Supplemental Material to
OPTIMALLY STUBBORN

By

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Online Appendix

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Pooling Equilibria with $K > 2$

Before proving the existence of pooling equilibria with $K > 2$, it is helpful to restate the following supplementary lemma and its proof:

Lemma 4. *There exists a pooling equilibrium with support $\{\alpha_1, \ldots, \alpha_K\}$ only if the offers α_1 through α_K along with probabilities q_1 through to q_K, and positive numbers μ_1 through to μ_K solve (5)–(8).*

Proof. Fix $z > 0$, and an equilibrium, specifying $\{\alpha_1, \ldots, \alpha_K\}$, $\mu_1, \ldots, \mu_K > 0$, and $q_1, \ldots, q_K > 0$. For any $k \leq K$, define

$$
\begin{aligned}
v_k^r = &\sum_{\substack{i \text{ s.t.} \\ \alpha_i \leq 1-\alpha_k}} q_i \left(\frac{\alpha_k + 1 - \alpha_i}{2} \right) \\
&+ \sum_{\substack{i \text{ s.t.} \\ \alpha_i > 1-\alpha_k}} q_i \left(\alpha_k \min\left\{ 0, 1 - \left(\frac{\mu_i}{\mu_k}\right)^{1-\alpha_i} \right\} + (1-\alpha_i) \min\left\{ 1, \left(\frac{\mu_i}{\mu_k}\right)^{1-\alpha_i} \right\} \right), \qquad (1)
\end{aligned}
$$

$$
v_k^s = v_k^r - \sum_{\substack{i \text{ s.t.} \\ \alpha_i > 1-\alpha_k}} q_i (1-\alpha_i) \max\left\{ \mu_i^{\alpha_k}, \left(\frac{\mu_i}{\mu_k}\right)^{1-\alpha_i} \mu_k^{\alpha_k} \right\}. \qquad (2)
$$

For a detailed derivation of these payoffs see the supplementary material on my website. For any $k, k' \leq K$, define

$$
\begin{aligned}
\Delta_{k,k'}^r &= v_k^r - v_{k'}^r, &(3)\\
\Delta_{k,k'}^s &= v_k^s - v_{k'}^s. &(4)
\end{aligned}
$$

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Given z and $\{\alpha_1, \dots, \alpha_K\}$, define the following system in (q_i, μ_i), $i = 1, \dots, K$:

$$\Delta^r_{k,k+1} = 0, \ \forall k < K, \tag{5}$$
$$\Delta^r_{k,k+1} - \Delta^s_{k,k+1} = 0, \ \forall k < K \tag{6}$$
$$\sum_{i=1}^{K} q_i \mu_i^{1-\alpha_i} = z, \text{ and} \tag{7}$$
$$\sum_{i=1}^{K} q_i = 1. \tag{8}$$

Note that there are $2K$ equations (and as many variables). For a candidate equilibrium with support $\{\alpha_1, \dots, \alpha_K\}$, both types need to be indifferent over all demands α_1 through to α_K, with probabilities $q_i > 0$, given an ex ante probability of a player being stubborn, z. Equation (5) shows the difference in payoff for a rational type between making a demand of α_k and making a demand of α_{k+1}, conditional on the opponent mixing over the offers α_1 through to α_K. Hence, equation (6) ensures indifference of the rational type between any two offers, α_k and α_{k+1}. In the same manner, equation (6) ensures indifference of the stubborn type between any two offers, simplified using the indifference of the rational type. Equation (8) ensures that the probabilities of being faced with a given offer add up to 1; and equation (7) ensures that the conditional probabilities of stubbornness, $\mu_i^{1-\alpha_i}$, are consistent with the ex ante probability of a player being stubborn, z.

Fix K demands (satisfying Lemmas 1 and 2). Suppose that for all $\bar{z} > 0$, there exists $z < \bar{z}$, such that there exist $q_i > 0$, and $\mu_i > 0$ for $i = 1, 2, \dots, K$ such that (z, α, q, μ) satisfies (5) to (8). Then there exists a sequence $(z^n, \alpha^n, q^n, \mu^n)_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} z^n \to 0$, solving (5)–(8), such that it is *not* the case that $\alpha_i^n - \alpha_{i+1}^n \to 0$ for all i, $i + 1 \leq \lceil K/2 \rceil - 1$ and all $i, i + 1 \geq \lceil K/2 \rceil$ with $i + 1 < K$. Recall, that $\alpha^n, q^n, \mu^n \in [0, 1]$. Hence, without loss, assume that α^n, q^n and μ^n converge. By continuity, $(z = 0, \lim_{z\to 0} \alpha, \lim_{z\to 0} q, \lim_{z\to 0} \mu)$ also solves (5)–(8). In the following, I drop the subscript n; limits are indicated explicitly by $\lim_{z\to 0}$ throughout.

In other words, if the system has a solution for small enough z, then for at least one $i \notin \{\lceil K/2 \rceil - 1, K\}$, $\alpha_i \neq \alpha_{i+1}$. □

Proof of Proposition 5. When $K = 3$, I can write (5) for $k = 1, 2$ as:

$$\begin{aligned} &- q_1 \frac{\alpha_2 - \alpha_1}{2} - q_2 \frac{1 - \alpha_1 - \alpha_2}{2} \\ &+ q_3 \left((\alpha_1 + \alpha_3 - 1) \left(1 - l_{3,1}^{1-\alpha_3}\right) - (\alpha_2 + \alpha_3 - 1) \left(1 - l_{3,2}^{1-\alpha_3}\right) \right) = 0, \end{aligned} \tag{9}$$
$$- q_1 \frac{1 - \alpha_1 - \alpha_2}{2} + q_3 (\alpha_2 + \alpha_3 - 1) \left(1 - l_{3,2}^{1-\alpha_3}\right) = 0, \tag{10}$$

and respectively, I can write (6) for $k = 1, 2$ as:

$$q_2 (1 - \alpha_2) \mu_2^{\alpha_2} - q_3 (1 - \alpha_3) \mu_3^{1-\alpha_3} \left(\mu_1^{\alpha_1+\alpha_3-1} - \mu_2^{\alpha_2+\alpha_3-1} \right) = 0, \tag{11}$$
$$q_1 (1 - \alpha_1) \mu_1^{\alpha_3} - q_2 (1 - \alpha_2) (\mu_2^{\alpha_2} - \mu_2^{\alpha_3}) - q_3 (1 - \alpha_3) \left(l_{3,2}^{1-\alpha_3} \mu_2^{\alpha_2} - \mu_3^{\alpha_3} \right) = 0. \tag{12}$$

The proof that follows is divided into the following steps. I first show that in any sequence of equilibria, $\mu_i \to 0$ for any $i > 1$ (Claim 1). I then show that if $\alpha_2 + \alpha_1 < 1$, an equilibrium with support $\{\alpha_1, \alpha_2, \alpha_3\}$ does not exist in the limit (Claim 2). Next, I show that if $\alpha_2 + \alpha_1 = 1$, an equilibrium with support $\{\alpha_1, \alpha_2, \alpha_3\}$ does exist in the limit (Claim 3). Finally, I show that if $\alpha_2 + \alpha_1 = 1$ and $K = 3$, the system (5)–(8) can be solved locally around $z = 0$, with $q_i \in (0, 1)$, and $\mu_i \in (0, 1)$ for $i = 1, 2, 3$ (Claim 4). Together these claims establish that fixing any 3 demands satisfying Lemmas 1 and 2 with $\alpha_2 = 1 - \alpha_1$, an equilibrium with support $\{\alpha_1, \alpha_2, \alpha_3\}$ does exist for z small enough.

Claim 1. *For (5)–(8) to be satisfied when $K = 3$, $\lim_{z\to 0} \mu_i = 0$ for $i = 2, 3$.*

Proof. By (7), either $\lim_{z\to 0} q_i = 0$ or $\lim_{z\to 0} \mu_i = 0$. By (8) and (7), there must exist $\lim_{z\to 0} \mu_i = 0$. Recall that by Lemma 1, $\mu_3 \le \mu_2 < \mu_1$. Hence, $\lim_{z\to 0} \mu_3 = 0$. Suppose $\lim_{z\to 0} \mu_2 \neq 0$. Then $\lim_{z\to 0} q_i = 0$, for $i = 1, 2$. But then (9) cannot be satisfied – since the last term is non-zero. Hence, it must be that $\lim_{z\to 0} \mu_2 = 0$. □

Claim 2. *If $\alpha_2 + \alpha_1 < 1$ and $K = 3$, the system (5)–(8) cannot be solved in the limit.*

Proof. Using (8), I can replace q_3 by $1 - q_1 - q_2$ in (9) and (10). I can then solve (9) and (10) for q_1 and q_2 as a function of μ_i, $i = 1, 2, 3$, only. I can then replace q_i, $i = 1, 2, 3$ in (11). I can then write (11) as a function of μ_i, $i = 1, 2, 3$ only, which allows me to solve for μ_3:

$$\mu_3^{1-\alpha_3} = -\frac{2(1-\alpha_2)(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\, l_{2,1}^{\alpha_2}\mu_1^{1-\alpha_3+\alpha_2-\alpha_1}}{(1-\alpha_1-\alpha_2)^2(1-\alpha_3) + k_0\mu_2^{\alpha_2-\alpha_1}}, \tag{13}$$

where

$$\begin{aligned} k_0 = &\left(-\alpha_1^2(1-\alpha_3) + (1-\alpha_2)\left(2\alpha_1(2\alpha_2+\alpha_3-1) - (\alpha_2+\alpha_3+\alpha_2\alpha_3-1)\right)\right) l_{2,1}^{\alpha_1+\alpha_3-1} \\ &+ 2(1-\alpha_2)(1-\alpha_1-\alpha_2)(\alpha_1+\alpha_3-1)\, l_{2,1}^{\alpha_1}. \end{aligned}$$

Note first that k_0 is only a function of the demands and $l_{2,1}$, all of which are bounded, and hence, k_0 is bounded. Since $\lim_{z\to 0} \mu_2^{\alpha_2-\alpha_1} = 0$, the denominator of (13) is positive for n large enough. Moreover, all terms in the numerator of (13) are positive. This implies there exists N (finite) such that for any $n > N$, $\mu_3 < 0$. But by definition this cannot be. Hence, if $\alpha_2 \neq 1 - \alpha_1$ and $K = 3$, (5)–(8) cannot be satisfied. □

Claim 3. *If $\alpha_2 = 1 - \alpha_1$ and $K = 3$, the system (5)–(8) has a solution in the limit.*

I first simplify (7)–(8) and (9)–(12) by $\alpha_2 = 1 - \alpha_1$. In particular, simplifying (10) to:

$$q_3(\alpha_2+\alpha_3-1)\left(1 - \left(\frac{\mu_3}{\mu_2}\right)^{1-\alpha_3}\right) = 0. \tag{14}$$

It follows immediately that $\mu_3 = \mu_2$. I can solve the simplified versions of (9) and (11) for q_1 and q_2:

$$q_1 = \frac{2\alpha_1(\alpha_1+\alpha_3-1)\left(\mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1} - \mu_2^{1-2\alpha_1} l_{2,1}^{\alpha_1}\right)}{(1-2\alpha_1)(1-\alpha_3) - (\alpha_3+\alpha_1-1)\left(2\alpha_1\mu_2^{1-2\alpha_1} l_{2,1}^{\alpha_1} - \mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1}\right)} \tag{15}$$

$$q_2 = \frac{(1-2\alpha_1)(1-\alpha_3)\left(1 - \mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1}\right)}{(1-2\alpha_1)(1-\alpha_3) - (\alpha_3+\alpha_1-1)\left(2\alpha_1\mu_2^{1-2\alpha_1} l_{2,1}^{\alpha_1} - \mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1}\right)}. \tag{16}$$

Note that it follows immediately from this that,

$$\lim_{z\to 0} q_1 = 0, \quad \lim_{z\to 0} q_2 = 1, \quad \text{and} \quad \lim_{z\to 0} q_3 = 0.$$

Replacing q_1 through q_3 in (12), dividing by $\alpha_1 \mu_2^{1-\alpha_1}$ and simplifying, I get:

$$\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)\left(\mu_1^{\alpha_3-\alpha_1} - \frac{\mu_1^{1-\alpha_1}}{\mu_2^{1-\alpha_3}}\right) + (1-2\alpha_1)(1-\alpha_3)\left(1-\mu_2^{\alpha_1+\alpha_3-1}\right)}{-(1-2\alpha_1)(1-\alpha_3) + (\alpha_1+\alpha_3-1)\left(2\alpha_1\mu_2^{1-2\alpha_1} l_{2,1}^{\alpha_1} - \mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1}\right)} = 0, \quad (17)$$

Note the second term of the denominator of (17) is 0 in the limit, and hence, the limit of the denominator is a constant. For (17) to be satisfied, it must then be that

$$\lim_{z\to 0} \frac{\mu_1^{1-\alpha_1}}{\mu_2^{1-\alpha_3}} = \frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}.$$

Put differently,

$$\mu_1 = k_2 \mu_2^{\frac{1-\alpha_3}{1-\alpha_1}} + O(\mu_2^{x_0}), \quad (18)$$

where $x_0 = \min\left\{\alpha_1+\alpha_3-1+\frac{1-\alpha_3}{1-\alpha_1}, \frac{1-\alpha_3}{1-\alpha_1}(1+\alpha_3-\alpha_1)\right\}$ and

$$k_2 = \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{\frac{1}{1-\alpha_1}}.$$

Using (18), I can rewrite (15) and (16):

$$q_1 = \frac{2\alpha_1(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)} k_2^{1-\alpha_3-\alpha_1} \mu_2^{\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}} + O(\mu_2^{x_2}), \quad (19)$$

$$q_2 = 1 - k_2^{1-\alpha_3-\alpha_1} \frac{(2\alpha_3-1)\alpha_1}{(1-2\alpha_1)(1-\alpha_3)} \mu_2^{\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}} + O(\mu_2^{x_2}), \text{ and similarly,} \quad (20)$$

$$q_3 = \frac{\alpha_1}{1-\alpha_3} k_2^{1-\alpha_1-\alpha_3} \mu_2^{\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}} + O(\mu_2^{x_1}), \quad (21)$$

where

$$x_1 = x_0 + \frac{(1-\alpha_1)^2 - (1-\alpha_3^2)}{1-\alpha_1},$$

and

$$x_2 = \min\left\{x_1, 2\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1}\right\}.$$

Equivalently,

$$x_1 = \min\left\{\frac{\alpha_3(\alpha_3-\alpha_1)}{1-\alpha_1}, \frac{1-\alpha_1(3-\alpha_1-\alpha_3)}{1-\alpha_1}\right\}, \text{ and}$$

$$x_2 = \min\left\{\frac{\alpha_3(\alpha_3-\alpha_1)}{1-\alpha_1}, \frac{1-\alpha_1(3-\alpha_1-\alpha_3)}{1-\alpha_1}, 2\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}\right\}.$$

Recall that using $\mu_2 = \mu_3$ and $\mu_1 = k_2 \mu_2^{\frac{1-\alpha_3}{1-\alpha_1}} + O(\mu_2^{x_0})$. With some abuse of notation, I can write (7) as:

$$q_1 \left(k_2 \mu_2^{\frac{1-\alpha_3}{1-\alpha_1}} + O(\mu_2^{x_0}) \right)^{1-\alpha_1} + q_2 \mu_2^{\alpha_1} + q_3 \mu_2^{1-\alpha_3} = z, \tag{22}$$

with q_1 through q_3 defined in (19) through (21). This simplifies to:

$$\frac{\alpha_1}{1-\alpha_3} \left(\frac{2(\alpha_1 + \alpha_3 - 1)}{1 - 2\alpha_1} k_2^{1-\alpha_1} + 1 \right) k_2^{1-\alpha_1-\alpha_3} \mu_2^{\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3} + \mu_2^{\alpha_1} + \mathcal{O}\left(\mu_2^{x_3}\right) = z, \tag{23}$$

where

$$x_3 = \min\left\{ \frac{1 - \alpha_1 - \alpha_3 + \alpha_3^2}{1-\alpha_1}, \frac{2 - \alpha_3 - \alpha_1(4 - \alpha_1 - 2\alpha_3)}{1-\alpha_1} \right\}.$$

If

$$\alpha_1 < \frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3,$$

then

$$\mu_2^{\alpha_1} + \mathcal{O}\left(\mu_2^{\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3} \right) = z.$$

Similarly, if

$$\alpha_1 > \frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3,$$

then

$$\frac{\alpha_1}{1-\alpha_3} \left(\frac{2(\alpha_1 + \alpha_3 - 1)}{1 - 2\alpha_1} k_2^{1-\alpha_1} + 1 \right) k_2^{1-\alpha_1-\alpha_3} \mu_2^{\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3} + \mathcal{O}\left(\mu_2^{\alpha_1}\right) = z.$$

Recall that $s_i = \frac{\mu_i^{1-\alpha_i} q_i}{z}$.

Case 1: $\alpha_1 < \frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3$. Taylor approximation as before yields:

$$s_1 = \frac{\alpha_1}{1-\alpha_1} k_2^{-\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}} \mu_2^{\frac{2(1-\alpha_1)^2 - \alpha_3(2-\alpha_1) + \alpha_3^2}{1-\alpha_1}} + O(\mu_2^{x_4}), \tag{24}$$

$$s_2 = 1 - \mathcal{O}\left(\mu_2^{\frac{2(1-\alpha_1)^2 - \alpha_3(2-\alpha_1) + \alpha_3^2}{1-\alpha_1}} \right) \tag{25}$$

$$s_3 = \frac{\alpha_1}{1-\alpha_3} k_2^{1-\alpha_3-\alpha_1} \mu_2^{\frac{2(1-\alpha_1)^2 - \alpha_3(2-\alpha_1) + \alpha_3^2}{1-\alpha_1}} + \mathcal{O}\left(\mu_2^{x_4}\right) \tag{26}$$

where

$$x_4 = \min\left\{ \frac{(1-\alpha_1)^2 - \alpha_3 + \alpha_3^2}{1-\alpha_1}, \frac{(1-2\alpha_1)(2-\alpha_1-\alpha_3)}{1-\alpha_1}, 2\frac{2(1-\alpha_1)^2 - \alpha_3(2-\alpha_1) + \alpha_3^2}{1-\alpha_1} \right\}.$$

Case 2: $\alpha_1 > \frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3$. Then we get,

$$s_1 = \frac{1-\alpha_3}{2-\alpha_1-\alpha_3} + O(\mu_2^{x_5}), \tag{27}$$

$$s_2 = \frac{(1-\alpha_1)(2\alpha_3-1)}{(1-2\alpha_1)(2-\alpha_1-\alpha_3)}\mu_2^{\alpha_1+\alpha_3-1} + O(\mu_2^{x_6}), \tag{28}$$

$$s_3 = \frac{1-\alpha_1}{2-\alpha_1-\alpha_3} + \mathcal{O}\left(\mu_2^{\alpha_1+\alpha_3-1}\right), \tag{29}$$

where

$$x_5 = \min\left\{\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1}, \frac{\alpha_3(2-\alpha_1) - 2(1-\alpha_1)^2 - \alpha_3^2}{1-\alpha_1}\right\},$$

$$x_6 = \min\left\{\frac{\alpha_3(\alpha_3-\alpha_1)}{1-\alpha_1}, \frac{\alpha_3(3-\alpha_3-2\alpha_1) - 3(1-\alpha_1)^2}{1-\alpha_1}\right\}.$$

To summarize:

Case 1 $\alpha_1 < \frac{1}{4}\left(4-\alpha_3-\sqrt{(8-7\alpha_3)\alpha_3}\right)$. Then:

$$\begin{aligned} &\lim_{z\to 0} s_1 = 0, \quad \lim_{z\to 0} s_2 = 1, \quad \lim_{z\to 0} s_3 = 0, \\ &\lim_{z\to 0} r_1 = 0, \quad \lim_{z\to 0} r_2 = 1, \quad \lim_{z\to 0} r_3 = 0. \end{aligned} \tag{30}$$

Case 2 $\alpha_1 > \frac{1}{4}\left(4-\alpha_3-\sqrt{(8-7\alpha_3)\alpha_3}\right)$. Then in the same fashion, I derive:

$$\begin{aligned} &\lim_{z\to 0} s_1 = \frac{1-\alpha_3}{2-\alpha_1-\alpha_3}, \quad \lim_{z\to 0} s_2 = 0, \quad \lim_{z\to 0} s_3 = \frac{1-\alpha_1}{2-\alpha_1-\alpha_3}, \\ &\lim_{z\to 0} r_1 = 0, \quad \lim_{z\to 0} r_2 = 1, \quad \lim_{z\to 0} r_3 = 0. \end{aligned} \tag{31}$$

When $\alpha_1 = \frac{1}{4}\left(4-\alpha_3-\sqrt{(8-7\alpha_3)\alpha_3}\right)$, then $\lim_{z\to 0} s_2 = \frac{1-\alpha_3}{\alpha_1}$. Hence, in this case all three probabilities of the stubborn player are strictly interior (everything else unchanged).

Claim 4. *If $\alpha_2 = 1-\alpha_1$ and $K = 3$, the system (5)–(8) can be solved locally around $z = 0$, with $q_i \in (0,1)$, and $\mu_i \in (0,1)$ for $i = 1,2,3$.*

Proof. Define

$$\begin{aligned} A(\mu_1,\mu_2) =& 2(1-\alpha_1)(\alpha_1+\alpha_3-1)\left(\mu_1^{\alpha_3-\alpha_1} - \mu_1^{1-\alpha_1}\mu_2^{-(1-\alpha_3)}\right) \\ &+ (1-2\alpha_1)(1-\alpha_3)\left(1-\mu_2^{\alpha_1+\alpha_3-1}\right), \end{aligned} \tag{32}$$

$$\begin{aligned} B(\mu_1,\mu_2,z) =& (1-2\alpha_1)\left((1-\alpha_3)\mu_2^{\alpha_1} + \mu_1^{1-\alpha_1-\alpha_3}\mu_2^{\alpha_3-\alpha_1}\left(\alpha_1\mu_2^{1-\alpha_3} - (1-\alpha_3)\mu_2^{\alpha_1}\right)\right) \\ &+ 2\alpha_1(\alpha_1+\alpha_3-1)\mu_1^{1-2\alpha_1}\left(\mu_1^{1-\alpha_3}\mu_2^{\alpha_3-\alpha_1} - \mu_2^{1-2\alpha_1}\right) - dz, \end{aligned} \tag{33}$$

where

$$d = (1-2\alpha_1)(1-\alpha_3) - (\alpha_1+\alpha_3-1)\left(2\alpha_1\mu_2^{1-\alpha_1}\mu_1^{-\alpha_1} - \mu_1^{1-\alpha_1-\alpha_3}\mu_2^{\alpha_3-\alpha_1}\right). \tag{34}$$

Following identical steps to the first part of the proof, I can reduce the system (5)–(8) to $A(\mu_1,\mu_2) = 0$ and $B(\mu_1,\mu_2,z) = 0$.

Case 1 Throughout Case 1, $\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1} > \alpha_1 + \alpha_3 - 1$. Let me introduce two auxiliary variables, y_b and n_b, where

$$y_b = \mu_1^{\frac{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}{1-\alpha_3}}, \text{ and} \tag{35}$$

$$n_b = \left(\mu_2 \mu_1^{-\frac{1-\alpha_1}{1-\alpha_3}} - \left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)} \right)^{\frac{1}{1-\alpha_3}} \right)^{\frac{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}{(1-\alpha_1)(\alpha_1+\alpha_3-1)}}. \tag{36}$$

Using the IFT, I will explicitly derive the derivatives:

$$\left.\frac{dy_b}{dz}\right|_{y_b=n_b=z=0} = \frac{(1-\alpha_3)}{\alpha_1} \left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)} \right)^{\frac{\alpha_1}{1-\alpha_3}} > 0, \text{ and} \tag{37}$$

$$\begin{aligned} \left.\frac{dn_b}{dz}\right|_{y_b=n_b=z=0} &= \left(\frac{1-\alpha_3}{\alpha_1} \right)^{1-\frac{2+\alpha_1^2-\alpha_1(3-\alpha_3)-\alpha_3(2-\alpha_3)}{(1-\alpha_1)(\alpha_1+\alpha_3-1)}} \\ &\cdot \left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)} \right)^{\frac{1-\alpha_1}{1-\alpha_3}+\frac{1-\alpha_1}{\alpha_1+\alpha_3-1}-\frac{1-2\alpha_1-\alpha_3}{1-\alpha_1}} > 0. \end{aligned} \tag{38}$$

In order to compute those derivatives, it is useful to further introduce y_a and n_a:

$$y_a = \mu_1^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{1-\alpha_3}}, \text{ and} \tag{39}$$

$$n_a = \mu_2 \mu_1^{-\frac{1-\alpha_1}{1-\alpha_3}} - \left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)} \right)^{\frac{1}{1-\alpha_3}}. \tag{40}$$

Note that

$$\begin{aligned} \left.\frac{dy_a}{dn_a}\right|_{y_b=n_b=z=0} &= \left. \frac{d\left(y_b^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}} \right)}{d\left(n_b^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}} \right)} \right|_{y_b=n_b=z=0} \\ &= \left. \left(\frac{y_b}{n_b} \right)^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}-1} \right|_{y_b=n_b=z=0} \left.\frac{dy_b}{dn_b}\right|_{y_b=n_b=z=0} \end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} \left.\frac{dy_b}{dn_b}\right|_{y_b=n_b=z=0} &= \left. \left(\frac{y_a}{n_a} \right)^{\frac{(1-\alpha_1)^2-(2-\alpha_1-\alpha_3)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)(\alpha_1+\alpha_3-1)}} \right|_{y_b=n_b=z=0} \left.\frac{dy_a}{dn_a}\right|_{y_b=n_b=z=0} \\ &= \left. \left(\frac{dy_a}{dn_a} \right)^{\frac{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)(\alpha_1+\alpha_3-1)}} \right|_{y_b=n_b=z=0}. \end{aligned} \tag{42}$$

Using the IFT on (32), I show that:

$$\left.\frac{dy_a}{dn_a}\right|_{y_a=n_a=o} = -\left.\frac{\frac{\partial A_1}{\partial n_a}}{\frac{\partial A_1}{\partial y_a}}\right|_{y_a=n_a=0} > 0. \tag{43}$$

I can rewrite (32) as a function of y_a and n_a, using (39) and (40). Denote this new function $A_1(y_a, n_a)$. Note that y_a is simply constructed such that the smallest exponent on y_a in A_1 is 1. Taking derivatives of A_1 w.r.t. y_a and n_a, and evaluating the derivative at $y_a = n_a = 0$, I get

$$\left.\frac{\partial A_1}{\partial y_a}\right|_{y_a=n_a=0} = (1-2\alpha_1)(1-\alpha_3)\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_3}} > 0, \tag{44}$$

$$\left.\frac{\partial A_1}{\partial n_a}\right|_{y_a=n_a=0} = -\frac{(1-2\alpha_1)(1-\alpha_3)^2}{\alpha_1}\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{-\alpha_1}{1-\alpha_3}} < 0. \tag{45}$$

Hence,

$$\left.\frac{dy_a}{dn_a}\right|_{y_a=n_a=o} = -\left.\frac{\frac{\partial A_1}{\partial n_a}}{\frac{\partial A_1}{\partial y_a}}\right|_{y_a=n_a=0} \tag{46}$$

$$= \frac{1-\alpha_3}{\alpha_1}\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{-\frac{2\alpha_1+\alpha_3-1}{1-\alpha_3}} > 0. \tag{47}$$

Similarly, I can rewrite (33) and (34) using (35) and (36). Denote these new functions $B_1(y_b, n_b, z)$, and d_1, where

$$\begin{aligned} d_1 =& (1-2\alpha_1)(1-\alpha_3) \\ & - 2\alpha_1(\alpha_1+\alpha_3-1)\left(n_b^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}} + k_2^{-\frac{1-\alpha_1}{1-\alpha_3}}\right)^{1-\alpha_1} y_b^{\frac{(1-\alpha_1)^2-\alpha_1(1-\alpha_3)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}} \\ & + (\alpha_1+\alpha_3-1)\left(n_b^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}} + k_2^{-\frac{1-\alpha_1}{1-\alpha_3}}\right)^{\alpha_3-\alpha_1} y_b^{1-\frac{(1-\alpha_3)(1-\alpha_1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}}. \end{aligned}$$

Note that, y_b is simply constructed such that the smallest exponent on y_b in $B_1 - d_1 z$. Taking derivatives of B_1 with respect to y_b, n_b and z, I get:

$$\left.\frac{\partial B_1}{\partial y_b}\right|_{y_b=n_b=z=0} = \alpha_1(1-2\alpha_1)\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{1-\alpha_1}{1-\alpha_3}} > 0, \tag{48}$$

$$\left.\frac{\partial B_1}{\partial n_b}\right|_{y_b=n_b=z=0} = 0, \tag{49}$$

$$\left.\frac{\partial B_1}{\partial z}\right|_{y_b=n_b=z=0} = -(1-2\alpha_1)(1-\alpha_3) < 0. \tag{50}$$

Since

$$\left.\frac{dy_b}{dz}\right|_{y_b=n_b=z=0} = -\left.\frac{\frac{\partial B_1}{\partial z}}{\frac{\partial B_1}{\partial y_b}}\right|_{y_b=n_b=z=0}, \text{ and} \tag{51}$$

$$\left.\frac{dn_b}{dz}\right|_{y_b=n_b=z=0} = -\left.\frac{\frac{\partial B_1}{\partial z}}{\frac{\partial B_1}{\partial n_b}+\frac{\partial B_1}{\partial y_b}\frac{\partial y_b}{n_b}}\right|_{y_b=n_b=z=0}, \tag{52}$$

(37) and (38) follow immediately. Note that the exponents on y_b in d_1 are less than 1. It is easy to verify that the derivative of d_1 w.r.t. y_b, evaluated at $y_b = n_b = z = 0$ is 0. I simply need to take into account the rate at which y_b goes to 0 relative to z. Recall that $\lim_{z\to 0} \frac{\mu_1^{1-\alpha_1}}{\mu_2^{1-\alpha_3}} = k_2^{1-\alpha_1}$. Moreover, (i) if $\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3 < \alpha_1$, then $\lim_{z\to 0} \frac{\mu_2^{\alpha_1}}{z} = 1$. Therefore,

$$\lim_{z\to 0} \frac{y_b^{\frac{(1-\alpha_1)\alpha_1}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}}}{z} = k_3,$$

where k_3 is some positive constant. It follows that[1]

$$\lim_{z\to 0} y_b^{\frac{(1-\alpha_1)^2-\alpha_1(1-\alpha_3)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}-1} z = 0, \text{ and}$$

$$\lim_{z\to 0} y_b^{-\frac{(1-\alpha_3)(1-\alpha_1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}} z = 0.$$

Similarly, (ii) if $\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3 < \alpha_1$, then

$$\lim_{z\to 0} \frac{\mu_2^{\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}+1-\alpha_3}}{z} = k_4,$$

where k_4 is some positive constant. Therefore, $\lim_{z\to 0} \frac{y_b}{z} = k_5$, where k_5 is some positive constant. Hence,

$$\lim_{z\to 0} y_b^{\frac{(1-\alpha_1)^2-\alpha_1(1-\alpha_3)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}-1} z = 0, \text{ and}$$

$$\lim_{z\to 0} y_b^{-\frac{(1-\alpha_3)(1-\alpha_1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}} z = 0.$$

Therefore, if $\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1} > \alpha_1 + \alpha_3 - 1$, $\alpha_2 = 1 - \alpha_1$ and $K = 3$, the system (5)–(8) can be solved locally around $z = 0$, with $q_i \in (0, 1)$, and $\mu_i \in (0, 1)$ for $i = 1, 2, 3$.

[1] Note that

$$0 > \left(\frac{(1-\alpha_1)\,\alpha_1}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}\right)^{-1}\left(-\frac{(1-\alpha_3)(1-\alpha_1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}\right)+1.$$

Case 2 Throughout Case 2, $\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1} < \alpha_1 + \alpha_3 - 1$. Case 2 has two subcases, where I will introduce a pair of auxiliary variables for each subcase. Let me first state the auxiliary variables, and the derivatives for each case before giving a sketch of the proof.

Case 2.1 Throughout Case 2.1, $\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3 < \alpha_1$. Let me introduce two auxiliary variables, y_b and m_b, where

$$y_b = \mu_2^{\frac{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}{1-\alpha_1}}, \text{ and} \tag{53}$$

$$n_b = \left(\mu_1 \mu_2^{-\frac{1-\alpha_3}{1-\alpha_1}} - \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{\frac{1}{1-\alpha_1}} \right)^{\frac{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}{(1-\alpha_1)}}. \tag{54}$$

Using the IFT, I will explicitly derive the derivatives:

$$\left.\frac{dy_b}{dz}\right|_{y_b=m_b=z=0} = \frac{1-\alpha_3}{\alpha_1} \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}} > 0, \text{ and} \tag{55}$$

$$\left.\frac{dm_b}{dz}\right|_{y_b=m_b=z=0} = \frac{1-\alpha_3}{\alpha_1} \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{\frac{1-\alpha_1}{1-\alpha_3}} > 0. \tag{56}$$

Case 2.2 Throughout Case 2.2, $\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3 > \alpha_1$. Again, let me introduce two auxiliary variables, y_b and m_b, where

$$y_b = \mu_2^{\frac{(1-\alpha_1)(\alpha_3-\alpha_1)+(1-\alpha_3)(2-2\alpha_1-\alpha_3)}{1-\alpha_1}}, \text{ and} \tag{57}$$

$$m_b = \left(\mu_1 \mu_2^{-\frac{1-\alpha_3}{1-\alpha_1}} - \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{\frac{1}{1-\alpha_1}} \right)^{\frac{(1-\alpha_1)(\alpha_3-\alpha_1)+(1-\alpha_3)(2-2\alpha_1-\alpha_3)}{(1-\alpha_1)(\alpha_3-\alpha_1)}} \tag{58}$$

Using the IFT, one can again explicitly derive the derivatives:

$$\left.\frac{dy_b}{dz}\right|_{y_b=m_b=z=0} = \frac{1-\alpha_1}{\alpha_1} \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}} > 0, \text{ and} \tag{59}$$

$$\left.\frac{dm_b}{dz}\right|_{y_b=m_b=z=0} = \frac{1-\alpha_1}{\alpha_1} \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{\frac{1-\alpha_1}{1-\alpha_3}} > 0. \tag{60}$$

For both Case 2.1 and Case 2.2, in order to compute the above derivatives, it is useful to further introduce y_a and m_a, where where

$$y_a = \mu_2^{\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1}} \tag{61}$$

$$m_a = \mu_1 \mu_2^{-\frac{1-\alpha_3}{1-\alpha_1}} - \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{\frac{1}{1-\alpha_1}} \tag{62}$$

As set out in great detail for Case 1, I can rewrite (32) as a function of y_a and m_a, using (61) and (62). As with Case 1, taking derivatives and using the IFT, this gives me

$$\left.\frac{dy_a}{dm_a}\right|_{y_a=m_a=0} = \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{-\frac{\alpha_3-\alpha_1}{1-\alpha_1}} > 0. \tag{63}$$

Following the identical steps of Case 1, we get the derivatives given in (55), (56), (59), and (60). Therefore, if $\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1} < \alpha_1+\alpha_3-1$, and $\alpha_2 = 1-\alpha_1$ and $K=3$, the system (5)–(8) can be solved locally around $z=0$, with $q_i \in (0,1)$, and $\mu_i \in (0,1)$ for $i=1,2,3$. □

□

Proof of Proposition 6.

Proposition 1 (Restatement of Proposition 6)**.** *(a) Fix demands* $\{\alpha_1,\ldots,\alpha_K\}$ *(satisfying Lemmas 1 and 2), with* $K>3$*. Then there exists* $\bar{z}>0$ *such that for any* $z<\bar{z}$*, there exist no* $q_i>0$*, and* $\mu_i>0$ *for* $i=1,2,\ldots,K$ *such that* (z,α,q,μ) *satisfies (5)–(8).*

(b) Fix a sequence $z^n \to 0$*, and a corresponding convergent sequence of equilibria* (α^n,q^n,μ^n)*, where* $\alpha^n = (\alpha_i^n)_{i=1}^K$ *with* $K>3$*. Then there exists* $a_1 \in (0,1/2]$ *and* $a_K \in (1-a_1,1]$ *such that*

$$\lim_{n\to\infty}(\alpha_1^n,\ldots,\alpha_{k-1},\alpha_k,\ldots,\alpha_K^n) = (\underbrace{a_1,\ldots,a_1}_{\lceil K/2\rceil-1\ terms},\underbrace{1-a_1,\ldots,1-a_1,a_K}_{K-\lceil K/2\rceil+1\ terms}),$$

where $k=\lceil K/2\rceil$*. Moreover, along any such sequence,*

$$\lim_{n\to\infty} q^n = (\underbrace{0,\ldots,0}_{K-2\ terms},1,0).$$

The proof that follows is divided into the following steps. First, I show that in any sequence of equilibria, $\mu_i \to 0$ for any $i>1$ (Claims 5 and 6). I then show that there must be more than one immediate concession in the limit (Claim 7)[2] , i.e., there exist i,j with $i>j>1$ such that

$$\frac{\mu_i}{\mu_j} \to 0.$$

Finally, I show that if there is more than one immediate concession in the limit, an equilibrium with support $\{\alpha_1,\ldots,\alpha_K\}$ does not exist for z small enough (Claim 8). Together these claims establish that fixing any K demands satisfying Lemmas 1 and 2, an equilibrium with support $\{\alpha_1,\ldots,\alpha_K\}$ does not exist for z small enough.

[2]Recall from Lemma 1 that $\mu_i \le \mu_j$ for any $i>j$. Hence, if there exist i,j with $i>j>1$ such that

$$\frac{\mu_i}{\mu_j} \to 0,$$

then

$$\frac{\mu_i}{\mu_1} \to 0.$$

Claim 5. *For (5)–(8) to be satisfied, it is necessary that*

$$\lim_{z\to 0} \mu_K = 0.$$

Proof. Recall that by Lemma 1, in order for (5) to be satisfied it must be that $\mu_{k+1} \leq \mu_k$, $\forall k$, $\forall z > 0$, and hence also $\mu_K \leq \mu_{K-k}$, $\forall z > 0$. Note that (8) implies that

$$q_i \geq \frac{1}{K},$$

for at least one i. Therefore, by (7), $\lim_{z\to 0} \mu_i = 0$ for some i. It then follows that $\lim_{z\to 0} \mu_K = 0$. □

Claim 6. *If* $\lim_{z\to 0} \mu_K = 0$*, then for (5)–(8) to be satisfied it is necessary that for any* $i > 1$*,*

$$\lim_{z\to 0} \mu_i = 0.$$

Proof. Suppose $\lim_{z\to 0} \mu_{k+1} = 0$ and $\lim_{z\to 0} \mu_k \neq 0$ for some $k > 1$. Then for every $i < k$, $\lim_{z\to 0} \mu_i \neq 0$, and hence, it follows from (7) that $\lim_{z\to 0} \sum_{i=1}^{k} q_i = 0$.

Case 1: $\alpha_k > 1/2$. Then I can write (5) for $k' = k-1$ (i.e., $\Delta^r_{k-1,k}|_{\alpha_k > 1/2}$) as:

$$\begin{aligned}
&\sum_{\substack{i \text{ s.t.}\\ \alpha_i < 1-\alpha_k}} \underbrace{q_i \frac{1}{2}\left(\alpha_{k-1} - \alpha_k\right)}_{\to 0} + \sum_{\substack{i \text{ s.t.}\\ 1-\alpha_k < \alpha_i \leq \min\{\alpha_k, 1-\alpha_{k-1}\}}} \underbrace{q_i \frac{1}{2}\left(\alpha_{k-1} + \alpha_i - 1\right)}_{\to 0} \\
&+ \sum_{\substack{i \text{ s.t.}\\ \alpha_k \geq \alpha_i > \max\{\alpha_{k-1}, 1-\alpha_{k-1}\}}} \underbrace{q_i \left(\alpha_{k-1} + l_{i,k-1}^{1-\alpha_i}\left(1-\alpha_i-\alpha_{k-1}\right)\right)}_{\to 0} \\
&+ \sum_{i>k}^{K} q_i \left(\alpha_{k-1} + \underbrace{l_{i,k-1}^{1-\alpha_i}}_{\to 0}\left(1-\alpha_i-\alpha_{k-1}\right) - \alpha_k - \underbrace{l_{i,k}^{1-\alpha_i}}_{\to 0}\left(1-\alpha_i-\alpha_k\right)\right) = 0.
\end{aligned} \tag{64}$$

Since $\lim_{z\to 0} q_i = 0$ for any $i \leq k$, the first three terms go to 0. Note that there must exist $\lim_{z\to 0} q_i > 0$ for some $i > k$. Moreover, $\alpha_{k-1} \neq \alpha_k$. For the last term to go to 0, it is necessary that $\lim_{z\to 0} l_{i,k} \neq 0$ for some i. By assumption $\lim_{z\to 0} \mu_k \neq 0$, yet $\lim_{z\to 0} \mu_i = 0$ for any $i > k$. Hence, if $\lim_{z\to 0} \mu_k \neq 0$ with $\alpha_k > 1/2$, (64) cannot be satisfied. Therefore, it is necessary that $\forall \alpha_k > 1/2$,

$$\lim_{z\to 0} \mu_k = 0.$$

Case 2: $\alpha_k \leq 1/2$. Then I can write (5) for $k' = k-1$ (i.e., $\Delta^r_{k-1,k}|_{\alpha_k \leq 1/2}$) as:

$$\sum_{i\le k} q_i \underbrace{\frac{1}{2}(\alpha_{k-1}-\alpha_k)}_{\to 0} + \sum_{\substack{i \text{ s.t.}\\ \alpha_k<\alpha_i\le 1-\alpha_k}} q_i \underbrace{\frac{1}{2}(\alpha_{k-1}+\alpha_k-1)}_{\le 0}$$
$$+ \sum_{\substack{i \text{ s.t.}\\ 1-\alpha_k<\alpha_i\le 1-\alpha_{k-1}}} q_i \left(\underbrace{\frac{\alpha_{k-1}+1-\alpha_i}{2} - \alpha_k}_{<0} + \underbrace{l_{i,k}^{1-\alpha_i}}_{\to 0} (\alpha_i+\alpha_k-1) \right) \tag{65}$$
$$+ \sum_{\substack{i \text{ s.t.}\\ \alpha_i>1-\alpha_{k-1}}} q_i \left(\alpha_{k-1} + \underbrace{l_{i,k-1}^{1-\alpha_i}}_{\to 0} (1-\alpha_i-\alpha_{k-1}) - \alpha_k - \underbrace{l_{i,k}^{1-\alpha_i}}_{\to 0} (1-\alpha_i-\alpha_k) \right) = 0.$$

As in Case 1, since $\lim_{z\to 0} q_i = 0$ for any $i \le k$, the first term goes to 0. The second term is strictly negative if $\lim_{z\to 0} q_i > 0$. Suppose the second term is strictly negative. Then either the third or fourth term need to be strictly positive for (65) to be satisfied. Regarding the third term, note that

$$\frac{\alpha_{k-1}+1-\alpha_i}{2} - \alpha_k = \frac{1}{2}\underbrace{(\alpha_{k-1}-\alpha_k)}_{<0} + \frac{1}{2}\underbrace{(1-\alpha_i-\alpha_k)}_{<0} < 0.$$

Hence, for either the third or fourth term to be weakly positive, it is necessary that $\lim_{z\to 0} l_{i,k} \neq 0$. Since $\lim_{z\to 0} \mu_i = 0$ for any $i > k$, this would require $\lim_{z\to 0} \mu_k = 0$. However, by assumption this does not hold. Recall there must be at least one i for which $\lim_{z\to 0} q_i > 0$. Therefore, at least one of the remaining three terms is strictly negative, and no term is strictly positive. Hence, if $\lim_{z\to 0} \mu_k \neq$ with $\alpha_k \le \frac{1}{2}$, $\Delta_{k-1} = 0$ cannot be satisfied. Therefore, it is necessary that for any $\frac{1}{2} \ge \alpha_k > \alpha_1$,

$$\lim_{z\to 0} \mu_k = 0.$$

Therefore, if $\lim_{z\to 0} \mu_K = 0$, then for (5)–(8) to be satisfied it is necessary that for any $i > 1$,

$$\lim_{z\to 0} \mu_i = 0.$$

□

Note that the system (5)–(8) is linear in the probability of being faced with an offer α_i, q_i. Hence, without loss, I normalize $q_{K-1} = 1$ for the remaining part of the proof.

Claim 7. *There exist i, j with $i > j > 1$ such that*

$$\lim_{z\to 0} l_{i,j} = 0.$$

Proof. Suppose not; i.e., suppose that for all i, j with $i > j > 1$, there exist $1 \ge \lim_{z\to 0} l_{i,j} > 0$, such that

$$\frac{\mu_i}{\mu_j} = l_{i,j}.$$

Evaluating (6) for $k = 1$ and rearranging, I get:

$$q_{K-1}\left(1-\alpha_{K-1}\right)\mu_2^{\alpha_2+\alpha_{K-1}-1}\mu_{K-1}^{1-\alpha_{K-1}} = q_K\left(1-\alpha_K\right)\mu_K^{1-\alpha_K}\left(\mu_1^{\alpha_1+\alpha_K-1}-\mu_2^{\alpha_2+\alpha_K-1}\right) \tag{66}$$

Suppose $\lim_{z\to 0} l_{K,1} > 0$. Then I can write (66) as:

$$\begin{aligned} q_{K-1}\left(1-\alpha_{K-1}\right)\mu_2^{\alpha_2}l_{K-1,2}^{1-\alpha_{K-1}} &= q_K\left(1-\alpha_K\right)\left(\mu_1^{\alpha_1}l_{K,1}^{1-\alpha_K}-\mu_2^{\alpha_2}l_{K,2}^{1-\alpha_K}\right),\text{ or} \\ q_{K-1}\left(1-\alpha_{K-1}\right)\underbrace{\mu_2^{\alpha_2-\alpha_1}}_{\to 0}\underbrace{l_{2,1}^{\alpha_1}l_{K-1,2}^{1-\alpha_{K-1}}}_{>0} &= q_K\left(1-\alpha_K\right)\underbrace{\left(l_{K,1}^{1-\alpha_K}-\mu_2^{\alpha_2-\alpha_1}l_{2,1}^{\alpha_1}l_{K,2}^{1-\alpha_K}\right)}_{>0} \end{aligned} \tag{67}$$

Since $q_{K-1} = 1$, it must be that $\lim_{z\to 0} q_K = 0$.

Suppose instead $\lim_{z\to 0} l_{K,1} = 0$. Since $\lim_{z\to 0} l_{K,2} > 0$, this implies that $\lim_{z\to 0} l_{2,1} = 0$. Hence, I can write (66) as:

$$q_{K-1}\left(1-\alpha_{K-1}\right)\underbrace{l_{2,1}^{\alpha_1+\alpha_K-1}\mu_2^{\alpha_2-\alpha_1}}_{\to 0}\underbrace{l_{K-1,2}^{\alpha_K-\alpha_{K-1}}}_{>0} = q_K\left(1-\alpha_K\right)l_{K,K-1}^{1-\alpha_K}\left(1-\underbrace{l_{2,1}^{\alpha_1+\alpha_K-1}\mu_2^{\alpha_2-\alpha_1}}_{\to 0}\right) \tag{68}$$

Since $q_{K-1} = 1$, it must be that $\lim_{z\to 0} q_K = 0$.

Evaluating (5) for $k = 1$ gives:

$$\begin{aligned} & q_K\left(\alpha_1\left(1-l_{K,1}^{1-\alpha_K}\right)-\alpha_2\left(1-l_{K,2}^{1-\alpha_K}\right)\right)-q_K(1-\alpha_K)\left(l_{K,2}^{1-\alpha_K}-l_{K,1}^{1-\alpha_K}\right) \\ & -q_{K-1}\left(\frac{1-\alpha_1-\alpha_{K-1}}{2}\left(\alpha_2+\alpha_{K-1}-1\right)\left(1-l_{K-1,2}^{1-\alpha_{K-1}}\right)\right)-\sum_{i=1}^{K-2}q_i\left(\frac{\alpha_2-\alpha_1}{2}\right)=0. \end{aligned} \tag{69}$$

Note that the last three terms in (69) are negative. However, $\lim_{z\to 0} q_K = 0$, and $q_{K-1} = 1$. Hence, (69) cannot be satisfied. Hence, it is necessary for there to exist i, j with $i > j > 1$ such that

$$\lim_{z\to 0} l_{i,j} = 0.$$

□

Claim 8. *Suppose that there exist i, j with $i > j > 1$ such that*

$$\lim_{z\to 0} l_{i,j} = 0.$$

Then for all $(z = 0, \lim_{z\to 0}\alpha, \lim_{z\to 0} q, \lim_{z\to 0}\mu)$, solving (5)–(8), it must be that

$$\lim_{z\to 0} \alpha_i - \alpha_{i+1} = 0$$

for all i, $i+1 \le \lceil K/2\rceil - 1$ and all $i, i+1 \ge \lceil K/2\rceil$ with $i+1 < K$.

Proof. Suppose there exist i, j with $i > j > 1$ such that

$$\lim_{z\to 0} l_{i,j} = 0.$$

Recall that by Lemma 1, $\mu_i \leq \mu_j$ for any $i > j$. Hence, without loss, there exists i such that

$$\lim_{z\to 0} l_{K,i} = 0.$$

It follows that

$$\lim_{z\to 0} l_{K,i} = 0$$

for $i = 1, 2$. Note that this implies that:

$$\lim_{z\to 0} q_K(1-\alpha_K)\left(l_{K,2}^{1-\alpha_K} - l_{K,1}^{1-\alpha_K}\right) = 0.$$

Therefore, the unique candidate solution to (69) is:

$$\lim_{z\to 0} q_i = 0, \text{ for any } i \neq K-1 \tag{70}$$

$$\lim_{z\to 0} l_{K-1,2} = 1, \text{ and} \tag{71}$$

$$\lim_{z\to 0} \alpha_1 + \alpha_{K-1} = 1. \tag{72}$$

Note that in this case, the rational type is indifferent between any two demands – α_{K-1} never concedes to a lower demand, and a player is faced with a demand of α_{K-1} with probability 1. Hence, the rational receives $1 - \alpha_{K-1}$ regardless of his demand.

Evaluating (6) at $k = 2$ and rearranging gives

$$\begin{aligned}
q_{K-2}\left(1-\alpha_{K-2}\right)\mu_3^{\alpha_3+\alpha_{K-2}-1}\mu_{K-2}^{1-\alpha_{K-2}} &= q_K\left(1-\alpha_K\right)\mu_K^{1-\alpha_K}\left(\mu_2^{\alpha_2+\alpha_K-1} - \mu_3^{\alpha_3+\alpha_K-1}\right) \\
&\quad + q_{K-1}\left(1-\alpha_{K-1}\right)\mu_{K-1}^{1-\alpha_{K-1}}\left(\mu_2^{\alpha_2+\alpha_{K-1}-1} - \mu_3^{\alpha_3+\alpha_{K-1}-1}\right), \\
&\iff \\
q_{K-2}\left(1-\alpha_{K-2}\right)\mu_3^{\alpha_3} l_{K-2,3}^{1-\alpha_{K-2}} &= q_K\left(1-\alpha_K\right)\left(l_{K,2}^{1-\alpha_K}\mu_2^{\alpha_2} - l_{K,3}^{1-\alpha_K}\mu_3^{\alpha_3}\right) \\
&\quad + q_{K-1}\left(1-\alpha_{K-1}\right)\left(l_{K-1,2}^{1-\alpha_{K-1}}\mu_2^{\alpha_2} - l_{K-1,3}^{1-\alpha_{K-1}}\mu_3^{\alpha_3}\right), \\
&\iff \\
q_{K-2}\left(1-\alpha_{K-2}\right) l_{3,2}^{\alpha_3}\mu_2^{\alpha_3-\alpha_2} l_{K-2,3}^{1-\alpha_{K-2}} &= q_K\left(1-\alpha_K\right)\left(l_{K,2}^{1-\alpha_K} - l_{K,3}^{1-\alpha_K} l_{3,2}^{\alpha_3}\mu_2^{\alpha_3-\alpha_2}\right) \\
&\quad + q_{K-1}\left(1-\alpha_{K-1}\right)\left(l_{K-1,2}^{1-\alpha_{K-1}} - l_{K-1,3}^{1-\alpha_{K-1}} l_{3,2}^{\alpha_3}\mu_2^{\alpha_3-\alpha_2}\right).
\end{aligned} \tag{73}$$

Recall that $\lim_{z\to 0} q_i = 0$, for any $i \neq K-1$. Hence, the LHS and the first term on the RHS of (73) go to 0. Further, recall that $\lim_{z\to 0} l_{K-1,2} = 1$ and $\lim_{z\to 0} \mu_2 = 0$. Hence, it follows from (73) that $\lim_{z\to 0} q_{K-1} = 0$. A contradiction since by (70),

$$\lim_{z\to 0} q_{K-1} = 1.$$

Hence, if there exist i, j with $i > j > 1$ such that

$$\lim_{z \to 0} l_{i,j} = 0,$$

then for all $(z = 0, \lim_{z\to 0} \alpha, \lim_{z\to 0} q, \lim_{z\to 0} \mu)$, solving (5)–(8), it must be that

$$\lim_{z \to 0} \alpha_i - \alpha_{i+1} = 0$$

for all $i,\ i+1 \leq \lceil K/2 \rceil - 1$ and all $i, i+1 \geq \lceil K/2 \rceil$ with $i+1 < K$. But by Claim 7 there exist i, j with $i > j > 1$ such that

$$\lim_{z \to 0} l_{i,j} = 0.$$

Hence, I have established that for all $(z = 0, \lim_{z\to 0} \alpha, \lim_{z\to 0} q, \lim_{z\to 0} \mu)$ solving (5)–(8), it must be that

$$\lim_{z \to 0} \alpha_i - \alpha_{i+1} = 0$$

for all $i,\ i+1 \leq \lceil K/2 \rceil - 1$ and all $i, i+1 \geq \lceil K/2 \rceil$ with $i+1 < K$. □

□

Proof of Proposition 7.

Claim 9. *Let $\underline{\alpha} = \inf \operatorname{supp} r$. Then $\underline{\alpha}$ must be a point of increase in the equilibrium distribution.*

Proof. The proof is by contradiction. In particular, I show that there exists $\underline{\alpha} + \epsilon > \underline{\alpha}$ such that $v^r(\underline{\alpha} + \epsilon) > v^r(\underline{\alpha})$. Define

$$\underline{h}(\alpha_k) = \int_{\underline{\alpha}}^{1-\alpha_k} \frac{1 - \alpha_i + \alpha_k}{2} dG(\alpha_i) + \int_{1-\alpha_k}^{\bar{\alpha}} (1 - \alpha_i) dG(\alpha_i), \tag{74}$$

$$\bar{h}(\alpha_k) = \int_{\underline{\alpha}}^{1-\alpha_k} \frac{1 - \alpha_i + \alpha_k}{2} dG(\alpha_i) + \int_{1-\alpha_k}^{\bar{\alpha}} \alpha_k dG(\alpha_i), \tag{75}$$

where $\underline{\alpha}$ and $\bar{\alpha}$ denote the lowest and highest offer in I respectively. Note that

$$\underline{h}(\alpha_k) \leq v^r(\alpha_k) \leq \bar{h}(\alpha_k).$$

Evaluating (74) and (75) at $\alpha_k = \underline{\alpha}$ gives:

$$\underline{h}(\underline{\alpha}) = \int_{\underline{\alpha}}^{\bar{\alpha}} \frac{1 - \alpha_i + \underline{\alpha}}{2} dG(\alpha_i) + \nu(\bar{\alpha})\left(1 - \bar{\alpha}\right), \tag{76}$$

$$\bar{h}(\underline{\alpha}) = \int_{\underline{\alpha}}^{\bar{\alpha}} \frac{1 - \alpha_i + \underline{\alpha}}{2} dG(\alpha_i) + \nu(\bar{\alpha})\underline{\alpha}. \tag{77}$$

Hence, $\underline{h}(\underline{\alpha}) = \bar{h}(\underline{\alpha})$, and thus, $v^r(\underline{\alpha}) = \underline{h}(\underline{\alpha})$. Note that $\underline{h}(\alpha_k)$ is differentiable in α_k, and hence I can write:

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{d\underline{h}(\alpha)}{d\alpha}\bigg|_{\alpha = \underline{\alpha}+\epsilon} &= \lim_{\epsilon \to 0} \left(\frac{1 - (1 - \underline{\alpha} - \epsilon) + \underline{\alpha} + \epsilon}{2} g(\bar{\alpha} + \epsilon) + \int_{\underline{\alpha}}^{\bar{\alpha}+\epsilon} \frac{1}{2} dG(\alpha_i) - \left(\underline{\alpha} + \epsilon\right) g(\bar{\alpha} - \epsilon) \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{2} G(\bar{\alpha} - \epsilon) > 0.
\end{aligned}$$

Therefore, since

$$\lim_{\epsilon \to 0} \left. \frac{d\underline{h}(\alpha)}{d\alpha} \right|_{\alpha = \underline{\alpha} + \epsilon} > 0,$$

there exists $\epsilon > 0$, such that

$$v^r(\underline{\alpha} + \epsilon) > v^r(\underline{\alpha}).$$

□

The rest of the proof of Proposition 7 is a straightforward modification of the proof of Proposition 6. □

Existence of semiseparating equilibria

In analogue to Lemma 4, we can define a corresponding system of equations for semiseparating equilibria. Recall that $\Delta^r_{k,k+1}$ and $\Delta^s_{k,k+1}$ are defined in (3) and (4). Given z and $\{\alpha_1, \ldots, \alpha_K\}$, define the following system in (q_i, μ_i), $i = 1, \ldots, K$:

$$\Delta^r_{k,k+1} = 0, \ \forall k \ \text{ such that } \alpha_k \in \operatorname{supp} r, \tag{78}$$

$$\Delta^r_{k,k+1} \leq 0, \ \forall k \ \text{ such that } \alpha_k \notin \operatorname{supp} r, \tag{79}$$

$$\Delta^s_{k,k+1} = 0, \ \forall k \ \text{ such that } \alpha_k \in \operatorname{supp} s \tag{80}$$

$$\Delta^s_{k,k+1} \geq 0, \ \forall k+1 \ \text{ such that } \alpha_{k+1} \notin \operatorname{supp} s, \tag{81}$$

$$\sum_{i=1}^{K} q_i \mu_i^{1-\alpha_i} = z, \text{ and} \tag{82}$$

$$\sum_{i=1}^{K} q_i = 1. \tag{83}$$

Proof of Proposition 8.

Claim 10. *Under (A1), there can be at most one offer which is made exclusively by the rational type, i.e.,*

$$|\operatorname{supp} r \setminus \operatorname{supp} s| \leq 1.$$

Proof. Suppose there were several separating offer by the rational type. Consider the two highest such offers, β and β', where $\beta < \beta'$. I denote any remaining offers α_1 through α_K.

Case 1: $\alpha_1 + \alpha_K \geq 1$**.** Then the rational type's payoff difference, (79), for $(k, k+1) = (\beta, \beta')$ is given by:

$$\begin{aligned}\Delta^r_{\beta,\beta'} = & \sum_{i=1}^{K} q_i (1 - \alpha_i) + q_\beta (1 - \beta) + q_{\beta'} \beta \\ & - \left(\sum_{i=1}^{K} q_i (1 - \alpha_i) + q_\beta (1 - \beta) + q_{\beta'} (1 - \beta') \right).\end{aligned}$$

Hence, $\Delta^r_{\beta,\beta'} > 0$. Therefore, under (A1), if $\alpha_1 + \alpha_K \geq 1$, then there exists at most one separating offer by the rational type.

Case 2: $\alpha_1 + \alpha_K < 1$. In this case, the payoff to a rational type and a stubborn type from making the demand α_1 is identical. Hence, by Lemma 2, it must be that α_1 is a separating offer by the stubborn type. This implies that α_1 must be exactly compatible with the lowest separating offer by the rational type (otherwise, the stubborn type is not willing to demand α_1 – and deviates to the complement of the lowest separating offer by the rational type). Hence, it must be that $\alpha_1 + \beta' > 1$. But if $\alpha_1 + \beta' > 1$, a rational player has an incentive to deviate from β' to $\beta' - \epsilon$, where $\beta' - \beta > \epsilon > 0$: conditional on facing β', the rational type increases his payoff from $1 - \beta'$ to $\beta' - \epsilon$; while the payoff conditional on facing any other offer remains unchanged. Therefore, under (A1), if $\alpha_1 + \alpha_K < 1$, there exists at most one separating offer by the rational type.

Hence, under (A1), there exists at most one separating offer by the rational type. □

Claim 11. *Under (A1), there exists no symmetric equilibrium with a separating offer by the rational type, i.e., in every symmetric equilibrium,* $\operatorname{supp} r \setminus \operatorname{supp} s = \emptyset$.

Proof. There are two cases to consider: (1) $|\operatorname{supp} r \cup \operatorname{supp} s| > 2$, and (2) $|\operatorname{supp} r \cup \operatorname{supp} s| = 2$.

Case 1: $|\operatorname{supp} r \cup \operatorname{supp} s| > 2$. Denote the separating offer by the rational type by β. By Lemma 1 and 2, and the results on semiseparating offers by the stubborn type, the separating offer must either be (i) incompatible with all other offers, or (ii) exactly compatible with the lowest offer (which then must be a separating offer by the stubborn type) and incompatible with all higher offers. Hence, the payoff from making the offer β is:

$$v_\beta^r = \sum_{i=1}^{K} q_i \left(1 - \alpha_i\right) + q_\beta \left(1 - \beta\right).$$

(i) If $\alpha_1 + \beta > 1$, then it is profitable for the rational type to deviate to $\beta - \epsilon$ for any $\epsilon \in (0, \min\{\beta - \alpha_K, \beta + \alpha_1 - 1\}]$ since:

$$v_{\beta-\epsilon}^r = \sum_{i=1}^{K} q_i \left(1 - \alpha_i\right) + q_\beta \left(\beta - \epsilon\right).$$

(ii) Suppose instead $\alpha_1 + \beta = 1$. The payoff from demanding α_2 in this case is identical for the rational and the stubborn type: the only offer incompatible with α_2 is the rational separating offer β, which concedes immediately with probability 1 to α_2:

$$v_2^r = v_2^s = \sum_{i=1}^{K} q_i \left(\frac{\alpha_2 + 1 - \alpha_i}{2}\right) + q_\beta \alpha_2.$$

But if the payoff to the stubborn type is the same from demanding α_2, it must be the same for any other offer, including α_1. The rational type's payoff from α_1 is:

$$v_1^r = \sum_{i=1}^{K} \left(\frac{\alpha_2 + 1 - \alpha_i}{2}\right) + q_\beta \alpha_1.$$

But the payoff from demanding α_K is strictly higher:

$$v^r_\beta = \sum_{i=1}^{K} q_i \left(1 - \alpha_i\right) + q_\beta \left(1 - \beta\right),$$

where recall that $\beta = 1 - \alpha_1$. Hence, there exists no equilibrium with $\alpha_1 + \beta = 1$.

Therefore, under (A1), if $|\operatorname{supp} r \cup \operatorname{supp} s| > 2$, there exists no symmetric equilibrium with a separating offer by the rational type.

Case 2: $|\operatorname{supp} r \cup \operatorname{supp} s| = 2$**.** Denote the rational separating offer by β, and the remaining offer by α.

(i) $\alpha + \beta \leq 1$, and α is a pooling offer. The rational type's payoff from α is:

$$v^r_\alpha = q_\alpha \frac{1}{2} + q_\beta \left(\frac{\alpha + 1 - \beta}{2}\right),$$

while the payoff from deviating to $1 - \alpha$ is:

$$v^r_{1-\alpha} = q_\alpha \left(1 - \alpha\right) + q_\beta \left(\frac{1 - \alpha + 1 - \beta}{2}\right).$$

Since $\alpha < \frac{1}{2}$, the rational type has an incentive to deviate to $1 - \alpha$.

(ii) $\alpha + \beta \leq 1$, and α is a separating offer by the stubborn type. The stubborn type's payoff from demanding α is:

$$v^s_\alpha = q_\alpha \frac{1}{2} + q_\beta \alpha.$$

By deviating to β, the stubborn type receives a payoff of:

$$v^s_\beta = q_\alpha \left(\frac{\beta + 1 - \alpha}{2}\right) + q_\beta \left(1 - \beta\right)$$

Since, $\beta > \alpha$, and $\alpha + \beta \leq 1$, the payoff to the stubborn type from β is higher than the payoff from α.

(iii) $\alpha + \beta > 1$. The rational type's payoff from demanding β is:

$$v^r_\beta = q_\alpha \left(1 - \alpha\right) + q_\beta \left(1 - \beta\right).$$

The rational type's payoff from deviating to $\beta - \epsilon$, where $\beta > \beta - \epsilon > \alpha$:

$$v^r_{\beta-\epsilon} = q_\alpha \left(1 - \alpha\right) + q_\beta \left(\beta - \epsilon\right).$$

Hence, the rational type has an incentive to deviate to $\beta - \epsilon$ for any $\epsilon \in (0, \beta - \alpha)$.

Therefore, under (A1), if $|\operatorname{supp} r \cup \operatorname{supp} s| = 2$, there exists no symmetric equilibrium with a separating offer by the rational type. Hence, under (A1), there exists no symmetric equilibrium with a separating offer by the rational type. □

□

Proof of Proposition 9. The proof is in three steps. I first show that the lowest separating offer must be incompatible with the highest offer for z small enough (Claim 12). In short, if the highest offer was compatible with the lowest separating offer, the stubborn type would be better off by deviating to a higher offer. The next step then is to show that there can be at most one separating offer (Claim 13): Note that any separating offer will be conceded to immediately (conditional on facing a rational opponent). Hence, when the probability of facing a rational opponent goes to 1, there is no incentive to make the lower separating offer. The last step is to prove that for sufficiently small probability of stubbornness, there exists no equilibrium with a separating offer by the stubborn type when there are more than two offers (Claim 14).

Claim 12. *Fix demands $\{\alpha_1, \ldots, \alpha_K\}$ (satisfying Lemmas 1 and 2), where $\alpha_1 \in \operatorname{supp} s \setminus \operatorname{supp} r$, $\alpha_1 + \alpha_K \leq 1$ and $K > 2$. Then there exists $\bar{z} > 0$, such that for any $z < \bar{z}$, there exist no $q_i > 0$, and $\mu_i > 0$, for $i = 1, 2, \ldots, K$ such that (z, α, q, μ) satisfy (78)–(83).*

Proof. Suppose $\alpha_1 \leq 1 - \alpha_K$, where $K > 2$, and α_1 is a separating offer by the stubborn type. Then clearly, the rational type does not want to mimic α_1. If $\alpha_1 < 1 - \alpha_K$, the stubborn type would also not be willing to mimic α_1: he would receive a strictly higher payoff regardless of the demand he faces by demanding $1 - \alpha_K$. Hence, if the separating demand, α_1, is compatible with every other demand, it must be that $\alpha_1 = 1 - \alpha_K$. Moreover, it must be that α_K is a pooling offer. If α_K was a separating offer by the rational type, then the payoff from demanding α_2 would be identical to the rational and the stubborn type (the only incompatible offer would be α_K which concedes w.p. 1 to α_2 at time 0). But then the payoff must be identical for any other offer, which we know cannot be if there are multiple offers. Hence, if $\alpha_1 + \alpha_K = 1$, then α_1 is a separating offer by the stubborn type and there exists no separating offer by the rational type.

If $\alpha_1 = 1 - \alpha_K$, the stubborn type's payoff difference between α_1 and α_2, i.e., equation (4) evaluated at $k = 1$, is given by:

$$\begin{aligned}\Delta^s_{1,2} = &-\sum_{i=1}^{K-1} q_i \frac{1}{2}\left(\alpha_K + \alpha_2 - 1\right) \\ &+ q_K\left(-\left(\alpha_K + \alpha_2 - 1\right)\left(1 - l_{K,2}^{1-\alpha_K}\right) + (1-\alpha_K) l_{K,2}^{1-\alpha_K} \mu_2^{\alpha_2}\right).\end{aligned} \tag{84}$$

But $\mu_i \to 0$ for any $i > 1$ such that $\alpha_i \in \operatorname{supp} r$ (by Claim 6, see proof of Proposition 6). Hence,

$$l_{K,2}^{1-\alpha_K} \mu_2^{\alpha_2} \to 0.$$

Recall that there must exist q_i such that $q_i \not\to 0$. Hence, there exist $\bar{z} > 0$ such that $\Delta^s_1 < 0$ for any $z < \bar{z}$, i.e., the stubborn type's payoff from demanding α_2 is strictly higher than the payoff from demanding α_1. □

Claim 13. *Fix demands* $\{\alpha_1, \dots, \alpha_K\}$ *(satisfying Lemmas 1 and 2), where* $K > 2$ *and where* $\alpha_1, \alpha_2 \in \operatorname{supp} s \setminus \operatorname{supp} r$*. Then there exists* $\bar{z} > 0$ *such that for any* $z < \bar{z}$*, there exists no* $q_i > 0$*, and* $\mu_i > 0$*, for* $i = 1, 2, \dots, K$ *such that* (z, α, q, μ) *satisfy (78)–(83).*

Proof. Evaluating the stubborn type's payoff difference, equation (4), for $k = 1$ gives:

$$\begin{aligned}\Delta^s_{1,2} = &\sum_{i \text{ s.t. } \alpha_i < 1-\alpha_2} q_i \left(\frac{\alpha_1 - \alpha_2}{2}\right) + q_{K-1}\left(\left(\frac{\alpha_1 + 1 - \alpha_{K-1}}{2}\right) - \left(1 - \mu_{K-1}^{1-\alpha_{K-1}}\right)\alpha_2\right) \\ &+ q_K\left(1 - \mu_K^{1-\alpha_K}\right)(\alpha_1 - \alpha_2).\end{aligned} \tag{85}$$

Note that the first and the last term are negative. Moreover, note that

$$\lim_{z \to 0}\left(\left(\frac{\alpha_1 + 1 - \alpha_{K-1}}{2}\right) - \left(1 - \mu_{K-1}^{1-\alpha_{K-1}}\right)\alpha_2\right) = \frac{\alpha_1 + 1 - \alpha_{K-1}}{2} - \alpha_2 < 0. \tag{86}$$

Hence,

$$\lim_{z \to 0} \Delta^s_1 < 0.$$

Hence, there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, there exists no q_i and μ_i, for $i = 1, 2, \dots, K$ such that (z, α, q, μ) satisfy (78)–(83). □

Claim 14. *Fix demands* $\{\alpha_1, \dots, \alpha_K\}$ *(satisfying Lemmas 1 and 2), where* $\alpha_1 \in \operatorname{supp} s \setminus \operatorname{supp} r$*. Then there exists* $\bar{z} > 0$ *such that for any* $z < \bar{z}$*, there exist no* $q_i > 0$*, and* $\mu_i > 0$*, for* $i = 1, 2, \dots, K$ *such that* (z, α, q, μ) *satisfy (78)–(83).*

Proof. Combining (79) and (80) for $k = 1$ gives

$$q_{K-1}(1 - \alpha_{K-1})\mu_{K-1}^{\alpha_2}\left(\frac{\mu_2}{\mu_{K-1}}\right)^{\alpha_2 + \alpha_{K-1} - 1} \geq q_K(1 - \alpha_K)\mu_K^{1-\alpha_K}\left(1 - \mu_2^{\alpha_2 + \alpha_{K-1} - 1}\right), \tag{87}$$

which we can rewrite as:

$$q_{K-1}(1 - \alpha_{K-1})\underbrace{\mu_2^{\alpha_2 + \alpha_{K-1} - 1}}_{\to 0}\underbrace{l_{K-1,2}^{\alpha_K - \alpha_{K-1}}}_{>0} \geq q_K(1 - \alpha_K)\, l_{K,K-1}^{1-\alpha_K}\left(1 - \mu_2^{\alpha_2 + \alpha_K - 1}\right). \tag{88}$$

Hence, either $\lim_{z \to 0} q_K = 0$ or $\lim_{z \to 0} l_{K,i} = 0$, $\forall i < K$. Therefore, (88) holds with equality. Hence, it must be that (79) for $k = 1$ holds with equality:

$$\begin{aligned}&q_K(1 - \alpha_K)\left(l_{K,2}^{1-\alpha_K} - \mu_K^{1-\alpha_K}\right) - q_K(\alpha_2 - \alpha_1) - q_{K-1}\left(\frac{1 - \alpha_1 - \alpha_{K-1}}{2}\right) \\ &- q_{K-1}(\alpha_2 + \alpha_{K-1} - 1)\left(1 - l_{K-1,2}^{1-\alpha_{K-1}}\right) - \sum_{i=1}^{K-2} q_i\left(\frac{\alpha_2 - \alpha_1}{2}\right) = 0.\end{aligned} \tag{89}$$

If $\lim_{z \to 0} q_K = 0$, or $\lim_{z \to 0} l_{K,i} = 0$, $\forall i < K$, then for (89) to be satisfied, it is necessary that

$$\begin{aligned}&\lim_{z \to 0} q_i = 0,\ \forall\, i \neq K - 1, \\ &\lim_{z \to 0} \alpha_1 = 1 - \alpha_{K-1}, \\ &\lim_{z \to 0} l_{K-1,2} = 1.\end{aligned}$$

Therefore, it must be that $\lim_{z \to 0} l_{K,i} = 0$, $\forall i < K$. However, by the same argument as in the Proof of 6 $\Delta^r_{2,3} \neq 0$ in this case. □

□

Proof of Proposition 10.

Claim 15. *Fix any* $\{\alpha_1, \alpha_2\}$*, with* $\alpha_1 + \alpha_2 \neq 1$*. Then there exists no symmetric equilibrium with* $\operatorname{supp} r = \{\alpha_2\}$ *and* $\operatorname{supp} s = \{\alpha_1, \alpha_2\}$*.*

Proof. Since strength is decreasing in any equilibrium and $\mu_1 = 1$, it follows that $\alpha_1 < \alpha_2$.

Case 1: $\alpha_1 > 1/2$. In this case, the rational type has an incentive to deviate to α_1:

$$v_1^r = q_1 (1 - \alpha_1) + q_2 \left(\left(1 - \mu_2^{1-\alpha_2}\right) \alpha_1 + \mu_2^{1-\alpha_2} (1 - \alpha_2) \right), \tag{90}$$

$$v_2^r = q_1 (1 - \alpha_1) + q_2 (1 - \beta). \tag{91}$$

Hence, $v_1^r > v_2^r$.

Case 2: $\alpha_2 \leq 1/2$. In this case, the rational type has an incentive to deviate to $1 - \alpha_1$: Conditional on facing a demand of α_1, demanding α_2 gives $1/2 (\alpha_2 + 1 - \alpha_1)$, while demanding $1 - \alpha_1$ gives $1 - \alpha_1 > \alpha_2$. Conditional on facing a demand of α_2, demanding α_2 gives $1/2$, while demanding $1 - \alpha_1$ gives at least $1 - \alpha_2$. Hence, the rational type has a strictly higher payoff from demanding $1 - \alpha_1$ than from demanding α_2.

Case 3: $\alpha_1 \leq 1/2 < \alpha_2$. If $\alpha_1 + \alpha_2 < 1$, then the stubborn type has an incentive to deviate from α_1 to $1 - \alpha_2$: Conditional on facing a demand of α_1, a stubborn type receives $1/2$ from demanding α_1 and receives $1/2 (1 - \alpha_1 + \alpha_2)$ from demanding $1 - \alpha_2$. Note that $\alpha > \alpha_1$. Conditional on facing a demand of α_2, a stubborn type receives $1/2 (\alpha_1 + 1 - \alpha_2)$ from demanding α_1, while demanding $1 - \alpha_2$ would give the stubborn type $1 - \alpha_2$. Hence, the stubborn type has a strictly higher payoff from demanding $1 - \alpha_2$ than from demanding α_1.

Hence, it must be that $\alpha_1 + \alpha_2 > 1$. A stubborn type's payoff from demanding α_2 and $\alpha_3 = 1 - \alpha_2$ is:

$$v_2^s = q_2 (1 - \alpha_2) (1 - \mu_2^{\alpha_2}), \tag{92}$$

$$v_3^s = q_1 \left(\frac{1 - \alpha_2 + 1 - \alpha_1}{2} \right) + q_2 (1 - \alpha_2). \tag{93}$$

Hence, $v_2^s < v_3^s$. □

Claim 16. *Fix any* $\alpha_1,\ \alpha_2$*, with* $\alpha_2 = 1 - \alpha_1 > 1/2$*. Then there exists an equilibrium with* $\operatorname{supp} r = \{\alpha_2\}$ *and* $\operatorname{supp} s = \{\alpha_1, \alpha_2\}$*.*

Proof. Evaluating (79) for $k = 1$ gives:

$$\Delta_{1,2}^r = -(1 - q_2) \left(\alpha_2 - \frac{1}{2} \right) < 0. \tag{94}$$

Hence, the rational type has no incentive to deviate to α_1. Evaluating (80) for $k = 1$, i.e. $\Delta^s_{1,2}$, gives:

$$q_2 (1 - \alpha_2) \mu_2^{\alpha_2} - (1 - q_2) \left(\alpha_2 - \frac{1}{2} \right) = 0. \tag{95}$$

Solving (95) for q_2 I get:

$$q_2 = \frac{2\alpha_2 - 1}{2\alpha - 1 + 2\mu_2^{\alpha_2} (1 - \alpha_2)} \in (0, 1]. \tag{96}$$

Plugging (96) into (82) and simplifying, I get:

$$z = \frac{2\mu_2^{\alpha_2} (1 - \alpha_2) + \mu_2^{1-\alpha_2} (2\alpha_2 - 1)}{2\mu_2^{\alpha_2} (1 - \alpha_2) + 2\alpha_2 - 1}. \tag{97}$$

Note that $\mu_2^{\alpha_2} < \mu_2^{1-\alpha_2}$, and hence,

$$\mu_2^{1-\alpha_2} / z \to 1. \tag{98}$$

□

□

Robustness: Refinements

Proposition 11.

1. *Every one-offer equilibrium satisfies D1.*
2. *A pooling equilibrium with support* $\{\alpha_1, \dots, \alpha_K\}$ *and* $K \geq 2$ *satisfies D1 iff* $\alpha_2 = 1 - \alpha_1$ *(and hence,* $K = 3$*).*
3. *Every semiseparating equilibrium with* $\operatorname{supp} r = \{\alpha_2\}$ *and* $\operatorname{supp} s = \{\alpha_1, \alpha_2\}$ *satisfies D1.*

Sketch of Proof of Proposition 11. Throughout denote a potential deviation by d, and the associated strength by μ_d. Moreover, denote the strength that makes the rational type indifferent between his equilibrium demand and d by μ_d^r.

1. One-offer equilibria. Consider a one-offer equilibrium with $\alpha = 1/2$. For any $d > 1/2$, the rational type is willing to deviate to d regardless of the belief of the opponent. For any $d < 1/2$, neither type is willing to deviate regardless of the belief of the opponent. Hence, there exists no d such that the stubborn type has a larger range of beliefs (and associated best responses by the opponent) which makes that deviation profitable.

   Consider the one-offer equilibrium with $\alpha_1 = 1$. This requires that each player puts probability 1 on any deviation coming from the rational type. Both types of player are then indifferent between demanding 1 and making any other demand. As a result, if a player (regardless of its type) is believed to be stubborn with non-zero probability following a deviation, he strictly benefits from deviating.

2. Pooling equilibria with multiple offers.

   (a) Two offer pooling equilibria. Consider a deviation to $d \in (\alpha_1, 1-\alpha_1)$. Note there exists $\mu_d < \mu_1$ such that the rational type is willing to deviate to d; and note that for any $\mu_d < \mu_2$, neither type is willing to deviate. The payoff to the rational type from deviating to d is given by:

$$v_d^r = q_1 \left(\frac{d+1-\alpha_1}{2} \right) + q_2 \left(d \left(1 - \left(\frac{\mu_2}{\mu_d} \right)^{1-\alpha_2} \right) + (1-\alpha_2) \left(\frac{\mu_2}{\mu_d} \right)^{1-\alpha_2} \right) \tag{99}$$

   Denote the strength that makes the rational type willing to deviate to d by μ_d^r:

$$v_1^r - v_d^r|_{\mu_d = \mu_d^r} = 0. \tag{100}$$

   If $\mu_d = \mu_d^r$, the stubborn type's payoff difference between α_1 and d is given by:

$$v_1^s - v_d^s|_{\mu_d = \mu_d^r} = -q_2 (1-\alpha_2) \mu_2^{1-\alpha_2} \left(\mu_1^{\alpha_1+\alpha_2-1} - (\mu_d^r)^{d+\alpha_2-1} \right). \tag{101}$$

   Note that $\mu_d^r < \mu_1$ and $d > \alpha_1$. Hence,

$$v_1^s - v_d^s|_{\mu_d = \mu_d^r} < 0.$$

   Hence, at the strength (and hence, correspondingly the belief and associated best response by the opponent) which makes the rational type indifferent, the stubborn type strictly prefers to deviate. Since both type's payoffs are continuous in strength μ, there must exist a strength μ_d such that the rational type prefers α_1 and the stubborn type prefers d. Hence, no pooling equilibrium with two demands satisfies D1.

   (b) Three offer pooling equilibria. Suppose $\alpha_2 \neq 1-\alpha_1$. Consider a deviation $d \in (\alpha_2, 1-\alpha_1)$, with an associated strength μ_d. I claim that there exists a belief μ_d at which the stubborn type strictly benefits from deviating to d and the rational type strictly prefers his equilibrium demand. Note that player's payoffs are continuous (and monotonous) in their strength. Hence, it is enough to show that at the belief that makes the rational type indifferent the stubborn type has a strict incentive to deviate. I can then write the stubborn type's payoff difference between α_2 and the deviation d at strength μ_d^r as:

$$\begin{aligned} v_2^s - v_d^s|_{\mu_d = \mu_d^r} = & - q_2 (1-\alpha_2) \left(\mu_2^{\alpha_2} - \mu_2^d \right) \\ & - q_3 (1-\alpha_3) \mu_3^{1-\alpha_3} \left(\mu_2^{\alpha_2+\alpha_3-1} - (\mu_d^r)^{d+\alpha_3-1} \right) < 0. \end{aligned} \tag{102}$$

   Hence, by continuity of the payoffs in μ, there must exist μ_d such that the rational type strictly prefers α_2 and the stubborn type strictly prefers d.

   Now suppose $\alpha_2 = 1-\alpha_1$. Clearly, neither type wants to deviate to $d < 1-\alpha_3$. Moreover, the rational type is always willing to deviate to $d > \alpha_3$ regardless of the opponent's belief.

If $d \in (1-\alpha_3, \alpha_1)$, then the stubborn type's payoff difference between α_1 and d evaluated at μ_d^r is given by:

$$v_1^s - v_d^s|_{\mu_d = \mu_d^r} = -q_3 (1-\alpha_3) \mu_3^{1-\alpha_3} \left(\mu_1^{\alpha_1+\alpha_3-1} - (\mu_d^r)^{d+\alpha_3-1} \right). \tag{103}$$

Since $\mu_d > \mu_1$ and $\alpha_1 > d$, $v_1^s - v_d^s|_{\mu_d=\mu_d^r} > 0$. The stubborn type prefers his equilibrium demand to d at μ_d^r.

If $d \in (\alpha_1, 1-\alpha_1)$, then the stubborn type's payoff difference between α_2 and d evaluated at μ_d^r is given by:

$$\begin{aligned} v_2^s - v_d^s|_{\mu_d=\mu_d^r} = &- q_2 (1-\alpha_2) \left(\mu_2^{\alpha_2} - (\mu_d^r)^{\alpha_2+d-1} \mu_2^{1-\alpha_2} \right) \\ &- q_3 (1-\alpha_3) \left(\mu_2^{\alpha_2} - (\mu_d^r)^{d+\alpha_3-1} \mu_3^{1-\alpha_3} \right). \end{aligned} \tag{104}$$

Since $\alpha_2 > d$ and $\mu_d^r > \mu_2 = \mu_3$, $v_2^s - v_d^s|_{\mu_d=\mu_d^r} > 0$.

If $d \in (1-\alpha_1, \alpha_3)$, then the rational type is willing to deviate regardless of the belief of the opponent.

Hence, any three offer pooling equilibrium with $\alpha_2 = 1-\alpha_1$ satisfies D1.

(c) Pooling equilibria with K demands. Consider a deviation $d \in (\alpha_{K-k}, 1-\alpha_1)$ with $k=2$ if $\alpha_{K-1} = 1-\alpha_1$ and $k=1$ otherwise. The stubborn type's payoff difference between α_{K-2} and d evaluated at μ_d^r is given by:

$$\begin{aligned} v_{K-k}^s - v_d^s|_{\mu_d=\mu_d^r} = &- \sum_{\alpha_i \ s.t.\ (K-k)^{\mathcal{W}}} q_i (1-\alpha_i) \left(\mu_i^{\alpha_{K-k}} - \mu_i^d \right) \\ &- \sum_{\alpha_i \ s.t.\ (K-k)^{\mathcal{S}}} q_i (1-\alpha_i) \left(\mu_{K-k}^{\alpha_i+\alpha_{K-k}-1} - (\mu_d^r)^{\alpha_i+d-1} \right) < 0. \end{aligned} \tag{105}$$

Hence, if the rational type is indifferent, the stubborn type strictly prefers to deviate. By continuity of the payoffs in strength, any equilibrium with more than three demands fails D1.

3. Semiseparating equilibria. Recall $\alpha_1 + \alpha_2 = 1$ (otherwise, a semiseparating equilibrium does not exist). Neither type has an incentive to deviate to $d < \alpha_1$. The rational type is willing to deviate to $d > \alpha_2$ regardless of the belief of the opponent. Hence, consider a deviation $d \in (1-\alpha_2, \alpha_2)$. The stubborn type's payoff difference between α_2 and d evaluated at μ_d^r is:

$$v_2^s - v_d^s|_{\mu_d=\mu_d^r} = -q_2 (1-\alpha_2) \mu_2^{1-\alpha_2} \left(\mu_2^{2\alpha_2-1} - (\mu_d^r)^{\alpha_2+d-1} \right) > 0. \tag{106}$$

Hence, if the rational type is indifferent, the stubborn type prefers the equilibrium demand α_2. Hence, any semiseparating equilibrium with $\operatorname{supp} r = \{\alpha_2\}$ and $\operatorname{supp} s = \{\alpha_1, \alpha_2\}$ satisfies D1.

□