SELLING IMPRESSIONS: EFFICIENCY VS. COMPETITION

By

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Selling Impressions: Efficiency vs. Competition*

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Abstract

In digital advertising, a publisher selling impressions faces a trade-off in deciding how precisely to match advertisers with viewers. A more precise match generates efficiency gains that the publisher can hope to exploit. A coarser match will generate a thicker market and thus more competition. The publisher can control the precision of the match by controlling the amount of information that advertisers have about viewers. We characterize the optimal trade-off when impressions are sold by auction. The publisher pools premium matches for advertisers (when there will be less competition on average) but gives advertisers full information about lower quality matches.

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1 Introduction

In the internet advertising market, it has become technologically feasible to match advertisers to viewers with ever-greater precision. But do publishers selling impressions have an incentive to do so? Finer matching generates efficiency gains that sellers can hope to exploit. But coarser matching generates market thickness, and so more market competition and less information rent for advertisers. Impressions are typically sold by auction and publishers can control the precision of the match by controlling the information that bidders have access to. We characterize the trade-off between efficiency and competition for the publisher.

We do this in two steps. First, we characterize what information a seller would choose to give buyers about their values in a second price auction in a standard independent private values setting. Second, we provide a model of the market for impressions in digital advertising markets, and show how our characterization applies in this setting. We now discuss these two steps in turn.

Consider a second price auction where bidders’ valuations are independently and symmetrically distributed, but initially unknown to the bidders. The seller can choose what information each bidder can learn about their own value. If the seller did not allow them to learn anything, then all bidders would bid their (common) expected value and the good would be randomly (and inefficiently) allocated among them. If the seller allowed bidders to learn their true value, then they would have a dominant strategy (under private values in a second price auction) to bid their values. The good would be allocated efficiently to the bidder with the highest value. The revenue of the seller would equal the value of the efficient allocation minus the bidders’ information rent. By permitting bidders to learn something but not everything about their values, the seller can trade off efficiency loss with information rent reduction. Our main result is a characterization of the optimal (among symmetric) information policies for the seller.

Conditional on having a low value, a bidder is likely to be competing with other bidders and earn low information rents. But conditional on having a high value, a bidder is likely to win (facing no competition) and thus can expect to win at a price significantly below his value, thus earning high information rents. Thus the gains from concealing information will be highest when valuations are high. In the optimal policy, high values are pooled and low values are revealed. There is a critical threshold described by a quantile above which all valuations are bundled together (Proposition 1). The threshold is given by a quantile of the distribution that depends only on the number of bidders (and not the distribution of valuations). The optimal quantile until which disclosure occurs is increasing in the number of participating bidders and goes towards 1 (i.e., full disclosure) as the number of bidders grows arbitrarily large. Thus, the information policy is influencing the distribution of bids, holding fixed the distribution of preferences among the bidders.
The assumption of a fixed finite number of bidders is extreme for many applications. We provide two results about what happens in large markets. First, we show that if the distribution of values has fat tails, then even as the number of bidders becomes large, and the quantile at which pooling starts approaches one, the gains from the optimal information policy relative to complete information or no information is high. We also consider the case where there is random entry into the auction. Suppose there is a prior probability that any bidder has a positive value for the object and a complementary probability that the bidder has zero value or no interest in the specific item. Then the optimal information structure intentionally invites the advertiser to bid on an item with positive probability even when the bidder has zero value for the item. That is, sometimes a market is made more competitive even when this lowers the expected value of the object (Proposition 3).

Our motivation for studying this problem is the market for impressions in digital advertising. A large share of digital advertising, whether in search, display advertising or social networks, is allocated by auction mechanisms. The second price auction is commonly used in digital advertising to form a match between competing advertisers (the *bidders*) and a viewer. A match between viewer and advertiser creates an *impression* (or search result) on the publisher’s website. The seller (a *publisher* or *intermediary*) *or* the publisher by an auction platform sells the attention (“eyeball”) of the viewer to competing advertisers. The viewer is thus the object of the auction. The viewers are typically heterogeneous in many attributes, their demographic characteristics, their preferences, their (past) shopping behavior, their browsing history and many other aspects, observable and unobservable. The advertisers therefore display a corresponding degree of heterogeneity in their willingness to pay for a match between their advertisement and a specific viewer. The private (and the social) value of any particular match is then determined jointly by a vector of attributes of the viewer and a vector of preferences for those attributes of the advertiser. In the presence of this heterogeneity on both sides of the match, viewer and advertiser, internet advertising has moved towards targeted advertising to join the information. The auction can therefore support highly targeted advertising that may increase the social efficiency in the match formation between viewer and advertiser. But - as discussed earlier - allowing for finely targeted bidding may also thin the market among the advertisers, and hence reduce the competition between advertisers. Publishers distinguish two schemes, or algorithms, for mapping preferences and attributes into bids, *automated bidding* and *manual bidding*. In automated bidding, *autobidding* for short, the seller offers a bidding algorithm that generates optimal bids for the advertisers given the disclosed information.¹ In manual bidding, the seller offers a disclosure

¹See Google Ads Help Center (2021b), Google Ads Help Center (2021a) or Facebook Business Help Center (2021) for summary descriptions of automated bidding mechanisms.
algorithm that generates information about the attributes, and in particular a bid recommendation, which
each bidder then manually adopts or modifies into a bid for the impression, also referred to as dashboard
mechanism in Hartline, Johnsen, Nekipelov, and Zoeter (2019). Autobidding has become increasingly
prevalent in digital advertising to convert the high-dimensional information across millions of impressions
into bids with minimal latency, see Aggarwal, Badanidiyuru, and Mehta (2019) and Deng, Mao, Mirrokni,
and Zuo (2021).

In this market, publishers can control the information that advertisers have about their values. While
there are many reasons why advertisers’ values might be correlated, they will not be if variation in viewers’
attributes is horizontal. A second contribution of the paper is to develop a stylized model of the market
for impressions, establish that our earlier model applies to this market, and show how the results apply
under reasonable assumptions on the market.

In our model of this market, a viewer is characterized a (perhaps high-dimensional) attribute. The
publisher (but not the advertisers) knows the attribute of the viewer. The advertiser (but not the publisher)
knows his preference over attributes. The viewer attribute can be combined with an advertiser’s preference
to generate a match quality, and the value of the viewer to the advertiser is an increasing function of the
match quality. A key feature of this model is that the advertiser’s private information is not informative
about his value of a viewer unless it is combined with information held by the publisher. The publisher
sells impressions (the allocation of the viewer to an advertiser) in a second price auction. Because only
the publisher knows the attributes of the viewer, the publisher can control the information that advertisers
have about their match quality with the viewer.

We present two results in our model. We first show that this model of two-sided information gives
rise to the setting of our main result: an independent private values setting, where the advertiser has
no information about his value of the viewer, but the combination of the publisher’s information about
attributes and advertiser’s information about preferences fully reveals the state. Now we can identify the
optimal information structure from our earlier analysis. But how can this be implemented in practise? We
consider autobidding. The advertiser reports his preferences to the publisher and the publisher commits to
bid optimally for the bidder as a function of the optimal information structure. We show that the bidder
will have an incentive to truthfully report his preferences. Thus autobidding implements the optimal
outcome in this market.
**Literature**  Levin and Milgrom (2010) suggested that the idea of conflation (central in many commodity markets) by which similar but distinct products are treated as identical in order to make markets thicker or reduce cherry-picking, may be relevant for the design on online advertising markets. The information structure (4) determines exactly when conflation should occur, in the upper interval, and when not, in the lower interval.

The paper relates to the literature studying optimal information disclosure in selling mechanisms. Ganuza (2004) studies the optimal information disclosure in a second-price auction, where bidder’s valuation are determined by the quality of a match between a bidder’s taste and the good’s characteristic represented by Hotelling model on a circle. The seller chooses the optimal public signal about the good’s characteristic. He shows that the equilibrium information provision is less than the surplus-maximizing one. His result is about costly public signals and so it does not address the trade-off between efficiency and market thickness that is central in our paper.

Bergemann and Pesendorfer (2007) analyze the joint optimal design of auction and information structure. In particular, they allow for asymmetric information structures and personalized reserve prices. Here, we fix the selling mechanism to be a second-price auction as it is a better fit for the markets for selling impressions.

Palfrey (1983) studies the bundling decision of a monopolist who sells $J$ goods to $N$ bidders via a second-price auction. He gives necessary and sufficient conditions for a single bundle to generate a higher revenue than $J$ independent auctions. The trade-off that governs the decision is similar as in our model, efficiency vs. market thickness, but the decision available for the seller is coarser as he compares only two options, to bundle or not to bundle. This is akin to comparing when no information generates higher revenue than complete information in our model.

Similar high-dimensional models with attribute (or features) and preferences have appeared recently in dynamic pricing literature. Here, the seller does not initially know the values of the different features, but can learn the values of the features based on whether products were sold at the posted prices in the past, see Cohen, Lobel, and Leme (2020).

## 2 Model

There are $N$ agents who bid for an indivisible good in an auction. Bidder $i$’s valuation is denoted by $v_i$. We assume that the valuations are independently and identically distributed across agents according to an absolutely continuous distribution, denoted by $F$. The assumptions that $F$ is absolutely continuous helps
simplify some of the expressions but all results go through unchanged if we relax this assumption.

The seller can choose how much information each bidder will have about his own valuation. An information structure is denoted by:

$$s_i : \mathbb{R}_+ \rightarrow \Delta \mathbb{R}_+,$$

where $$s_i(v_i)$$ is the signal observed by bidder $$i$$ when his valuation is $$v_i$$. After observing $$s_i$$, the bidder forms his beliefs about his valuation. An agent’s expected valuation is denoted by:

$$w_i \triangleq \mathbb{E}[v_i \mid s_i].$$

We denote by $$G_i$$ the distribution of expected valuations. Note that we are making two assumptions about the information structure. First, each bidder only observes information about his own valuation, which is reflected by the fact that $$s_i$$ takes as an argument $$v_i$$ only (instead of $$(v_1, \ldots, v_N)$$). Additionally, there is no common source of randomization in the signals. Hence, the signals will be independently distributed across agents. Finally, we assume that the seller is restricted to symmetric information structures, i.e., $$s_i(\cdot) = s_j(\cdot)$$.

The objective of the seller is to maximize revenue. Since agents are bidding in a second-price auction it is a dominant strategy to bid their expected valuation. Hence, revenue is equal to the second-highest expected valuation across bidders. We denote the $$k$$-th highest valuation by $$w_{(k)}$$. The objective of the seller is to solve:

$$R \triangleq \max_{\{s : \mathbb{R} \rightarrow \Delta \mathbb{R}\}} \mathbb{E}[w_{(2)}]. \tag{1}$$

3 Optimal Information Structure

Since the expected revenue is equal to the expectation of second-highest valuation, the distribution of expected valuations generated by the signal is a sufficient statistic to compute the seller’s expected revenue. Hence, instead of studying explicitly the signal chosen by the seller, we frequently refer to the distribution of expected valuations generated by a signal (recall that this is denoted by $$G$$).

The second-order statistic of $$N$$ symmetrically and independently distributed random variables is distributed according to

$$\mathbb{P}(w_{(2)} \leq t) = NG^{N-1}(t)(1 - G(t)) + G^N(t).$$

The expected revenue of the auctioneer is therefore:

$$\mathbb{E}[w_{(2)}] = \int_0^\infty td(NG^{N-1}(t)(1 - G(t)) + G^N(t)).$$
We now characterize the set of feasible distributions $G$.

By Blackwell (1951), Theorem 5, there exists a signal $s$ that induces a distribution of expected valuations if and only if $F$ is a mean preserving spread of $G$. $F$ is defined to be a mean preserving spread of $G$ if

$$\int_v^\infty dF(t) \leq \int_v^\infty dG(t), \forall v \in \mathbb{R}^+$$

and

$$\int_0^\infty dF(t) = \int_0^\infty dG(t).$$

If $F$ is a mean preserving spread of $G$ we write $F \prec G$.

We can now express the seller’s problem as finding an optimization over a distribution $G$ subject to a mean-preserving restriction. The choice of the optimal information structure can be written as the following maximization problem:

$$R = \max_G \int_0^\infty td(NG^{N-1}(t)(1 - G(t)) + G^N(t))$$

subject to $F \prec G$. 

This problem consists of maximizing over feasible distributions of expected valuations. However, the objective function is non-linear in the probability (or density) of the optimization variable $G$. Moreover, the non-linearity cannot be confined to be either concave or convex on $G$.

The key step in our argument comes from a change of variables, re-writing the above in terms of the quantile $q$ of the second order statistic. We denote by $S_N(q)$ the cumulative distribution function of the quantile of the second-highest valuation: $S_N(q) \triangleq \mathbb{P}(F(w_{(2)}) \leq q)$. We index by $N$ to highlight the dependence on the number of buyers. We observe that $S_N(q)$ is given by:

$$S_N(q) = Nq^{N-1}(1 - q) + q^N.$$ 

The quantile distribution $S_N$ is independent of the underlying distribution $F$ or $G$. Just as the quantile of any random variable is uniformly distributed, the quantile of the second-order statistic of $N$ symmetric independent random variables is distributed according to $S_N$ for any underlying distribution. Hence, the revenue can be computed by taking the expectation over quantiles using measure $S_N(q)$: the revenue given the quantile of the second-order statistic is $G^{-1}$. So maximization problem (2) can be transformed into:

$$\max_{G^{-1}} \int_0^1 S_N'(q)G^{-1}(q)dq$$

subject to $G^{-1} \prec F^{-1}$. 

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The corresponding constraint states that the seller can choose any distribution of expected valuations whose quantile function $G^{-1}$ is a mean-preserving spread of the quantile function $F^{-1}$ of the initial distribution of valuations. This uses a well-known property of the distribution function, see Shaked and Shanthikumar (2007), Chapter 3, stating that $F \prec G$ if and only if $G^{-1} \prec F^{-1}$. Hence, we have a linear (in $G^{-1}$) maximization problem subject to a majorization constraint, which will allow us to solve the problem with known methods.

**Proposition 1 (Optimal Information Structure)**

The unique optimal symmetric information structure is given by:

$$s(v_i) = \begin{cases} v_j & \text{if } F(v_i) \leq q^*_N \\ \mathbb{E}[v_j \mid F(v_j) \geq q] & \text{if } F(v_i) \geq q^*_N \end{cases}$$

where the critical quantile $q^*_N \in [0,1)$ is independent of $F$. In particular, $q^*_N = 0$; $q^*_N$ is increasing in $N$; $q^*_N \to 1$ as $N \to \infty$; and for each $N \geq 3$, $q^*_N$ is the unique solution in $(0,1)$ to:

$$S'_N(q)(1-q) = 1 - S_N(q).$$

Note that (5) is an $N$–th degree polynomial in $q$. Thus, the optimal information structure is to reveal the valuation of all those bidders who have a valuation lower than some threshold determined by a fixed quantile $q^*_N$ and otherwise reveal no information beyond the fact that the valuation is above the threshold. The threshold in terms of the valuation is given by $F^{-1}(q^*_N)$, but the quantile $q^*_N$ is independent of the distribution $F$ of valuations. The optimal information structures thus supports more competition at the top of the distribution at the expense of an efficient allocation. The information structure bundles for every bidder all valuations above the threshold $F^{-1}(q^*_N)$ into a single mass point. It therefore fails to distinguish in the allocation between any two valuations that are in the upper tail of the distribution $[F^{-1}(q^*_N), \infty)$. The benefit accrues through more competitive bids among the high value bidders. Namely, if the second highest bid is in the interval, then its competitive bid matches exactly the bid of the winning bid, and thus the information rent of the winning bidder is depressed considerably with a corresponding gain in the revenue for the seller.

The pooling at the top ensures competition when values are high. A natural question to ask is how many bidders will be in the pooling region. Note that the number of bidders with valuations above the threshold has a binomial distribution with parameters $(N, 1 - q^*_N)$. We can evaluate numerically the expected number of bidders who have values above the threshold: It is always in the interval $[1.75, 2.25]$
and is decreasing in $N$ for $N \geq 3$. Results in the following Section imply that the expectation converges down to 1.79 as $N \to \infty$.

Before we prove the result, we will provide some intuition for the critical quantile by confirming that it must be given by equation (5) as long as information takes the form given in the proposition. Suppose that we fix a quantile threshold $q$ and write $v = F^{-1}(q)$ for the corresponding value. At the cutoff $v$ there is a discontinuity in the bids as a function of the bidders’ valuation. Bidders with values marginally below $v$ will bid essentially $v$; bidders with values marginally above $v$ will bid $\mathbb{E}_F [t|t \geq v]$. The difference is given by:

$$\Delta \triangleq \mathbb{E}_F [t|t \geq v] - v.$$

Now what happens to revenue if we decrease the threshold by $dq$? Now with probability $S_N'(q)\ dq$ the second-highest bid was not in the pooling zone before the decrease and is after the decrease; and revenue increases by $\Delta$. With probability $1 - S_N(q)$, the second-highest bid was in the pooling zone before the decrease and there is a loss of revenue of $\frac{\Delta}{dq}$. But one can show that $\frac{\Delta}{dq} = \frac{1}{1-q}\Delta$, so equating expected loss and gains requires

$$S_N'(q)\ dq\Delta = (1 - S_N(q))\ \frac{1}{1-q}\Delta\ dq$$

giving (5).

To prove Proposition 1, we state a result of Kleiner, Moldovanu, and Strack (2021) in terms of our maximization problem (3).

**Proposition 2 (Kleiner, Moldovanu, and Strack (2021), Proposition 2)**

Let $G^{-1}$ be such that for some countable collection of intervals $\{(\underline{x}_i, \overline{x}_i) | i \in I\}$,

$$G^{-1}(q) = \begin{cases} 
F^{-1}(q) & q \notin \bigcup_{i \in I}[\underline{x}_i, \overline{x}_i) \\
\int_{x_i}^{F^{-1}(q)} \frac{F^{-1}(t)dt}{x_i - \underline{x}_i} & q \in [x_i, \overline{x}_i) 
\end{cases}$$

If $\text{conv} \ S_N$ is affine on $[x_i, x_i)$ for each $i \in I$ and if $\text{conv} S_N = S_N$ otherwise, then $G^{-1}$ solves problem (3). Moreover, if $F^{-1}$ is strictly increasing the converse holds.

Here, $\text{conv} S_N$ is the convexification of $S_N$, i.e., the largest convex function that is smaller than $S_N$. With this result we can prove our main result.

**Proof of Proposition 1.** The second derivative of the distribution $S_N$ of the quantile of the second order statistic is given by:

$$S''_N(q) = q^{N-3}(N-1)N(N-2-q(N-1)).$$
Hence, $S_N(q)$ is concave if and only if

$$q \geq (N-2)/(N-1),$$

and convex otherwise. Thus, the convex hull of $S_N$ for $N \geq 3$ is given by:

$$\text{conv}S_N(q) = \begin{cases} S_N(q) & \text{if } q \leq q^*_N; \\ S'_N(q^*_N)(q - q^*_N) + S(q^*_N) & \text{otherwise.} \end{cases}$$

where $q^*_N$ is defined as in (5) for $N \geq 3$. In Figure 1 we illustrate $S_N$ and $\text{conv}S_N$ for $N = 3$.

For $N = 2$, we have

$$\text{conv}S_2(q) = q$$

and can define $q^*_2 = 0$. Now let $G^{-1}$ be given by:

$$G^{-1}(q) = \begin{cases} F^{-1}(q) & q < q^*_N; \\ \frac{\int_{q_N^-}^{q} F^{-1}(t) dt}{1-q_N^-} & q \in [q^*_N, 1] \\ 1 & q = 1 \end{cases}$$  \text{(6)}$$

Then, $G^{-1}$ satisfies all the assumptions of Proposition 2, so it is the unique optimal solution to (3). For all valuations below $G^{-1}(q^*_N)$ the distribution over expected valuations is the same as that of the real

\[\text{conv}C(q)\]

Figure 1: Convexification of $S(q)$ for $N = 3$. 

$C(q)$

$\text{conv}C(q)$
valuations. Hence, types below $G^{-1}(q_N^*)$ know their own values. On the other hand, for valuations above $G^{-1}(q_N^*)$ the distribution over expected valuations is a mass point at the expected valuation conditional on being above $G^{-1}(q_N^*)$. Hence it is clear that this distribution is induced by information structure (4).

To check that $q_N^*$ is strictly increasing in $N$ we define:

$$\psi(q, N) \triangleq S_N'(q)(1 - q) - (1 - S_N(q)).$$

By definition, $\psi(q_N^*, N) = 0$. We now note that:

$$\psi(q(N + 1)) - \psi(q, N) = N(q - 1)^2(N(q - 1) + 1)q^{N-2}.$$ 

so $\psi(q(N + 1)) - \psi(q, N) \geq 0$ if and only if $q \geq (N - 1)/N$. As previously argued, $q_N^* < (N - 2)/(N - 1)$ so $q_N^* < (N - 1)/N$, which implies that:

$$\psi(q_N^*, N + 1) < 0.$$

(7)

We also have that $\psi(0, N) = -1$ and $\psi(1 - \varepsilon, N) > 0$ for $\varepsilon$ small enough, where the last part can be verified by noting that

$$\psi(1, N) = \frac{\partial \psi(1, N)}{\partial q} = 0 \quad \text{and} \quad \frac{\partial^2 \psi(1, N)}{\partial q^2} = N(N - 1) > 0.$$

As previously argued $\psi(q(N + 1))$ has a unique root in $(0, 1)$, so (7) implies that $q_N^* < q_{N+1}^*$.

Finally, if $N$ diverges to infinity and $\lim_{N \to \infty} q_N^* < 1$, then in the limit we would have that

$$S_N(q_N^*), S_N'(q_N^*) \to 0.$$

So (5) would not be satisfied. We thus must have that $\lim_{N \to \infty} q_N^* = 1$. $\blacksquare$

The information structure (4) that emerges here for every bidder is sometimes referred to as "upper censorship" in the Bayesian persuasion literature, as it pools all the states above a cutoff and reveals all the states below the cutoff, see Proposition 3 in Alonso and Camara (2016) or Theorem 1 in Kolotilin, Mylovanov, and Zapechelnyuk (2021). It is useful to compare our problem to a Bayesian persuasion model where the objective function permits a nonlinear evaluation $u(x)$ of an outcome but linear in the probability, thus

$$\max_G \int_0^1 u(t) dG(t),$$

subject to $F \prec G$,

as, for example in Dworczak and Martini (2019). Our original maximization problem (2) did not take this form as it was non-linear in probabilities. However, we reformulated the problem to one that is linear.
in the new optimization variable $G^{-1}$, changing the direction of the constraint. For this problem, the convexification of $S_N$ was key to identifying the optimal information structure. The fact that $S_N$ is always convex-concave then generated the upper censorship information structure.

4 Large Markets

We now develop some implications of the optimal information structure in markets with a large number of (possible) bidders which is arguably the prevailing condition in digital advertising. We first consider how the information responds to the random participation of bidders. We then consider the revenue performance of the auction with the optimal information structure when the actual number of participating bidders grows large. Here we analyze the a class of distribution with heavy tails that have been proposed by Arnosti, Beck, and Milgrom (2016) in their analysis of internet advertising.

Random Number of Bidders We now assume that the valuation of bidders is distributed according to $F$ with probability $1 - p$ and with probability $p$ the valuation is equal zero. To analyze the optimal information structure, we analyze the optimal quantile in the limit as $N \to \infty$ and $p \to 1$. We keep the expected number of bidders who have strictly positive values constant at:

$$\lambda \triangleq N(1 - p).$$

In this limit, the number of bidders who have a strictly positive values is randomly distributed according to a Poisson distribution with parameter $\lambda$ (this is called the law of rare events or the Poisson limit theorem). Hence, in the limit, it is as if the number of bidders is randomly distributed.

To characterize the optimal information structure, let $\rho$ be the unique solution larger than zero to the following equation:

$$\rho^2 + \rho + 1 = e^\rho,$$

with $\rho \approx 1.793$. We denote the expected valuation conditional on $v$ being drawn from the distribution $F$ by

$$v_F = \mathbb{E}_F [v].$$
Proposition 3 (Optimal Information)

In the limit as $N \to \infty$, $p \to 1$, the optimal information structure is:

1. If $\lambda \leq \rho$, then bidders observe binary signals and their expected valuation is either 0 or $v_F \lambda / \rho$.

2. If $\lambda > \rho$, bidders with valuation $F(v_i) \leq (\lambda - \rho) / \lambda$ learn their value, and bidders with valuation $v_i \in [F^{-1}((\lambda - \rho) / \lambda), 1]$ only learn that their valuation is in this interval.

Proof of Proposition 3. For any fixed $N$, we define the expected number of bidder who have value above the $q^*_N$ quantile as:

$$\rho \triangleq N(1 - q^*_N).$$

In the limit $N \to \infty$, (5) converges to the following equation (in terms of $\rho$):

$$\rho^2 e^{-\rho} = 1 - e^{-\rho} - \rho e^{-\rho},$$

We then get the result by applying Proposition 1. ■

The distribution of bidders whose expected valuation is above the cutoff quantile is distributed according to a Poisson distribution with parameter $\rho$ (regardless of $\lambda$):

$$\rho = N(1 - q^*_N),$$

So the expected number of bidders whose expected valuation is above the cutoff quantile is equal to $\rho$. It is also interesting that the probability that there is just one bidder above the cutoff is approximately 0.3, while the probability that there are at least 3 bidders is approximately 0.27. In the former case, there is not enough competition to extract this bidder’s surplus, in the latter case there are excess bidders so it would be best to have less bidders with higher valuation. The optimal information structure approximately equates the probability of these two type of errors.

In the model with a random number of bidders, we also get a more nuanced analysis when there are few bidders. We can see that if expected number of “serious” bidders (i.e., bidders with non-zero valuation) is small, it is optimal to attract bidders whose valuation is zero and disclose no information. This is intuitive, when there are few bidders the priority is increasing market thickness, which comes about at the cost of lower expected valuations. In terms of the random entry, this means that it is part of the optimal information structure to present objects to the bidder that have zero value to the bidder as long as the bundle also includes objects that have positive value, but the bidder is bidding for them without being able to make the distinction. This practice thus bundles low value and high value impression to maintain a competitive market.
The optimal information structure thus supports a match process between advertiser and viewer that is often referred to as "broad match" in digital advertising, see Dar, Mirrokni, Muthukrishnan, Mansour, and Nadav (2009) and Eliaz and Spiegler (2016). By broad matching which is common in ad auctions, the seller provides matches not only on the exact and narrow matches for specific keywords and characteristics but also for a larger, hence broad set of matches, see Google Ads Help Center (2021c). Proposition 3 then establishes that broad matching is an important instrument to maintain competition in the ad auction. The first part of the Proposition 3 suggests that with thin markets, it is even optimal to include irrelevant matches with value $0$ and thus lower the expected value as it increases the probability of competitive bids.

**Revenue Performance with Large Number of Bidders** We now examine the difference between the revenue generated under the optimal information structure and under complete information when the number of bidders become large. Throughout this subsection we assume that the density of distribution of valuations has regularly varying tails with index $\alpha - 1$, that is,

$$\lim_{t \to \infty} \frac{f(kt)}{f(t)} = k^{\alpha-1}, \text{ for all } t > 0.$$  

We assume that $\alpha < -1$. For example, the Pareto distribution satisfies this assumption, $f(v) = -\alpha v^{\alpha-1}$ for $v \geq 1$. Intuitively, the densities that have regularly varying tails have fat tails that decay as a Pareto distribution with shape parameter $\alpha$. As argued by Arnosti, Beck, and Milgrom (2016), the Pareto distribution, with its fat tail provides a good fit for demand in the digital advertising market, our leading application.

We denote by $R_c$ the expected revenue in the second price auction under complete disclosure of information:

$$R_c \triangleq \mathbb{E}[v(2)].$$

We now compare the revenue under the optimal information structure, $R$ with the revenue under complete disclosure, $R_c$ for large $N$.

**Proposition 4 (Revenue Gain with Large Number of Bidders)**

As the number of bidders grows, there exists $z \in (1, \infty)$ such that:

$$\lim_{N \to \infty} \frac{R}{R_c} = z. \quad (11)$$

Furthermore, in the limit $\alpha \to -1$, $z \to \infty$.

---

3The condition $\alpha < -1$ is necessary and sufficient to guarantee that the distribution of valuations has finite first moment.
As the number of bidders grows, the gains from using an optimal information structure does not vanish. The reason is that despite there being many bidders, there always remains a sufficiently high probability of there being a bidder with a disproportionately high valuation. Under complete information, it is not possible to extract the surplus from this bidder because there is not enough competition. Formally, when the distribution has fat tails:

$$\mathbb{E}[v_{(1)}] - \mathbb{E}[v_{(2)}] \to \infty, \text{ as } N \to \infty. \quad (12)$$

By contrast, the optimal information structure thickens the market at the tail of the distribution and can thus provide a revenue improvement even as the numbers of bidders becomes arbitrarily large.

5 Market for Impressions

We now translate the earlier results into a market for impressions with two-sided information. This translation recasts the optimal information design as bidding mechanisms in the world of digital advertising. The choice of the optimal information structure can then interpreted in terms of the information policy of the publisher who matches the viewer with the advertisers.

The viewer has attribute $x \in X \subset \mathbb{R}^J$ distributed according to $F_x$. Each advertiser $i$ has a preference for the attributes described by, $y \in Y \subset \mathbb{R}^J$, distributed according to $F_y$, identically and independently distributed across advertisers.

An impression is a match between an advertiser and a viewer. The value $v_i$ of advertiser $i$ from attracting a viewer is determined by a function $u$:

$$u : X \times Y \rightarrow \mathbb{R}_+,$$

such that:

$$v_i \triangleq u(x, y),$$

and we refer to $u$ as the valuation function. The distribution of characteristics $(x, y)$ and the valuation function $u$ induce a distribution of the bidder $i$’s value $v_i$, which we denote by $F$.

We assume that the unconditional distribution of values $(v_1, ..., v_N)$ generated by $(x, y)$ are independent across bidders. We further assume that the unconditional distribution of values are the same as the conditional distribution of values $(v_1, ..., v_N)$ conditional on either $x$ or $y$. That is, $(x, v_1, ..., v_N)$ and $(y, v_1, ..., v_N)$ are random vectors consisting of independently distributed random variables. Of course, $(x, y_i, v_i)$ are not independently distributed. In other words, the preference vector $y_i$ provides only information about the

---

4This result is immediately implied by the analysis in the proof of Proposition 4.
value $v_i$ when combined with information about the attributes $x$ of the viewer. Moreover, each advertiser $i$’s belief about the valuation of their competitors is unaffected by attribute $x$ or preference $y_i$. Thus, the present model of attributes and preferences generates a model of independent private values.

These assumptions implicitly impose restrictions on the set of valuations functions and distributions of attributes and characteristics that we consider. In other words, not every valuation function $u$ and distributions $F_x, F_y$ will generate joint distributions of values that satisfy these assumptions. The attributes of viewer therefore reflect an aspect of horizontal differentiation with value implications that depend on the preferences of the advertiser. Conversely, an aspect of vertical differentiation that has similar implications across all preferences would fail the equivalence between unconditional and conditional value distribution.

We now briefly describe two classes of models that satisfy the above conditions, one high-dimensional and one low-dimensional set of models.

Our leading example is given by the following specification. Let there be a vector of attributes $x \in \{-1, 1\}^J$, so attribute $j$ takes values $-1$ or $1$. Each advertiser $i$ has a preference for attribute $j$, $y_{ij} \in \{-1, 1\}$. Thus $y_i \in \{-1, 1\}^J$ is advertiser $i$’s preference, and $y = (y_1, ..., y_N) \in \{-1, 1\}^{NJ}$ is the profile of the preferences of the advertisers. The attributes and preferences are uniformly and independently distributed across components and bidders. The valuation function is given by:

$$u(x, y_i) = u \left( \frac{1}{\sqrt{J}} \sum_{j=1}^{J} x_j y_{ij} \right),$$

(13)

for some strictly increasing function $u$.

An alternative class of models is given the following Hotelling location model that has only a one-dimensional space of uncertainty. Suppose that $x, y_i \in [0, 1]$ are positions in a circle of perimeter $1$ uniformly and independently distributed. Let $d_i$ be the shortest distance between $x, y_i$ on the circle and

$$u(x, y_i) = u(d_i(x_i, y))$$

(14)

for some strictly decreasing function $u$.

Both of these classes of models satisfy the independence conditions above.

We will now analyze automated bidding in the second-price auction based on a signal of the value of the impression. Thus, the publisher commits: (i) to complement the advertiser’s information with a signal regarding the match quality; and (ii) to set the advertiser-optimal bid. In turn, the advertiser submits his preference $y$ (and thus a description of the attributes he cares about). The central aspect of automated bidding is that the publisher complements the advertiser’s private information $y$ with information about the viewer’s attribute $x$ that is unknown to the advertiser.
Formally, the publisher chooses a signal
\[ s_i : \mathbb{R} \rightarrow \Delta \mathbb{R} \]
as a function of the advertiser’s reported value \( v_i(x, y_i) \). While the publisher cannot directly observe \( v_i \), she elicits the advertiser’s preferences and knows the viewer attributes, so she can infer \( v_i \). The publisher submits a bid \( b_i : Y \times \mathbb{R} \rightarrow \mathbb{R} \) satisfying that:
\[
b_i(y_i, s_i) = \mathbb{E}[v_i \mid y_i, s_i(v_i)]
\]
That is, the publisher submits a bid on behalf of advertiser \( i \) equal to the advertiser’s expected valuation given his preferences \( y_i \) and the additional information \( s_i \) provided by the publisher. This is a advertiser’s dominant strategy given the available information \((y_i, s(v_i))\). Because \((v_i, y_i)\) are independently distributed, we have that:
\[
\mathbb{E}[v_i \mid y_i, s_i(v_i)] = \mathbb{E}[v_i \mid s_i(v_i)].
\]
Hence, the publisher’s problem reduces to find an information structure \( s \) that solves the original problem stated earlier in (1):
\[
R \overset{\Delta}{=} \max_{s: \mathbb{R} \rightarrow \Delta \mathbb{R}} \mathbb{E}[b(2)].
\]
We now verify that under the automated bidding in the second price auction, it is optimal for each advertiser to truthfully report their preferences to the publisher. A reporting strategy for bidder \( i \) is denoted by:
\[
\tilde{y}_i : Y \rightarrow \Delta Y.
\]
Given the reported preferences, the seller discloses to the bidder a signal \( s(\tilde{v}_i) \), where \( \tilde{v}_i \) is the valuation of an advertiser with preferences \( \tilde{y}_i \). The induced bid is denoted by \( \tilde{b}_i \), while \( b_i \) denotes the bid when preferences are reported truthfully.

**Proposition 5 (Truthful Reporting)**

Under the optimal information structure, it is a dominant strategy for an advertiser to report truthfully his preferences to the publisher.

**Proof.** By assumption, the unconditional and the conditional distribution of \( v_i \), thus \( v_i \) and \( v_i \mid y_i \) are identical. This implies that \((v_i, y_i)\) are independently distributed. Thus the distribution of bids \( \tilde{b}_i \) is the same for every reported strategy, and every reported preference \( \tilde{y}_i \). Of course, the joint distribution
of \((v_i, \tilde{b}_i)\) does depend on the reporting strategy. The difference in the expected bidder’s surplus under truthful reporting and misreporting is then given by:

\[
\Delta V \triangleq \mathbb{E}[(v_i - b_i)\mathbb{I}_{\{b_i \geq b_{-i}\}}] - \mathbb{E}[(v_i - \tilde{b}_i)\mathbb{I}_{\{\tilde{b}_i \geq b_{-i}\}}] = \mathbb{E}[(v_i\mathbb{I}_{\{b_i \geq b_{-i}\}}) - \mathbb{E}[v_i\mathbb{I}_{\{\tilde{b}_i \geq b_{-i}\}}],
\]

where \(b_{-i}\) is the highest bid of the other \(N - 1\) advertisers and \(\mathbb{I}_{\{b_i \geq b_{-i}\}}\) is the indicator function:

\[
\mathbb{I}_{\{b_i \geq b_{-i}\}} = \begin{cases} 
0 & \text{if } b_i < \max_{j \neq i} b_j, \\
1 & \text{if } b_i \geq \max_{j \neq i} b_j.
\end{cases}
\]

We define:

\[
p(b) \triangleq \text{Pr}(b \geq b_{-i}) = \mathbb{E}[\mathbb{I}_{\{b \geq b_{-i}\}}]
\]

and write \(\Delta V\) as follows:

\[
\mathbb{E}[v_ip(b_i)] - \mathbb{E}[v_ip(\tilde{b}_i)].
\]

Clearly \(p(b)\) is an increasing function of \(b\), so \(v_ip(b_i)\) is supermodular in \((v_i, b_i)\). We now show that truth-telling generates the highest correlation among all joint distributions \((v_i, \tilde{b}_i)\) with fixed marginals. Thus the supermodular function is maximized when reporting truthfully.

We say that two random variables \(X\) and \(Y\) are positive quadrant dependent (PQD) if

\[
\text{Pr}(X \geq x, Y \geq y) \geq \text{Pr}(X \geq x)\text{Pr}(Y \geq y),
\]

see Lehmann (1966). We note that for all \(v', b' \in \mathbb{R}\):

\[
\text{Pr}(v_i \leq v', \tilde{b}_i \leq b') \leq \min\{\text{Pr}(v_i \leq v'), \text{Pr}(\tilde{b}_i \leq b')\} = \min\{\text{Pr}(v_i \leq v'), \text{Pr}(b_i \leq b')\} = \mathbb{P}(v_i \leq v', b_i \leq b'),
\]

The first inequality is true for any bivariate distribution, the equality is because both distributions have the same marginals, and the last equality follows from the fact that:

\[
\text{Pr}(v_i \leq v', b_i \leq b') = \begin{cases} 
F(v_i') & \text{if } v_i' \leq b_i' \text{ or } b_i' = \mathbb{E}[v_i \mid v_i \geq F^{-1}(q)], \\
F(b) & \text{otherwise}.
\end{cases}
\]

Hence, \((v_i, b_i)\) is greater than \((v_i, \tilde{b}_i)\) in the positive quadrant dependent order, which implies that

\[
\mathbb{E}[v_ip(b_i)] \geq \mathbb{E}[v_ip(\tilde{b}_i)]
\]

(see (9.A.18.) in Shaked and Shanthikumar (2007)).
The proposition states that advertisers are willing to submit their preferences honestly to the publisher. The intuition for the proof is that misreporting would not change the distribution of bids, but instead, only decrease the correlation between an advertiser’s bid and his valuation. The automated bidding algorithm can alternatively be interpreted as a restriction on the bidding language imposed by a publisher. The publisher then runs a second price auction conditional on the realized attributes and the bidding rules.

We can illustrate this in our leading example of binary characteristics described above in (13). The optimal information structure can be implemented in a straightforward manner in the model of characteristics. Namely, the seller informs each bidder about the number of matched characteristics as long as this number is smaller than a threshold number $n^*$ implied by the optimal quantile $q_N^*$. If the number of matched characteristics exceeds $n^*$, then the seller only reports that the realized matches exceed the threshold number $n^*$. Thus, a restriction in terms of the bidding language would allow the bidder to place bids as a function of the matched characteristics up to $n^*$, but not beyond that. This restriction in terms of the bidding language would then lead to the optimal bids as described by (15).

In the context of digital advertising, the information structure can be given a second interpretation. In the standard auction setting, we fix an object and then the value of the bidder for the object is random and given by a distribution $F$. An alternative interpretation is that the valuation of the bidder, his taste, is fixed and the objects are drawn at random. A particular expected value for each bidder is then attained by bundling objects of different characteristics so that a given expected value is attained. In this interpretation, the seller has to offer the bidder a particular distribution of impressions to attain a particular expectation. Namely, all objects that offer more matches than the threshold level $n^*$ are sold in package, whereas all objects with fewer than $n^*$ matches are sold and priced in fine segments equal to the number of matches.

An important qualification is that our results suggest that in the presence of heterogeneous bidders, the conflation of objects is personalized, and optimally dependent on the preferences of each bidder. Thus, the items are not conflated uniformly across bidders, but in a manner that depends on the preferences elicited from the bidders.

6 Discussion and Conclusion

We presented a model of attributes of an object and preferences of the bidders that generated a model of independent private values. The seller controls the revenue in the second price auction through the elicitation of preferences and the flow of information into the bids. By disclosing only limited information,
the seller can increase the revenue at the expense of some efficiency losses. The optimal information design can be interpreted as a form of personalized conflation.

**Manual Bidding and Obedience** We discussed how the optimal information structure can be implemented by automated bidding algorithms. These algorithms generate bids for the advertisers as a function of the preferences elicited from the advertisers and the attributes of the viewer. We established in Proposition 5 that automated bidding supports truthful revelation of the preference information by the advertisers. A different class of algorithms is frequently described as manual bidding algorithms. Here, the bidders are asked to reveal their preferences first, and then are invited to bid on the basis of bid recommendations that take into account both preference and attribute information. These algorithms implicitly require both truth-telling and obedience constraints to be satisfied. The additional restrictions imposed by the obedience constraints, namely that the bidder wishes to follow the bid recommendation, may sometimes prevent the implementation of the optimal information structure for a given number \( N \) of bidders. Yet, we can show that in the presence of a large number of bidders we can approximate the revenue of the optimal information structure even when we impose the dual incentive constraints of truth-telling and obedience on the information design. Namely, even if the seller has to elicit the private taste vector of the bidders in an incentive compatible manner, the optimal revenue can be approximated arbitrarily closely. Thus, there is a class of information structures that can appropriately balance revenue-maximization and incentive compatibility when there is a large number of bidders.

**Horizontal vs Vertical Differentiation** An interpretation of the characteristics model is that it generates horizontal differentiation among bidders. By contrast, other models of characteristics would generate vertical differentiation and correlated values. For example, if each dimension of attributes and tastes were represented by the positive orthant, then a higher realization of any attribute would represent an element of vertical differentiation, and the ex-ante distribution of values would be correlated across bidders, thus introducing an element of common values. If valuations are correlated among bidders, then bidders would be subject to the winner’s curse. This would introduce new trade-offs that relate the bidders’ information to the winner’s curse. From a technical perspective, the second price would no longer have a dominant-strategy equilibrium, so the characterization of the revenue in terms of the order statistic of expected valuations would no longer hold. While extending the analysis formally is not trivial, the linkage principle suggests that the seller would publicly disclose any shocks about common shocks affecting correlations, thus making the valuations conditionally independent (which would allow us to go back to our analysis). Bergemann, Brooks, and Morris (2017), (2019) analyze in detail how correlated information...
across bidders can improve the revenue performance of standard auctions, such as first and second price auctions.

**Auction Format**  In the current environment with independent private values, the revenue equivalence result holds. Thus all classic auction formats generate the same expected revenue. Hence, while we formally study the second-price auction, the results extend to all classic auction formats, e.g., first-price auction, all-pay auction as long as we maintain to sell the object with probability one.

**Reserve Price**  Our main analysis focuses on the second-price auction without reserve price. However, the analysis of the optimal information structure can be extend to the auctions with a reserve price $r$ with minor modifications. With a reserve price $r$, the optimal information structure now displays two pooling regions. Next to the pooling region at the top that remains, there is now a pooling region of value such that the expected valuation in this lower pool exactly matches the reserve price $r$. The details are spelt out in the Appendix. There remains an intermediate interval in which the bidders learn their valuation. In the presence of the reserve price, the logic of the lower pooling region is similar to the pooling region at the top. Given that there is a reserve price $r$, which acts as if there is a competing bid, the information structure create a thicker market by pooling the valuations even if the pooling leads to a loss in efficiency.

**Asymmetric Information Structures**  We focus on describing the optimal symmetric information structure. While we do not have a general result showing that the optimal information structure is always symmetric, we have some partial results that point to this direction. The symmetric information structure that we derive is indeed the unique optimal information structure when there are two or three bidders, thus $N = 2$ or $N = 3$. We can also show that, if the information structure is the optimal symmetric information for $N - 1$ bidders, then it is optimal for the remaining bidder to also observe the optimal symmetric information. Hence, there is no improvement that involves changing the information structure of only one bidder. The detailed results are in the Appendix.
References


7 Appendix

Proof of Proposition 4. Before we begin, it is useful to establish the rate at which $\rho$ as defined earlier in (9) converges to 1 as $N$ diverges to infinity. In the limit, the optimal quantile satisfies:

$$
\lim_{N \to \infty} \frac{1 - q_N^*}{1/N} = \rho,
$$

with $\rho$ satisfying (8).\(^5\) Hence, for $N$ large enough, $q \approx 1 - \rho/N$, for some $\rho \in \mathbb{R}$.

In what follows, for any two positive functions $H, \tilde{H}$,

$$H(t) \sim \tilde{H}(t), \text{ as } t \to \infty$$

means that

$$\lim_{t \to \infty} \frac{H(t)}{\tilde{H}(t)} = 1$$

Throughout the proof, we use the following results about regularly-varying functions. First, for any regularly varying function $H(t)$ with index $\gamma$, there exists a slowly varying function $l(t)$ such that:

$$H(t) = l(t)t^\gamma.$$

Second, the slowly varying function behaves as a constant under integration of the tail:

$$\int_t^\infty l(y)y^\gamma dy \sim -l(t)t^{\gamma+1}(\gamma+1)^{-1}, \text{ as } t \to \infty$$

whenever $\gamma < -1$.

We can then write the density as follows:

$$f(t) = -\alpha l(t)t^{\alpha-1}.$$ 

In the limit $t \to \infty$, the hazard rate satisfies:

$$\tilde{F}(t) \Delta = 1 - F(t) \sim l(t)t^\alpha, \text{ as } t \to \infty.$$

We thus have that $\tilde{F}$ is also a slowly varying function.

\(^5\)We can verify this claim, by noting that if we replace $q = 1 - \rho/N$ in equation (??) and take the limit, we get this expression for $\rho$. Similarly, if $q$ converges to 1 at a faster or slower rate than $1/N$, then clearly (??) cannot be satisfied in the limit.
We consider the following upper-pooling information structures:

\[ G^{-1}(q) = \begin{cases} F^{-1}(q) & q \leq 1 - \frac{\rho}{N}; \\ \int_{F^{-1}(1 - \frac{\rho}{N})}^{\infty} \frac{tdF(t)}{1/N} & \text{if } q > 1 - \frac{\rho}{N}, \end{cases} \tag{17} \]

with \( \rho \in \mathbb{R} \). The quantile threshold is not necessarily the same as in the optimal information structure. We define:

\[ V(N) \triangleq F^{-1}(1 - \frac{\rho}{N}), \]

which is the value at which the pooling zone begins. We denote by \( R' \) the expected revenue generated by this information structure.

We denote the difference between revenue generated under information structure (17) and under complete information as follows:

\[ \Delta R \triangleq R' - R_c. \]

Since \( w_{(2)} = v_{(2)} \) whenever \( v_{(2)} < V(N) \), we can write the difference as follows:

\[
\Delta R = \mathbb{P}(v_{(2)} \geq V(N))\mathbb{E}[v_i | v_i \geq V(N)] - \mathbb{E}[v_{(2)}1_{v_{(2)} \geq V(N)}] \\
= \int_{V(N)}^{\infty} \frac{tdF(t)}{\rho/N} (1 - S(1 - \rho/N)) - \int_{V(N)}^{\infty} tN(N - 1)F^{N-2}(t)(1 - F(t))dF(t).
\]

Finally, we can bound the difference as follows:

\[
\Delta \tilde{R} \triangleq \int_{V(N)}^{\infty} \frac{tdF(t)}{\rho/N} (1 - S(1 - \rho/N)) - \int_{V(N)}^{\infty} tN(N - 1)(1 - F(t))dF(t) \leq \Delta R,
\]

where we omitted the term \( F^{N-2} \) in the integral and denoted the lower bound by \( \Delta \tilde{R} \). Finally, we give a bound on the revenue generated under complete information:

\[
\tilde{R}_c \triangleq S(1 - \rho/N)V(N) + \int_{V(N)}^{\infty} tN(N - 1)(1 - F(t))f(t)dt \\
\geq \int_{0}^{V(N)} tN(N - 1)(1 - F(t))F^{N-2}(t)f(t)dt + \int_{V(N)}^{\infty} tN(N - 1)(1 - F(t))F^{N-2}(t)f(t)dt = \mathbb{E}[v_{(2)}],
\]

where we obtained the upper bound by omitting the term \( F^{N-2} \) in the integral, replaced \( t \) in the integral with \( V(N) \), and denoted the lower bound by \( \tilde{R}_c \). We note that:

\[
\frac{R}{R_c} - 1 \geq \frac{\Delta R}{R_c} \geq \frac{\Delta \tilde{R}}{\tilde{R}_c}.
\]

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We prove that the right-hand-side of the inequalities does not converge to 0 as $N$ grow large.

In the limit $N \to \infty$,

\[
\int_{V(N)}^{\infty} t dF(t) \sim \alpha \frac{l(V(N))(V(N))^{(\alpha+1)}}{(\alpha + 1)}
\]

\[
\int_{V(N)}^{\infty} tN(N-1)(1-F(t))dF(t) \sim \alpha \frac{N(N-1)}{2(\alpha + 1)} l(V(N))^2 V(N)^{2\alpha+1} (1-S(1-\rho/N)) \sim \frac{e^\rho - \rho - 1}{e^\rho}
\]

We now recall that:

\[
l(V(N)) V(N)^\alpha \sim \tilde{F}(1-\rho/N) = \frac{\rho}{N}, \text{ as } N \to \infty.
\] (18)

We thus have the following approximations:

\[
\int_{V(N)}^{\infty} t dF(t) \sim \alpha \frac{V(N)}{N(\alpha + 1)};
\]

\[
\int_{V(N)}^{\infty} tN(N-1)(1-F(t))dF(t) \sim \alpha \frac{N(N-1) \rho^2}{2(\alpha + 1) N^2} V(N).
\]

So we conclude that:

\[
\Delta \tilde{R} \sim V(N)(\frac{\alpha}{(\alpha + 1)} \frac{e^\rho - \rho - 1}{e^\rho} - \frac{\alpha \rho^2}{(2\alpha + 1)}), \text{ as } N \to \infty.
\]

Using the same calculations as before, in the limit $N \to \infty$:

\[
\tilde{R}_c \sim V(N)\left(\frac{1 + \rho}{e^\rho} + \alpha \frac{1}{2(\alpha + 1)}\right)
\]

So, we have that:

\[
\frac{\Delta \tilde{R}}{\tilde{R}_c} \sim \frac{\left(\frac{\alpha}{(\alpha + 1)} \frac{e^\rho - \rho - 1}{e^\rho} - \frac{\alpha \rho^2}{(2\alpha + 1)}\right)}{\left(\frac{1 + \rho}{e^\rho} + \alpha \frac{1}{2\alpha + 1}\right)}, \text{ as } N \to \infty
\]

Finally, in the limit $\rho \to 0$,

\[
\lim_{\rho \to 0} \frac{e^\rho - \rho - 1}{e^\rho \rho^2} = \frac{1}{2}.
\]

However, we also have that:

\[
\left(\frac{\alpha}{(\alpha + 1)} \frac{1}{2} - \frac{\alpha}{2(\alpha + 1)}\right) > 0.
\]

Hence, $\alpha\left(\frac{1}{(\alpha + 1)} \frac{e^\rho - \rho - 1}{e^\rho} - \frac{\rho^2}{(2\alpha + 1)}\right) > 0$ for a small enough $\rho$, so get that:

\[
\lim_{N \to \infty} \frac{R}{\tilde{R}_c} - 1 > 0.
\]
This proves that the ratio (11) converges to a number larger than 1. Furthermore, in the limit \( \alpha \to -1 \), we have that:

\[
\lim_{\alpha \to -1} \lim_{N \to \infty} \frac{R}{R_c} - 1 = \infty.
\]

This limit holds uniformly for every \( \rho \). This proves that the ratio diverges in the limit \( \alpha \to -1 \).

Finally, we prove that \( z \) in (11) exists (i.e., it is not infinite). For this, we now define:

\[
\Delta \tilde{R}' \triangleq \frac{\int_{V(N)}^\infty tdf(t)}{1/N} S(1 - 1/N) - F^{N-2}(V(N)) \int_{V(N)}^\infty tN(N - 1)(1 - F(t))dF(t)
\]

\[
\tilde{R}'_{c}^{N-2}(V(N)) \int_{V(N)}^\infty Nt(1 - F(t))f(t)dt.
\]

and note that:

\[
\frac{R}{R_c} - 1 \leq \frac{\Delta \tilde{R}'}{\tilde{R}'_c}.
\]

The difference with the lower bounds previously calculated is that instead of omitting the term \( F^{N-2} \), we evaluate it at the lower limit of the integral. Following similar steps as before, we have that, as \( N \to \infty \):

\[
\Delta \tilde{R}' \sim -\alpha V(N) \frac{1}{(\alpha + 1)} \frac{e^\rho - \rho - 1}{e^\rho} - \frac{\rho^2}{e^\rho} \frac{1}{(2\alpha + 1)}
\]

\[
\tilde{R}'_c \sim -\frac{1}{e^\rho} \frac{1}{2\alpha + 1} V(N).
\]

We thus conclude that, in the limit \( N \to \infty \):

\[
\frac{R}{R_c} - 1 \leq \frac{\Delta \tilde{R}'}{\tilde{R}'_c} \sim \frac{-\alpha \left( \frac{1}{\alpha + 1} \frac{e^\rho - 1 - \rho}{e^\rho} - \frac{\rho^2}{e^\rho} \frac{1}{2\alpha + 1} \right)}{\frac{1}{e^\rho} \frac{1}{2\alpha + 1}} < \infty,
\]

for all \( \rho \in \mathbb{R} \). Finally, we note that the optimal quantile converges to 1 at a rate of \( \rho/N \), so the upper pooling information structure we are studying converges to the optimal one for some \( \rho \). This proves that the limit \( z \) does not converge.
8 Appendix with Extensions

8.1 Manual Bidding

We now examine a model of manual bidding. We suppose that the seller needs to elicit the preference $y_i$ and then present the bidder a signal as a function of the realized match quality (based on the reported preference). The buyer can then freely choose his bid based on the information presented to him. As before, the buyers bid in a second-price auction without reserve, and the seller’s objective is to maximize revenue. While the optimal information structure may not be incentive compatible, we provide another information structure that is incentive compatible and generates a revenue approximately optimal when the number of bidders is large.

8.1.1 Model of Manual Bidding

To analyze the incentives of bidders to truthfully report their preferences to the publisher, we need to make further assumptions about the payoff environment. In particular, we will assume that the valuation function is as in (13). As before, attributes and preferences take values in $\{-1, 1\}$, are uniformly and independently distributed across components and bidders.

As with autobidding, a reporting strategy for bidder $i$ is denoted by:

$$\tilde{y}_i : \{-1, 1\}^J \to \Delta\{-1, 1\}^J.$$  

Given the reported preferences, the seller discloses to the bidder a signal $s(\tilde{v}_i)$, where

$$\tilde{v}_i \triangleq u\left(\frac{1}{\sqrt{J}} \sum_{j=1}^{J} \tilde{y}_{ij}(y_{ij})x_j\right).$$

That is, $\tilde{v}_i$ is the valuation of an advertiser with preferences $\tilde{y}_i$.

We denote by $\hat{w}_i$ the expected value of $v_i$ conditional on $s(\tilde{v}_i)$:

$$\hat{w}_i \triangleq \mathbb{E}[v_i \mid s(\tilde{v}_i), y_i].$$

and by $\hat{G}_i$ the distribution of expected valuations. As before, $w_i$ denotes the expected valuation when a bidder reports truthfully, and $G_i$ denotes the respective distribution. The seller’s problem is then to find an information structure that solves:

$$R \triangleq \max_{\{s: \mathbb{R} \to \Delta\mathbb{R}\}} \mathbb{E}[w(2)];$$  

$$\mathbb{E}[\max\{w_i - \max\{w_j\}_{j \neq i}, 0\}] \geq \mathbb{E}[\max\{\hat{w}_i - \max\{w_j\}_{j \neq i}, 0\}], \text{ for all } \tilde{y}_i$$  

(20)
Here the constraint is the incentive compatibility constraint: the expected bidder surplus is weakly larger when reporting truthfully than any other reporting strategy. Here we require Bayesian incentive compatibility to keep the notation more compact, but this will play no role.

### 8.1.2 Incentive Compatibility Under Manual Bidding

Since the preferences and attributes are symmetrically distributed, a sufficient statistic for the bidder’s strategy is the fraction of preferences truthfully reported:

\[ \rho_i \triangleq \sum_{j=1}^{J} \tilde{y}_{ij} y_{ij}. \]

Note that \( \rho_i \) is the correlation between the reported preference and the real preference. In other words, for any reporting strategy \( \tilde{y}_i, \tilde{y}'_i \) satisfying \( \rho_i = \rho'_i \), the induced distribution of expected valuations will be the same. In this case, \( \tilde{y}_i \) and \( \tilde{y}'_i \) are equivalent from the perspective of the information generated for the bidder. If \( \rho_i = 1 \) then the preference has been correctly reported; if \( \rho_i = 0 \) then half of all preference components have been misreported; if \( \rho_i = -1 \) then every preference component has been incorrectly reported.

Since \( \tilde{v}_i \) is a noise signal about \( v_i \), a natural conjecture is that bidders will want to report their preference truthfully. However, misreporting every preference component (i.e., \( \rho_i = -1 \)) may sometimes be profitable. In this case, the bidder will observe signal \( s(u(-m_i)) \) (instead of \( s(u(m_i)) \)). In this case, the distribution of expected valuations will be:

\[
\tilde{G}^{-1}(t) \triangleq \begin{cases} 
F^{-1}(t) & \text{for all } t \geq 1 - q_N^* \\
\int_{a_{-q_N^*}}^{1-q_N^*} F^{-1}(q) dq & \text{for all } t < 1 - q_N^*
\end{cases}
\]

with \( q_N^* \) defined in (5). This expression is akin to (6), but the pooling section is at the lower quantile instead of the upper quantile.

We begin by establishing that the only relevant incentive constraints are those induced by reporting the exact opposite preference.

**Lemma 1 (Informativeness of Signals)**

Let \( s \) be the optimal information structure. The generated signal for every \( \rho_i \in [0,1) \) is less informative than the signal generated for \( \rho_i = 1 \). The generated signal for every \( \rho_i \in [-1,0) \) is less informative than the signal generated for \( \rho_i = -1 \).
Proof of Lemma 1. We prove the statement for $\rho_i \in [0, 1]$. We denote by $\tilde{F}$ the distribution of the expected value of $v_i$ conditional on $\tilde{v}_i$ (i.e., $\mathbb{E}[v_i | \tilde{v}_i] \sim \tilde{F}$). We denote by $\tilde{G}$ the distribution of $\tilde{w}_i$, which is given by:

$$
\tilde{G}^{-1}(t) = \begin{cases} 
\tilde{F}^{-1}(t) & \text{for all } t < q_N^* \\
\int_{q_N^*}^1 \frac{\tilde{F}^{-1}(q) dq}{1-q_N^*} & \text{for all } t \in [q_N^*, 1)
\end{cases}
$$

with $q_N^*$ defined in (5) (this is simply (6) but replacing $F$ with $\tilde{F}$).

We first observe that $\tilde{v}_i$ is a noisy signal of $v_i$. Hence, $\tilde{F}$ is a mean preserving contraction of $F$, which is equivalent to stating that:

$$
\int_0^t F^{-1}(q) dq \leq \int_0^t \tilde{F}^{-1}(q) dq
$$

for all $t$ with equality for $t = 1$. We thus have that:

$$
\int_0^t G^{-1}(q) dq \leq \int_0^t \tilde{G}^{-1}(q) dq
$$

for all $t \leq q_N^*$ (in this range $\tilde{G}^{-1}(t) = \tilde{F}^{-1}(t)$ and $G^{-1}(t) = F^{-1}(t)$). Since

$$
\int_0^1 \tilde{G}^{-1}(q) dq = \int_0^1 G^{-1}(q) dq
$$

and $\tilde{G}^{-1}(t), G^{-1}(t)$ are constant for $t > q_N^*$ we must have that:

$$
\int_{q_N^*}^1 \frac{\tilde{F}^{-1}(q) dq}{1-q_N^*} \leq \int_{q_N^*}^1 \frac{F^{-1}(q) dq}{1-q_N^*}
$$

and

$$
\int_0^t \tilde{G}^{-1}(q) dq \leq \int_0^t G^{-1}(q) dq
$$

for all $t \geq q_N^*$, with equality only for $t = 1$. This proves the result for $\rho_i \in [0, 1]$.

The case $\rho_i \in [-1, 0]$ can be proved in a completely analogous way except for the fact that the distribution of expected valuations under signals $s(-m_i)$ is:

$$
\tilde{G}^{-1}(t) = \begin{cases} 
\tilde{F}^{-1}(t) & \text{for all } t \geq 1 - q_N^* \\
\int_{q_N^*}^1 \frac{\tilde{F}^{-1}(q) dq}{q_N^*} & \text{for all } t < 1 - q_N^*
\end{cases}
$$

and analogously for $s(\tilde{v}_i)$. This is because the signals generate pooling for low quantiles instead of high quantiles. The rest of the proof proceed in a completely analogous way. ■
The lemma shows that there are two strategies that lead to the most informative signals: reporting truthfully and reporting the opposite preference.

It is possible to verify that reporting truthfully may fail to be an equilibrium for some model parameters (in particular, for some $N, u$). The reason is that by misreporting his preference, an advertiser may gain better information about his own valuation when this is high. This is because when he misreports his preferences the distribution of expected valuations is fully informative at the top but has a pooling section at the bottom. Depending on the shape of $u$ and the number of advertisers, reporting truthfully may fail to be an equilibrium (in fact, it will not be an equilibrium unless $u$ is sufficiently concave).

### 8.1.3 Approximately Optimal Mechanism

Our analysis here pursues a limited objective. We do not attempt to characterize the set of feasible and incentive compatible mechanism which would be an interesting avenue to pursue. Rather, we suggest a small modification of the optimal mechanism of Proposition 1 and show that the modified mechanism is indeed incentive compatible. Moreover, as the number of bidders becomes large, the revenue of the modified mechanism approximates the mechanism of the optimal information structure.

While truthtelling will not be an equilibrium for every $N, u$ there is a class of information structures that can appropriately balance revenue-maximization and incentive compatibility when there is a large number of bidders. Consider the following information structure.

$$s(v_i) = \begin{cases} 
\mathbb{E}[v_j | F(v_j) \leq 1 - q] & \text{if } F(v_j) \leq 1 - q_N^* \\
v_j & \text{if } 1 - q_N^* \leq F(v_j) \leq q_N^* \\
\mathbb{E}[v_j | F(v_j) \geq q] & \text{if } F(v_j) \geq q_N^* 
\end{cases}$$

(21)

with $q_N^*$ defined in (5). In other words, the information structure is as the optimal information structure (4) but in addition to the pooling at the top, there is pooling at the bottom.

**Proposition 6 (Incentive Compatibility)**

*Under manual bidding and information structure (21), it is a dominant strategy for the advertiser to report his preference truthfully.*

**Proof of Proposition 6.** It is easy to check that Lemma 1 also holds for the two-sided pooling information structures. We now notice that $s(v_i) = s(-v_i)$, so reporting truthfully or the exact opposite
generates the same signal. Hence, reporting truthfully generates the most informative signal about an bidder’s own type, so this is a dominant strategy.

Under the optimal information structure, truthtelling might not be an equilibrium because bidders have an incentive to report the exact opposite of their preferences. Because the two-sided pooling information is symmetric – that is, there is the same pooling zone for high and low quantiles – the incentive to misreport the tastes disappears. In particular, reporting truthfully and reporting the exact opposite preference generates the same signal for the bidder.

Information structure (21) will not maximize revenue across all information structure, but the loses from having pooling at the bottom as the number bidders becomes large is negligible. This is because the probability that the valuation of the second-highest bidder is smaller than $F^{-1}(1 - q_N^*)$ converges to zero as $N$ becomes large. Note that there are two complementary effects by which this probability converges to 0. First, $(1 - q_N^*)$ converges to 0 as $N$ becomes large, so the probability that the valuation of any bidder is below this threshold converges to 0. Second, for any fixed quantile $q$ the probability that the valuation of the second-highest bidder is smaller than $F^{-1}(1 - q_N^*)$ converges to zero as $N$ becomes large. Hence, the revenue losses from having pooling at the bottom vanish due to both effects, and are expected to be small.

**Proposition 7 (Approximate Optimality)**

Under the two-sided pooling information structure the revenue converges to the one under the optimal information structure when the number of bidders grows large:

$$\lim_{N \to \infty} (\mathbb{E}[w(2)] - R) = 0$$

**Proof of Proposition 7.** We denote by $H$ (resp. $G$) the distribution of expected valuations induced by the two-sided pooling information structure (resp. optimal information structure) and write the difference as follows:

$$(\mathbb{E}[w(2)] - R) = \int_0^1 H^{-1}(q)S'(q)dq - \int_0^1 G^{-1}(q)S'(q)dq$$

Whenever $v_i \geq G^{-1}(1 - q)$ the two-sided pooling information structure and the optimal information structures are the same, so $H^{-1}(q) = F^{-1}(q)$. We thus have that,

$$(\mathbb{E}[w(2)] - R) = \int_0^{1-q} (H^{-1}(q) - G^{-1}(q))S'(q)dq.$$ 

In the limit $N \to \infty$, we have that $q \to 0$ and $S'(q) \to 0$ for every fixed $q$. Since $H^{-1}(q)$ and $G^{-1}(q)$ are bounded in the interval $[0, 1 - q]$, we must have that

$$\lim_{N \to \infty} (\mathbb{E}[w(2)] - R) = 0$$
which establishes the result. ■

The proposition states that the two-sided pooling information structure can effectively balance the trade-off between incentive compatibility and revenue maximization when there is a large number of bidders. This result is specially relevant when the distribution of valuations has a thick tail because in this case the gains from using the optimal information structure (instead of complete information) do not vanish. Finding the optimal incentive compatible information structure is an interesting open question that is left for future work.

8.2 Reserve Price

We now assume that the second-price auction has reserve price $r > 0$. As before, we can assume that the bid is equal to an advertiser’s expected valuation $w_i$.

Proposition 8 (Optimal Information Structure)

There are quantiles $q_1, q_2, q_3$ such that an optimal information structure is given by:

$$s(v_i) = \begin{cases} 
  s_1 & \text{if } F(v_i) \in [0, q_1) \\
  s_2 & \text{if } F(v_i) \in [q_1, q_2) \\
  v_i & \text{if } F(v_i) \in [q_2, q_3] \\
  s_3 & \text{if } F(v_i) \in [q_3, 1] 
\end{cases} \quad (22)$$

The intervals may be degenerate, and $E[v_i \mid s = s_2] = r$.

Proof. The expected revenue is given by:

$$R = P\{w_{(1)} \geq r \text{ and } w_{(2)} < r\}r + E[w_{(2)}1_{w_{(2)} \geq r}].$$

For any distribution of expected valuations $G$, the distribution of $w_{(2)}$ is given by $(nG^{n-1}(x)(1 - G(x)) + G^n(x))$ and

$$P\{w_{(1)} \geq r \text{ and } w_{(2)} \leq r\} = nG^{n-1}(r)(1 - G(r)),$$

so we can write the revenue as follows:

$$R = 1 - rG(r)^n - \int_r^1 (nG^{n-1}(x)(1 - G(x)) + G^n(x))dx.$$
Let \( \hat{G} \) be the distribution of expected valuations induced by the optimal information structure. Let \( \hat{\psi}_r = \hat{G}(r) \) and \( \hat{Q} = \hat{G}^{-1} \). Then, we can write it as follows

\[
R = 1 - \hat{\psi}_r \hat{Q}(\hat{\psi}_r) - \int_{\hat{\psi}_r}^1 (nq^{n-1}(1 - q) - q^n)dQ(q)
\]

The majorization constraint states that for all \( x \in [0, 1] \):

\[
\int_x^1 \hat{Q}(q) dq \leq \int_x^1 F^{-1}(q) dq.
\]

Let \( \psi \in [0, 1] \) be such that,

\[
\int_{\hat{\psi}_r}^1 \hat{Q}(q) dq = \int_{\hat{\psi}_r}^1 1_{\{q \geq \psi\}} F^{-1}(q) + 1_{\{q < \psi\}} \hat{Q}(\hat{\psi}_r) dq
\]

(23)

Let \( \bar{F}^{-1} : [\hat{\psi}_r, 1] \to [0, 1] \) be defined as follows:

\[
\bar{F}^{-1}(q) = 1_{\{q \geq \psi\}} F^{-1}(q) + 1_{\{q < \psi\}} \hat{Q}(\hat{\psi}_r)
\]

Consider the following maximization problem:

\[
Q_r \triangleq \arg \max_{Q: [\hat{\psi}_r, 1] \to [\hat{Q}(\hat{\psi}_r), 1]} - \int_{\hat{\psi}_r}^1 (nq^{n-1}(1 - q) - q^n) dQ(q)
\]

subject to: for all \( q \geq \hat{\psi}_r \),

\[
\int_x^1 Q(q) dq \leq \int_x^1 \bar{F}^{-1}(q) dq \text{ with equality when } x = \hat{\psi}_r.
\]

(25)

We first note that \( \hat{Q}(x) = Q_r(x) \). To verify this, consider the following function:

\[
\hat{Q}'(q) = \begin{cases} 
\hat{Q}(q) & \text{if } q \leq \hat{\psi}_r \\
Q_r(q) & \text{otherwise}
\end{cases}
\]

The revenue under \( \hat{Q}' \) is given by:

\[
R = 1 - \hat{\psi}_r \hat{Q}'(\hat{\psi}_r) - \int_{\hat{\psi}_r}^1 (nq^{n-1}(1 - q) - q^n)d\hat{Q}'(q)
\]

\[
= 1 - \hat{\psi}_r \hat{Q}(\hat{\psi}_r) - \int_{\hat{\psi}_r}^1 (nq^{n-1}(1 - q) - q^n)dQ_r(q)
\]

\[
\geq 1 - \hat{\psi}_r \hat{Q}(\hat{\psi}_r) - \int_{\hat{\psi}_r}^1 (nq^{n-1}(1 - q) - q^n)d\hat{Q}(q)
\]
where the last term is the revenue under $\hat{Q}$. Hence, $\hat{Q}'$ is optimal. We also note that:
\[
\int_{x}^{1} \hat{Q}'(q)dq \leq \int_{x}^{1} Q(q)dq,
\]
with equality for $x \leq \hat{q}_r$. Hence, $\hat{Q}'$ satisfies the majorization constraint. The inequality follows from the fact that $Q_r$ satisfies (25) and $\psi$ is such that (23) is satisfied.

We now note that (24) is the same problem that we solved to prove Propositions 1 but restricted to $[\hat{q}_r, 1]$. Hence, $Q_r(q)$ restricted to $[\hat{q}_r, 1]$ is
\[
Q_r(q) = \begin{cases} 
\bar{F}^{-1}(q) & \text{if } q \in [\hat{q}_r, q^*_N] \\
\mathbb{E}[v_j \mid F(v_j) \geq q] & q \in [q^*_N, 1) \\
1 & q = 1
\end{cases}
\]
for some $q^*_N$. By definition, $\hat{Q}(q) < r$ for all $q < \hat{q}_r$, so the expected valuation induced by these quantiles is less than the reserve price. Hence, the distribution of expected valuations below this quantile is irrelevant. Finally, we note that the following information structure generates this distribution of expected valuations:
\[
s(v_i) = \begin{cases} 
s_1 & \text{if } F(v_i) \in [0, \hat{q}_r) \\
s_2 & \text{if } F(v_i) \in [\hat{q}_r, \psi) \\
v_i & \text{if } F(v_i) \in [\psi, q^*] \\
s_3 & \text{if } F(v_i) \in [q^*, 1]
\end{cases}
\]
It is straightforward that information structure (22) generates the distribution of expected valuations (where we just replaced the specific quantiles with generic variables $q_1, q_2, q_3$).

The proposition shows that the information structure has three pooling intervals and one interval of full disclosure. The first interval consists of bidders who know their valuation is below $r$ so they do not buy the good (i.e., interval $[0, q_1]$). The second interval consists of the bidders whose conditional expected valuation is $r$, so they buy the good. The third interval are the bidders who know their valuation. The fourth interval is the bidders who know their valuation is the highest possible. The last two intervals are the same as in the case without reserve price.