_{t=1}^n \left\{ f^{-1}(z_t) \sum_{s=1}^n x_s x_s' [\beta(z_s) - \beta(z_t)] K_{st} - f^{-1}(z) \sum_{s=1}^n x_s x_s' [\beta(z_s) - \beta(z)] K_{sz} \right\} K_{tz} \\ &+ \frac{1}{n^2 h} \left(\int B_x B_x' \right)^{-1} \frac{1}{n} \sum_{t=1}^n \left\{ f^{-1}(z_t) \sum_{s=1}^n x_s u_s K_{st} - f^{-1}(z) \sum_{s=1}^n x_s u_s K_{sz} \right\} K_{tz} \\ &= \left(\int B_x B_x' \right)^{-1} \frac{1}{n^2 h} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n} \sum_{s=1}^n x_s x_s' [\beta(z_s) - \beta(z_t)] K_{st} \\ &- \left(\int B_x B_x' \right)^{-1} f^{-1}(z) \frac{1}{n^2 h} \sum_t K_{tz} \frac{1}{n} \sum_s x_s x_s' [\beta(z_s) - \beta(z)] K_{sz} \\ &+ h \left(\int B_x B_x' \right)^{-1} \frac{1}{n^2 h} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{nh} \sum_{s=1}^n x_s u_s K_{st} \\ &- h \left(\int B_x B_x' \right)^{-1} f^{-1}(z) \frac{1}{n^2 h} \sum_{t=1}^n K_{tz} \frac{1}{nh} \sum_{s=1}^n x_s u_s K_{sz} \\ &=: \Pi_{2n1}^* - \Pi_{2n2}^* + \Pi_{2n3}^* - \Pi_{2n4}^*. \end{aligned} \quad (3.15)$$

In analyzing these terms we employ similar arguments to those in the stationary case. For the first term, using conditional expectations $\mathbb{E}_{z,t}$ given z_t , we have

$$\begin{aligned} \Pi_{2n1}^* &= \left(\int B_x B_x' \right)^{-1} \frac{1}{n} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n^2 h} \sum_{s \neq t}^n x_s x_s' [\beta(z_s) - \beta(z_t)] K_{st} \\ &= \left(\int B_x B_x' \right)^{-1} \frac{1}{n} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n^2 h} \sum_{s \neq t}^n x_s x_s' \mathbb{E}_{z,t} \xi_{\beta st} \end{aligned}$$

$$\begin{aligned}
& + \left(\int B_x B'_x \right)^{-1} \frac{1}{n} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n^2 h} \sum_{s \neq t}^n x_s x'_s [\xi_{\beta st} - \mathbb{E}_{z,t} \xi_{\beta st}] \\
& =: \Pi_{2n1}^{*a} + \Pi_{2n1}^{*b}
\end{aligned} \tag{3.16}$$

where $\xi_{\beta st} := [\beta(z_s) - \beta(z_t)]K_{st}$. For the first term Π_{2n1}^{*a} we have $\Pi_{2n1}^{*a} \sim_a h^{L^*+1} D_L(z)$ following the same lines as that of Π_{2n1} in the stationary case. For the second term Π_{2n1}^{*b} , first as in Lemma B.2 we can verify that $\mathbb{E}_{z,t} \xi_{\beta st} = O(h^{L^*+1})$ and $\mathbb{E}_{z,t} \xi_{\beta st}^2 = O(h^{2L+1})$. As a result, $\sum_{s \neq t}^n x_s x'_s [\xi_{\beta st} - \mathbb{E}_{z,t} \xi_{\beta st}] = O_p(n\sqrt{nh^{2L+1}}) \sim_a n\sqrt{nh^{2L+1}} \int B_x B'_x dB_{\xi, z_t}$ where B_{ξ, z_t} is dependent on z_t . Then we have

$$\begin{aligned}
\Pi_{2n1}^{*b} & = \left(\int B_x B'_x \right)^{-1} \frac{1}{n} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n^2 h} \sum_{s \neq t}^n x_s x'_s [\xi_{\beta st} - \mathbb{E}_{z,t} \xi_{\beta st}] \\
& \sim_a \left(\int B_x B'_x \right)^{-1} h \frac{1}{nh} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{n^2 h} n \sqrt{nh^{2L+1}} \int B_x B'_x dB_{\xi, z_t} \\
& = h \frac{1}{n^2 h} n \sqrt{nh^{2L+1}} \left(\int B_x B'_x \right)^{-1} f^{-1}(z) f(z) \int B_x B'_x dB_{\xi, z} \\
& = \sqrt{\frac{h^{2L+1}}{n}} \left(\int B_x B'_x \right)^{-1} \int B_x B'_x dB_{\xi, z}
\end{aligned} \tag{3.17}$$

$$= O_p \left(\sqrt{\frac{h^{2L+1}}{n}} \right), \tag{3.18}$$

where $B_{\xi, z}$ means localize z_t at z in B_{ξ, z_t} . Note that $A_n(z)^{-1} (\sum_t x_t x'_t) \Pi_{2n1}^{*b} = O_p \left(\frac{1}{n^2 h} n^2 \sqrt{\frac{h^{2L+1}}{n}} \right) = O_p \left(\sqrt{\frac{h^{2L-1}}{n}} \right)$, which has the same order as the first term in (A.13) in the main paper. Therefore it may contribute to the limit distribution and should be retained in the bias corrected expression $\hat{\beta}(z) - \beta(z) - \hat{B}(z)$.

For Π_{2n2}^* we have

$$\begin{aligned}
\Pi_{2n2}^* & = \left(\int B_x B'_x \right)^{-1} f^{-1}(z) \frac{1}{nh} \sum_{t=1}^n K_{tz} \frac{1}{n^2} \sum_s x_s x'_s [\beta(z_s) - \beta(z)] K_{sz} \\
& = \left(\int B_x B'_x \right)^{-1} f^{-1}(z) \frac{1}{nh} \sum_{t=1}^n K_{tz} \frac{1}{n^2} \sum_s x_s x'_s \mathbb{E} \xi_{\beta s} \\
& + \left(\int B_x B'_x \right)^{-1} f^{-1}(z) \frac{1}{nh} \sum_{t=1}^n K_{tz} \frac{1}{n^2} \sum_s x_s x'_s (\xi_{\beta s} - \mathbb{E} \xi_{\beta s}) \\
& =: \Pi_{2n2}^{*a} + \Pi_{2n2}^{*b},
\end{aligned}$$

where $\xi_{\beta_s} := [\beta(z_s) - \beta(z)]K_{sz}$. For the first term Π_{2n2}^{*a} we have

$$\begin{aligned}\Pi_{2n2}^{*a} &= \left(\int B_x B_x' \right)^{-1} f^{-1}(z) \frac{1}{nh} \sum_{t=1}^n K_{tz} \frac{1}{n^2} \sum_s x_s x_s' \mathbb{E} \xi_{\beta_s} \\ &\sim_a f^{-1}(z) f(z) \mathbb{E} [\beta(z_s) - \beta(z)] K_{sz} \sim_a \mathbb{E} \xi_{\beta t}.\end{aligned}\quad (3.19)$$

For the second term Π_{2n2}^{*b} , first following Lemma B.2 $\mathbb{E} \xi_{\beta_s} = O(h^{L^*+1})$ and $\mathbb{E} \xi_{\beta_s}^2 = O(h^{2L+1})$. Therefore $\sum_s x_s x_s' (\xi_{\beta_s} - \mathbb{E} \xi_{\beta_s}) = O_p(n\sqrt{nh^{2L+1}}) \sim_a n\sqrt{nh^{2L+1}} \int B_x B_x' dB_\xi$. Then we have

$$\begin{aligned}\Pi_{2n2}^{*b} &= \left(\int B_x B_x' \right)^{-1} f^{-1}(z) \frac{1}{nh} \sum_{t=1}^n K_{tz} \frac{1}{n^2} \sum_s x_s x_s' (\xi_{\beta_s} - \mathbb{E} \xi_{\beta_s}) \\ &\sim_a \left(\int B_x B_x' \right)^{-1} f^{-1}(z) f(z) \frac{1}{n^2} n\sqrt{nh^{2L+1}} \int B_x B_x' dB_\xi \\ &= \sqrt{\frac{h^{2L+1}}{n}} \left(\int B_x B_x' \right)^{-1} \int B_x B_x' dB_\xi\end{aligned}\quad (3.20)$$

$$= O_p\left(\sqrt{\frac{h^{2L+1}}{n}}\right).\quad (3.21)$$

Note that the leading variation term of Π_{2n1}^* and Π_{2n2}^* , namely Π_{2n1}^{*b} and Π_{2n2}^{*b} , cannot be cancelled although they share the same order. So they both should be retained in the bias corrected expression $\hat{\beta}(z) - \beta(z) - \hat{B}(z)$.

The remaining two terms of (3.15) involve the difference

$$\begin{aligned}\Pi_{2n3}^* - \Pi_{2n4}^* &= h \left(\int B_x B_x' \right)^{-1} \frac{1}{n^2 h} \sum_{t=1}^n f^{-1}(z_t) K_{tz} \frac{1}{nh} \sum_{s=1}^n x_s u_s K_{st} \\ &\quad - h \left(\int B_x B_x' \right)^{-1} f^{-1}(z) \frac{1}{n^2 h} \sum_{t=1}^n K_{tz} \frac{1}{nh} \sum_{s=1}^n x_s u_s K_{sz},\end{aligned}\quad (3.22)$$

which has zero mean and does not contribute to bias centering. As in the stationary x_t case, we need to examine this term's contribution to the variance and limit distribution. For ease of presentation and calculation, we again set $g = f^{-1}$, take the scalar x_t case and let $u_t \sim_{iid} (0, \sigma_u^2)$ and z_t be *iid*. Write (3.22) as

$$\Pi_{2n3}^* - \Pi_{2n4}^* = \frac{1}{n^2} \left(\int B_x^2 \right)^{-1} \frac{1}{\sqrt{nh}} \sum_{s=1}^n x_s u_s \left(\frac{1}{\sqrt{nh}} \sum_{t=1}^n K_{tz} [K_{st} g(z_t) - K_{sz} g(z)] \right), \quad (3.23)$$

and since $x_s u_s$ is a martingale difference, this expression has mean zero and conditional variance

given \mathcal{F}_x

$$\begin{aligned}
& \frac{\sigma_u^2}{n^4} \left(\int B_x^2 \right)^{-2} \frac{1}{nh} \sum_{s=1}^n x_s^2 \mathbb{E} \left(\frac{1}{\sqrt{nh}} \sum_{t=1}^n K_{tz} [K_{stg}(z_t) - K_{szg}(z)] \right)^2 \\
& \sim_a \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-2}}{n^3 h} \left(\int B_x^2 \right) \mathbb{E} \left(\frac{1}{\sqrt{nh}} \sum_{t=1}^n K_{tz} [K_{stg}(z_t) - K_{szg}(z)] \right)^2 \\
& = \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-1}}{n^4 h^2} \mathbb{E} \left(\sum_{t,r=1}^n K_{tz} K_{rz} [K_{stg}(z_t) - K_{szg}(z)] [K_{sr g}(z_t) - K_{szg}(z)] \right) \\
& \sim_a \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-1}}{n^3 h^2} \mathbb{E} \left(K_{tz}^2 [K_{stg}(z_t) - K_{szg}(z)]^2 \right) \\
& + \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-1}}{n^2 h^2} \mathbb{E} (K_{tz} K_{rz} [K_{stg}(z_t) - K_{szg}(z)] [K_{sr g}(z_t) - K_{szg}(z)]) \\
& =: D_1^* + D_2^*. \tag{3.24}
\end{aligned}$$

Proceeding in the same way as the argument leading to (3.11) and (3.12), we find that

$$D_1^* = \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-1}}{n^3} \int_{-1}^1 \int_{-1}^1 K(p)^2 [K(p-q) - K(q)]^2 dpdq, \tag{3.25}$$

and

$$\begin{aligned}
D_2^* &= \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-1}}{n^2 h^2} \mathbb{E} (K_{tz} K_{rz} [K_{stg}(z_t) - K_{szg}(z)] [K_{sr g}(z_r) - K_{szg}(z)]) \\
& \sim_a \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-1}}{n^2 h^2} \int \int \int K \left(\frac{z_t - z}{h} \right) K \left(\frac{z_r - z}{h} \right) \left[K \left(\frac{z_s - z_t}{h} \right) g(z_t) - K \left(\frac{z_s - z}{h} \right) g(z) \right] \\
& \times \left[K \left(\frac{z_s - z_r}{h} \right) g(z_r) - K \left(\frac{z_s - z}{h} \right) g(z) \right] f(z_s) f(z_t) f(z_r) dz_s dz_t dz_r \\
& = h \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-1}}{n^2} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K(p) K(w) [K(p-q) g(z+ph) - K(q) g(z)] \\
& \times [K(q-w) g(z+hw) - K(q) g(z)] f(z+qh) f(z+ph) f(z+wh) dpdqdw \\
& \sim_a h \frac{\sigma_u^2 \left(\int B_x^2 \right)^{-1}}{n^2} f(z) \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K(p) K(w) [K(p-q) - K(q)] [K(q-w) - K(q)] dpdqdw. \tag{3.26}
\end{aligned}$$

It follows that $\Pi_{2n3}^* - \Pi_{2n4}^*$ has zero mean and conditional asymptotic variance $D_1^* + D_2^* = O_p(\frac{h}{n^2})$ in view of (3.24) - (3.26), so that $\Pi_{2n3}^* - \Pi_{2n4}^* = O_p(\frac{h^{1/2}}{n})$. The implication for the bias is that $A_n(z)^{-1} (\sum_t x_t x_t') (\Pi_{2n3} - \Pi_{2n4}) = O_p(\frac{1}{n^2 h} n^2 \sqrt{\frac{h}{n^2}}) = O_p(1/\sqrt{n^2 h})$, which is of the same order as the asymptotic variance term in the nonstationary case. It therefore affects the limit distribution and should be retained in the bias corrected formula $\hat{\beta}(z) - \beta(z) - \hat{B}(z)$.

Letting $A_{n,h}^* := \frac{1}{n^2 h} \sum_{t=1}^n x_t x_t' K_{tz}$ we have the bias corrected estimation error

$$\begin{aligned}
\hat{\beta}(z) - \beta(z) - \hat{B}(z) &= A_n^{-1}(z) \sum_{t=1}^n x_t x_t' [\beta(z_t) - \beta(z)] K_{tz} + A_n^{-1}(z) \sum_{t=1}^n x_t u_t K_{tz} - \hat{B}(z) \\
&= A_n^{-1}(z) \sum_{t=1}^n x_t x_t' [\beta(z_t) - \beta(z)] K_{tz} + A_n^{-1}(z) \sum_{t=1}^n x_t u_t K_{tz} - A_n^{-1}(z) \left(\sum_{t=1}^n x_t x_t' \right) \left(\frac{1}{n} \sum_{s=1}^n [\hat{\beta}(z_s) - \hat{\beta}(z)] K_{sz} \right) \\
&= A_{n,h}^{*-1} \frac{1}{n^2 h} \sum_{t=1}^n x_t x_t' [\beta(z_t) - \beta(z)] K_{tz} - A_{n,h}^{*-1} \frac{1}{n^2} \sum_{t=1}^n x_t x_t' \left(\frac{1}{nh} \sum_{s=1}^n [\hat{\beta}(z_s) - \hat{\beta}(z)] K_{sz} \right) + A_n^{-1}(z) \sum_{t=1}^n x_t u_t K_{tz} \\
&= A_{n,h}^{*-1} \frac{1}{n^2 h} \sum_{t=1}^n x_t x_t' \mathbb{E} \xi_{\beta t} + A_{n,h}^{*-1} \frac{1}{n^2 h} \sum_{t=1}^n x_t x_t' \eta_{\beta t} - A_{n,h}^{*-1} \frac{1}{n^2} \sum_{t=1}^n x_t x_t' \left(\frac{1}{nh} \sum_{t=1}^n \{\beta(z_t) - \beta(z)\} K_{tz} \right) \\
&\quad - A_{n,h}^{*-1} \frac{1}{n^2} \sum_{t=1}^n x_t x_t' \left(\frac{1}{nh} \sum_{t=1}^n \left\{ \hat{\beta}(z_t) - \beta(z_t) - [\hat{\beta}(z) - \beta(z)] \right\} K_{tz} \right) + A_n^{-1}(z) \sum_{t=1}^n x_t u_t K_{tz} \\
&= o(h^{L^*}) + A_{n,h}^{*-1} \frac{1}{n^2 h} \sum_{t=1}^n x_t x_t' \eta_{\beta t} - A_{n,h}^{*-1} \frac{1}{n^2} \sum_{t=1}^n x_t x_t' \left(\frac{1}{h} \Pi_{2n}^* \right) + A_{n,h}^{*-1} \frac{1}{n^2 h} \sum_{t=1}^n x_t u_t K_{tz} \\
&= o(h^{L^*}) + A_{n,h}^{*-1} \frac{1}{n^2 h} \sum_{t=1}^n x_t x_t' \eta_{\beta t} - A_{n,h}^{*-1} \frac{1}{n^2} \sum_{t=1}^n x_t x_t' \frac{1}{h} \left(\Pi_{2n1}^{*b} - \Pi_{2n2}^{*b} \right) \\
&\quad - A_{n,h}^{*-1} \frac{1}{n^2} \sum_{t=1}^n x_t x_t' \frac{1}{h} \left(\Pi_{2n3}^* - \Pi_{2n4}^* \right) + A_{n,h}^{*-1} \frac{1}{n^2 h} \sum_{t=1}^n x_t u_t K_{tz}, \tag{3.27}
\end{aligned}$$

where $\xi_{\beta t} = [\beta(z_t) - \beta(z)] K_{tz}$ and $\eta_{\beta t} = \xi_{\beta t} - \mathbb{E} \xi_{\beta t}$.

In view of the above analysis, we find that the bias estimator $\hat{B}(z)$ is not consistent. Furthermore, it introduces additional variation that affects the final limit distribution through $\Pi_{2n1}^{*b} - \Pi_{2n2}^{*b}$ and $\Pi_{2n3}^* - \Pi_{2n4}^*$. In particular, note that the variance of Π_{2n1}^{*b} and Π_{2n2}^{*b} depends on the unknown value of L and $\beta^{(L)}(z)$, like that of $B_{\eta,L}(\cdot)$ in Lemma B.2 (a). So the asymptotic variance of $\hat{\beta}(z) - \beta(z) - \hat{B}(z)$ includes two parts: one part involves $\beta^{(L)}(z)$ and L through the first two terms of (3.27), and the other involves the variation entering through the last two terms of (3.27). Therefore, use of the inconsistent estimator $\hat{B}(z)$ produces both bias and variance complications that lead to a non-pivotal limit theory for the adaptive statistic. ■

References

Phillips, P. C. B. and Wang, Y. (2020). When bias contributes to variance: true limit theory in functional coefficient cointegrating regression. *Cowles Foundation Discussion Paper No. 2250, Yale University.*